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'STUDIES IN STABILITY AND CONTROL ANALYSIS OF AIRFRAMES HAVING NONLINEAR AERODYNAMIC CHARACTERISTICS'

by

P. A. T. Christopher

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'Studies in Stability and Control Analysis of Airframes Having Nonlinear Aerodynamic Characteristics'

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P.A.T. Christopher

SUMMARY

The problem of longitudinal stability and control of an airframe, having nonlinearity in its principal aerodynamic characteristics, is considered. It is shown that the equation describing the response in w, and thus the incidence, is a nonlinear differential equation of the fourth order. This equation, and its degenerate forms, is used as an example to demonstrate various nonlinear techniques and their shortcomings.

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Notation

The system of aerodynamic derivatives used in this paper is that defined in Ref. 5. All other symbols are defined in the text.

1. Introduction

The linear theory of airframe dynamic stability is based on the assumption that, during a disturbance of the steady motion, terms of second or higher degree in the dynamic variables u, v, etc. may be neglected. See Ref. 1. The resulting equations of motion are linear differential equations with constant coefficients, and all the powerful, elegant, methods used for determining the stability of linear systems stem directly from the fact that the form of the general solution to these equations is known. These linear equations may always be put in the vector form

$$\dot{x} = Ax + Q(t), \qquad (1.1)$$

where x is an n-vector whose components are the dynamic variables and their time derivatives. A is a constant $n \times n$ matrix, Q is an n-vector known explicitly in terms of t and $\dot{x} = dx/dt$. Provided $|Q(t)|$ has a finite bound, then all questions of stability of the solutions of (1.1) may be answered by consideration of the degenerate autonomous equation

$$\dot{x} = Ax \qquad (1.2)$$

If all the characteristic roots of A have negative real parts then the solutions of (1.2) and (1.1) are asymptotically stable. See Ref. 2, p. 314. Much of airframe stability theory is concerned with linking this simple criterion with changes in geometric and flight parameters, which alter the coefficients of A, and seeking simple ways of predicting the changes in dynamic stability arising therefrom.

If the above assumption, used to simplify the equations of motion, is not made then the resulting equations are nonlinear. Explicit general solutions are known for only a very restricted class of nonlinear equations (see Refs. 3 and 4) and the present equations are of a more complicated type. This lack of explicit solutions results in an almost complete absence of elegant techniques for determining the stability and response and certainly none with the simplicity of those existing for linear systems e.g. Routh's discriminant, Nyquist criterion.

Nonlinearity arises from both aerodynamic and inertial terms. In order to make the discussion of nonlinearity more explicit, consider the following example of airframe longitudinal motion influenced by aerodynamic nonlinearity associated with the velocity W. It is assumed that the aerodynamic configuration is symmetric and is, therefore, likely to have characteristics for the normal force Z, pitching moment M and longitudinal force X similar to those shown in Fig. 1. Of these Z(W) and M(W) are taken to be of 'odd' form and X(W) of 'even' form. In many cases these curves may be taken to be analytic in W and can, therefore, be represented by power series

$$\left.\begin{aligned} X(W)/m &= (x_o)_w + x_2 W^2 + x_4 W^4 + \ldots \\ Z(W)/m &= z_w W + z_3 W^3 + z_5 W^5 + \ldots \\ M(W)/B &= m_w W + m_3 W^3 + m_5 W^5 + \ldots \end{aligned}\right\} \qquad (1.3)$$

The justification for and mutual consistency of the series representing X(W)/m and Z(W)/m may be exhibited in the following way. By definition

$$X = L \operatorname{Sin} \alpha - D \operatorname{Cos} \alpha \qquad (1.4)$$

and

$$Z = - L \operatorname{Cos} \alpha - D \operatorname{Sin} \alpha \qquad (1.5)$$

For symmetric configurations experimental evidence indicates an 'odd' form for the L, α characteristic and an 'even' form for the D, α characteristic. Also Sin α and Cos α are odd and even power series in α, respectively. Thus if L and D are analytic

$$\begin{aligned} X &= (\text{Odd series})(\text{Odd series}) - (\text{Even series})(\text{Even series}) \\ &= \text{Even series} \\ &= X_o + X_2\alpha^2 + X_4\alpha^4 + \ldots, \text{ say,} \end{aligned} \qquad (1.6)$$

and

$$\begin{aligned} Z &= -(\text{Odd series})(\text{Even series}) - (\text{Even series})(\text{Odd series}) \\ &= \text{Odd series} \\ &= Z_1 + Z_3\alpha^3 + Z_5\alpha^5 + \ldots, \text{ say} \end{aligned} \qquad (1.7)$$

