Nonlinear dynamics of a flapping rotary wing: Modeling and optimal wing kinematic analysis

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Abstract The analysis of the passive rotation feature of a micro Flapping Rotary Wing (FRW) applicable for Micro Air Vehicle (MAV) design is presented in this paper. The dynamics of the wing and its influence on aerodynamic performance of FRW is studied at low Reynolds number (~10^3). The FRW is modeled as a simplified system of three rigid bodies: a rotary base with two flapping wings. The multibody dynamic theory is employed to derive the motion equations for FRW. A quasi-steady aerodynamic model is utilized for the calculation of the aerodynamic forces and moments. The dynamic motion process and the effects of the kinematics of wings on the dynamic rotational equilibrium of FRW and the aerodynamic performances are studied. The results show that the passive rotation motion of the wings is a continuous dynamic process which converges into an equilibrium rotary velocity due to the interaction between aerodynamic thrust, drag force and wing inertia. This causes a unique dynamic time-lag phenomena of lift generation for FRW, unlike the normal flapping wing flight vehicle driven by its own motor to actively rotate its wings. The analysis also shows that in order to acquire a high positive lift generation with high power efficiency and small dynamic time-lag, a relative high mid-up stroke angle within 7°–15° and low mid-down stroke angle within ~40° to ~35° are necessary. The results provide a quantified guidance for design option of FRW together with the optimal kinematics of motion according to flight performance requirement.

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1. Introduction

The Micro Air Vehicle (MAV) has become an active research area due to the potentiality for the civil and military application. The typical characteristics of MAV are small dimension (wing spans within 15 cm), low weight (gross take-off weight ranging from 100 to 200 g) and low flight speed (between 10 and 15 m/s). In recent two decades, a variety of MAV layouts, which mainly include fixed wing, rotary wing, and flapping wing, had been put forward. However, due to the extremely small dimension and high lift and efficiency requirements at
low Reynolds numbers $Re$, few practical MAVs with load carrying capabilities has been accomplished. Research efforts for new and practical designs of MAVs have never been stopped. In 2004, Vandenberghe et al. employed the experimental method and found that a pair of wing flapping up and down can freely rotate spontaneously around the horizontal shaft as a critical frequency was exceeded. Based on this discovery, Guo et al. proposed a design of Flapping Rotary Wing (FRW) flight vehicle as a new configuration of MAV. Similar concept was also proposed and applied in full-scaled helicopter rotor by Van Holten et al. As shown in Fig. 1, a pair of anti-symmetrically mounted wings, which can flap along the vertical direction by a drive shaft, is fixed on the rotary rigid base. The thrust generated by the wings’ vertically flapping motion drives them to rotate around the shaft, resulting in a flapping and simultaneously rotating kinematics. Combined with tuning the pitch angles of the wings asymmetrically in the up-stroke and down-stroke, the high lift force is produced to make FRW take-off and hover.

Recently, experimental works were used to measure the force produced and proved that the lift from flapping rotary wing was larger than that from conventional rotary wing in the range of $Re$ from 2600 to 5000. Wu et al. conducted a computational fluid dynamics method to research the unsteady aerodynamic behavior of FRW. It is observed that the leading-edge vortex attached on the wing surface during the whole flapping period, which is the main reason for the high lift generation by FRW. Unlike the ordinary Flapping Wing (FW) flight vehicle which is driven by its own motor to rotate, the flapping rotary wing is driven by the aerodynamic force to rotate passively. Previous works on FRW have mostly assigned a constant rotation velocity by assuming an ‘equilibrium’ state. However, for a practical wing, the inertia forces associated with the complicated kinematics will essentially interact with the aerodynamic force production. The influence of the wing inertia and the dynamic process as the wing converges to the equilibrium status will necessarily have a nontrivial effect on the aerodynamic performance of FRW. The...
varying rotation velocity conversely affects the aerodynamics and flow structure of the flapping rotary wing, resulting in a coupling between the passively rotary motion and aerodynamic force. Therefore, the nonlinear dynamic model, especially for FRW, is needed to analyze its aerodynamic performance.

To date, only a few studies have focused on the dynamics of FRW. However, many studies have been relevant to flapping wing flight vehicle. In these studies, dynamics of FW are generally investigated using standard aircraft equations with six degrees of freedom. However, this approach neglects the inertial effects of the mass of the wings. Recently, some studies have investigated the dynamics from the aspect of multiple-body nonlinear system, such as Gebert and Sun et al. Orlowski and Girard modeled a flapping wing micro air vehicle as a system of three rigid bodies, a body and two wings, and studied the influence of the mass of the wings to the dynamics. Mahjoubi and Byl developed the dynamic multi-body model using Lagrangian method and the proposed control approach to optimize the wings’ mass and mechanical impedance properties of the joints. These studies have indicated that the multiple-body dynamic theory may be used to analyze the dynamics of FRW.

In this paper, a simplified FRW is modeled as three rigid bodies, one for the body of rotary base and others for each wing. The wing pitching motion is assumed to be actively driven through a control servo, as shown in Fig. 1. Thereby, each flapping wing owns three degrees of freedom: the actively flap- ping, pitching and the passively rotating. Using the D’Alembert’s Principle given in Ref.15, a multi-body dynamic model is derived for FRW. In addition, a quasi-steady aerodynamic model is utilized for the calculation of the aerodynamic forces and moments. The motion process of wings is simulated in a selected typical parameter set to understand the coupling with the lift/thrust production. Finally, the effects of the kinematics of wings on the dynamic rotational equilibrium of FWR and the aerodynamic performances are presented.

