Abstract: Robust control is challenging to achieve for air-breathing missiles operating in a high Mach number regime, such as at high supersonic speeds ($M > 3$). The challenge arises because of strong couplings, significant non-linearities and large uncertainties in the aerodynamics and propulsion system. The feasibility of achieving robust control in such applications is strongly linked to the development of an appropriate control design structure. The purpose of this paper is to illustrate that in order to stabilise a highly unstable airframe and achieve the required performance, a hybrid of two control schemes may be used to achieve best results. A state feedback linear quadratic regulator is used to stabilise the plant and a forward path $H_{\infty}$ optimal controller is used to achieve the required performance and robustness. We also highlight the complementary attributes of the two control schemes that together can generate a more robust controller; LQR is used since it can achieve good gain and phase margins, whereas, the $H_{\infty}$ control method is better equipped to deal with uncertainties.

Keywords: Missile, H-Infinity, Linear Quadratic Regulator, LQR, Optimal Control, Asymmetric

1. INTRODUCTION

Classical single-input, single-output (SISO) design techniques known for their intuitive nature have been used in the development of missile autopilots for decades. However, as system complexities such as high non-linear missile characteristics and strong airframe/propulsion system couplings arise, it often becomes very cumbersome to design controllers using such methods. It is also well known that multiple-input, multiple-output (MIMO) systems are handled better with optimal multi-variable control methods. The Linear, Quadratic and Gaussian (LQG) (Anderson and Moore (1989)) with Linear Quadratic Regulators (LQR) is one such method. These techniques have been around since the 1960s and are widely used in academia and industry alike. LQG uses white noise to approximate the model uncertainties and disturbances to the system, which in practice may not be very meaningful and can even be too conservative at times. However, LQR (assuming all the states are available) provides an optimal controller with guaranteed stability and good phase and gain margins (Safonov and Athans (1976)).

Therefore, in this paper, we combine two control methods to capitalize on their strengths. First we describe the liner time-invariant state space model of the missile airframe. Then, we use a two loop control structure to design a LQR controller on the inner loop and a mixed-sensitivity $H_{\infty}$ optimal controller on the outer loop. Autopilot transient responses and controller effort are then shown and discussed.

2. MISSILE DYNAMICS

The developed model is a 180° Bank-To-Turn (BTT) missile, as this form of steering aligns best with the incidence constraints of the air-breathing motor; and also BTT steering can help
keep aerodynamic cross-coupling (caused by the asymmetric configuration of the airframe) to a lower level than if the steering were skid-to-turn (STT). The linear airframe dynamics of the missile are described by the state space equations (1).

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

We ignore gravity and the state vector \( x \) is selected as \([p \ q \ r \ w \ v]^T\), the control input vector \( u \) as \([\xi \ \eta \ \zeta]^T\), the plant output vector \( y \) as \([p \ q \ r \ a_y \ a_z]^T\) and the matrices \( A_{5 \times 5}, B_{5 \times 3}, C_{5 \times 5} \) and \( D_{5 \times 3} \) are as follows:

\[
A = \begin{bmatrix}
    l_p & l_q & l_r & l_w & l_v \\
    m_p & m_q & m_r & m_w & m_v \\
    n_p & n_q & n_r & n_w & n_v \\
    y_p + \alpha U & y_q & (y_r - U) & y_w & y_v \\
    z_p - \beta U & (z_q + U) & z_r & z_w & z_v
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
    l_\xi & l_\eta & l_\zeta \\
    m_\xi & m_\eta & m_\zeta \\
    n_\xi & n_\eta & n_\zeta \\
    \eta_\xi & \eta_\eta & \eta_\zeta \\
    z_\xi & z_\eta & z_\zeta
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
    1 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & y_w & y_v \\
    0 & 0 & 0 & z_w & z_v
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    y_\eta & y_\eta & y_\eta & z_\xi & z_\eta \\
    z_\xi & z_\eta & z_\zeta & z_\xi & z_\eta
\end{bmatrix}
\]

\( l_i, m_i \) and \( n_i \) are partial derivatives of the roll, pitch and yaw aerodynamic angular accelerations with respect to the appropriate \( i \) about the respective axis. \( y_i \) and \( z_i \) are the partial derivatives of aerodynamic translational acceleration with respect to the appropriate \( i \) along the respective axis. \( U \) is the forward velocity and \( \alpha \) is angle of attack and \( \beta \) the sideslip angle.

3. AUTOPILOT DESIGN

The outputs being controlled are the roll angle \( \phi \) which is acquired by integrating the roll rate \( p \), the acceleration in y-direction and z-direction, \( a_y \) and \( a_z \), respectively.