Now $\sin \alpha = W/U_\infty$, where U_∞ is the flight vector velocity, or

$$\alpha = \sin^{-1}(W/U_\infty) = \text{Odd series in } W \qquad (1.8)$$

Substituting for α in (1.6) and (1.7) then produces series for X and Z of the form assumed in (1.3). For further simplicity in the subsequent analysis it will be assumed that adequate representation is obtained by retaining only the first two terms in each series.

In the example the airframe is assumed to be disturbed by an incremental elevator deflection, η, from a trimmed, straight line, climb at an angle γ to the horizontal and having a trimmed incidence α_o. The linearised equations of motion for this problem are

$$\dot{u} - x_u u - x_w w + (W_o + x_q)\dot{\theta} + g \cos \Theta_o . \theta = 0 \qquad (1.9)$$

$$- z_u u + \dot{w} - z_w w - (U_o + z_q)\dot{\theta} + g \sin \Theta_o . \theta = z_\eta \eta \qquad (1.10)$$

$$- m_u u - m_{\dot{w}} \dot{w} - m_w w + \ddot{\theta} - m_q \dot{\theta} = m_\eta \eta \qquad (1.11)$$

where

$$\Theta_o = \gamma + \alpha_o, \quad U = U_o + u, \quad W = W_o + w,$$

$$\Theta = \Theta_o + \theta \text{ and } \sin \alpha_o = W_o/U_\infty .$$

If now the nonlinear description of X(W), Z(W) and M(W) is used the linear increments $x_w w$, $z_w w$ and $m_w w$ will be replaced by increments

$$\Delta X(W)/m = [X(W) - X(W_o)]/m$$

$$\Delta Z(W)/m = [Z(W) - Z(W_o)]/m$$

and

$$\Delta M(W)/B = [M(W) - M(W_o)]/B,$$

respectively. Upon substitution from (1.3), expansion, and retention of terms in w, w^2 and w^3 only, the equations of motion become

$$\dot{u} - x_u u - (2x_2 W_o w + x_2 w^2) + (W_o + x_q)\dot{\theta} + g \operatorname{Cos}\Theta_o.\theta = 0 \qquad (1.12)$$

$$- z_u u + \dot{w} - [(z_w + 3z_3 W_o{}^2)w + 3z_3 W_o.w^2 + z_3 w^3]$$

$$- (U_o + z_q)\dot{\theta} + g \operatorname{Sin}\Theta_o.\theta = z_\eta \eta \qquad (1.13)$$

$$- m_u u - m_{\dot{w}}\dot{w} - [(m_w + 3m_3 W_o{}^2)w + 3m_3 W_o.w^2 + m_3 w^3] + \ddot{\theta} - m_q\dot{\theta} = m_\eta \eta$$

$$(1.14)$$

Elimination of u and θ between these equations then gives

$$f_4 \overset{(4)}{w} + f_3(w)\dddot{w} + f_2(w,\dot{w}).\ddot{w} + f_1(w,\dot{w}).\dot{w} + f_o(w) = F(\eta) \qquad (1.15)$$

where

$$f_4 = z_u,$$

$$f_3(w) = - z_u K_1 + K_{10} - K_6(U_o+z_q) - 6z_u z_3 W_o.w - 3z_u z_3.w^2$$

$$f_2(w,\dot{w}) = z_u K_2 - K_1 K_{10} - K_7(U_o+z_q) + K_4 K_6 - m_u g \operatorname{Sin}\Theta_o.$$