2. Reference definition

To describe the motion of rotary base in the FWR body frame, and the motion of wings with respect to rotary base, four reference frames are used. The body frame $O_{b}x_{b}y_{b}z_{b}$ is attached to the center of the body of FWR. As shown in Fig. 2, the positive $x_{b}$ axis is along the longitudinal axis of the central body. The $y_{b}$ axis locates in the vertical symmetry plane of body and is perpendicular to the $x_{b}$ axis with a positive upward. The $z_{b}$ axis is perpendicular to the $xO_{b}y$ plane. The unit vectors of the body frame are presented by $e_{x_{b}}, e_{y_{b}}$ and $e_{z_{b}}$.

After rotating an angle $\psi$ about the $y_{b}$ axis of the body frame for the rotary base, it becomes the rotary plane frame $O_{r}x_{r}y_{r}z_{r}$ (shown as the subscript “$r$”). The rotate plane frame defines the rotary motion of two connected wings.

The wing-fixed frames $O_{w}x_{w}y_{w}z_{w}$ are two fixed frames attached to the wings. The initial orientation of the wing-fixed frames is parallel to the rotate plane frame with an origin coincident with the rotation of the wings joint. The orientation of the wings with respect to the rotary plane is determined by the pitch angle $\gamma_{w}$ and flap angle $\alpha_{w}$ of the wings. Here we use the subscripts L, R to represent the left and right wings, respectively.

The rotation matrix from the body frame to rotation plane frame is

$$R_{br} = R_y(\psi_b)$$

As shown in Fig. 2, the wings successively rotate about the $x_r$ and $z_w$ axis with the angles of $\gamma_w$ and $\alpha_w$ to reach the ultimate position. The rotation matrices for the right wing are

$$R_{\alpha R} = R_z(\alpha_{wR})R_x(\gamma_{wR})$$

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$$R_{\alpha R} = R_z(\alpha_{wR})R_x(\gamma_{wR})$$
The rotation matrices for the left wing, with respect to the rotary base, are combined in the same manner as in Eq. (2). The only difference that the signs of \( \gamma_u \) and \( \phi_u \) are interchanged for the right and left wings.

Combining Eqs. (1) and (2), the rotation matrices from the body frame to right and left wing-fixed frames are

\[
R_{bwr} = R_{wr}R_{br}, \quad R_{bwL} = R_{wr}R_{br}
\]  

(3)

3. Dynamic model of FRW

3.1. Method and assumption

The flapping wing vehicle is modeled as a system of three rigid bodies: a central body of rotary base with two rigid wings attached at ideal hinges. The method chosen to derive the equations of motion is D’Alembert’s Principle Extended to Multiple Rigid Bodies.\(^{15}\) The functions of generalized inertia forces are described as

\[
Q_i^j = \sum_{i=1}^{3} (F_i^j \gamma_i^j + M_i^j \beta_i^j)
\]  

(4)

where \( i \) presents the number of rigid bodies and \( j \) denotes the number of generalized coordinates, \( \gamma_i^j \) represents velocity coefficients, \( \beta_i^j \) are angular velocity coefficients. The inertia force \( F_i^j \) and moment \( M_i^j \) of the \( i \)th rigid body are given as

\[
F_i^j = m_i (v_i + \dot{p}_i) \quad M_i^j = I_i \omega_i + \alpha_i \times I_i \omega_i + m_i p_i \times \dot{v}_i
\]  

(5)

where \( m_i, v_i, p_i, \alpha_i \) and \( I_i \) denote the mass, velocities, reference vectors, angular velocities and the resulting mass moments of inertia matrices of the \( i \)th rigid body. In this study, the two wings are assumed to be attached to the rotary base body by joints that allow two degrees of freedom respectively with a common rotation degree of freedom. To simplify the derivation, firstly, the inertia tensors for the individual bodies are calculated with respect to the reference point and they do not need to be calculated at the time-varying center of mass of the system. Then, the body of FRW is assumed to be always fixed on the ground, thereby its motions relative to inertial space are neglected and the body frame is equal to inertial frame. As a result, a dynamic system of three rigid bodies: one for the rotary base, the other two for each wing, are considered. The five degrees of freedom are selected to be described by the generalized coordinates \( x_u \), listed together as

\[
x_u = [\psi_f, \theta_{wR}, \gamma_{wR}, \theta_{wL}, \gamma_{wL}]
\]  

(6)

The related quasi-velocities of coordinates, expressed in inertia frame, are

\[
u_u = [\alpha_{x,f}, \alpha_{x,wR}, \alpha_{x,wR}, \alpha_{x,wL}, \alpha_{x,wL}]
\]  

(7)

The variables \( \alpha_{x,f} \), \( \alpha_{x,wR} \) and \( \alpha_{x,wL} \) describe the angular velocity of the each selected rigid bodies in the body frame. Especially, \( \alpha_{x,f} \) denotes the rotation angular velocity of the rotary base with two wings, and the flapping and pitching angular velocities of each wing are expressed as \([\alpha_{x,wR}, \alpha_{x,wL}]\) and \([\alpha_{x,wL}, \alpha_{x,wL}]\), respectively.

