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Bilinear Modelling and Attitude Control of a Quadrotor

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Abstract: The design of a bilinear controller for a quadrotor and its subsequent stability and performance are presented. A Carleman bilinearization technique is applied to the the nonlinear equations of motion of a quadrotor to obtain a bilinear model which is used as the basis for a bilinear PD controller design. For comparison purposes, a linear model of the quadrotor is also developed and used as the basis for PD controller design. Results for a transient simulation of the proposed BPD controller are presented and compared with that of the PD controller. The results show that the bilinear PD controller gives more improved responses over a broader operating range with respect to stability and performance compared to the PD controller.

Keywords: Quadrotor, Bilinear modelling, Attitude control, Bilinear PD

1. INTRODUCTION

Advances in electronics and electrical motor technology in recent years have led to development of small, low-cost Unmanned Aerial Vehicles (UAVs). Among various types of UAVs, the quadrotor (or quadcoptor) as shown in Fig. 1, is one of the most common. They are a very popular platform amongst hobbyists and research laboratories and are increasingly being used as observation platforms for a number of roles. The main reasons for this appears to be their mechanical simplicity (compared to traditional rotorcraft) which means very low cost.

Despite lacking inherent stability, the simplicity of quadrotors means they are relatively easy to control with simple Proportional plus Derivative (PD) feedback controllers, at least for non-aggressive manoeuvres in calm conditions. However, the problem is much more challenging when taking into account different practical issues (such as parametric uncertainty, external disturbances, motor failures, etc.) and maintaining flight path tracking performance and stability over a wide range of operating conditions. Hence a wide variety of control methods have been attempted. For example, Bouabdallah et al. (2004a) achieved satisfactory results for a PID controller. Using a quarternion description, the attitude can be controlled using just PD control (Tayebi and McGilvray, 2006). More recently, PD control (Marks et al., 2012) has been shown in simulation to achieve good responses even for high upset angles. Satisfactory trajectory tracking and attitude control can be obtained with LQR control (Cowling et al., 2006, 2010; Rinaldi et al., 2013). Aside from the linear control techniques, nonlinear control techniques such as sliding mode and backstepping have also been applied by, for example, Shaik and Whidborne (2016) and Madani and Benallegue (2006), respectively. In general, linear control techniques may be unable to sustain required degree of control over the full range of operation (particularly for aggressive manoeuvres), whilst nonlinear controllers require greater design time and computational effort (Raptis and Valavanis, 2011, p 13). Bilinear controllers provide a compromise between these two types (Martineau et al., 2004). This paper proposes their use for quadrotors.

Fig. 1. Draganfly X-pro quadrotor (developed by Draganfly Innovations Inc)

For linear control design, the aircraft dynamics are usually approximated as linear models that are obtained by a first order Taylor series approximation of the nonlinear model at a particular point of operation. It is clear that such linear models may be inaccurate over a wider range of operation, hence bilinear models have been proposed to more accurately describe the nonlinear systems (see, for example, Bruni et al. (1974); Schwarz and Dorissen (1989)). Bilinear models can characterize nonlinear properties more accurately than linear models, and hence broaden the range of adequate performance. In this paper, a bilinear model of the quadrotor is developed by applying a Carleman Bilinearization technique, described in Ghasemi et al. (2014), on the nonlinear equations of motion (EOM) of the quadrotor. The generalized state space representation of a multiple-input multiple-output (MIMO) bilinear system is expressed as (Kim and Lim, 2003):

$$\dot{x} = Ax + \left(B + \sum_{i=1}^{N} x_i M_i \right) u \qquad (1)$$

where A, B and M_i are constant matrices of suitable dimensions, $u \in \mathbb{R}^{m\times 1}$ denotes the control vector, $x \in \mathbb{R}^{n\times 1}$ represents the vector of state variables and N denotes the number of expansion terms and augmented states.

In addition, this paper presents a design for a Bilinear Proportional Derivative (BPD) controller for the attitude control of the quadrotor. This controller simply incorporates a single term (Martineau et al., 2004) to extend a PD controller. This controller has the advantages of simple implementation compared to non-linear controllers but with improved performance and stability over a broader operating limit compared to a PD controller.

This paper is structured so that the bilinear and linear models of the quadrotor are presented in Section 2. Section 3 describes the designs of PD and BPD controllers for the attitude control of the quadrotor. The transient simulation results of both PD and BPD controllers are presented in Section 4. The conclusions of the paper are provided in the last section.