$$+ 2[z_u K_3 - 3K_{10} z_3 W_o - K_8(U_o+z_q)]w$$

$$+ 3[z_u x_u z_3 - K_{10} z_3 - K_9(U_o+z_q)]w^2 - 18z_u z_3.w\dot{w} - 18z_u z_3 W_o.\dot{w},$$

$$f_1(w,\dot{w}) = K_2 K_{10} + K_4 K_7 + K_5 K_6 + K_1 m_u g \operatorname{Sin}\Theta_o$$

$$+ 2[K_3 K_{10} + K_4 K_8 + 3z_3 m_u g \operatorname{Sin}\Theta_o.W_o]w$$

$$+ 3[K_4 K_9 + K_{10} x_u z_3 + z_3 m_u g \operatorname{Sin}\Theta_o.W_o]w^2$$

$$+ 6[z_u x_u z_3 - K_{10} z_3 - K_9(U_o+z_q)]w\dot{w}$$

$$+ 2[z_u K_3 - K_8(U_o+z_q) - 3K_{10} z_3.W_o]\dot{w} - 6z_u z_3(\dot{w})^2,$$

$$f_o(w) = (K_5 K_7 - K_2 m_u g \operatorname{Sin}\Theta_o)w + (K_5 K_8 - K_3 m_u g \operatorname{Sin}\Theta_o)w^2$$

$$+ (K_5 K_9 - x_u z_3 m_u g \operatorname{Sin}\Theta_o)w^3,$$

$$F(\eta) = \{z_\eta z_u D^3 + [z_\eta(K_{10} - z_u x_u) - K_{11}(U_o + z_q)]D^2$$

$$+ [K_4K_{11} - z_\eta(K_{10}x_u + m_u g \text{ Sin } \Theta_o)]D + K_5K_{11} + z_\eta x_u m_u g \text{ Sin } \Theta_o\}\eta,$$

$$K_1 = x_u + z_w + 3z_3W_o^2,$$

$$K_2 = x_u z_w - 2z_u x_2 W_o + 3x_u z_3 W_o^2,$$

$$K_3 = - z_u x_2 + 3x_u z_3 W_o,$$

$$K_4 = x_u(U_o + z_q) + z_u(W_o + x_q) + g \text{ Sin } \Theta_o,$$

$$K_5 = g(z_u \text{ Cos } \Theta_o - x_u \text{ Sin } \Theta_o),$$

$$K_6 = m_u + z_u m_{\dot{w}},$$

$$K_7 = z_u(m_w + 3m_3W_o^2) - m_u(z_w + 3z_3W_o^2),$$

$$K_8 = 3(z_u m_3 - m_u z_3)W_o,$$

$$K_9 = z_u m_3 - m_u z_3,$$

$$K_{10} = (U_o + z_q)m_u - z_u m_q$$

and

$$K_{11} = m_u z_\eta - z_u m_\eta$$

In order to obtain the usual linear equation from (1.15), only the constant terms in the coefficients are retained and $2x_2W_o$ must be identified with x_w. Other important special cases arise when $m_u = 0$, which is often a good approximation; when $\text{Sin } \Theta_o = 0$, corresponding to horizontal flight; and when W_o, i.e. the initial trimmed incidence, is sufficiently small compared to w to be able to neglect terms containing W_o or W_o^2.

Unlike the linear problem, the general solution to (1.15) is not known and it may reasonably be assumed that the form of solution is dependent on the form of $F[\eta(t)]$. For this reason it is convenient to restrict the range of possible forms by considering only the standard response test functions:

Step function: $\eta(t) = 0,\ t \leq 0;\ \eta(t) = \eta,\ t > 0,$ (1.16)

Sinusoid: $\eta(t) = \eta_a \text{ Sin } \omega t$ (1.17)

The step function may also include the special case $\eta = 0$, i.e. the stability of equilibrium.