3.2. Velocities and reference vectors

The angular velocity vector of the wing related to rotary base and expressed in rotation plane frame can be obtained by the time derivative of the two Euler angles \( \gamma_u \) and \( \phi_u \), which are assumed to be known as the command input. For right wing, the equation is defined as

\[
\omega^i_{wR} = \begin{bmatrix} \dot{\gamma}_{wR} \\ 0 \\ 0 \end{bmatrix} + R_{(\gamma_{wR})}^T \begin{bmatrix} 0 \\ 0 \\ \dot{\phi}_{wR} \end{bmatrix}
\]  

(8)

Related to body mass center, the joint point owns an angular velocity

\[
\omega_1 = \omega^b_{u,b} = \begin{bmatrix} 0 & \dot{\psi}_f & 0 \end{bmatrix}^T
\]  

(9)

With the combination of Eqs. (8) and (9), the angular velocity of the right and left wings with respect to the body frame, and expressed in the body frame, are

\[
\omega_2 = \omega^b_{u,b} + R_{w} \omega^i_{wR}, \quad \omega_3 = \omega^b_{wL} + R_{w} \omega^i_{wL}
\]  

(10)

The reference vectors denote the position of the center of mass of the \( i \)th body with respect to the reference point. For the rotary base, the reference point is chosen to be its respective center of mass, thereby the reference vector \( \rho_i \) equals zero. In each wing-fixed frame, the position of mass center owns two components along \( x_w \) axis and \( z_w \) axis directions:

\[
\begin{align*}
c_{wR} &= \begin{bmatrix} c_{x,wR} & 0 & c_{z,wR} \end{bmatrix}^T \\
c_{wL} &= \begin{bmatrix} c_{x,wL} & 0 & c_{z,wL} \end{bmatrix}^T
\end{align*}
\]  

The related reference vectors are transformed from the wing-fixed frames according to

\[
\begin{align*}
\rho_2 &= R_{wr}c_{wR} \\
\rho_3 &= R_{wr}c_{wL}
\end{align*}
\]  

(11)

Since the translational velocity of the rotary base, the reference velocity \( v_f \) equals zero. And for each of the wings, the reference point of translational velocity is its joint point. The vectors from the center of rotary base to wing joint points, are expressed as \( r_{wR} \) and \( r_{wL} \). As shown in Fig. 1, the vectors defined in body frame own two components along \( y_b \) axis and \( z_b \) axis directions:

\[
\begin{align*}
r_{wR} &= \begin{bmatrix} 0 & r_y & -r_z \end{bmatrix}^T, & r_{wL} &= \begin{bmatrix} 0 & r_y & r_z \end{bmatrix}^T
\end{align*}
\]  

(12)

The reference velocity, for each of the wings, is the velocity of the respective wing joint in the inertia frame. The velocities of the wings are

\[
\begin{align*}
\dot{\rho}_2 &= \dot{\omega}_1 \times r_{wR} + \omega_1 \times (\dot{\omega}_1 \times r_{wR}) \\
\dot{\rho}_3 &= \dot{\omega}_1 \times r_{wL} + \omega_1 \times (\dot{\omega}_1 \times r_{wL})
\end{align*}
\]  

(13)

3.3. Coefficients

The angular velocity coefficients \( \beta_{ij} \) are necessary for the derivation of dynamic model, which arise from the calculation
of virtual work performed by moments. Each coefficient is vector and is determined for each rigid body and velocity combination. The angular velocity coefficients are defined as

$$\beta_y = \frac{\partial \omega_y}{\partial u_y}$$

The coefficients of center body of FWR are

$$\beta_{ij} = [e_{j,b} 0_{3 \times 1} 0_{3 \times 1} 0_{3 \times 1} 0_{3 \times 1}]$$

The angular coefficients of right and left wings are

$$\beta_{ij} = [R_{ib}^T e_{j,b} R_{ib}^T e_{j,b} R_{ib}^T e_{j,b} 0_{3 \times 1} 0_{3 \times 1}]$$

3.4. Mass moments of inertia

For the rotary base, the mass symmetry for XOy and XOz planes is assumed. No planes of mass symmetry are assumed for either wing during the model development. As a result, the resulting mass moments of inertia matrices for each rigid body are

$$I_1 = \begin{bmatrix} I_{x,b} & 0 & 0 \\ 0 & I_{y,b} & 0 \\ 0 & 0 & I_{z,b} \end{bmatrix}$$

$$I_2 = \begin{bmatrix} I_{x,bl} & -I_{y,bl} & -I_{z,bl} \\ -I_{y,bl} & I_{x,bl} & 0 \\ -I_{z,bl} & 0 & I_{x,bl} \end{bmatrix}$$

3.5. Motion equations of rotary base

The derived equations of passive rotation motion, with all of the individual pieces together, are presented in vector notation.

$$Q_1 = (I_{i,b} \dot{\omega}_i + \omega_i \times I_{i,b} \omega_i) e_{j,b} + (I_{i,b} \dot{\omega}_i + \omega_i \times I_{i,b} \omega_i + m_2 \omega_i \times \dot{v}_i) (R_{ib}^T e_{j,b})$$

$$+ (I_{i,b} \dot{\omega}_i + \omega_i \times I_{i,b} \omega_i + m_3 \omega_i \times \dot{v}_i) (R_{ib}^T e_{j,b})$$

The rotations of the right wing and the left wings are described by Eqs. (18) and (19), respectively.