2. QUADROTOR MODELS

The general planform of a quadrotor has four propellers in a cross configuration driven by electric motors, as shown in Fig. 1. The pair of rotors on the same axis rotate in the same direction, but one of the pairs spins clockwise while the other counter-clockwise. By varying the rotation speeds, moments can be generated about each axis. Moments about the horizontal axes are produced by varying the difference in the speeds of an axis pair, whilst the moment about the vertical axis is obtained from the difference in the drag torques between the clockwise and counter-clockwise rotating motors.

A lot of work has been done on the mathematical modelling of a quadrotor and the equations of motion are well established Madani and Benallegue (2006); Shaik and Whidborne (2016); Bouabdallah et al. (2004a). In this research, the Newton-Euler approach is used (Bouabdallah et al., 2004b; Mian and Wang, 2008; Michini et al., 2011; Xu et al., 2016, for example) with the following assumptions:

- the structure is rigid and symmetric,
- the propellers are rigid,
- the rotor thrust is proportional to the square of the speed of the rotor,
- the rotor axes are parallel and lie in the z direction,
- aerodynamic drag and ground effects are neglected,
- the inertia matrix is diagonal,
- the rotor Coriolis force and wind forces are not included, and
- the motor dynamics are ignored.

The basic vehicle configuration, Earth frame, E, and body frame, B, are shown in Fig. 2. The body frame has the axes originating at the center of mass of the vehicle. An inertial coordinate frame is fixed to the Earth and has axes in the conventional North-East-Down arrangement. It is assumed that the Earth is flat and stationary. Each rotor provides a thrust force, f_i, and torque, τ_i. These combine to a vector of moments about the body axis, $\mathbf{M} = [L, M, N]^T$ and a thrust force in the negative z direction, $-T$.

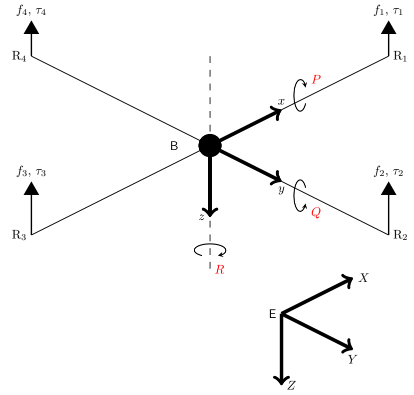

Fig. 2. Quadrotor schematic

The orthogonal rotation matrix $\mathbf{S}_b$ to transform from body frame to Earth frame is (Cook, 2013):

$$\mathbf{S}_b = \begin{bmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi c_\psi + s_\phi s_\theta s_\psi & c_\theta s_\phi \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\theta c_\phi \end{bmatrix} \quad (2)$$

where c_θ denotes $\cos\theta$, s_θ denotes $\sin\theta$, etc and (ϕ, θ, ψ) is the standard Euler angle roll-pitch-yaw triplet. The gravitational force vector, $\mathbf{F}_g$, in the body axis is

$$\mathbf{F}_g = m\,\mathbf{S}_b \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = mg \begin{bmatrix} -s_\theta \\ c_\theta s_\phi \\ c_\theta c_\phi \end{bmatrix} \quad (3)$$

where g is gravitational field constant which is taken as $g = 9.81\text{N kg}^{-1}$.

The Newton-Euler equations of motion of the body axes frame are

$$\mathbf{F} = m\dot{\mathbf{V}} + \boldsymbol{\omega} \times m\mathbf{V} \quad (4)$$

$$\mathbf{M} = \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} \quad (5)$$

where $\mathbf{V} = [U, V, W]^T$ is the vector of velocities in the body frame, $\boldsymbol{\omega} = [P, Q, R]^T$ is the vector of angular rates in the body frame, $\mathbf{I} = \text{diag}(I_x, I_y, I_z)$ is the moments of inertia matrix, m is the mass of the vehicle, $\mathbf{F} = \mathbf{F}_g + [0, 0, -T]^T$ is the vector of the forces acting on the center of mass, and $\mathbf{M} = [L, M, N]^T$ is the vector of moments acting about the center of mass.

Expanding and rearranging (4) gives

$$\begin{bmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{bmatrix} = \frac{1}{m} \begin{bmatrix} 0 \\ 0 \\ -T \end{bmatrix} + g \begin{bmatrix} -s_\theta \\ c_\theta s_\phi \\ c_\theta c_\phi \end{bmatrix} - \begin{bmatrix} QW - RV \\ RU - PW \\ PV - QU \end{bmatrix} \quad (6)$$

Similarly expanding and rearranging (5) gives

$$\begin{bmatrix} \dot{P} \\ \dot{Q} \\ \dot{R} \end{bmatrix} = \begin{bmatrix} L/I_x \\ M/I_y \\ N/I_z \end{bmatrix} - \begin{bmatrix} QR(I_z - I_y)/I_x \\ RP(I_x - I_z)/I_y \\ PQ(I_y - I_x)/I_z \end{bmatrix} \quad (7)$$