Consider first the response to a step function. Although the solution is not known, the points of equilibrium of the system may be obtained from the steady state equation

$$f_o(w_s) = (K_5K_{11} + z_\eta x_u m_u g \text{ Sin } \Theta_o)\eta,$$

which may be written as the cubic

$$(K_5K_9 - x_u z_3 m_u g \text{ Sin } \Theta_o)w_s^3 + (K_5K_8 - K_3 m_u g \text{ Sin } \Theta_o)w_s^2$$
$$+ (K_5K_7 - K_2 m_u g \text{ Sin } \Theta_o)w_s - (K_5K_{11} + z_\eta x_u m_u g \text{ Sin } \Theta_o)\eta = 0 \tag{1.18}$$

In the linear case this reduces to

$$w_s = (K_5K_{11} + z_\eta x_u m_u g \text{ Sin } \Theta_o)/(K_5K_7 - K_2 m_u g \text{ Sin } \Theta_o).\eta, \tag{1.19}$$

and if, additionally, $m_u = 0$ then (1.19) becomes

$$w_s = K_{11}/K_7.\eta = - m_\eta/m_w.\eta, \tag{1.20}$$

a well known result.

With all the other airframe and flight parameters fixed there will be, in the linear case, only one value of w_s corresponding to each value of incremental elevator angle, η, and this will be finite provided the denominator in (1.19) is not zero. This means, approximately, that the centre of gravity margin shall not be zero. In the nonlinear case there will be, at most, three equilibrium values for which w_s is real. Typical w, η trim curves, which are the solution curves of (1.18), are shown in Fig. 2. Now the essential problem in the step function response analysis is to determine which of the equilibrium, or singular, points the solution curves of (1.15) are going to arrive at after a sufficiently long time. To do this it is necessary to know which of the singular points are asymptotically stable, i.e. solution curves lying within a certain neighbourhood of the singular point all tend to the point as $t \to \infty$, and which of the points are unstable. If there is more than one stable singular point it is then necessary to determine which one of these the solution curves finally move into. In the case of equation (1.15) the answers to these questions are far from complete; however, in the degenerate second order problem, where only the nonlinear short-period motion is considered, the technique is well developed (see Ref. 5) and will be discussed in Section 2.

The other main problem is the influence of nonlinearity on the frequency response, i.e. when $\eta(t)$ has the form of (1.17). When the nonlinearity is small it may be shown that in the appropriate circumstances, governed essentially by Theorem 1.1 and 3.1, Chapter 14 of Ref. 2, there exists a periodic solution of (1.15) of greatest period $2\pi/\omega$. Also, in the analytic case, the coefficients of the Fourier series used to describe this solution, may be readily evaluated. Using a variant on this technique the author, in Ref. 6, has demonstrated how the approximate nonlinear frequency response may be obtained for the degenerate short-period problem.

Having obtained these periodic solutions, it is necessary to distinguish the physically realizable solutions, which are those having asymptotic stability, from the others. In order to do this, use is made of the equation of first variation of (1.15), which will be a linear differential equation with periodic coefficients. The stability is then governed by the characteristic exponents (see Theorem 2.1, Chapter 13 of Ref. 2) which may be evaluated by Cesari's method given in Ref. 7, Chapter 8 and employed by the present author in Refs. 8 and 9.

When the nonlinearity is not small there are no generally applicable existence theorems guaranteeing periodic solutions as there were above. Nevertheless, in many cases, if the solution is taken to be a Fourier series and the coefficients evaluated by a direct substitution and comparison of coefficients procedure, periodic solutions are obtained which agree well with analogue and digital computer solutions. This process has been put on a sound basis by Cesari in Ref. 10, using a functional analytic technique.

There is no doubt that this method offers considerable possibilities, but it has not yet been employed on equations of the complexity of (1.15). The problem of periodic solutions will be considered in Section 3.

A very powerful method for determining the stability of the singular points of systems of any finite dimension is that known as Lyapunov's direct method. Its application to a system of second order will be described in Section 4 and the extension to systems of higher order will be discussed.