$$Q_2 = (I_{r,b} \dot{\omega}_r + \omega_r \times I_{r,b} \omega_r + m_2 \omega_r \times \dot{v}_l) (R_{ib}^T e_{j,b})$$

$$Q_3 = (I_{l,b} \dot{\omega}_l + \omega_l \times I_{l,b} \omega_l + m_2 \omega_l \times \dot{v}_l) (R_{ib}^T e_{j,b})$$

$$Q_4 = (I_{r,b} \dot{\omega}_r + \omega_r \times I_{r,b} \omega_r + m_3 \omega_r \times \dot{v}_l) (R_{ib}^T e_{j,b})$$

$$Q_5 = (I_{l,b} \dot{\omega}_l + \omega_l \times I_{l,b} \omega_l + m_3 \omega_l \times \dot{v}_l) (R_{ib}^T e_{j,b})$$

Here, $Q_1$ is the rotational moment acted on rotary base. The generalized forces $Q_2, Q_3$ are the control moment for the right wing, and $Q_4, Q_5$ are the control moments for the left wing.

The rotation moment, expressed in body frame, can be divided as aerodynamic moments $M_{aero}$ produced by flapping wings and gravity moments $M_{mass}$ due to the mass of wings. In this study, the quasi-steady theory is used to calculate the aerodynamic moments $M_{aero}$ produced by flapping wings. The calculation model is given in the following chapter. For each wing, the $M_{mass}$ is calculated according to

$$M_{massR} = (p_2 + r_{wR}) \times \begin{bmatrix} 0 \\ m_2 g \\ 0 \end{bmatrix}$$

$$M_{massL} = (p_3 + r_{wL}) \times \begin{bmatrix} 0 \\ m_3 g \\ 0 \end{bmatrix}$$

where $R_{ib}$ denotes the transfer matrix from inertial frame to body frame. As the assumption of this study, we have $R_{ib} = I$. As a result, the $M_{mass}$ along the $y_b$ axis equals zero. That means the gravity of wings will not produce the rotation moment, if the body of FWR does not have angular motion in inertial frame. Then, the rotation moment has an expression as

$$Q_1 = (M_{aeroR} + M_{aeroL}) e_{j,b}$$

4. Aerodynamic model

In the numerical study of Wu et al. on FRW, a strong span-wise flow on the wing was observed, and the LEV on the FRW wing merged with the tip vortex and the Trailing Edge Vortex (TEV), forming a vortex ring structure that stayed attached on the wing throughout the flapping cycle. These findings suggest that the quasi-steady model used in this study is applicable for modeling the aerodynamic forces of FRW. As a result, in this study, we extended the quasi-steady aerodynamic model to the application of the flapping and simultaneously rotating wing kinematics of FRW.

Firstly, a geometric model of the FWR wing is chosen and the detailed shape and definition of geometric parameters of the wing are given in Appendix A. For blade element analysis, it is convenient to write down the velocity and acceleration of a 2D wing chord due to the gyration of the wing at span-wise location $r$. The resultant velocity and acceleration vector when expressed in the wing-fixed frame are planar vectors with only two nontrivial indices, i.e. the $x_w$ and $y_w$ components:

$$v_w(r) = \omega_w \times r + \omega_w \times (\omega_w \times r)$$

and

$$\dot{v}_w(r) = (\dot{\omega}_w, \dot{\omega}_w) = \omega_w \times r + \omega_w \times (\omega_w \times r)$$

where $e_{x,w}$ and $e_{y,w}$ are the unit vectors of right wing frame; $\omega_w$ is the angular velocity vector of the right wing related to inertia frame and expressed in wing frame. For two wings, the related $\omega_w$ and $\omega_{wL}$ are obtained as

$$\omega_w = R_{ib}^T \omega_z, \quad \omega_{wL} = R_{ib}^T \omega_z$$

Since the velocity and acceleration of wing are expressed in wing frame, the effective Angle of Attack (AOA) of the wing $\alpha_e$ can be easily found by inverse trigonometric function of the velocity components ratio of the wing:

$$\alpha_e = \arctan \left( \frac{v_{wL}}{v_{wR}} \right)$$
In quasi-steady theory of flapping wing, this relationship holds, except that the force vector acts perpendicular to the wing chord. The quasi-steady forces are divided by translational forces, rotational forces and virtual mass forces, and the corresponding coefficients are experimentally measured. Here, we will use this definition. The corresponding equations for translational force $F_t$ and rotational force $F_r$ for two wings are expressed as:

$$dF_t = (dF_{t1}, dF_{t2}) = \frac{1}{2} C_t \rho \| \nu(r) \|^2 c(r) dr$$ (27)

And

$$dF_r = dF_{r1} = -C_p \| \nu(r) \| \omega_{u,v} c(r) dr$$ (28)

where $\rho$ is the density of the surrounding air. $C_t$ is the rotational force coefficient due to wing pitching, and the value of this coefficient is chosen as $C_t = 1.6$ in our calculation. The vector $C_t$ is the translational force coefficient and can be treated as a unit force vector acting on the wing. In the velocity direction, $C_t$ is expressed as life coefficient $C_l$ and drag coefficient $C_d$, and can be approximated by the following equations:

$$\begin{bmatrix} C_L = C_{L_{\text{max}}} \sin(2\alpha) \\ C_D = 0.5(C_{D_{\text{max}}} + C_{D_{\text{p}}}) - 0.5(C_{D_{\text{max}}} - C_{D_{\text{p}}}) \cos(2\alpha) \end{bmatrix}$$ (29)

where the constant coefficients $C_{L_{\text{max}}}, C_{D_{\text{max}}}$ and $C_{D_{\text{p}}}$ at the specific Reynolds number ($Re \approx 4000$) are valued from 3D CFD case calculations result. The values are given as: $C_{L_{\text{max}}} = 0.18$, $C_{D_{\text{max}}} = 3.4$, $C_{D_{\text{p}}} = 0.05$.

