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A New Approach to Optimal Capital Allocation for RORAC Maximization in Banks

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Abstract

We introduce a new model for optimal internal capital allocation, which would allow banks to maximize their Return on Risk-Adjusted Capital (RORAC) under regulatory and capital constraints. We extend the single period model of Buch et al. (2011) to a multi-period model and improve its forecasting accuracy by including the debt effect and Bayesian learning innovations. The empirical application shows that our model significantly improves the RORAC of a sample of banks listed in the S&P 500 index.

Keywords: Regulatory risk, Economic capital, Optimal capital allocation, Banks, Euler principle

JEL Codes: C11, D81, E22, G21, G28, G31

1. Introduction

It is well established in the extant literature that allocation of risk capital is critical for measuring the financial performance and risk-return optimization in banks (Buch et al., 2011). Risk capital is the equity capital required to safeguard the commitments to creditors, customers and contract counterparties (Erel et al., 2015). Banks need to maintain the required risk capital because they deal in credit-sensitive assets and liabilities. Hence, the creditworthiness of banks is crucial to their ability to write many types of contracts (Perold, 2005). For banks, efficient allocation of risk capital to sub-businesses is a key for ensuring a favorable risk-return trade-off. Risk capital is costly and therefore it must be efficiently allocated for investment (Erel et al., 2015) and for managing market, operational and credit risks (Embrechts et al., 2003; Alessandri and Drehmann, 2010; Breuer et al., 2010; Rosen and Saunders, 2010). We introduce a closed-form solution for optimal capital allocation in order to maximize the Return on Risk-adjusted Capital (RORAC). Our work extends Buch et al.'s (2011) single period model to a multi-period setting and makes several improvements. The newly developed multi-period risk capital allocation model is then empirically tested using data from a sample of banks listed in the S&P 500 index.

In theory, it is assumed that markets are frictionless and informationally efficient and a manger's task is to simply distribute the required funds to the divisions, treating each of them as autonomous entities. In practice, however, the capital allocation process in banks is quite challenging, as different business units compete with one another for securing capital allocation from the headquarters. Therefore, efficient allocation of capital is one of the key features of the capital budgeting process in banks (Stoughton and Zechner, 2007). The existing literature is mostly theoretical and provides only limited guidance on how to best allocate capital to multiple divisions in the presence of asymmetric information and capital constraints. As far as we are aware, no study has provided evidence of empirical performance of theoretically derived capital allocation models for banks with multiple divisions.

Previous research has used different notions of risk capital. Merton and Perold (1993) define risk capital as the smallest amount that can be invested to insure the value of the firm's net assets against a loss in value relative to a risk-free investment. They consider risk capital as the value of a put option on the US Treasury Bonds. They find that risk capital is uniquely determined by the net assets' riskiness and its full allocation can distort the true profitability of individual businesses. On the other hand, Perold (2005) uses the VaR and Adjusted Net Present Value (APV) as risk and profit measures respectively for capital allocation process assuming normally distributed returns. His model predicts that opaque financial firms will diversify across businesses which will reduce the cost of risk capital and create more profitable investment opportunities.

Despite a thorough scrutiny of technical aspects of risk capital allocation in the existing literature, the economic rationale for capital allocation is still being debated. Some argue that capital allocation is essential for monitoring performance of business units by controlling risks ex ante (Buch et al., 2011). Others argue that firms should not use capital allocation as it is a useless exercise (Gründl and Schmeiser, 2007). Stoughton and Zechner (2007) are the first to show that optimization of capital allocation pursued via Economic Value Added (EVA) at the headquarters is consistent with maximization of EVA of the allocated capital at the business unit level. They find that divisions with increasing (decreasing) outside managerial opportunities tend to have lower (higher) information asymmetry requiring lower (higher) capital charge and over (under)-invest in risky projects. Whilst their model incorporates asymmetric information and outside managerial opportunities, it relies on only very specific incremental VaR allocation rule. Therefore, their work has limited practical use (Buch et al., 2011).

There is research that examines risk capital allocation in the insurance industry where the aim is to minimize losses. For example, Myers and Read (2001) use the option pricing method to allocate capital based on the marginal contribution of each business unit to the overall option based default value. They find that marginal contribution to default risk vary across the business lines. While their model is restricted under continuous state lognormal and normal loss distribution, Sherris (2006) generalizes their model to make it applicable under any loss distribution assumptions under risk neutral probability. He shows how the management allocates capital to business lines using the insolvency exchange option value by ranking of outstanding claim payments in the event of insolvency. Kim and Hardy (2009) further extend Sherris's (2006) model by using limited liability of the shareholders under real world probability measure that is applied in an incomplete market setting. They argue that it is the owners of the company, not the line managers, who determine the capital allocation. Furman and Zitikis (2008) take a different allocation approach where the weighted risk capital allocation is based on weighted premium calculation principle that is well suited for insurance pricing. They calculate the weighted premium using weighted loss distribution derived from the weight function that is deterministic, non-negative and Borel-measurable. They find that their model can capture many risk capital allocation approaches, which are special cases of their weighted allocation model. Dhaene et al. (2012) propose a generalized risk capital allocation model that can account for a variety of risk measures and allocation principles. Their model attempts to minimize the weighted sums of the deviations of business unit's losses with respect to their allocated capital. Erel et al. (2015) improve Myers and Read's (2001) model by extending it to other types of financial firms. They use marginal default value, adjusted present value (APV) and credit quality measured by the ratio of firm's default option to default free value of its liabilities for allocating capital to business units for the maximization of firm value.

Capital allocation problem has also been a topic of interest in the realm of mathematical finance. For example, Denault (2001) introduces the allocation principle by using the gradient allocation (i.e., Euler allocation) method that allocates capital according to each entity's relative contribution to the overall risk. He tests his model under two different settings. One in which a firm is considered indivisible (coalitional game setting) and the other where a firm is considered divisible (fractional players game setting). He finds the his capital allocation model works better under fractional players game setting because it requires less restrictive conditions on the risk measure used. Tasche (2004) uses the same approach but extends Euler allocation such that more capital is allocated if a business unit generates above average risk-adjusted returns. On the other hand, Kalkbrener (2005) proposes an axiomatic approach to capital allocation with three principles: linear aggregation, diversification and continuity [1]. He argues that in contrast to Value-at-Risk (VaR), Expected Shortfall (ES) and risk measures based on standard deviations are subadditive and positively homogeneous. He shows that for these two classes, it is possible to derive explicit formulae, which specify linear, diversifying capital allocations.

Buch et al. (2011) argue that the Euler allocation method is suboptimal as it can lead to over expansion or reduction of business lines resulting in the overall decline of RORAC. They introduce an additional risk correction term that considers the risk interdependencies among the business lines and maximizes the overall RORAC without over-expansion or over-reduction as in Tasche (2004). However, Buch et al.'s (2011) model is limited to single period

[1] Kalkbrener (2005)'s three allocation axioms are:

- Linear aggregation: the risk capital of portfolio equals the sum of the risk capital of its sub-portfolios.
- Diversification: the risk capital of the sub-portfolio does not exceed the risk capital of the stand-alone portfolio.
- Continuity: small changes in the portfolio have limited effect on the risk capital of its sub-portfolio.

and only considers static expected profit and risk. Given these limitations, their model cannot be empirically tested in a multi-period setting.

We make several important contributions to the existing literature. First, we extend Buch et al.'s (2011) single period model to a multi-period model thus enhancing its practical utility. Second, we allow more capital to be allocated to those business units that have higher debt but are profitable. Third, we incorporate the influence of managerial decisions in capital allocation via the Bayesian learning process. Finally, we also provide an empirical test of our theoretical model using both risk and regulatory capital constraints.

In terms of the theory, our model minimizes the impact of information asymmetry. This is achieved through the Bayesian learning process that recalculates the optimal capital for allocation over time by including the accumulated difference between the expected profit before allocation and the actual profit after allocation. Our model also considers the debt effect, something that has been ignored in the extant literature. The debt effect recalculates the optimal capital that could be allocated to divisions with relatively higher proportion of debt. In terms of practical contributions, we are the first to offer empirical evidence of shareholder value effects of capital allocation by considering both risk and regulatory capital. Our paper has significant implications for both the existing theory and practice of capital allocation in banks. We show that our multi-period capital allocation model can be empirically applied while ensuring that the regulatory capital requirements are met. Additionally, we also examine our model's capital allocation efficiency using the required risk/economic capital which is calculated using the ES measure.

We empirically test our capital allocation model by using data of fourteen U.S. banks from the S&P500 index over the period 2006 to 2014. Our results show that the average RORAC improves by 0.305 percentage points per quarter when using regulatory capital (under Basel^{+} regime) and by 0.331 percentage points when risk capital is used (under Economic

Capital+ regime) [2]. Further, we find that improvement in the RORAC is greater for large banks. Overall, we show that our model improves the RORAC of the banks in our sample.