The rotation matrix, $\mathbf{S}_b$, from (2) is used to express the movement of the vehicle in the Earth axes once the body-centric velocities are known:

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \mathbf{S}_b^T \begin{bmatrix} U \\ V \\ W \end{bmatrix} \quad (8)$$

$$= \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} \quad (9)$$

The Euler angle rates are related to the body angle rates through:

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} 1 & 0 & -s_\theta \\ 0 & c_\phi & s_\phi c_\theta \\ 0 & -s_\phi & c_\theta c_\phi \end{bmatrix} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} \tag{10}$$

giving:

$$\begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} = \begin{bmatrix} 1 & t_\theta s_\phi & t_\theta c_\phi \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi/c_\theta & c_\phi/c_\theta \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \tag{11}$$

In order to relate the translational motion from the body frame to inertial frame, (9) is differentiated and $\dot{\mathbf{S}}_b^T$ is neglected, which gives

$$\begin{bmatrix} \ddot X \\ \ddot Y \\ \ddot Z \end{bmatrix} = \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} \begin{bmatrix} \dot U \\ \dot V \\ \dot W \end{bmatrix} \tag{12}$$

Assuming that the Coriolis terms are negligible and substituting (6) in (12), the translational EOMs represented in the inertial frame is given as

$$\begin{bmatrix} \ddot X \\ \ddot Y \\ \ddot Z \end{bmatrix} = \begin{bmatrix} -T/m(c_\phi s_\theta c_\psi + s_\phi s_\psi) \\ -T/m(c_\phi s_\theta s_\psi + s_\phi c_\psi) \\ -T/m(c_\phi c_\theta) + g \end{bmatrix} \tag{13}$$

Furthermore, in order to relate the rotational motion from the body frame to inertial frame, (11) is differentiated and (10) is substituted, giving

$$\begin{bmatrix} \ddot\phi \\ \ddot\theta \\ \ddot\psi \end{bmatrix} = \begin{bmatrix} 0 & \dot\phi t_\theta & \dot\theta/c_\theta \\ 0 & 0 & -\dot\phi c_\theta \\ 0 & \dot\psi/c_\theta & \dot\theta t_\theta \end{bmatrix} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} + \begin{bmatrix} 1 & t_\theta s_\phi & t_\theta c_\phi \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi/c_\theta & c_\phi/c_\theta \end{bmatrix} \begin{bmatrix} \dot P \\ \dot Q \\ \dot R \end{bmatrix} \tag{14}$$

Assuming that the Coriolis terms in (7) are negligible, substituting (7) in (14) gives

$$\begin{bmatrix} \ddot\phi \\ \ddot\theta \\ \ddot\psi \end{bmatrix} = \begin{bmatrix} \dot\theta\dot\phi t_\theta + \dot\theta\dot\psi/c_\theta + L/I_x + M/I_y s_\phi t_\theta + N/I_z c_\phi t_\theta \\ -\dot\phi\dot\psi c_\theta + M/I_y c_\phi - N/I_z s_\phi \\ \dot\phi\dot\theta/c_\theta + \dot\theta\dot\psi t_\theta + M/I_y s_\phi/c_\theta + N/I_z c_\phi/c_\theta \end{bmatrix} \tag{15}$$

This paper focuses on the attitude control of the quadrotor, hence, the x and y axes of (13) are ignored, while the z axis is required for the altitude control as the control inputs are coupled to obtain the for voltage for each rotor. Therefore, the simplified quadrotor attitude model with respect to the inertial frame is given as

$$\ddot Z = -T/m(c_\phi c_\theta) + g \tag{16}$$

$$\ddot\phi = \dot\theta\dot\phi t_\theta + \dot\theta\dot\psi/c_\theta + L/I_x + M/I_y\, s_\phi t_\theta + N/I_z\, c_\phi t_\theta \tag{17}$$

$$\ddot\theta = -\dot\phi\dot\psi c_\theta + M/I_y\, c_\phi - N/I_z\, s_\phi \tag{18}$$

$$\ddot\psi = \dot\phi\dot\theta/c_\theta + \dot\theta\dot\psi t_\theta + M/I_y\, s_\phi/c_\theta + N/I_z\, c_\phi/c_\theta \tag{19}$$