The nature of the airframe happens to be inherently unstable. In our approach we first stabilise the airframe using LQR, which forms the inner closed loop system. Once stability is achieved with good margins, we then solve the optimisation problem by using the mixed-sensitivity optimal control method to get the desired tracking performance. LQR generates a static gain matrix \( K_s \) and this is ideal because we want to achieve robust stability without increasing the order of the inner-closed loop system. Figure (3) shows the isolated inner loop, which is equivalent to \( G_s \) in figure (5).

The mixed-sensitivity \( H_{\infty} \) optimal control method is chosen for performance because it has the framework to explicitly take uncertainties into account, which the traditional LQG method lacks (S. Skogestad (1988)). The reference tracking autopilot topology with the \( H_{\infty} \) controller in the forward path of the outer loop and a state-feedback LQR inner controller is shown in figure (2).

Fig. 1. Missile Airframe (with Body Axes superimposed)

\[
\frac{dx}{dt} = A x(t) + B u(t) \\
y(t) = C x(t) + D u(t)
\]

3.1 Linear Quadratic Regulator

The LQR controller is generated by minimizing the cost function \( J \) shown in (2). Detailed derivations can be found in Zhou and Doyle (1998), Skogestad and Postlethwaite (1996). Proofs for stability are found in Safonov and Athans (1976) and Anderson and Moore (1989).

\[
J = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) \ dt
\]

\( Q \) and \( R \) are both positive definite weighting matrices chosen by the designer. Therefore, the controller generated is optimal only to the chosen weights.

\[
\begin{align*}
-1 & \quad u \\
G & \quad y
\end{align*}
\]

Fig. 2. Control Topology

\[
K_s \quad u \\
G \quad y
\]

Fig. 3. LQR Regulator

The control law \( u(t) = -K_s x(t) \), where,

\[
K_s = R^{-1} B^T X
\]

and \( X = X^T \geq 0 \) is the unique solution to the Ricatti equation (4). Doyle et al. (1989)

\[
A^T X + X A - X B R^{-1} X + Q = 0
\]

It is known that for the dynamic system described in (1), given that \( (A, B) \) is stabilisable and \( (C, A) \) is detectable, then the solution \( X \) to the ricatti equation (4), which minimises the cost function (2) is always stable. (Doyle et al. 1989): lemma 3 and Cimen (2008): lemma 1)

3.2 \( H_{\infty} \) optimal control

The \( H_{\infty} \) control optimization problem was developed by Zames (1981) and further work done by Doyle et al. (1989).
Unlike, LQR where the weighting matrices essentially apply a constant upper gain limit across the frequency response, the $H_\infty$ optimal control method allows the designer to shape the frequency response with transfer functions, making it a somewhat less conservative and more tailored approach.

The system described by (1) with the control topology shown in figure 2 (which can be recast to the general control configuration (Figure 4), so that it complies with the $H_\infty$ framework. Figure (4) is described by (5).

$$
\begin{bmatrix}
 z \\
 w
\end{bmatrix} = P(s) \begin{bmatrix}
 u \\
 w
\end{bmatrix} = \begin{bmatrix}
 P_{11}(s) & P_{12}(s) \\
 P_{21}(s) & P_{22}(s)
\end{bmatrix} \begin{bmatrix}
 u \\
 w
\end{bmatrix}$$

$$u = K_p(s)v$$

Where, $P(s)$ is the state-space realisation of the generalised plant, which consists of the stable closed loop LQR plant $G_z(s)$ and the performance and control effort weights, $W_z(s)$ and $W_u(s)$, respectively. Topology of $P(s)$ is shown in figure (5).

"The $H_\infty$ optimal control problem solves for all stabilising controllers $K_p(s)$ which minimises $\|F_1(P(s), K_p(s))\|_\infty$." (Zhou and Doyle (1998))

Solving for the optimal controller is numerically and theoretically complicated and the characteristics of optimal controllers are sometimes undesired (higher order). Therefore a sub-optimal solution can be sought after.

"Given $\gamma > 0$, find all admissible controllers $K_p(s)$, if there are any, such that $\|F_1(P(s), K_p(s))\|_\infty < \gamma$." (Zhou and Doyle (1998))

Detailed solution to the $H_\infty$ optimal control problem can be found in Doyle et al. (1989) and further summaries in Skogestad and Postlethwaite (1996).

The design approach taken to solve the $H_\infty$ norm of the plant is called the mixed-sensitivity optimal control.

$$\|F_1(P, K_p)\|_\infty = \left\| \begin{bmatrix} W_z S & W_u K_p S \end{bmatrix} \right\|_\infty < \gamma$$

Where, the sensitivity function $S = (1 + G_z K_p)^{-1}$, $W_z$ is the performance weight and $W_u$ the control weighting function. $W_z$ and $W_u$ are both stable proper transfer functions.