The rest of the paper is organized as follows. Section 2 explains our model development process. Section 3 explains the data and the empirical approach. Section 4 presents results, and Section 5 concludes.

2. Model Development

Our starting point is the model for optimal risk capital allocation proposed by Buch et al. (2011) that aims to maximise a firm's overall return-to-risk measured by RORAC.[3] They use the Euler principle to allocate risk capital according to each business unit's marginal RORAC. The marginal RORAC is the partial derivative of the overall RORAC with respect to each business unit's risk capital.[4] It describes the marginal effect of the business unit on the firm's overall RORAC. The decision rule is to expand or contract a business unit by allocating more or less risk capital. For example, a bank would expand a business unit by allocating more capital if its marginal RORAC before capital allocation is greater than the overall RORAC, until the marginal RORAC is the same as the overall RORAC. However, the model does not consider creation of new business units or closure of the existing ones.[5]

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[2] In our work, when we internally allocate regulatory or risk capital we call it Basel or Economic Capital regime, respectively. Furthermore, when we incorporate debt effect and the Bayesian learning process in our model, we call it Basel+ or Economic Capital+ regime, respectively.

[3] RORAC is the expected profit over the allocated risk-adjusted capital. Buch et al. (2011) argue that their model is also applicable when Risk-Adjusted Return on Capital (RAROC) and Risk-Adjusted Return on Risk Adjusted Capital (RARORAC) are used.

[4] The business unit in our work means a division. We use these two terms interchangeably throughout the paper.

[5] The headquarters calculates the optimal capital to be allocated based on the overall risk of the portfolio made up of different business units.

Buch et al.'s (2011) allocation of capital to sub-businesses is based on two assumptions. First, the risk measure should be a homogeneous function[6] in order to be differentiable. Second, managers have superior knowledge about the financial implications of expansion and contraction of their business units. Buch et al. (2011) acknowledge that their model is not a multi-period model since the expected profit, risk and capital are not time varying. Moreover, their model does not consider the debt effect and information asymmetry.

Our work extends Buch et al.'s (2011) single period model by relaxing their assumptions and by incorporating the debt effect to account for higher debt and the Bayesian learning process to minimise the errors arising from inaccuracies in estimating the next period's return on capital. Further, for calculating RORAC, we use both regulatory and risk capital. Our model calculates the optimal capital weights to be assigned to each business unit at time *t*. We then use these optimal capital weights at time *t* to calculate the corresponding RORAC in the next period ($t+\Delta t$) using the new ROC and total capital in this period ($t+\Delta t$) which are unknown at time *t* and reflect the forecasting uncertainty. Further, we do not allow any external capital injection and reallocate only the existing capital among the business units.

2.1. Optimal Capital Allocation Model

2.1.1. Time-Varying Risk Measurement

Definition 2.1. We follow Lönnbark (2013) and develop a time-varying risk measure by using one-step-ahead ES ρ_t.[7] Specifically, we use the absolute risk measure of the one-step-ahead ES by assuming that the expected profit is zero. This eliminates the possibility of negative

[6] A function f such that all points (x_1, ……, x_n) in its domain of definition and all real t>0, the equation $f(tx_1, \ldots, tx_n) = t^{\lambda} f(x_1, \ldots, x_n)$ holds, where λ is a real number (Kudryavtsev, 2002).

[7] ES is a measure of the average of all potential losses exceeding the VaR at a given confidence level. The Basel committee advocates using ES over VaR as a measure of risk (Basel Committee on Banking Supervision, 2013).

Economic Capital.

$$\rho_t[u_t^w] = \frac{\phi(\Phi_\alpha^{-1})}{\alpha} \times \underbrace{\sqrt{u_t^w \times \sigma_t^{\text{cov}} \times u_t^{w'}}}_{\text{Square root of weighted conditional covariance}} \quad (1)$$

N is the number of business units, $u_{\mathrm{i,t}}$ and $u_{i,t}^w$ $(= \frac{u_{i,t}}{\sum_{i=1}^{N} u_{i,t}})$ are the capital amount and weight of business unit *i* (where *i=1, ⋯, N*) at time *t*, respectively. u_t^w is a (*1* × *N*) vector with elements of capital weights $u_{i,t}^w$ of all business units at time *t*. $u_t^{w'}$ is a transpose of the vector u_t^w. $\rho_t[u_t^w]$ is the ES dependent on capital weight u_t^w with confidence interval $(1-\alpha)$ at time *t*. Φ_α^{-1} is an inverse of the cumulative distribution function (CDF) (i.e., quantile function) of normal distribution evaluated a the confidence level $(1-\alpha)$. We use 97.5% confidence level, which is the BIS suggested requirement for the ES. $\phi(\bullet)$ is the probability density function (PDF) of standard normal distribution. σ_t^{cov} is the square root of the conditional covariance at time *t* using all business units' (*i=1, ⋯, N*) return on capital (ROC) in a (*N* × *N*) matrix form.

Our one-step-ahead ES measure captures the time varying ROC and the capital to estimate risk in a dynamic manner. The weight feature of the realized covariance matrix allows estimating the dynamic risk arising not only from the correlation among the business units but also from their relative weights where each business unit's value itself is time varying. The model assumes normally distributed risk as shown in $\phi(\bullet)$ and Φ_α^{-1}.

2.1.2. Time-Varying Expected Profit Process

Definition 2.2. We assume the time-varying expected profit process follows a Stochastic Differential Equation (SDE):[8]

[8] The SDE is a time-varying version of the stochastic profit process used by Buch et al. (2011). However, its drift parameter is non-time varying.

$$\underbrace{Y_{i,t}[u_t]}_{\text{Profit}} = \underbrace{\mu_{i,t}[u_{i,t}]}_{\text{Expected Profit}} + \underbrace{\sigma_{i,t}[u_{i,t}]W_t}_{\text{Profit Fluctuation}} \tag{2}$$

The profit for business unit *i* at time *t* is its expected profit plus profit fluctuations, where both the drift μ and the diffusion σ are time varying. W_t is the standard Wiener process. All business units within a firm follow this SDE process. Instead of calculating the time-varying drift parameter $\mu_{i,t}[u_{i,t}]$ itself, we indirectly measure this through discretization as shown in our proof of lemma 2.3 of the appendix.

2.1.3. The Closed-Form Solution

We derive a closed-form solution for optimal internal capital allocation by implementing our time-varying risk measure (definition 2.1) and expected profit (definition 2.2) into the following equation (3) which equates marginal RORAC and overall RORAC.

$$\underbrace{r[\varepsilon_{i,t}^{w}|u_{i,t}^{w}]}_{\substack{\text{Marginal RORAC with}\\ \text{additional capital weight } \varepsilon_{i,t}^{w}}} = \underbrace{r[u_t^{w}]}_{\text{Overall RORAC}} \tag{3}$$

We consider weights of capital, $\varepsilon_{i,t}^{w}$ (additional capital weight) and $u_{i,t}^{w}$ (existing capital weight), while allowing the total capital to vary over time. In other words, our model reallocates the existing total capital amount among the business units to maximize the overall RORAC. The additional capital weight $\varepsilon_{i,t}^{w}$ is optimal when it satisfies equation (3) which maximizes bank's overall RORAC. Therefore, we express the $\varepsilon_{i,t}^{w}$ in equation (3) which includes the time-varying risk and expected profit measures. The derivation process is provided in our proof of lemma 2.3 of the appendix.

Lemma 2.3. We develop a closed from solution as follows:

$$\varepsilon_{i,t}^{w} = \frac{-a_{i,t} + \rho_t[u_t^{w}]}{\Lambda_t} \tag{4}$$

$\frac{d\rho_t[u_{i,t}^{w}]}{du_{i,t}^{w}} = a_{i,t}$ is the risk contribution of business unit *i* at time *t*, $\rho_t[u_t^{w}]$ is the time-varying risk measure (i.e., one-step ahead ES) using capital weight u_t^{w} as the input and Λ_t is the maximum eigenvalue of the Hessian matrix $\frac{d^2\rho_t[u_{i,t}^w]}{d^2u_{i,t}^w}$ with the business units' capital weights $u_{i,t}^w$.

In our model, the capital is allocated to (subtracted from) division *i* when the marginal RORAC of division *i* is greater (less) than the overall RORAC. The additional capital weight to business unit i, $\varepsilon_{i,t}^w$, which equates its marginal RORAC to the overall RORAC is considered as optimal since it maximizes the overall RORAC as shown in equation (3). The closed-form solution for the optimal capital weight $\varepsilon_{i,t}^w$ in equation (4) is deduced from this relationship in equation (3). The closed-form solution in equation (4) shows that the optimal capital allocation to a business unit *i* decreases with its risk contribution $a_{i,t}$ but increases with the overall risk $\rho_t[u_t^w]$ at time *t*. Furthermore, when the maximum directional strength Λ_t (i.e., the effect of the magnitude of capital allocation on the overall RORAC) from additional capital input is large (small), less (more) capital is allocated. Another innovation we make in our model is that we consider ex-ante ROC. We estimate the optimal capital weight for each division for the next period under the conditions when the ROC and total capital are unknown.