2.1 Bilinear Model

With (16) decoupled and built separately in the full quadrotor model where the rotor dynamics are considered (see Seah (2016) for more details), the state of the quadrotor attitude model is defined as

$$\mathbf{x} = \begin{bmatrix} \phi & \dot\phi & \theta & \dot\theta & \psi & \dot\psi \end{bmatrix}^T \tag{20}$$

while the control inputs are

$$\mathbf{u} = [L \;\; M \;\; N]^T \tag{21}$$

Therefore, the quadrotor attitude model can be put in the general state space form

$$\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x}) + \mathbf{b}(\mathbf{x})\mathbf{u} \tag{22}$$

where

$$\mathbf{a}(\mathbf{x}) = \begin{bmatrix} \dot\phi \\ \dot\theta\dot\phi t_\theta + \dot\theta\dot\psi/c_\theta \\ \dot\theta \\ -\dot\phi\dot\psi c_\theta \\ \dot\psi \\ \dot\phi\dot\theta/c_\theta + \dot\theta\dot\psi t_\theta \end{bmatrix}, \tag{23}$$

$$\mathbf{b}(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 1/I_x & s_\phi t_\theta/I_y & c_\phi t_\theta/I_z \\ 0 & 0 & 0 \\ 0 & c_\phi/I_y & -s_\phi/I_z \\ 0 & 0 & 0 \\ 0 & 1/I_y\, s_\phi/c_\theta & 1/I_z\, c_\phi/c_\theta \end{bmatrix} \tag{24}$$

The Carleman Bilinearization technique described in Krener (1974) and Ghasemi et al. (2014) is applied to the nonlinear quadrotor attitude model to obtain a bilinear quadrotor attitude model in the form of

$$\dot{\delta x} = A\delta x + \left(B + \sum_{i=1}^{N} \delta x_i M_i\right) u \tag{25}$$

To proceed, Jacobians of $\mathbf{a}(\mathbf{x})$, $\mathbf{b}(\mathbf{x})$ and Hessians of $\mathbf{a}(\mathbf{x})$ are required. The Jacobians of each term in $\mathbf{a}(\mathbf{x})$ of (23) (denoted by $a_i'(\mathbf{x})$) are

$$a_1'(\mathbf{x}) = [0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0]$$

$$a_2'(\mathbf{x}) = \left[0 \;\; \dot\theta t_\theta \;\; \left(\frac{\dot\theta(\dot\phi + \dot\psi s_\theta)}{c_\theta^2}\right) \;\; \left(\dot\phi t_\theta + \frac{\dot\psi}{c_\theta}\right) \;\; 0 \;\; \frac{\dot\theta}{c_\theta}\right]$$

$$a_3'(\mathbf{x}) = [0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0]$$

$$a_4'(\mathbf{x}) = [0 \;\; -\dot\psi c_\theta \;\; \dot\phi\dot\psi s_\theta \;\; 0 \;\; 0 \;\; \dot\phi c_\theta]$$

$$a_5'(\mathbf{x}) = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1]$$

$$a_6'(\mathbf{x}) = \left[0 \;\; \frac{\dot\theta}{c_\theta} \;\; \left(\frac{\dot\theta(\dot\phi s_\theta + \dot\psi)}{c_\theta^2}\right) \;\; \left(\frac{\dot\phi + \dot\psi s_\theta}{c_\theta}\right) \;\; 0 \;\; \dot\theta t_\theta\right]$$

The Hessians of each term in $\mathbf{a}(\mathbf{x})$ of (23) (denoted by $a_i''(\mathbf{x})$) are

$$a_2''(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\dot\theta}{c_\theta^2} & t_\theta & 0 & 0 \\ 0 & \frac{\dot\theta}{c_\theta^2} & \left(\frac{\dot\theta(\dot\psi + \dot\psi s_\theta^2 + 2\dot\phi s_\theta)}{c_\theta^3}\right) & \frac{\dot\phi + \dot\psi s_\theta}{c_\theta^2} & 0 & \frac{\dot\theta s_\theta}{c_\theta^2} \\ 0 & t_\theta & \left(\frac{\dot\phi + \dot\psi s_\theta}{c_\theta^2}\right) & 0 & 0 & \frac{1}{c_\theta^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\dot\theta s_\theta}{c_\theta^2} & \frac{1}{c_\theta^2} & 0 & 0 \end{bmatrix}$$

$$a_4''(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dot\psi s_\theta & 0 & 0 & -c_\theta \\ 0 & \dot\psi s_\theta & \dot\phi\dot\psi c_\theta & 0 & 0 & \dot\phi s_\theta \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_\theta & \dot\phi s_\theta & 0 & 0 & 0 \end{bmatrix}$$

$$
a_6''(\mathbf{x}) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \dfrac{\dot{\theta} s_\theta}{c_\theta^2} & \dfrac{1}{c_\theta} & 0 & 0 \\
0 & \dfrac{\dot{\theta} s_\theta}{c_\theta^2} & \left(\dfrac{\dot{\theta}(\dot{\phi} + \dot{\phi} s_\theta^2 + 2\dot{\psi} s_\theta)}{c_\theta^3}\right) & \dfrac{\dot{\phi} s_\theta + \dot{\psi}}{c_\theta^2} & 0 & \dfrac{\dot{\theta}}{c_\theta^2} \\
0 & \dfrac{1}{c_\theta} & \dfrac{\dot{\phi} s_\theta + \dot{\psi}}{c_\theta^2} & 0 & 0 & t_\theta \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \dfrac{\dot{\theta}}{c_\theta^2} & t_\theta & 0 & 0
\end{bmatrix}
$$