Since we calculate the force and moment in the wing frame, the translational force coefficient $C_t$ can be obtained as

$$C_t = \begin{bmatrix} C_H \\ C_V \end{bmatrix} = \begin{bmatrix} \cos \gamma_c & - \sin \gamma_c \\ \sin \gamma_c & \cos \gamma_c \end{bmatrix} \begin{bmatrix} C_D \\ C_L \end{bmatrix}$$ (30)

where $C_H$ is the translational force coefficient along $x_u$ axis, and $C_V$ is the translational force coefficient along $y_u$ axis.

For the calculation of the aerodynamic torque, the location of the Centre of Pressure (CP) at a chord-wise location $r$ is defined as $r_{CP} = x_{CP} c_{x,u} + y_{CP} c_{y,u}$. As a result, the aerodynamic torque of the above two forces can be decided by the following equation:

$$dM_a = r_{CP} \times (dF_t + dF_r)$$ (31)

The virtual mass force and moment are calculated using Sedov’s formula, which is suitable for our coordinate definition:

$$dF_v = (dF_{v1}, dF_{v2}) = \omega_a (\lambda_a v_r + \lambda_{au} \omega_r) e_{x,u} \, dr$$

$$dM_v = r_{CP} \times dF_v$$ (32)

where $v_r$ and $\omega_r$ can be obtained from the vectors $\omega_{x,u}, v_{x,u}$ of the each wing. $\lambda_a$ and $\lambda_{au}$ are the added mass force coefficients, which are obtained as

$$\lambda_a = \frac{\pi \rho c(r)^3}{2}$$

$$\lambda_{au} = \frac{\pi}{4} \rho c(r)^3$$ (33)

After integrating Eqs. (22), (28), (31) and (32) along the wing span orientation, as a result, the total aerodynamic forces and moments, expressed in body frame, are obtained as

$$F_{aero} = R_{aero}^b (F_t + F_r + F_v)$$

$$M_{aero} = R_{aero}^b (M_t + M_r + M_v)$$ (34)

The forces and moments for each wings are calculated based on its velocity and angular velocity respectively. Then, the necessary aerodynamic moments $M_{aero_{ch}}$ and $M_{aero_{flap}}$ in Eq. (22) are obtained.

5. Simulation conditions

5.1. Kinematic functions of wings

In this study, we use simple harmonic functions to describe the flapping and pitching motion of the wing, as previous studies for insects flight. The kinematic functions of the wing is specified by giving the variation functions:

$$\gamma_w = -\frac{\Delta \gamma_w}{2} \sin(f_w t)$$ (35)

$$\vartheta_a = \Delta \vartheta \sin \left( f_a t + \frac{\pi}{2} \right) + \vartheta_0$$ (36)

where $f_w$ is the flapping frequency, $\Delta \gamma_w$ is the flapping amplitude angle, and $\Delta \vartheta$ is the pitching amplitude; for modeling the asymmetric pitching, the angle $\vartheta_0$ is introduced. By this definition, the calculation functions between $\Delta \gamma_w$, $\vartheta_0$, with the geometric AOA of the wing at mid up-stroke angle $\gamma_U$ and mid down-stroke angle $\gamma_D$ are given as

$$\Delta \gamma_w = \frac{\gamma_D - \gamma_U}{2}$$

The time history of flapping motion and pitching motion is plotted as an example in Fig. 3 to illustrate the relationship between flapping motion $\gamma_w$ and pitching motion $\vartheta_a$ ($\Delta \gamma_w = 30^\circ$, $f_w = 22$ Hz, $\gamma_U = -30^\circ$, $\gamma_D = 20^\circ$).

5.2. Nondimensional coefficients

We used the mean chord length of the wing $\bar{c}$ and the mean flapping velocity at the wingtip $v_t = 2\Delta \gamma_w f_w \bar{c}$ as the reference length and reference velocity. The aerodynamic lift and rotation moment coefficients are thus defined as

$$C_L = \frac{F_t}{0.5 \rho \bar{c}^2 \bar{S}}; \quad C_R = -\frac{M_{aero_{rot}}}{0.5 \rho \bar{c} \bar{S} \bar{c}}$$ (38)

![Fig. 3 Kinematic pattern and parameter definition of FRW wing.](image-url)
Here, $F_L$ and $M_{aero}$ mean aerodynamic lift and rotational moment, $\bar{C}_L$ and $\bar{C}_R$ are the averaging aerodynamic lift and rotational moment coefficients during one flapping period, which are obtained by

$$\bar{C}_L = \frac{\int_{t_0}^{t_0+T_F} C_L dt}{T_F}, \quad \bar{C}_R = \frac{\int_{t_0}^{t_0+T_F} C_R dt}{T_F} \quad (39)$$

where $t_0$ is the initial time at the beginning of one flapping period and $T_F$ means the flapping period. If the effect of geometric shape on aerodynamics is not considered, then the flapping period $T_F$ only depends on frequency $f_F$. The energetic cost of the FRW’s wings can be calculated by the time averaged power efficiency coefficient over a flapping period $T_F$. For a practical MAV design, elastic storage is desirable for energy efficiency, of which the order is decided by the design property of the mechanical system.