Definition 2.4. We add the self-evolving existing capital weight $u_{i,t}^w$ to the additional capital weight $\varepsilon_{i,t}^w$ derived from our closed-form solution in lemma 2.3 to produce the new capital weight $u_{new,i,t+1}^w$.

$$u_{new,i,t+1}^w = \varepsilon_{i,t}^w + u_{i,t}^w \tag{5}$$

Equation (5) shows that we add the optimal capital weight $\varepsilon_{i,t}^w$ from equation (4) to the existing capital weight $u_{i,t}^w$ to derive the new capital weight $u_{new,i,t+1}^w$ for division *i* at time *t+1*. We use the new capital weight $u_{new,i,t+1}^w$ to calculate the overall RORAC at time *t+1* when the ROC and total capital are unknown.

2.2. Optimal Allocation Under Constraints

The above closed form solution (equation 5) has two shortcomings. First, the additional capital weight $\varepsilon_{i,t}^{w}$ does not necessarily sum to zero. This implies that at every iteration, there will be either capital injection or reduction. Second, the new capital weight $u_{new,i,t+1}^{w}$ can be negative which would suggest a short position in one or more business units. However, this is not possible in practice. With an aim to overcome these challenges, we optimize a firm's return-to-risk ratio by reallocating the existing capital while not allowing negative capital for any business unit. We re-inject any negative capital values back into our model within the same allocation period until we derive non-negative capital values. To ensure no change in the total capital, we require the capital weights $u_{new,i,t+1}^{w}$ to always sum up to one.[9]

2.3. Debt Effect

In our model, we consider risk that may be incurred due to excessive debt at the divisional level. For this purpose, we calculate the relative debt amount of division i, $D_{i,t}$, compared to the firm's total debt D_t at time *t*, which is the debt weight of division *i* at time *t*, $d_{i,t}^{w} = \frac{D_{i,t}}{D_t}$. We add this to the new capital weights in equation (5) to produce the updated new capital weight with debt effect, $u_{new,i,t+1}^{w}$.

$$\underbrace{u_{new,i,t+1}^{w}}_{\substack{\text{new optimal capial weight} \\ \text{with debt effect}}} = \underbrace{\varepsilon_{i,t}^{w}}_{\substack{\text{additional} \\ \text{optimal capital weight}}} + \underbrace{u_{i,t}^{w}}_{\substack{\text{existing} \\ \text{capital weight}}} + \underbrace{d_{i,t}^{w}}_{\text{debt effect}} \quad (6)$$

The debt effect enables our model to provide more capital to a division with relatively high debt compared to the overall firm. Thus, our model considers the risk arising from excessive debt at the divisional level.

[9] Please see the illustration of definition 2.4 using capital constraints in the appendix for details.

2.4. Bayesian Learning

In order to improve the forecasting accuracy, we incorporate the Bayesian learning process in our model. It minimizes forecasting errors that arise from the information asymmetry between the 'belief' about the expected profit after capital allocation and the actual profit before the capital allocation. As the ROC of next period is unknown, this may cause a significant forecasting error leading to inaccurate optimal capital weights for RORAC maximization. The forecasting error may increase significantly if there is a large information asymmetry between the 'belief' and 'actual' ROC of the next period that Buch et al. (2011) have not considered. Our Bayesian learning process considers historic ROCs' over time to update the 'belief' about the next period's ROCs and the corresponding expected profits after capital allocation. As a result, the forecasting error arising from the difference between the 'belief' and 'actual' profits for the next period is minimized through this accumulated learning process. This leads to an enhanced optimal capital allocation for RORAC maximization.

Lemma 2.5. The closed-form solution for optimal capital weight reflecting the Bayesian learning process is as follows.

$$\varepsilon_{new,i,t}^{w} = \frac{-a_{i,t} + \rho_t[u_t^w] - \rho_t \int_0^t \sum_{i=1}^{N}((u_{new,t-dt}^{w} \times ROC_t) - (u_{new,t}^{w} \times ROC_t))du}{\Lambda_t} \tag{7}$$

$\varepsilon_{new,i,t}^{w}$ is the new optimal capital weight of division *i* at time *t* reflecting the Bayesian learning process. The divisional capital weights before and after capital allocation are $u_{new,t-dt}^{w}$ and $u_{new,t}^{w}$, respectively. The integral part of equation (7) accumulates the difference between the actual and expected profit over time, scaled by ES risk measure ρ_t. The proof for this definition is provided in our proof of lemma 2.5 of the appendix.

3. Data and Method

We use data for selected U.S. banks from the S&P 500 index as of December 31, 2014 based on the following criteria:

(1) Bank has more than one business unit

(2) Bank has published financial statements with information regarding the regulatory capital

(3) Bank has published financial statements which provide asset and net income data at the business unit level

Fourteen U.S. banks met these criteria. Quarterly data were collected from Bloomberg, starting with the most recent quarter for each bank going back to the year when the business units of each bank as of December 31, 2014 were still operational.

3.1. Risk and Regulatory Capital

3.1.1. Regulatory Capital

Regulatory capital that we consider in our paper is the capital required by banks to maintain a minimum capital adequacy ratio of 8% between the total capital (i.e. Tier 1 capital + Tier 2 capital) and risk-weighted assets (Basel Committee on Banking Supervision, 1988). A revised capital adequacy framework in June 2004 was released under Basel II which retains the capital adequacy ratio of 8%, however, three new pillars were added (Basel Committee on Banking Supervision, 2015). The first pillar considered operational risk in addition to the credit and market risks in quantifying risk more precisely for the purpose of calculating the minimum capital requirement. The second pillar requires banks to use internal risk assessment measures that includes pension risk and goodwill risk. The third pillar requires banks to report the risk and capital structure of banks (Wernz, 2014). Basel III released in 2010 revises and strengthens the three pillars laid out in Basel II. It reflects the Basel Committee's response to the global financial crisis by addressing a number of shortcomings in the pre-crisis regulatory framework. Basel III provides a foundation for a resilient banking system to avoid the systemic vulnerabilities. It aims at improving the quality of bank regulatory capital by placing a greater focus on going-concern loss-absorbing capital in the form of Common Equity Tier 1 capital to ensure that banks are sufficiently resilient to withstand losses in times of stress. Additionally, Basel III also requires a minimum leverage ratio to constrain excess leverage in the banking system and complement the risk-weighted capital requirements (Basel Committee on Banking Supervision, 2010a, 2010b).

3.1.2. Risk/Economic Capital

We define Risk/Economic capital as the minimum capital needed to ensure that a bank could survive insolvency resulting from an unexpected loss (Zaik et al., 1996). Since we use ES as a measure of risk, our empirical tests reflect not only the regulatory requirements of Basel II and III but also to some extent, the capital adequacy norms as laid down under the Internal Capital Adequacy Assessment Process (ICAAP).[10] In addition to the quantitative capital requirements under Pillar I, the BIS also provides requirements/expectations from the supervisory authorities for internal capital requirements which are mainly principle based under Pillar II (Basel Committee on Banking Supervision, 2013).

3.1.3. Calculation of Business Unit Level Risk and Regulatory Capital

We generate the risk and regulatory capital data for each business unit, as these are not disclosed at the business unit level in the consolidated financial statements. However, the financial statements do provide the asset value of each business unit. Thus, we derive risk and regulatory capital data using the relative asset volatility of each business unit compared to the

[10] According to the Bank for International Settlement (BIS), January 2016 report, ES must be computed on a daily basis for the bank-wide internal model for regulatory capital purposes. The BIS demands that the ES must also be computed on a daily basis for each trading desk that a bank wishes to include within the scope for the internal model for regulatory capital purposes.

sum of all the business units' asset volatility up to time *t*, measured by realized standard deviations. We multiply this by the total capital at time t, u_t to calculate business unit *i*'s capital, $u_{t,i}$ as shown in equation (8).

$$\frac{\sqrt{\frac{\sum_{t=1}^{m}(A_{t,i}-A_{t-dt,i})^2}{t}}}{\sqrt{\frac{\sum_{i=1}^{N}\sum_{t=1}^{m}(A_{t,i}-A_{t-dt,i})^2}{t}}} \times u_t = u_{t,i} \tag{8}$$

$A_{t,i}$ is the asset value for business unit *i* at time *t*, N is the total number of business units, and *m* is the number of quarters. Similarly, we derive the risk capital for each business unit by multiplying the total capital by the risk ratio as given in equation (8) and call it as 'Economic Capital regime'. The empirical results derived with the regulatory capital are reported under the title 'Basel regime'. In order to avoid excessive risk, our model ceases capital allocation if our Economic Capital amount (minimum required capital) exceeds the available total equity capital.