The Jacobians of each row of $\mathbf{b}(\mathbf{x})$ of (24) are

$$
b_2'(\mathbf{x}) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\dfrac{c_\phi t_\theta}{I_{yy}} & 0 & \dfrac{s_\phi}{I_{yy} c_\theta^2} & 0 & 0 & 0 \\
\dfrac{-s_\phi t_\theta}{I_{zz}} & 0 & \dfrac{c_\phi}{I_{zz} c_\theta^2} & 0 & 0 & 0
\end{bmatrix}
$$

$$
b_4'(\mathbf{x}) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\dfrac{-s_\phi}{I_{yy}} & 0 & 0 & 0 & 0 & 0 \\
\dfrac{-c_\phi}{I_{zz}} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
b_6'(\mathbf{x}) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\dfrac{c_\phi}{I_{yy} c_\theta} & 0 & \dfrac{s_\phi s_\theta}{I_{yy} c_\theta^2} & 0 & 0 & 0 \\
\dfrac{-s_\phi}{I_{zz} c_\theta} & 0 & \dfrac{c_\phi s_\theta}{I_{zz} c_\theta^2} & 0 & 0 & 0
\end{bmatrix}
$$

Expanding as a truncated Taylor series and substituting (20) gives

$$
\dot{\delta x}_1 = \delta x_2 \tag{26}
$$

$$
\begin{aligned}
\dot{\delta x}_2 = {} & \frac{x_4 s_{x_3}}{c_{x_3}} \delta x_2 + \frac{x_4 (x_2 + x_6 s_{x_3})}{c_{x_3}^2} \delta x_3 + \frac{x_4}{c_{x_3}} \delta x_6 \\
& + \left(x_2 t_{x_3} + \frac{x_6}{c_{x_3}}\right) \delta x_4 + \frac{2 x_4}{c_{x_3}^2} \delta x_2 \delta x_3 + \frac{2 s_{x_3}}{c_{x_3}} \delta x_2 \delta x_4 \\
& + \frac{x_4 (x_6 + x_6 s_{x_3}^2 + 2 x_2 s_{x_3})}{c_{x_3}^3} (\delta x_3)^2 + \frac{2(x_2 + x_6 s_{x_3})}{c_{x_3}^2} \delta x_3 \delta x_4 \\
& + \frac{2(x_4 s_{x_3})}{c_{x_3}^2} \delta x_3 \delta x_6 + \frac{2}{c_{x_3}} \delta x_4 \delta x_6 + \frac{c_{x_1} t_{x_3}}{I_y} \delta x_1 M \\
& - \frac{s_{x_1} t_{x_3}}{I_z} \delta x_1 N + \frac{s_{x_1}}{I_y c_{x_3}^2} \delta x_3 M + \frac{c_{x_1}}{I_z c_{x_3}^2} \delta x_3 N
\end{aligned} \tag{27}
$$

$$
\dot{\delta x}_3 = \delta x_4 \tag{28}
$$

$$
\begin{aligned}
\dot{\delta x}_4 = {} & -x_6 c_{x_3} \delta x_2 + x_2 x_6 s_{x_3} \delta x_3 + x_2 c_{x_3} \delta x_6 \\
& + x_2 x_6 c_{x_3} (\delta x_3)^2 + 2 x_6 s_{x_3} \delta x_2 \delta x_3 - 2 c_{x_3} \delta x_2 \delta x_6 \\
& + 2 x_2 s_{x_3} \delta x_3 \delta x_6 - \frac{s_{x_1}}{I_y} \delta x_1 M - \frac{c_{x_1}}{I_z} \delta x_1 N
\end{aligned} \tag{29}
$$