2. The response of second-order systems to a step function

The principal method to be used in this section is that known as Poincaré's theory of singular points in the phase-plane, which will now be described. Consider the real equation

$$\ddot{x} + B(x).\dot{x} + C(x) = 0, \tag{2.1}$$

which, upon taking $x \equiv x_1$, may be written as the equivalent pair of first order equations

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= - C(x_1) - B(x_1).x_2 \end{aligned}\right\} \tag{2.2}$$

The solution curves of this equation in the x_1, x_2 plane, known by engineers as the phase plane, are referred to as 'integral curves' and, provided B and C are analytic in x_1, through each ordinary point in the plane there passes only one such curve. The stationary positions of equilibrium of (2.2), defined by $C(x_1) = 0$, $x_2 = 0$, correspond with the singularities of the equivalent equation

$$dx_2/dx_1 = \{- C(x_1) - B(x_1).x_2\}/x_2, \tag{2.3}$$

and analysis of the character of these singularities gives considerable insight into the nature of the integral curves near these points and, thereby, information on whether the equilibrium points are asymptotically stable or otherwise.

More generally, consider the singularities of the equation

$$dx_2/dx_1 = P(x_1, x_2)/Q(x_1, x_2), \tag{2.4}$$

defined by $P = Q = 0$, where P and Q are analytic in x_1 and x_2. Since the origin can always be changed to the singular point, then analysis can be restricted to singularities at the origin. When (2.4) has a singularity at the origin then it was shown by Poincaré that in the neighbourhood of the origin the integral curves of (2.4) may be accurately represented by the integral curves of

$$dx_2/dx_1 = (ax_1 + bx_2)/(cx_1 + dx_2), \tag{2.5}$$

provided $ad-bc \neq 0$, and where a, b, c, d are the coefficients of the leading terms of power series representations made possible by the analytic nature of P and Q. The nature of the integral curves of (2.5) fall into four categories (see Ref. 2, Chapter 15) which may be determined from the roots of the characteristic equation

$$\lambda^2 - \lambda(b + c) - (ad - bc) = 0 \tag{2.6}$$

In the particular case of equation (2.3), taking

$$\left.\begin{aligned} B(x) &= b_o + b_1x + b_2x^2 + \ldots\ldots \\ \text{and} \quad C(x) &= c_1x + c_2x^2 + c_3x^3 + \ldots\ldots \end{aligned}\right\} \tag{2.7}$$

then the equation corresponding to (2.5) is

$$dx_2/dx_1 = (- c_1x_1 - b_ox_2)/x_2 \tag{2.8}$$

and the types of singularity may be classified by means of the diagram shown in Fig. 3.

If, in the linearized equations of motion (1.9) to (1.11), the longitudinal velocity is assumed to be constant, then these equations degenerate to two equations in w and θ which approximately describe the 'short-period' motion, i.e. the phugoid motion has been eliminated from the equations. The same assumption reduces equations (1.12) to (1.14) to two equations in w and θ describing, what may be called, the 'nonlinear short-period' motion. Eliminating θ between these equations then gives

$$\ddot{w} - \{[z_w + 3z_3W_o^2 + m_q + m_{\dot{w}}(U_o + z_q)] + 6z_3W_o.w + 3z_3w^2\}\dot{w}$$

$$+ [m_q(z_w + 3z_3W_o^2) - (U_o+z_q)(m_w + 3m_3W_o^2)]w + 3W_o[m_qz_3 - m_3(U_o+z_q)]w^2$$

$$+ [m_qz_3 - m_3(U_o+z_q)]w^3 = [z_\eta D - z_\eta m_q + m_\eta(U_o+z_q)]\eta \qquad (2.9)$$

Further simplification is possible if W_o is small compared with w and may be neglected, and it was in this form that the equation was taken in Ref. 5. The results of that study will now be discussed.