3.2. How is the Capital Optimally Allocated?

In practice, the firm's headquarters chooses the capital allocation mechanism and elicits investment information from divisional managers to determine the optimal capital amount to be allocated to each division (Stoughton and Zechner, 2007). Nevertheless, banks have to consider meeting the minimum capital adequacy requirements as laid down in Basel II, III and the ICAAP.

The required economic capital calculations may be affected as bank can choose between various risk measures such as standard deviation, VaR, ES and spectral or distorted risk measures. There are several criteria for selecting an appropriate risk measures. These include, intuitive, stable, easy to compute, easy to understand, coherent, simple and meaningful. Amongst the various risk measures, VaR and ES are the most widely used. While each of them has advantages and disadvantages, VaR is easier to understand than ES but does not satisfy the coherence criteria. On the other hand, ES is coherent but relatively less easy to interpret than VaR, and its link with desired target credit is not clear (Basel Committee on Banking Supervision, 2009). The Bank for International Settlement (BIS) proposed moving its standardized risk measure from VaR to ES in 2012 after identifying a number of deficiencies with VaR (Basel Committee on Banking Supervision, 2013). The spectral or distorted risk measure is effective in assigning different weights to the quantiles of loss distribution, which makes it more powerful than other risk measures. However, it is highly dependent on the underlying loss distribution, and its computation method is complicated to use and understand (Basel Committee on Banking Supervision, 2009, 2013). The BIS too advocates using ES. For these reasons, we use ES as the risk measure.

In the context of RORAC, the Euler allocation principle which we use is considered as most appropriate (Tasche, 2008; Buch et al., 2011). It allocates capital according to the risk contributions of each business unit. The risk contributions are calculated using partial derivatives of the risk measure with respect to the capital of each business unit (Tasche, 2008). We empirically estimate the RORACs by using our model and compare these with the benchmark RORACs.

4. Empirical Results

4.1. Preliminary Findings

We begin by illustrating the empirical applicability of our theoretically derived model using data from Bank of America Corp. Table 1 presents results of capital allocation derived by using our model under both Basel^{+} and Economic Capital^{+} regimes. The results provide a comparison of RORAC obtained via our model vis-a-vis the benchmark with no capital allocation. The quarterly capital allocation begins in March 31, 2012 and finishes in December 31, 2014. The results show that RORAC with capital allocation using our model under both the Basel^{+} and the Economic Capital^{+} regimes exceed the benchmark RORAC in almost all quarters except quarter 10 (September 30, 2013). The average RORAC using our model is significantly higher than the benchmark RORAC. This confirms that our model is able to enhance the efficiency of the capital allocation process.

>> Insert Table 1<<

Figure 1 shows the evolution of the overall RORAC and the capital allocated to different business units. The red dotted and green dashed lines in Panel A represent the RORACs achieved through our model with enhancements. RORAC in red dotted line is calculated when we consider capital requirements as per the Basel regime. The RORAC in green dashed line is estimated when we consider economic capital. Panel A clearly shows that both red dotted and green dashed RORAC lines are above the blue solid line indicating that our model is able to enhance the RORAC consistently across all quarters, except for quarter 10.

Panel A shows the changes in the overall RORAC are driven primarily by the changes in the expected overall net income shown in Panel B. Panels C and D demonstrate the quarterly capital allocation to various business units. In general, the pattern of capital allocation is similar under both Basel^{+} and Economic Capital^{+} regimes. We find the initial periods for capital allocation (e.g., approximately from quarter 4 to 8) show large capital shifts among divisions (Panel C and D) compared to the latter periods (e.g., from quarter 9 to 15). Specifically, our model significantly reduces capital to the Consumer Real Estate Services and All Other divisions and gives more capital to the Consumer & Business Banking, Global Banking and Global Markets divisions during the initial periods. After this large initial adjustment, the capital allocations among the divisions are relatively stable during the latter periods. Since the capital shifts during the latter allocation periods are not large, the overall RORAC evolution in the latter periods resembles the benchmark RORAC (with no capital allocation). However, the RORAC achieved through optimal capital allocation using our model is higher.

>> Insert Figure 1<<

4.2. Descriptive Statistics

We now extend our empirical analysis to all fourteen U.S. banks included in our sample. Table 2 shows the descriptive statistics of banks in our sample. As our sample period is from June 30, 2006 to December 31, 2014, it includes the introduction of Basel II and III. Panel A of the table shows average, minimum, maximum, standard deviation, total number of observations, total asset, total risk capital, regulatory capital and total net income of our sample banks. The average total assets for our sample banks are US$614,519m with average risk capital and regulatory capital of US$67,672m and US$65,448m respectively. The average total income is US$1,368m over the entire sample period. Panel B of the table shows total numbers of quarters available for estimation and total number of business units for each bank.

>>Insert Table 2<<

4.3. Capital Allocation Efficiency

In Table 3, we report the results obtained by using our model. We include the '+' superscript to our regime names when we include both debt effects and the Bayesian learning process in our model (i.e., Basel^{+} and Economic Capital^{+} regimes). Panels A and B report results under the Basel regime whereas Panels C and D report results under the Economic Capital regime.

>>Insert Table 3<<

In Panel A, we find that the average quarterly RORAC improves by 0.036 percentage points using our model with no enhancements. This demonstrates that our model improves the capital allocation efficiency even when we do not consider the debt and Bayesian learning effects. We report results with enhancements in Panel B. We find that the average quarterly RORAC increases by 0.305 percentage points.[11] The results show that the RORAC is higher when we include the debt effects and the Bayesian learning process in our model.

Panels C and D show empirical results when we use Economic Capital for capital allocation. Panel C shows that the average quarterly RORAC reduces by 0.017 percentage points without enhancements. However, Panel D shows that the average quarterly RORAC improves by 0.331 percentage points with enhancements[12] indicating that the debt effect and the Bayesian learning process do indeed improve the capital allocation efficiency. Overall, we find capital allocation using our model increases the average RORAC with the enhancements under both Basel and Economic Capital regimes.[13]

Using our model, the difference between the enhanced RORAC and the observed RORAC is approximately 0.3 percentage points (i.e., 0.305 and 0.331 percentage points per quarter under Basel^{+} and Economic Capital^{+} regimes, respectively in Table 3). In other words,

[11] We have five exceptional cases, BB&T Corp, Comerica Inc., KeyCorp, People's United Financial Inc. and Wells Fargo & Co., where average RORAC after allocation is lower.

[12] There are again same five cases, BB&T Corp, Comerica Inc., KeyCorp, People's United Financial Inc. and Wells Fargo & Co., where the RORAC is lower than the benchmark.

[13] Although our model, on average improves the overall RORAC of the banks in our sample, some banks show a slight decline in the RORAC. This may be because our sample period includes the global financial crisis that began with severe problems in the US subprime mortgage market in 2007 and which eventually resulted in the collapse of Lehman Brothers in 2008. During this period, many American banks experienced financial difficulties that may have affected their ROCs.

this implies that the capital allocation using our model generates additional 0.3 percentage point returns per quarter. The median capital level of our sample banks is approximately US$20,000m, (i.e., US$19,414m and US$22,251m regulatory and risk capitals, respectively). This suggests that banks could benefit to the extent of US$60m quarterly (0.3 percentage points of US$20,000m) through more efficient capital allocation using our model.

4.4. Capital Allocation Efficiency and Bank Size

The extant literature provides evidence that larger banks benefit from economies of scale (Altunbas et al., 2007; Abedifar et al., 2013). Further, Tran et al. (2016) suggest that small banks' profitability tends to be more capital dependent compared to the large banks that are more likely to engage in off-balance sheet or fee-based activities. Therefore, we test how the bank size affects the RORAC.

We take categorize banks into large and small banks based on whether the average total assets of the bank is above or below the median of the total assets of all banks in our sample. Table 4 presents the results of RORAC improvements based on size under both Basel^{+} and Economic Capital^{+} regimes. We find that large size banks experience higher RORAC improvements on average compared to smaller banks. These results appear to be consistent with the previous studies discussed above.

>>Insert Table 4<<

5. Conclusions

We introduce a new capital allocation model for banks by extending Buch et al.'s (2011) model. We develop a multi-period model allowing for the time-varying risk and expected profit. Under dynamic financial conditions, we derive a closed-form solution for the optimal capital allocation to different business units with an aim to maximize overall RORAC. We impose capital constraints by not allowing external capital input. Further, our model includes two additional enhancements − debt effect and the Bayesian learning process. The debt effect overcomes distortions caused by excessive debt at business unit level and the Bayesian learning process help reduce the forecasting error arising from information asymmetry between the 'belief' and 'actual' profit.

This is the first study to provide empirical evidence of the effects of internal capital allocation on the RORAC. We use data of fourteen banks listed in S&P 500 index. We maximize RORAC while ensuring that both the required regulatory and risk capital requirements are satisfied. Our findings show that the average quarterly RORAC improves by 0.305 percentage points under the Basel$^+$ regime and by 0.331 percentage points under the Economic Capital$^+$ regime compared to the benchmark. We also find that larger banks with above median size of total assets show greater improvements in the RORAC.