In the current study, we consider that the mechanical system of the FRW can fully store the input power. The instantaneous aerodynamic power efficiency coefficient for hovering flight due to gyration equals directly minus dot product of the angular velocity vector $\omega_w$ with the aerodynamic torque $M_{aero}$:

$$P_t = -\omega_w \cdot M_{aero} = -\omega_w \cdot M_{aero}. \quad (40)$$

then, the average power efficiency coefficient is given by

$$\bar{P}_t = \frac{\int_{t_0}^{t_0+T_F} (P_t) dt}{T_F} \quad (41)$$

A nondimensional variable $\hat{t} = t/T_F$ is defined to describe clearly the time courses of wing motion during a flapping period. As shown in Fig. 3, $\hat{t} \in [0, 0.5]$ indicates that the wing’s motion is in the phase of up-stroke, whereas $\hat{t} \in [0.5, 1.0]$ indicates that the wing’s motion is in the phase of down-stroke.

5.3. Validation

To validate the compatibility of the aerodynamic model, we compare the calculation results of our model with 3D CFD results presented by Wu et al., who studied the aerodynamic characteristic of FRW at a low Reynolds Number. In Wu’s work, the CFD mode employed a boundary fitted dynamic grid to orientate the wing boundary at a different time with the prescribed kinematics. An OH type mesh was used for flow simulation. Grid 1 has dimensions of $31 \times 33 \times 37$ (in normal, chordwise, and spanwise directions, respectively); and grids 2 and 3 have dimensions of $51 \times 57 \times 61$ and $81 \times 81 \times 91$, respectively. The outer boundary for these grids is located 30c away from the wing surface and 15c away from the wing-tip. The first grid spacings from the wing surface of the three grids are 0.002, 0.001, and 0.0005.

In the benchmark case, the flapping and pitching motions may be defined by the previous descriptions. In the example in CFD results of Wu et al., the dynamic of rotation motion is ignored and the rotation speed is assumed to be constant, and we use the same model to describe the rotation motion in this case.

$$\psi_f = -\psi_R \hat{t} \quad (42)$$

where $\psi_R$ is the rotating speed.

Since the kinematics of FWR is combined by steady rotation and reciprocal flapping motion, a nondimensional rotation speed $k_R$ is defined to measure the deflection of the effective AOA:

$$k_R = \frac{\psi_R}{f_F} \quad (43)$$

Based on Eq. (43), $\psi_R$ can be obtained.

The necessary parameters in Eqs. (35), (37) and (43) for validation case are $\Delta \gamma_0 = 30^\circ$, $f_F = 22$ Hz, $z_U = -30^\circ$, $z_D = 20^\circ$, $k_R = 0.25$. The Reynolds number for flapping flight may be defined by $Re = v_c/c/\nu$, where $v$ is the kinematic viscosity of the air. In this case, we have $Re = 4058$. As shown in Fig. 4, a reasonable agreement is achieved between the lift and rotation moment trends obtained from the present model and CFD results. The averaging coefficients of the present model are $\bar{C}_L = 0.99$ and $\bar{C}_R = 0.173$, while $\bar{C}_L = 0.972$ and $\bar{C}_R = 0.168$ in CFD results. Thereby, the case study shows that the model employed in the present study is credible.

6. Results and analysis

6.1. Analysis of a typical case

A typical case ($\Delta \gamma_0 = 30^\circ$, $f_F = 22$ Hz, $z_U = -30^\circ$, $z_D = 20^\circ$, $Re = 4058$) is selected and discussed in this section to investigate the rotation performance of the FRW with the structural parameters listed in Table 1.

The result of the rotation angular velocity varying with time is presented in Fig. 5, and the rotational moment coefficient $C_R$ and its averaging value $\bar{C}_R$ in each flapping periods are given in Fig. 6. It can be clearly seen that the aerodynamic rotational moment in the up-stroke decreases to negative value with the increase of the rotational velocity, and the aerodynamic average rotational moment finally decreases close to zero after fifteen flapping periods, and then maintains at a small value continually. It differs from other existing kinematics with prescribed wing motion, such as rotary wing and insects flapping wing.

However, note that $\bar{C}_R$ of each period cannot converge to zero strictly and $\omega_w$ oscillates continuously with an amplitude of $\pm 0.5 \text{r/s}$, even after a lot of flapping periods. It is caused by inertia coupling phenomenon of two wings, which are not symmetrical about the axis of rotation. According to Table 1, if the inertial products $I_{x\gamma\omega}$, $I_{z\omega\omega}$ and $I_{z\gamma\omega}$ are small and ignored, according to Eqs. (17) and (22), the inertia coupling rotation moment $M_{\text{coupled}}$ can be expressed as

$$F_R$$

Fig. 4 Comparisons of instantaneous lift and rotational moment coefficients.
Similar to coefficients $C_R$ and $\bar{C}_R$, the coefficients of $M_{\text{coupl}}$ are defined as:

$$C_{\text{Rcoupl}} = \frac{M_{\text{coupl}}}{\frac{\mu}{2}r_p^2 \omega^2_z}$$
$$\bar{C}_{\text{Rcoupl}} = \frac{\int_{0}^{T_f} C_{\text{Rcoupl}} dt}{T_f}$$

Fig. 7 gives the result of rotation coefficients $C_{\text{Rcoupl}}$ and $\bar{C}_{\text{Rcoupl}}$ in each flapping period. After the comparison with Fig. 7, it can be seen that the inertia coupling rotation moment always exists even because of the high-speed flapping and pitching motions of two wings. As a result, the rotational moment is balanced with not only the resistance drag of the fluid, but also inertia coupling moment. However, the inertia coupling moment is small compared with the aerodynamic moment, and the total moment is still zero. Consequently, an FWR would reach and stay in an equilibrium rotation speed. In this study, we define the motion status of $\bar{C}_R$ near zero as the Equilibrium Rotational Status (ERS) for FWR.