It is convenient to refer the w co-ordinate to the final equilibrium value w_s. Thus

$$w = w_s + \xi \qquad (2.10)$$

and the equation describing the ξ motion becomes

$$\ddot{\xi} - [B_1 + B_3(w_s+\xi^2)]\dot{\xi} - [A_1(w_s+\xi) + A_3(w_s+\xi)^3]$$

$$= [z_\eta D + m_\eta(U_o+z_q) - z_\eta m_q]\eta \qquad (2.11)$$

where

$$\left.\begin{aligned} A_1 &= (U_o+z_q)m_w - m_qz_w \\ A_3 &= (U_o+z_q)m_3 - m_qz_3 \\ B_1 &= (U_o+z_q)m_{\dot{w}} + m_q + z_w \\ B_3 &= 3z_3 \end{aligned}\right\} \qquad (2.12)$$

Now w_s is defined by

$$- A_1w_s - A_3w_s^3 = [m_\eta(U_o+z_q) - z_\eta m_q]\eta, \qquad (2.13)$$

which, upon subtraction from (2.11), gives

$$\ddot{\xi} - [B_1 + B_3(w_s+\xi)^2]\dot{\xi} - [(A_1+3A_3w_s^2)\xi + 3A_3w_s\xi^2 + A_3\xi^3] = z_\eta\dot{\eta} \qquad (2.14)$$

The importance of the term $z_\eta\dot{\eta}$ in (2.14) (this is the term which, for rear-control configurations, produces the 'negative kick' on the response curve) will depend on the airframe configuration, and its influence on the stability has been determined in Ref. 5. In the present discussion only the case $z_\eta = 0$ will be considered.

As a result of the previous assumption (2.14) reduces to the same form as (2.1) with (2.7) and the equation corresponding to (2.8) is

$$d\xi_2/d\xi_1 = \{(A_1+3A_3w_s^2)\xi_1 + (B_1+B_3w_s^2)\xi_2\}/\xi_2 \qquad (2.15)$$

A comparison between (2.15) and (2.8) then shows that the types of singularity in the ξ_1, ξ_2 plane may be classified by the use of Fig. 4. Two cases are relevant to the aeroplane stability and response problem.

Case 1. $A_1 < 0,\ A_3 < 0,\ B_1 < 0.$

This corresponds to an airframe which is statically stable at low incidence and for which $dC_M/d\alpha$ becomes more negative as α increases. See Fig. 1(b), Curve 2. The associated incremental trim curve is shown in Fig. 5(a). This is typical of tail controlled configurations whose tail efficiency increases with incidence. B_3 may be either sign, positive and negative B_3 corresponding to Fig. 1(a), Curves 2 and 3, respectively. Negative B_3 often arises with very low aspect-ratio configurations, whilst positive B_3 indicates a conventional lift curve containing a stall.

With w_s zero, the corresponding point on Fig. 4 is A, implying that the initial trimmed condition is one of asymptotic stability and that the settling down motion near this point is normally oscillatory. The curves AB and AC represent the variation with w_s. Only one value of w_s exists for a given η_s and the nature of the equilibrium point would normally be a stable spiral, as shown in Fig. 5(b). If, however, B_3 or w_s were very large the equilibrium point could move into the region of unstable spirals beyond the point B. With $B_3 < 0$ no instability is possible, but the settling down motion becomes non-oscillatory at C. The instability boundary at B is determined by

$$B_1 + B_3(w_s)_B^2 = 0$$

or

$$(w_s)_B = (-B_1/B_3)^{\frac{1}{2}},\ B_1 < 0,\ B_3 > 0, \qquad (2.16)$$

which from (1.3) and (2.12) must correspond to an incidence above the stall.

Graphical constructions, such as those of Lienard (See Ref. 11, pp. 217-220) can be used to determine the solution curves of (2.14) in the phase-plane, Fig. 5(b). From these may be obtained the normalized response curves. Fig. 5(c) shows a sketch of typical results. With η_s small the response curve differs little from the linear case, whilst with η_s large the 'rise time' is reduced and the settling down frequency increased. The interesting practical result here is that in many situations the linear normalized response curve gives a good approximation to the exact result.