The findings reported in the paper have several implications. First, given the scarcity of capital, banks can and indeed do benefit by optimizing internal capital allocation. Second, though most of our sample banks initially show relatively large capital shifts among divisions, the changes in the latter period are smaller indicating the Bayesian learning process has an influence on the optimal capital allocation process. Finally, the model shows that banks can achieve consistently higher RORAC by allocating capital to more profitable divisions that do not significantly increase the overall risk.

However, the findings reported in the paper should be interpreted with caution. Our model assumes normally distributed returns and may therefore under or overestimate risk. Further, the internal capital allocation process itself can incur costs and may increase capital charge, thus affecting the risk taking behavior of the divisional managers. Finally, in the real world, decision to close a business unit not may not be dependent entirely on RORAC maximization criterion.

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Table 1. Comparison of Results under Our Model using Bank of America Corp.

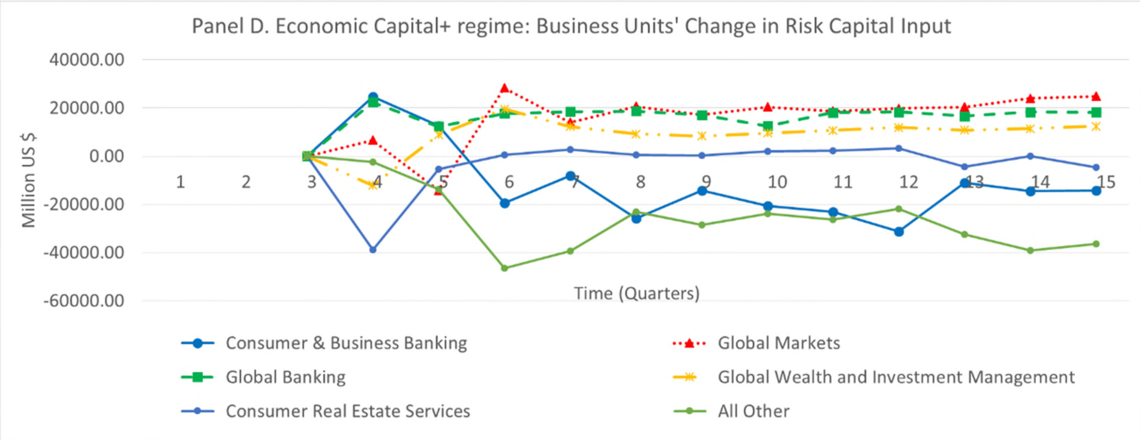
This table shows the our model's results using Bank of America Corp. for its overall RORAC and expected net income with no allocation (i.e. the benchmark), with allocation under the Basel$^+$ regime and with allocation under the Economic Capital$^+$ regime. The last two columns show the total available regulatory capital and risk capital amount that do not change during allocation, while we use only the available regulatory capital in our overall RORAC formula for the purpose of comparison. The capital allocation starts from March 31, 2012. The units for the overall RORAC are in percentage. All units other than the overall RORAC are in million U.S. dollars.

Time (Quarters)	Benchmark Overall RORAC	Our Model's Overall RORAC –Basel$^+$ regime	Our Model's Overall RORAC –Economic Capital$^+$ regime
2011-06-30			
2011-09-30			
2011-12-31	0.93%	0.93%	0.93%
2012-03-31	0.31%	2.34%	2.10%
2012-06-30	1.18%	2.09%	2.08%
2012-09-30	0.17%	1.82%	1.78%
2012-12-31	0.37%	1.37%	1.28%
2013-03-31	0.74%	2.15%	2.03%
2013-06-30	2.04%	3.04%	3.01%
2013-09-30	1.26%	1.32%	1.23%
2013-12-31	1.72%	2.32%	2.25%
2014-03-31	-0.15%	0.65%	0.37%
2014-06-30	1.16%	2.86%	2.78%
2014-09-30	-0.12%	1.02%	0.84%
2014-12-31	1.89%	3.22%	3.21%
Average	**0.88%**	**2.02%**	**1.91%**

$$\text{Overall RORAC} = \frac{\text{Expected Overall Net Income}}{\text{Available Regulatory Capital}}$$

Figure 1. Preliminary Findings using Bank of America Corp.

These figures show the overall RORAC, expected net income, and business units' new capital (i.e. risk or regulatory capital) after allocation using our model for Bank of America Corp. We compare these with the benchmark with no capital allocation against the Basel$^+$ and Economic Capital$^+$ regimes in terms of overall RORAC (Panel A), expected net income (Panel B), and change in business units' regulatory or risk capital input (Panel C and D). There are four business units and the allocation starts at time 4 (fourth quarter or on March 31, 2012) until time 15 (the 15th quarter or on December 31, 2014). All units other than the overall RORAC are in million U.S. dollars.



Panel D. Economic Capital+ regime: Business Units' Change in Risk Capital Input
40000.00
20000.00
0.00
-20000.00
-40000.00
-60000.00
Million US $
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
Time (Quarters)
Consumer & Business Banking
Global Markets
Global Banking
Global Wealth and Investment Management
Consumer Real Estate Services
All Other

Table 2. Descriptive Statistics

This table shows the descriptive statistics of the data used. Panel A shows the average, minimum, maximum, standard deviation (i.e. Std.), and total count (i.e. N) of the total asset, total risk capital, regulatory capital (i.e. risk-based capital) and total net income. The units for the financial variables (total asset, total risk capital, total regulatory capital, and total net income) are in million US dollars. The dates are quarterly from December 31, 2006 to December 31, 2014. Panel B shows each bank's name, total quarters of estimation, and total number of business units used in the empirical test.

Panel A. Financial Data

	Total Asset ($M)	Total Risk Capital ($M)	Regulatory Capital ($M)	Total Net Income ($M)
Average	614518.60	67671.79	65448.18	1367.86
Min	30400.10	4568.40	3133.30	-1928.00
Max	2573126.00	243471.00	215101.00	6061.00
Std.	742823.04	77852.43	70839.15	1686.17
N	223	223	223	223

Panel B. Banks' Characteristics

Banks	Total Quarters of Estimation	Total Number of Business Units
Bank of America Corp	13	5
BB&T Corp	9	7
Citigroup Inc.	20	3
Comerica Inc.	23	5
Fifth Third Bancorp	14	4
Huntington Bancshares Inc./OH	15	4
JPMorgan Chase & Co	6	5
KeyCorp	20	3
People's United Financial Inc.	5	3
PNC Financial Services Group Inc.	21	5
Regions Financial Corp	5	3
SunTrust Banks Inc.	9	4
US Bancorp/MN	30	5
Wells Fargo & Co	33	3

Table 3. Empirical Results

These tables show the empirical results of our model using fourteen U.S. banks listed in the S&P 500 index. We compare the benchmark RORAC without using our model and the optimized RORAC using our model for each bank under four different regimes—the Basel (Panel A), Basel+ (Panel B), Economic Capital (Panel C), and Economic Capital+ (Panel D) regimes. We show the benchmark quarterly average RORAC, our model's quarterly average RORAC, average improvement per quarter, and total quarters of estimation.

		Panel A. Basel regime			**Panel B. Basel+ regime**		
	Benchmark Quarterly Average RORAC	Our Model's Quarterly Average RORAC	Average Improvem -ent Per Quarter	Total Quarters of Estimation	Our Model's Quarterly Average RORAC	Average Improvem -ent Per Quarter	Total Quarters of Estimation
Bank of America Corp	0.881%	1.562%	0.681%	12	2.019%	1.138%	12
BB&T Corp	2.465%	2.001%	-0.464%	8	1.957%	-0.509%	8
Citigroup Inc.	1.706%	1.670%	-0.036%	19	2.112%	0.407%	19
Comerica Inc.	2.023%	2.271%	0.248%	22	1.976%	-0.048%	22
Fifth Third Bancorp	1.772%	1.579%	-0.193%	13	1.884%	0.112%	13
Huntington Bancshares Inc./OH	2.125%	2.030%	-0.095%	14	2.225%	0.100%	14
JPMorgan Chase & Co	2.620%	2.194%	-0.427%	5	3.043%	0.422%	5
KeyCorp	1.775%	1.505%	-0.270%	19	1.618%	-0.157%	19
People's United Financial Inc.	1.820%	1.804%	-0.016%	4	1.708%	-0.112%	4
PNC Financial Services Group Inc.	1.845%	1.879%	0.034%	20	1.907%	0.062%	20
Regions Financial Corp	2.093%	1.879%	-0.214%	4	3.081%	0.988%	4
SunTrust Banks Inc.	2.161%	3.381%	1.220%	8	3.881%	1.720%	8
US Bancorp/MN	3.088%	3.243%	0.155%	29	3.252%	0.163%	29
Wells Fargo & Co	2.927%	2.804%	-0.123%	32	2.908%	-0.019%	32
Average	**2.093%**	**2.129%**	**0.036%**	**14.93**	**2.398%**	**0.305%**	**14.93**