Fig. 8 gives the results of lift coefficients $C_L$ and $\bar{C}_L$ in each flapping period. As a comparison, constant rotational velocity model with $\kappa_R = 0.47$ is used to calculate the same case. It can be seen from Fig. 8(a) that the lift coefficient vastly increases with the increasing rotational velocity, especially in the up-stroke, where the large negative lift becomes positive at the rotational equilibrium state. As a result, as shown in Fig. 8(b), the variance of $\bar{C}_L$ is increasing gradually and reaches the equilibrium value finally. Only in this time, the lift force of FRW keeps stable with certain kinematic parameters input. Notably, the lift generation of FRW presents first-order inertia system characteristics for a new kinematics of wings input because of the inertia damping in passive rotation motion. That is different from the motions of rotary wing and insect flapping wing, the life force of whom changes and reaches stable immediately with the variance of wings kinematics.4,13 Thereby, the dynamic inertia time-lag phenomena of lift generation due to passively induced rotational velocity is a unique feature of the FRW configuration.

In order to assess the dynamic system for lift generation, a time-lag constant parameter $\tau_{\text{aero}}$ is defined to present the time cost of arriving ERS. In this case, the $\tau_{\text{aero}}$ is obtained as 0.68 s, which equivalent to 15 flapping period.

The following figures present a comparison between the simulation results of $C_R$ and $\bar{C}_R$ with the full, 2 times, 4 times and 1/2 times rotational moment of inertia of wings $I_{x,w}$. Fig. 9 shows that as the rotational moment of inertia of the wings relative to the central body increase, the inertia time-lag phenomena of lift generation becomes serious. $\tau_{\text{aero}}$ changes from 0.41 s, corresponding to 1/2 times $I_{x,w}$ to 2.72 s, corresponding to 4 times $I_{x,w}$. However, once FWR reaches and stays in an

### Table 1 Basic parameters of FRW.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of wing $m_2, m_3$ (g)</td>
<td>0.15</td>
</tr>
<tr>
<td>Mass of rotation body $m_1$ (g)</td>
<td>0.38</td>
</tr>
<tr>
<td>Inertia moment of wing $I_{x,w}$ (10$^{-6}$ kg m$^2$)</td>
<td>2.225</td>
</tr>
<tr>
<td>$I_{y,w}$ (10$^{-6}$ kg m$^2$)</td>
<td>1.308</td>
</tr>
<tr>
<td>$I_{z,w}$ (10$^{-7}$ kg m$^2$)</td>
<td>9.020</td>
</tr>
<tr>
<td>$I_{x,y,w}$ (10$^{-7}$ kg m$^2$)</td>
<td>1.128</td>
</tr>
<tr>
<td>$I_{x,z,w}$ (10$^{-8}$ kg m$^2$)</td>
<td>5.516</td>
</tr>
<tr>
<td>$I_{y,z,w}$ (10$^{-8}$ kg m$^2$)</td>
<td>1.478</td>
</tr>
</tbody>
</table>

The inertial moment of rotary base $I_{y,b}$ (10$^{-8}$ kg m$^2$) is 4.8
equilibrium rotational status, the lift generation is stable and not affected by the variance of $I_y$. Thus, a small rotational moment of inertia of wings is useful to decrease time-lag constant.

6.2. Effects of the kinematics of wings

As given in Eq. (36), the pitching kinematics of wings is described as sinusoidal wave functions. For a certain flapping frequency $f_F$, the pitch angle of the wing at any instantaneous time is decided by AOAs parameters: mid up-stroke $a_U$ and mid down-stroke $a_D$. The two parameters, donated pitching kinematics, may be selected to analyze the influence of wings kinematics to aerodynamic performance.

In order to present the aerodynamic performance of FRW in the equilibrium rotational status, the period average lift coefficient $C_{L,stab}$, power efficiency coefficient $\tilde{P}_{e,stab}$ and nondimensional rotational velocity $\bar{\omega}_{r,stab}$ are defined as

$$
C_{L,stab} = \frac{\sum_{i=1}^{j} C_{L}}{j}
$$

$$
\tilde{P}_{e,stab} = \frac{\sum_{i=1}^{j} P_{e}}{j}
$$

$$
\bar{\omega}_{r,stab} = \frac{\sum_{i=1}^{j} \omega_{r}}{j} \frac{1}{\Delta_i f_F}
$$

where $i$ means the flapping period while FRW has been in the ERS, $j$ is the total number of flapping period used to calculate coefficients, and $\tilde{P}_{e,stab}$ is defined as positive always. In this case, we let $j = 10$ to acquire accurate description of aerodynamic performance for FRW.