$$
\dot{\delta x}_5 = \delta x_6 \tag{30}
$$

$$
\begin{aligned}
\dot{\delta x}_6 = {} & \frac{x_4}{c_{x_4}} \delta x_2 + \frac{x_4 (x_2 s_{x_3} + x_6)}{c_{x_3}^2} \delta x_3 + \frac{x_2 + x_6 s_{x_3}}{c_{x_3}} \delta x_4 \\
& + x_4 t_{x_3} \delta x_6 + \frac{2 x_4 s_{x_3}}{c_{x_3}^2} \delta x_2 \delta x_3 + \frac{2}{c_{x_3}} \delta x_2 \delta x_4 \\
& + \frac{2 x_4}{c_{x_3}^2} \delta x_3 \delta x_6 + \frac{x_4 (x_2 + x_2 s_{x_3}^2) + 2 x_6 s_{x_3}}{c_{x_3}^3} (\delta x_3)^2 \\
& + \frac{2(x_2 s_{x_3} + x_6)}{c_{x_3}^2} \delta x_3 \delta x_4 + 2 t_{x_3} \delta x_4 \delta x_6 + \frac{c_{x_1}}{I_y c_{x_3}} \delta x_1 M \\
& - \frac{s_{x_1}}{I_z c_{x_3}} \delta x_1 N + \frac{s_{x_1} s_{x_3}}{I_y c_{x_3}^2} \delta x_3 M + \frac{c_{x_1} s_{x_3}}{I_z c_{x_3}^2} \delta x_3 N
\end{aligned} \tag{31}
$$

Using (26)-(31), the rates $\mathrm{d}(\delta x_i \delta x_j)/\mathrm{d}t = \dot{\delta x}_i \delta x_j + \delta x_i \dot{\delta x}_j$ where $i, j = 1,2,3...6$ are evaluated as (the omitted equations are trivially obtained)

$$
\begin{aligned}
\frac{\mathrm{d}(\delta x_1)^2}{\mathrm{d}t} &= 2 \dot{\delta x}_1 \delta x_1 \\
\frac{\mathrm{d}(\delta x_1 \delta x_2)}{\mathrm{d}t} &= \dot{\delta x}_1 \delta x_2 + \delta x_1 \dot{\delta x}_2 \\
\vdots \qquad & \qquad \vdots \\
\frac{\mathrm{d}(\delta x_6)^2}{\mathrm{d}t} &= 2 \dot{\delta x}_6 \delta x_6
\end{aligned} \tag{32}
$$

Defining an augmented state vector for Carleman bilinearization as:

$$
\mathbf{x}^{\otimes} = \left[\delta x_1 \ \delta x_2 \ \delta x_3 \ \delta x_4 \ \delta x_5 \ \delta x_6 \ (\delta x_1)^2 \ \delta x_1 \delta x_2 \cdots (\delta x_6)^2\right]^T
$$

and evaluating (26)-(31) and (32) about the hover condition

$$
\mathbf{x} = [x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6]^T = [0 \ 0 \ 0 \ 0 \ 0 \ 0]^T \tag{33}
$$

gives the bilinear quadrotor attitude model

$$
\dot{\delta x}_1 = \delta x_2 \tag{34}
$$

$$
\dot{\delta x}_2 = 2 \delta x_4 \delta x_6 + \delta x_3 N / I_z \tag{35}
$$

$$
\dot{\delta x}_3 = \delta x_4 \tag{36}
$$

$$
\dot{\delta x}_4 = 2 \delta x_2 \delta x_6 - \delta x_1 N / I_z \tag{37}
$$

$$
\dot{\delta x}_5 = \delta x_6 \tag{38}
$$

$$
\vdots \qquad \vdots
$$

$$
\dot{(\delta x_6)^2} = 2 \delta x_1 \delta x_6 M / I_y \tag{39}
$$

2.2 Linear Model

To obtain a linear model, (16) - (19) are linearised using Taylor series approximation in the usual way. At equilibrium hover condition, the values of the states are

$$
\phi_0 = \dot{\phi}_0 = \theta_0 = \dot{\theta}_0 = \psi_0 = \dot{\psi}_0 = 0 \tag{40}
$$

$$
\ddot{z}_e = \frac{m(g - \ddot{z}_0)}{\cos \phi_0 \cos \theta_0} \tag{41}
$$

Putting (40) and (41) into (16) - (19), the linear quadrotor attitude model is given as

$$
\ddot{Z} = -\frac{T}{m} \ ; \ \ddot{\phi} = \frac{L}{I_x} \ ; \ \ddot{\theta} = \frac{M}{I_y} \ ; \ \ddot{\psi} = \frac{N}{I_z} \tag{42}
$$

3. ATTITUDE CONTROLLER DESIGN

The proposed BPD controller design, shown in Fig. 3, where $\mathbf{p} = [\phi, \theta, \psi]^T$, is a combination of a bilinear compensator and a standard linear PD controller. The structure is based on that proposed by Inyang et al. (2016) for application to automatic oilwell drilling systems. The bilinear compensator is incorporated to account for the nonlinearities in the quadrotor attitude model.