		Panel C. Economic Capital regime			Panel D. Economic Capital+ regime		
	Benchmark Quarterly Average RORAC	Our Model's Quarterly Average RORAC	Average Improvem -ent Per Quarter	Total Quarters of Estimation	Our Model's Quarterly Average RORAC	Average Improvem -ent Per Quarter	Total Quarters of Estimation
Bank of America Corp	0.881%	1.402%	0.521%	12	1.913%	1.033%	12
BB&T Corp	2.465%	2.001%	-0.465%	8	2.264%	-0.201%	8
Citigroup Inc.	1.706%	1.914%	0.209%	19	2.242%	0.536%	19
Comerica Inc.	2.023%	1.394%	-0.629%	22	1.907%	-0.116%	22
Fifth Third Bancorp	1.772%	1.576%	-0.197%	13	1.916%	0.144%	13
Huntington Bancshares Inc./OH	2.125%	2.023%	-0.102%	14	2.261%	0.136%	14
JPMorgan Chase & Co	2.620%	2.183%	-0.437%	5	3.034%	0.414%	5
KeyCorp	1.775%	1.502%	-0.273%	19	1.616%	-0.159%	19
People's United Financial Inc.	1.820%	1.826%	0.006%	4	1.715%	-0.105%	4
PNC Financial Services Group Inc.	1.845%	1.884%	0.039%	20	1.904%	0.058%	20
Regions Financial Corp	2.093%	1.879%	-0.214%	4	3.077%	0.985%	4
SunTrust Banks Inc.	2.161%	3.378%	1.217%	8	3.882%	1.721%	8
US Bancorp/MN	3.088%	3.295%	0.207%	29	3.294%	0.205%	29
Wells Fargo & Co	2.927%	2.804%	-0.123%	32	2.903%	-0.024%	32
Average	**2.093%**	**2.076%**	**-0.017%**	**14.93**	**2.424%**	**0.331%**	**14.93**

Improvement = Our Model's RORAC - Benchmark RORAC
Benchmark: Overall RORAC without risk capital allocation under the Basel regime
Total quarters of estimation: Number of quarters prior to January 1, 2015

Table 4. Empirical Results of Different Bank Size

The following tables show the average RORAC levels with capital allocation under the Basel^{+} and the Economic Capital^{+} regimes. We classify the average RORAC with capital allocation into high/low (or large/small) categories in bank size (i.e. total assets)

Bank Size	Banks	Bank Size ($M)	RORAC Improvement (Basel^{+})	RORAC Improvement (Economic Capital^{+})
Large	JPMorgan Chase & Co	2495967.833	0.422%	0.414%
	Bank of America Corp	2165895.231	1.138%	1.033%
	Citigroup Inc	1621407.950	0.407%	0.536%
	Wells Fargo & Co	1172933.333	-0.019%	-0.024%
	US Bancorp/MN	305364.100	0.163%	0.205%
	Average	**1552313.689**	**0.422%**	**0.433%**
Small	PNC Financial Services Group Inc	198832.714	0.062%	0.058%
	BB&T Corp	184222.556	-0.509%	-0.201%
	SunTrust Banks Inc	174251.222	1.720%	1.721%
	Fifth Third Bancorp	131673.429	0.112%	0.144%
	Regions Financial Corp	84293.400	0.988%	0.985%
	KeyCorp	83584.550	-0.157%	-0.159%
	Comerica Inc.	61291.640	-0.048%	-0.116%
	Huntington Bancshares Inc/OH	47238.710	0.100%	0.136%
	People's United Financial Inc	32406.960	-0.112%	-0.105%
	Average	**110866.131**	**0.372%**	**0.372%**

Appendix. Proofs and Illustrations

Illustration of Definition 2.1. We calculate the conditional covariance σ_t^{cov} of return on capital (ROC) among *N* divisions at time *t*, which is conditional on the past covariance as follows:

$$\sigma_t^{\text{cov}} = \begin{pmatrix} \sigma^2_{(1,1),t} & \cdots & \sigma^2_{(1,N),t} \\ \vdots & \ddots & \vdots \\ \sigma^2_{(N,1),t} & \cdots & \sigma^2_{(N,N),t} \end{pmatrix} \tag{A.1}$$

where division index *i* ranges from *1* to *N*.

This one-step-ahead ES function satisfies all four coherent risk measure properties for the Euler allocation used in our model (Acerbi and Tasche, 2002).[14] Thus, the Economic Capital, EC_t at time *t*, is calculated by multiplying our one-step ahead ES by the total asset of the bank $\sum_{i=1}^{N} A_{i,t}$, where $A_{i,t}$ is the total asset of division *i* at time *t*.

$$EC_t = \sum_{i=1}^{N} A_{i,t} \times \rho_t[u_t^w] \tag{A.2}$$

Proof of Lemma 2.3. We assume the time-varying expected profit process follows a Stochastic Differential Equation (SDE):[15]

[14] Following Artzner et al. (1999), when X and Y are portfolios and ρ is the coherent risk measure, then it satisfies the following properties.

- Monotonicity: for all X and Y, if X ≤ Y, I have ρ(X) ≤ ρ(Y).
- Subadditivity: for all and, ρ(X+Y) ≤ ρ(X) + ρ(Y)
- Positive homogeneity: for all λ ≥ 0 and all X, ρ(λ X) = λ ρ(X).
- Translation invariance: for all X and all real numbers α, if A is a portfolio with guaranteed return α, I have ρ(X+A) = ρ(X) – α

[15] This SDE is a time-varying version of the stochastic profit process used by Buch et al. (2011).

$$\underbrace{Y_{i,t}[u_t]}_{\text{Profit}} = \underbrace{\mu_{i,t}[u_{i,t}]}_{\text{Expected Profit}} + \underbrace{\sigma_{i,t}[u_{i,t}]W_t}_{\text{Profit Fluctuation}} \quad (A.3)$$

The profit for business unit *i* at time *t* is its expected profit plus profit fluctuations, where both the drift μ and the diffusion σ are time varying. W_t is the standard Wiener process. All business units within a firm follow this SDE process. We then discretize the SDE process from time *t-dt* to *t* as follows:

$$\begin{aligned} & Y_{i,t}[u_t] = \mu_{i,t}[u_{i,t}] + \sigma_{i,t}[u_{i,t}]W_t \\ & \Rightarrow Y_{i,t}[u_t] - Y_{i,t-dt}[u_{t-dt}] \\ & \qquad = \left(\mu_{i,t}[u_{i,t}] - \mu_{i,t-dt}[u_{i,t-dt}]\right) + \left(\sigma_{i,t}[u_{i,t}]W_t - \sigma_{i,t-dt}[u_{i,t-dt}]W_{t-dt}\right) \end{aligned} \quad (A.4)$$

Here, *dt* is the minimum time step of the data.

The profit $Y_t[u_t]$ is a function of capital u_t and return on capital at time *t* (ROC_t), i.e.

$$\begin{aligned} & \Leftrightarrow \mu_{i,t}[u_{k,t}] \\ & = \mu_{i,t-dt}[u_{k,t-dt}] \\ & \quad +(u_{i,t} \times ROC_{i,t} - u_{i,t-dt} \times ROC_{i,t-dt}) - (\sigma_{i,t}[u_{i,t}]W_t - \sigma_{i,t-dt}[u_{i,t-dt}]W_{t-dt}) \end{aligned} \quad (A.5)$$

Since the expected value of the Weiner process is zero by definition, in expectation form, equation (A.5) becomes:

$$\mathrm{E}\left[\mu_{i,t}[u_{i,t}]\right] = E[\mu_{i,t-dt}[u_{i,t-dt}] + \left(u_{i,t} \times ROC_{i,t} - u_{i,t-dt} \times ROC_{i,t-dt}\right)] \quad (A.6)$$

$$\Leftrightarrow \mu_{i,t}[u_{k,t}] = \mu_{i,t-dt}[u_{k,t-dt}] + (u_{i,t} \times ROC_{i,t} - u_{i,t-dt} \times ROC_{i,t-dt})$$

We express this equation (A.6) in terms of capital weights $u_{i,t}^w$

$$\begin{aligned} & \mu_{i,t}[u_{i,t}^w] \\ & = \mu_{i,t-dt}[u_{i,t-dt}^w] + \left(\textstyle\sum_{i=1}^N u_{i,t} \times u_{i,t}^w \times ROC_{i,t} - \sum_{i=1}^N u_{i,t-dt} \times u_{i,t-dt}^w \times ROC_{i,t-dt}\right) \end{aligned} \quad (A.7)$$

where $u_{i,t}^w = \frac{u_{i,t}}{\sum_{i=1}^N u_{i,t}}$.