In a typical case of $\Delta_{f_w} = 30^\circ$, $f_F = 22$ Hz, $Re = 4058$, Fig. 10 presents the results of coefficients $C_{L,stab}$, $\tilde{P}_{e,stab}$, $\bar{\omega}_{r,stab}$ and $\tau_{aero}$, while the mid-up stroke $a_U$ varies from $0^\circ$ to
\[ -90^\circ \text{ with increments of } -2.5^\circ, \text{ and the mid-down stroke } z_D \text{ is fixed to } z_D = 10^\circ, 20^\circ, 30^\circ. \]

As shown in Fig. 10(a), notice that the lift of FRW is near zero when the \( \alpha_U \) equals \( -z_D \), e.g. the lift produced in up-stroke and down-stroke may cancel each other. In order to acquire a high positive lift generation of FRW, we need to increase \( \alpha_U \) and decrease \( \alpha_D \). Since in the up-stroke, the high value of \( \alpha_U \) makes the wing surface close to the airflow direction decided by rotation motion and flapping motion together. As a result, the negative effective AOA of wing \( \alpha_e \) decreases, or even changes to be positive. The negative lift generated in up-stroke phase decreases consequently. On the contrary, the small value of \( z_D \) increases the \( \alpha_e \) at down-stroke phase, which produces more positive lift force. The power efficiency does not increase with the lift performance improving, while compared with Fig. 10(a) and (b). Since the high drag force follows as high lift generation, the optimal lift generation does not correspond to the optimal power efficiency. In addition, a high equilibrium rotation velocity occurs at small AOA parameters while both \( \alpha_U \) and \( \alpha_D \) are within \( 20^\circ \) to \( -20^\circ \). However, as shown in Fig. 10(c) and (d), high \( \dot{\theta}_{stab} \) means more time is required to arrive ERS, thereby the \( \tau_{aero} \) becomes large for a small \( \alpha_U \) and \( z_D \).

In order to present the effect of pitching kinematic on FRW, three zones are defined, which include high lift force zone (\( C_{L,stab} > 1.5 \)), high power efficiency zone

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**Fig. 9** Averaging rotation moment coefficient \( \bar{C}_m \) and averaging lift coefficient \( \bar{C}_L \) in each flapping period.

**Fig. 10** Period average lift coefficient \( \bar{C}_{L,stab} \), power coefficient \( \bar{P}_{f,stab} \), nondimensional rotational velocity \( \dot{\theta}_{stab} \) and time-lag constant parameter \( \tau_{aero} \) variations with mid-up stroke \( \alpha_U \) and mid-down stroke \( z_D \).
The pitching kinematics of wings greatly affects the equilibrium rotational characteristics, thus the aerodynamic performance of FRW. The lift force generation, power efficiency, equilibrium rotational velocity and dynamic time-lag are studies for various AOA parameters of wings. The result shows that in order to acquire a high positive lift generation with high power efficiency and small dynamic time-lag, a relative high mid up-stroke \( \alpha_U \) and low mid down-stroke \( \alpha_D \) are necessary. In the zone where \( \alpha_D \) is within \( 7\text{–}15^\circ \) and \( \alpha_U \) is within \( -35^\circ \text{ to } -40^\circ \), the performance of FRW is optimal.

### Appendix A.

The geometry of the FRW wing is modeled by keeping the morphological parameters of quasi-static analysis similar with available insects’ data. In generation, we assume that the thickness of the wing is small enough and has little effect on the wing’s aerodynamic. Thereby, a 2D wing with a span length \( R \) is shown in Fig. A1. Along the span direction, the wing is dived by infinite small strip with width \( dr \). The chord length \( c(r) \) is shown at a chord-wise location \( r \); \( h \) refers to the coordinate of the major axis of the ellipse in \( x_a \) axis. The Center of Gravity (CG) of wing has two components \( x_{CG}, z_{CG} \) in the wing frame. The AR is within 3–5; the shape parameters, including wing aspect ratio AR, the first, second and third radii of nondimensional moment of wing area \( (r_1(s), r_2(s) \) and \( r_3(s) \) are: \( AR = 3.6, \quad r_1(s) = 0.55, \quad r_2(s) = 0.59, \quad r_3(s) = 0.63. \)

The chord-wise location of Centre of Pressure (CP) is in the \( x_a \) axis. Referring to Dickinson et al., the \( x_{CP} \) with respect to location \( r \) varies linearly with the change of the effective AOA \( \alpha_e \) (given in Eq. (26)) and the linear relation can be expressed as

\[
x_{CP}(r) = \left( \hat{h} + \frac{0.82}{\pi} |\alpha_e(r)| - 0.45 \right) c(r)
\]

where \( \hat{h} \) is the nondimensional local coordinate of the chord: \( \hat{h} = h/c(r) \).

### Table 2  Maximum nondimensional parameters and the corresponding geometric AOAs.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Maximum value</th>
<th>Related parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{L_{stab}} )</td>
<td>1.975</td>
<td>( \alpha_U ) (( ^\circ ))</td>
</tr>
<tr>
<td>( P_{a_{stab}} )</td>
<td>3.14</td>
<td>-20</td>
</tr>
</tbody>
</table>

### Fig. 11  Zones of high lift, power efficiency and less time-lag distribution variations with mid-up stroke \( \alpha_U \) and mid-down stroke \( \alpha_D \).

### Fig. A1  Geometric parameters definition of FWR wing.
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Nonlinear dynamics of a flapping rotary wing: Modeling and optimal wing kinematic analysis

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