3.1 PD Controller

The PD control for the Z, ϕ, θ and ψ control channels are respectively:

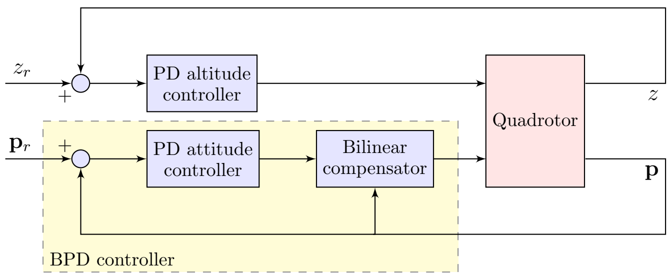

Fig. 3. BPD control architecture (simplified)

$$T = (Z_{\text{ref}} - Z)k_{pz} + (\dot{Z}_{\text{ref}} - \dot{Z})k_{dz}\frac{\tau_z}{s+\tau_z} \tag{43}$$

$$\tilde{L} = (\phi_{\text{ref}} - \phi)k_{p\phi} + (\dot{\phi}_{\text{ref}} - \dot{\phi})k_{d\phi}\frac{\tau_\phi}{s+\tau_\phi} \tag{44}$$

$$\tilde{M} = (\theta_{\text{ref}} - \theta)k_{p\theta} + (\dot{\theta}_{\text{ref}} - \dot{\theta})k_{d\theta}\frac{\tau_\theta}{s+\tau_\theta} \tag{45}$$

$$N = (\psi_{\text{ref}} - \psi)k_{p\psi} + (\dot{\psi}_{\text{ref}} - \dot{\psi})k_{d\psi}\frac{\tau_\psi}{s+\tau_\psi} \tag{46}$$

where Z_{ref}, ϕ_{ref}, θ_{ref} and ψ_{ref} are referenced altitude, roll, pitch and yaw angles, respectively; k_p and k_d are the proportional and derivative gains for the respective control channels. The gains for the PD controller are manually tuned to achieve desired performances in terms of zero steady state error, minimal overshoot and fast settling time. Notice that compensator, $\frac{\tau}{s+\tau}$ is incorporated in the respective control channels to filter noise; where τ is the tuning parameter of the compensator.

3.2 Bilinear Compensator

In order to account for the nonlinearity in (16) - (19), bilinear compensators are proposed. The bilinear compensators are proposed based on (35) and (37) which have the bilinear terms with the coupling of states and control input, N. The bilinear compensators for the ϕ and θ feedback loops are respectively:

$$\frac{L}{\tilde{L}} = \frac{1}{I_z \delta x_3} \text{ and } \frac{M}{\tilde{M}} = \frac{1}{I_z \delta x_1} \tag{47}$$

The bilinear compensator enhances the performance of the PD controller. The bilinear compensator, in combination with PD, facilitates the ensuing controller (BPD) to sustain a required degree of control throughout a broader scope of operation about the tuning point compared to that obtained with the PD controller.

4. SIMULATION RESULTS

To show the accuracy of the proposed bilinear quadrotor attitude model, an open loop simulations of the nonlinear, bilinear and linear models of the quadrotor are carried out. Furthermore, to demonstrate the effectiveness of the proposed controller, simulations of the proposed BPD controller with the dynamics of (16) – (19) are performed based on the BPD control architecture shown in Fig. 3. The rotor voltage combination, rotor dynamics, angular velocity to thrust conversion, forces and moments are implemented based on the works of Seah (2016). For comparison purposes, the simulation responses of the PD controller are also provided. The same values of k_p and k_d are used for both the PD and BPI controllers. The parameter set used for the simulations is listed in Table 1.

Table 1. Design Parameters and Values

Parameter	Value
ϕ, θ, ψ	20°, 25°, 35°, 40°, 45°, 50°, 55°
k_{pz}	-300
k_{dz}	-200
$k_{p\phi}$, $k_{p\theta}$	100
$k_{d\phi}$, $k_{d\theta}$	40
$k_{p\psi}$	300
$k_{d\psi}$	80
τ_ϕ, τ_θ	1000
τ_z, τ_ψ	100
I_x, I_y	0.167 kgm^2
I_z	0.2974 kgm^2
g	9.81 ms^{-2}
m	2.3535 kg
l	0.5 m