The overall RORAC $r[u_t^w]$ is defined as follows:

$$\text{Overall RORAC: } r[u_t^w] = \frac{\mu_t[u_t^w]}{EC_t} = \frac{\sum_{i=1}^N u_{i,t} \times ROC_{i,t}}{\sum_{i=1}^N A_{i,t} \times \rho_t[u_t^w]} \quad (A.8)$$

Since we use the absolute risk measure, our overall RORAC calculation does not subtract the expected profit from the risk in the denominator as in Buch et al. (2011).

However, similar to Buch et al. (2011), we use the Euler optimization method to derive the additional capital allocation to business unit *i* such that its marginal RORAC after allocation equals the overall RORAC before allocation. However, we use capital weights rather than the capital.

$$\underbrace{r[\varepsilon_{i,t}^{w}|u_{i,t}^{w}]}_{\substack{\text{marginal RORAC} \\ \text{after capital allocation}}} = \underbrace{r[u_{t}^{w}]}_{\substack{\text{overall RORAC} \\ \text{before capital allocation}}} \tag{A.9}$$

The unknown variable to solve from this equation $\varepsilon_{i,t}^{w}$ is the additional optimal capital weight for business unit *i* at time *t*. $u_{i,t}^{w}$ is the existing capital weight for the business unit *i* at time *t*. u_{t}^{w} is the total capital weight equal to one.

We define the marginal RORAC $r[\varepsilon_{i,t}^{w}|u_{i,t}^{w}]$ as follows:

$$r[\varepsilon_{i,t}^{w}|u_{i,t}^{w}] = \frac{\mu_{i,t}[u_{i,t}^{w}+\varepsilon_{i,t}^{w}]-\mu_{i,t}[u_{i,t}^{w}]}{\sum_{i=1}^{N} A_{i,t}\times\rho_t[u_{i,t}^{w}+\varepsilon_{i,t}^{w}]-\sum_{i=1}^{N} A_{i,t}\times\rho_t[u_{i,t}^{w}]} \tag{A.10}$$

We simplify the numerator of equation (A.10) using equation (A.7) and with additional capital weight $\varepsilon_{i,t}^{w}$ as follows:

$$\begin{aligned} &\mu_{i,t}[u_{i,t}^{w}+\varepsilon_{i,t}^{w}]-\mu_{i,t}[u_{i,t}^{w}] \qquad \text{(A.11)} \\ &= \mu_{i,t-dt}[u_{i,t-dt}] + \left(\textstyle\sum_{i=1}^{N} u_{i,t}\times(u_{i,t}^{w}+\varepsilon_{i,t}^{w})\times ROC_{i,t} - \sum_{i=1}^{N} u_{i,t-dt}\times u_{i,t-dt}^{w}\times ROE_{i,t-dt}\right) \\ &\quad - \mu_{i,t-dt}[u_{i,t-dt}] + \left(\textstyle\sum_{i=1}^{N} u_{i,t}\times u_{i,t}^{w}\times ROC_{i,t} - \sum_{i=1}^{N} u_{i,t-dt}\times u_{i,t-dt}^{w}\times ROE_{i,t-dt}\right) \\ &= \textstyle\sum_{i=1}^{N} u_{i,t}\times\varepsilon_{i,t}^{w}\times ROC_{i,t} \end{aligned}$$

We then simplify the denominator of equation (A.10) by subtracting the Taylor expansion forms for $\rho_t[u_{i,t}^{w}+\varepsilon_{i,t}^{w}]$ and $\rho_t[u_{i,t}^{w}]$. We first describe $\rho_t[u_{i,t}^{w}]$ as a general expansion form.

$$\rho_t[u_{i,t}^{w}+\varepsilon_{i,t}^{w}] = a_0 + a_1(u_{i,t}^{w}+\varepsilon_{i,t}^{w}) + a_2(u_{i,t}^{w}+\varepsilon_{i,t}^{w})^2 + a_3(u_{i,t}^{w}+\varepsilon_{i,t}^{w})^3 \cdots$$

$$\rho_t[u_{i,t}^{w}] = a_0 + a_1 u_{i,t}^{w} + a_2 {u_{i,t}^{w}}^2 + a_3 {u_{i,t}^{w}}^3 + \cdots$$

$$\rightarrow \rho_t\left[u_{i,t}^w + \varepsilon_{i,t}^w\right] - \rho_t\left[u_{i,t}^w\right] = d\rho_t\left[u_{i,t}^w\right] = a_1\varepsilon_{i,t}^w + a_2{\varepsilon_{i,t}^w}^2 + \cdots$$

$$\varepsilon_{i,t}^w = du_{i,t}^w$$

$$\rightarrow \rho_t\left[u_{i,t}^w + du_{i,t}^w\right] - \rho_t\left[u_{i,t}^w\right] = d\rho_t\left[u_{i,t}^w\right] = a_1 du_{i,t}^w + a_2{du_{i,t}^w}^2 + \cdots$$

$$\frac{d(\rho_t\left[u_{i,t}^w + du_{i,t}^w\right] - \rho_t\left[u_{i,t}^w\right])}{du_{i,t}^w} = a_1 + 2a_2 du_{i,t}^w + 3a_3{du_{i,t}^w}^2 + \cdots$$

$$\frac{d\,\frac{d(\rho_t\left[u_{i,t}^w + du_{i,t}^w\right] - \rho_t\left[du_{i,t}^w\right])}{du_{i,t}^w}}{du_{i,t}^w} = 2a_2 + 6a_3 du_{i,t}^w + \cdots$$

The first two terms of the Taylor expansion subtraction form are as follows:

$$\frac{d\rho_t[u_{i,t}^w]}{du_{i,t}^w} = \lim_{du_{i,t}^w \to 0} \frac{d(\rho_t[u_{i,t}^w + du_{i,t}^w] - \rho_t[u_{i,t}^w])}{du_{i,t}^w} = a_1 \;\rightarrow\; a_1 = \frac{d\rho_t[u_{i,t}^w]}{du_{i,t}^w} \tag{A.12}$$

$$\frac{d^2\rho_t[u_{i,t}^w]}{d^2u_{i,t}^w} = \lim_{du_{i,t}^w \to 0} \frac{d\frac{d(\rho_t\left[u_{i,t}^w + du_{i,t}^w\right] - \rho_t\left[du_{i,t}^w\right])}{du_{i,t}^w}}{du_{i,t}^w} = 2a_2 \rightarrow a_2 = \frac{1}{2}\frac{d^2\rho_t[u_{i,t}^w]}{d^2u_{i,t}^w} \tag{A.13}$$

We retain only the first two terms from the Taylor expansion subtraction form as follows:

$$\rho_t\left[u_{i,t}^w + \varepsilon_{i,t}^w\right] - \rho_t\left[u_{i,t}^w\right] = \frac{d\rho_t[u_{i,t}^w]}{du_{i,t}^w}\varepsilon_{i,t}^w + \frac{1}{2}\frac{d^2\rho_t[u_{i,t}^w]}{d^2u_{i,t}^w}{\varepsilon_{i,t}^w}^2 \tag{A.14}$$

Since we only use the first two terms of the Taylor expansion, equation (A.14) is subject to an approximation error given by:

$$\mathrm{f}\,[\varepsilon_{i,t}^w]\text{-}\,P_n\,[\varepsilon_{i,t}^w] = \frac{f^{(n+1)}(c)}{(n+1)!}(\varepsilon_{i,t}^w)^{n+1} \tag{A.15}$$

where f $[\varepsilon_{i,t}^w]$ is the actual value, $P_n[\varepsilon_{i,t}^w]$ is the estimated value from equation (A.14) and n is the degree of polynomial, which in our case is two. c in equation (A.15) is an arbitrary value that should be between zero and $\varepsilon_{i,t}^w$ for a satisfactory approximation.

$\frac{d\rho_t[u_{i,t}^w]}{du_{i,t}^w} = a_{i,t}$ in equation (A.12) is the risk contribution of business unit *i* at time *t* and $\frac{d^2\rho_t[u_{i,t}^w]}{d^2u_{i,t}^w}$ in equation (A.13) is the Hessian matrix with the business units' weights. Following Buch et al.

(2011), we use the highest eigenvalue Λ_t of this Hessian matrix, since it provides the maximum directional strength among the business units with the additional allocation input.