$S(j\omega)$ is shaped as a high pass filter so that the low-frequency steady state error is minimised to get the desired tracking performance. Therefore, $W_z$ is designed so that the inequality $\|W_z S\|_\infty \leq 1$ is met.

$K_p S(j\omega)$ is shaped to be a low pass filter to minimise the cost of the control effort and reject the high-frequency sensor noise. Therefore, $W_u$ is designed so that the inequality $\|W_u K_p S\|_\infty \leq 1$ is met (Skogestad and Postlethwaite (1996)). Therefore, for a physically realisable solution, the design metric $\gamma$ has to obey the inequality given in (8).

$$\gamma < 1$$

Using the weights from (9), we achieve a $\gamma$ of 0.7585. Figure (6) is the response to the plant described by (1) when the desired input pitch acceleration $a_z = 100 \text{ m/s}^2$ and figure (7) the corresponding controller effort. The plant is allowed the first 5 seconds to settle to 0 from any initial conditions the plant may be on due to trim conditions. For a step input at $t = 5s$ the missile reference tracks with a transient response under approximately 1 second with no over shoot and no steady-state error. An elevator fin deflection of $0.5 \text{ m/s}$ is required, which is expected for a high acceleration demand, it however settles back to 0. The roll and yaw body rates ($\phi$ and $\psi$, respectively) are regulated, and the pitch rate ($q$) exhibits the expected steady-state offset for this manoeuvre. Aileron and rudder deflections show the missile counteracting the coupling effects of the missile.

The following results disregard the propulsion constraints that impose $180^\circ$ BTB steering and we have assumed STT steering.

Fig. 4. The General Configuration

Fig. 5. Mixed-Sensitivity Optimisation Problem

4. RESULTS AND DISCUSSION

The weights $Q, R, W_z$ and $W_u$ were designed according to specification and guidelines given in Skogestad and Postlethwaite (1996) and Zhou and Doyle (1998).

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0.01 \end{bmatrix}$$

$$R = I_{3 \times 3}$$

$$W_z(s) = diag\left[ \frac{s + 10}{2s + 0.1}, 0, 0, \frac{s + 4}{2s + 0.04}, \frac{2s + 0.04}{2s + 0.04} \right]$$

$$W_u(s) = \frac{s + 0.0031}{0.00001s + 94.25} I_{3 \times 3}$$
this is done only to illustrate the response of the hybrid controller to aerodynamic coupling. The response to a desired pure yaw acceleration step of $100 \text{ m/s}^2$ at $t = 5s$ is shown in figures (8) and (9). Characteristics of the response are somewhat similar to the pitch accelerations, but in the opposite direction. Since it is a non-axisymmetric missile, we see a steady-state fin deflection in both aileron and the rudder control to hold the yaw acceleration step demand.

Figures (10) and (11) are the responses when we demand a step of $100 \text{ m/s}^2$ at pitch and yaw accelerations ($a_z$ and $a_y$ respectively) simultaneously. The autopilot is able to meet both demands and the transient response times for $a_z$ and $a_y$ are still similar. The aileron fin angle reaches maximum deflection (i.e. saturation) briefly, but it still manages to regulate the roll angle $\phi$ to 0 rad. As expected, the elevator and rudder deflection angles are larger in amplitude than compared to those of the previous results but are within expected bounds ($+/- 0.35$ rad).

5. CONCLUSION

In conclusion, we tackle the issue of highly unstable airframe aero-propulsion dynamics by using state-feedback LQR to guarantee stability of the nominal plant. However, LQR has no explicit means of dealing with the varying parameters and uncertainties. Therefore, we incorporate a $H_\infty$ optimal controller for robust stability. The LQR controller is on the feedback loop providing disturbance rejection and stability with no increase in the number of closed-loop states and forms the inner loop of the overall control architecture. The $H_\infty$ controller forms the outer loop and is on the forward path, for reference tracking. The $H_\infty$ framework enables the designer to deal with uncertainties, making the overall two-loop hybrid controller a preferable choice to achieve robustness and stability.

REFERENCES


Fig. 6. Step Response to desired pitch acceleration command $A_{zd} = 100 \text{ m/s}^2$

Fig. 7. Controller effort to desired pitch acceleration command $A_{zd} = 100 \text{ m/s}^2$
Fig. 8. Step Response to desired yaw acceleration command $A_{yd} = 100 \, m/s^2$

Fig. 9. Controller effort to desired yaw acceleration command $A_{yd} = 100 \, m/s^2$

Fig. 10. Step Response to both desired pitch and yaw acceleration command $A_{zd} = A_{yd} = 100 \, m/s^2$

Fig. 11. Controller effort to both desired pitch and yaw acceleration command $A_{zd} = A_{yd} = 100 \, m/s^2$