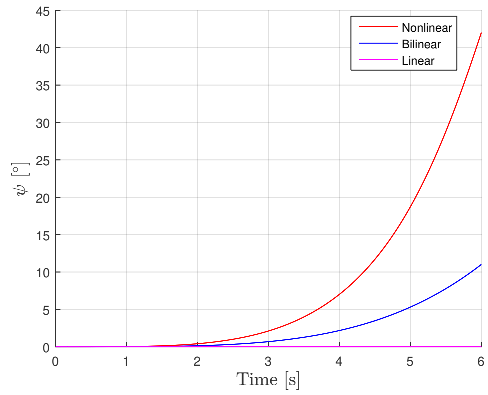

Fig. 4. Models comparison - ψ attitude responses

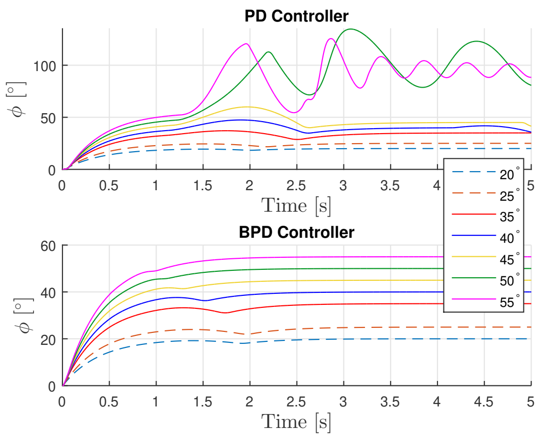

Fig. 5. ϕ attitude responses for PD and BPD controllers

As an example of the improved accuracy of the bilinear model, Fig. 4 shows the evolution of the ψ attitude response with 0.01 Nm input steps on L and M and no input on N, of the nonlinear, bilinear and linear models. For this particular case, the ϕ and θ attitude responses of the bilinear and linear models are identical, but the ψ attitude response of the bilinear model is closer to the nonlinear model response than the linear model response as shown in Fig. 4.

Figs. 5, 6 and 7 show the simultaneous step demand responses of ϕ, θ and ψ, respectively, for various angles of 20°, 25°, 35°, 40°, 45°, 50° and 55°, for PD and BPD controllers. Comparing the responses of the PD and BPD controllers, it can be seen that as the values of the angles increases (35° and above), the PD controller exhibits poor

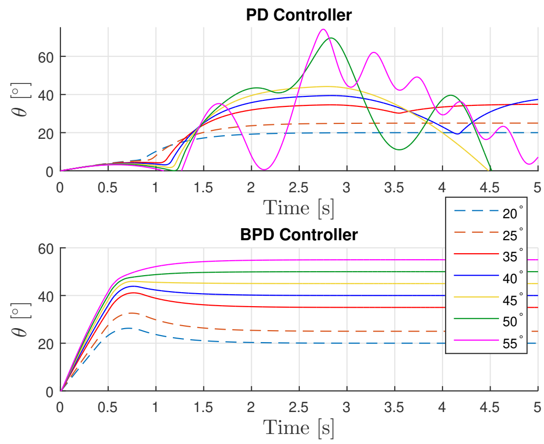

Fig. 6. θ attitude responses for PD and BPD controllers

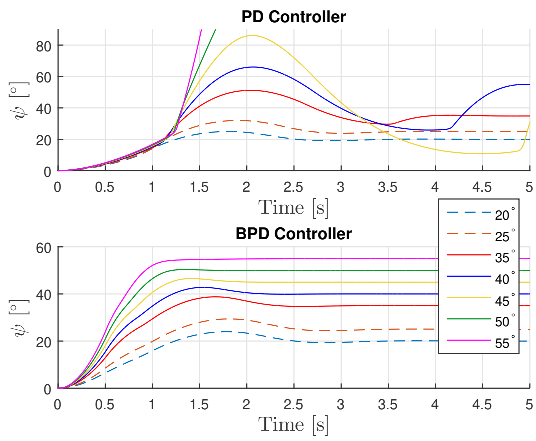

Fig. 7. ψ attitude responses for PD and BPD controllers

performance, while the BPD is still able to sustain the required degrees of control and improved performances in terms of zero steady state error, minimal overshoot and fast settling time. Hence, the proposed BPD controller gives more improved responses over a broader operating range with respect to stability and performance of the quadrotor compared to the PD controller.

5. CONCLUSIONS

This paper proposes a bilinear model of the quadrotor, and also highlights the design of a bilinear controller (BPD) for attitude control of the quadrotor. The proposed controller simply incorporates a single term to extend a PD controller, and it is simpler for implementation and for practicing engineers, starting from PD control. The accuracy of the proposed bilinear model of the quadrotor is demonstrated. This effective implementation shows the possible beneficial aspects of the proposed BPD controller through the improved responses over a broader operating range with respect to stability and performance of the quadrotor compared to the PD controller but, importantly, without a large increase in implementation complexity (in particular there are no transcendental functions to evaluate). The proposed BPD is able to sustain the required degrees of control throughout a broader scope of operation about the tuning point compared to that obtained with the PD controller.

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