We then combine equations (A.11) and (A.14) in (A.10) to derive the expression for marginal RORAC with capital allocation.

$$r\left[\varepsilon_{i,t}^{w} \middle| u_{i,t}^{w}\right] = \frac{\sum_{i=1}^{N} u_{i,t} \times \varepsilon_{i,t}^{w} \times ROC_{i,t}}{\sum_{i=1}^{N} A_{i,t} \times \left(\frac{d\rho_t\left[u_{i,t}^{w}\right]}{du_{i,t}^{w}} \varepsilon_{i,t}^{w} + \frac{1}{2} \frac{d^2\rho_t\left[u_{i,t}^{w}\right]}{d^2u_{i,t}^{w}} \varepsilon_{i,t}^{w\,2} \right)} = \frac{\sum_{i=1}^{N} u_{i,t} \times \varepsilon_{i,t}^{w} \times ROC_{i,t}}{\sum_{i=1}^{N} A_{i,t} \times (a_{i,t}\, \varepsilon_{i,t}^{w} + \frac{1}{2}\Lambda_t \varepsilon_{i,t}^{w\,2})} \tag{A.16}$$

We then equate the overall RORAC (A.8) and the marginal RORAC with capital allocation (A.16) and solve for the $\varepsilon_{i,t}^{w}$

$$\frac{\sum_{i=1}^{N} u_{i,t} \times \varepsilon_{i,t}^{w} \times ROC_{i,t}}{\sum_{i=1}^{N} A_{i,t} \times (a_{i,t}\, \varepsilon_{i,t}^{w} + \frac{1}{2}\Lambda_t \varepsilon_{i,t}^{w\,2})} = \frac{\sum_{i=1}^{N} u_{i,t} \times ROC_{i,t}}{\sum_{i=1}^{N} A_{i,t} \times \rho_t[u_t^{w}]}$$

$$\Leftrightarrow \quad \varepsilon_{i,t}^{w} = \frac{-2(a_{i,t} - \rho_t[u_t^{w}])}{\Lambda_t} \tag{A.17}$$

We account for the Taylor approximation error by choosing $0.5\varepsilon_{i,t}^{w}$ instead of $\varepsilon_{i,t}^{w}$ as the value for c in equation (A.15). Hence, we have the expression for $\varepsilon_{i,t}^{w}$:

$$\varepsilon_{i,t}^{w} = \frac{-a_{i,t} + \rho_t[u_t^{w}]}{\Lambda_t} \tag{A.18}$$

Illustration of Definition 2.4 using Capital Constraints. We optimize a firm's return-to-risk by reallocating the existing capital amount with no additional capital input. To achieve this, we ensure that no change is made in the total capital amount by making the sum of additional capital weights $\varepsilon_{i,t}^{w}$ add up to zero. This is equivalent to making the new total capital weight $u_{new,i,t+1}^{w}$ add up to one all the time. We provide a minimum constraint so that a business unit's capital weight does not fall below zero and new capital weights are greater than or equal to zero. We set the minimum threshold value to zero to avoid negative capital weight.

$$u^w_{new,i,t+1} = \text{Max}(\, u^w_{new,i,t+1}, 0) \tag{A.19}$$

Through this process, in equation (A.19), there are business units which have either zero or non-zero capital weight. If there are any business units that did not have zero weight before but turned out to have zero weight in equation (A.19), then the total sum of weights in equation (A.19) exceeds one. In this case, we renormalize the capital weights such that for a business unit i, the new capital weight becomes $\frac{u^w_{new,i,t+1}}{\sum_{i=1}^{N} u^w_{new,i,t+1}}$. Then we compare this with zero and select a larger value. This process is shown in equation (A.20):

$$u^w_{new,j,t+1} = \text{Max}\,(\,\frac{u^w_{new,i,t+1}}{\sum_{i=1}^{N} u^w_{new,i,t+1}}, 0\,) \tag{A.20}$$

We then re-inject a business units' capital weights from equation (A.20) back into equation (A.18) and (A.19) and re-estimate capital weights. If there are any business units that have zero weight, than our model allocates capital to other business units which have non-zero weights. We continue re-injecting until all business units' capital weights are identical using equations (A.19) and (A.20) and capital weights from equations (A.19) and (A.20) must add up to one simultaneously. When we reach this stage, the model achieves the optimal point and business units' capital weights add up to one. This enables our model to optimally allocate original capital and avoid injection of new capital.

Proof of Lemma 2.5. In our model, the belief is updated overtime through an accumulating learning process. Following Cvitanić et al. (2004, pp. 140–141), we develop the following SDE model:

$$\begin{aligned} Y_t &= \overline{\mu_t} + \sigma_t \overline{W_t} \\ \overline{W_t} &= W_t + \int_0^t (\mu_u - \overline{\mu_u}) du, (u \le t) \end{aligned} \tag{A.21}$$

Y_t is the stochastic profit process we use in our model that has time-varying drift $\overline{\mu_t}$ and time-varying diffusion σ_t. μ_t is the expected profit before capital allocation at time *t*. $\overline{\mu_t}$ is the 'belief' of the expected profit after capital allocation at time *t*. We start by inputting the innovation process $\overline{W_t}$ into the following stochastic profit process.

$$\begin{aligned} &Y_t = \overline{\mu_t} + \sigma_t^{\text{cov}}\overline{W_t} \\ &\Leftrightarrow Y_t = \overline{\mu_t} + \sigma_t^{\text{cov}}(W_t + \int_0^t (\mu_u - \overline{\mu_u})du) \end{aligned} \tag{A.22}$$

We use the square root of weighted realized covariance portion $\sqrt{u_t^w \times \sigma_t^{\text{cov}} \times u_t^{w'}}$ of the ES risk measure ρ_t in section 2.1.1 for our diffusion process σ_t^{cov} in equation (A.22). We then express $\mu_t - \overline{\mu_t}$ using the same procedure as in equation (A.11).

$$\begin{aligned} &\mu_t - \overline{\mu_t} \\ &= \mu_t[u_t] - \mu_t[u_t + \varepsilon_t] \\ &= \mu_{t-dt}[u_{t-dt}] + (u_t \times ROC_t - u_{t-dt} \times ROC_{t-dt}) \\ &-\mu_{t-dt}[u_{t-dt}] + ((u_t + \varepsilon_t) \times ROC_t - u_{t-dt} \times ROC_{t-dt}) \\ &= (u_t \times ROC_t) - (u_t + \varepsilon_t) \times ROC_t \end{aligned} \tag{A.23}$$

Substituting equation (A.23) into equation (A.22):

$$\begin{aligned} Y_t &= \overline{\mu_t} + \rho_t(W_t + \int_0^t (\mu_u - \overline{\mu_u})du) \\ &= \overline{\mu_t} + \rho_t W_t + \rho_t \int_0^t ((u_t \times ROC_t) - (u_t + \varepsilon_t) \times ROC_t)du \end{aligned} \tag{A.24}$$

We take the expected value on both sides of equation (A.24) to derive the expected profit with the Bayesian learning process:

$$E[Y_t] = E[\overline{\mu_t} + \rho_t W_t + \rho_t \int_0^t ((u_t \times ROC_t) - (u_t + \varepsilon_t) \times ROC_t)du] \tag{A.25}$$

$$\overline{\mu_{t,Bayesian}}$$

$$= \overline{\mu_t} + \rho_t \int_0^t ((u_t \times ROC_t) - (u_t + \varepsilon_t) \times ROC_t)du$$

$$= \overline{\mu_t} + \rho_t \int_0^t ((u_t \times ROC_t) - (u_t + \varepsilon_t) \times ROC_t)du$$

$$= \overline{\mu_t} + \rho_t \int_0^t \left(\left(\sum_{i=1}^N u_{i,t} \times u_{new,t-dt}^w \times ROC_t\right) - \left(\sum_{i=1}^N u_{i,t} \times u_{new,t}^w \times ROC_t\right)\right)du$$

$$= \overline{\mu_t} + \rho_t \int_0^t \sum_{i=1}^N u_{i,t} \left((u_{new,t-dt}^w \times ROC_t) - (u_{new,t}^w \times ROC_t)\right)du \tag{A.26}$$

We reflect this in our time-varying risk measure in equation (1).

$$\rho_t[u_t^w] - \frac{\overline{\mu_{t,Bayesian}}}{u_t} \tag{A.27}$$

The $\overline{\mu_{t,Bayesian}}$ is divided by the total capital weight u_t to have the consistent percentage term as our time-varying risk measure expression $\rho_t[u_t^w]$.

Since we are considering our time-varying risk measure to be an absolute risk measure which has the expected profit portion $\overline{\mu_t}$ to be zero, we are only left with the remaining part $\rho_t \int_0^t \sum_{i=1}^N u_{i,t} \left((u_{new,t-dt}^w \times ROC_t) - (u_{new,t}^w \times ROC_t) \right) du$ to use in our time-varying risk measure. Then the equation A.27 becomes:

$$\rho_t[u_t^w] - \frac{\rho_t \int_0^t \sum_{i=1}^N u_{i,t}((u_{new,t-dt}^w \times ROC_t) - (u_{new,t}^w \times ROC_t))du}{u_t}$$

$$\Leftrightarrow \rho_t[u_t^w] - \rho_t \int_0^t \sum_{i=1}^N((u_{new,t-dt}^w \times ROC_t) - (u_{new,t}^w \times ROC_t))du \tag{A.28}$$

We reflect this in equation A.28 in our closed-form solution in equation A.18 which becomes the following.

$$\varepsilon_{new,i,t}^w = \frac{-a_{i,t} + \rho_t[u_t^w] - \rho_t \int_0^t \sum_{i=1}^N((u_{new,t-dt}^w \times ROC_t) - (u_{new,t}^w \times ROC_t))du}{\Lambda_t} \tag{A.29}$$