Case 2. $A_1 < 0,\ A_3 > 0,\ B_1 < 0.$

The condition $A_3 > 0$ might well be typical of a canard configuration whose nonlinear body lift is forward of the centre of gravity. Starting with the same value of A_1 and the same initial conditions as the previous case, then curves AE and AD of Fig. 4 represent the variation in the nature of the equilibrium points with w_s. Three equilibrium points exist for values of η_s lying between the maximum and minimum of Fig. 5(d). Only $(w_s)_1$ is stable, the points $(w_s)_2$ and $(w_s)_3$ lying in the region of saddle points on Fig. 4. The corresponding phase-plane diagram is shown in Fig. 5(e). With η sufficiently large points 1 and 2 merge leaving only two saddle points, implying that the airframe is unstable for this and larger values of η_s. The stability boundary is given by

$$A_1 + 3A_3(w_s)^2_{E,D} = 0$$

or

$$(w_s)_{E,D} = [- A_1/(3A_3)]^{\frac{1}{2}}, \qquad (2.17)$$

corresponding with the minimum on Fig. 5(d).

The normalized response curves may be constructed in a similar manner to Case 1, typical curves being shown in Fig. 5(f). With η_s small the curve is nearly that of the linear case, whilst with increase of η_s the curve becomes more damped and the settling down frequency becomes less. When η_s is sufficiently large for the corresponding point on Fig. 4 to lie between E or D and the = 0 curve, then the response is non-oscillatory in the region of $w/w_s = 1$ as $t \to \infty$. Finally, when η_s corresponds to $w_s \geqslant (w_s)_{E,D}$ the response curves are unstable.

3. Periodic response of nonlinear systems

Writing $w = w_1$, $\dot{w}_1 = w_2$, ... etc. then (1.15) may be expressed in the vector form

$$\dot{\bar{w}} = A\bar{w} + G(\bar{w}) + f(t), \tag{3.1}$$

where $\bar{w}$ is the column vector col.(w_1, w_2, w_3, w_4), A is the constant matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -h_o/h_4 & -h_1/h_4 & -h_2/h_4 & -h_3/h_4 \end{bmatrix} \tag{3.2}$$

$G(\bar{w})$ is the column vector function

$$\text{col.}\{0,0,0,-\frac{1}{h_4}[g_o(w_1) + g_1(w_1,w_2).w_2 + g_2(w_1,w_2).w_3 + g_3(w_1).w_4]\}, \tag{3.3}$$

f(t) is the column vector function

$$\text{col.}\{0,0,0,-\frac{1}{h_4}F[\eta(t)]\}, \tag{3.4}$$

where h_o, h_1, ..., h_4 are the constant parts of f_o, f_1, ..., f_4, respectively and $g_o(w_1) = f_o(w_1) - h_o$, ..., $g_3(w_1) = f_3(w_1) - h_3$. When $\eta(t) = \eta_a \operatorname{Sin} \omega t$ and $|G(\bar{w})|$ is sufficiently small, then it may be shown that (3.1) has a unique periodic solution of period $2\pi/\omega$. This is proved in Ref. 2, Chapter 14, Theorems 1.1 and 3.1, but requires the prior reduction of (3.1) to the form

$$\dot{\bar{u}} = C\bar{u} + q(\bar{u},t) \tag{3.5}$$

by the substitution

$$\bar{w} = \bar{u}_o + \bar{u}, \tag{3.6}$$

where $\bar{u}_o$ is any periodic solution of the linear equation

$$\dot{\bar{u}} = A\bar{u} + f(t) \tag{3.7}$$

An alternative proof is given in Ref. 2, pp. 64-70.

If it is assumed that the non-constant terms in f_o, ..., f_4 are sufficiently small, then from the above theorem it follows that there exists a periodic solution of

$$f_4 \overset{(4)}{w} + f_3(w)\dddot{w} + f_2(w,\dot{w})\ddot{w} + f_1(w,\dot{w})\dot{w} + f_o(w) = F(\eta_a \text{ Sin } \omega t) \tag{3.8}$$

which may be written as the Fourier series

$$w = \sum_{n=0}^{\infty} (a_n \text{Sin } n\omega t + b_n \text{ Cos } n\omega t) \tag{3.9}$$

The approximate values of the coefficients a_n, b_n may be obtained by truncating the series, substituting into (3.8) and comparing coefficients. For $n > 3$ the labour of trigonometric manipulation is considerable. It is, therefore fortunate that in most practical cases the amplitude of the higher harmonics decrease rapidly with n and permits truncation at quite small values of n.

The full periodic solution of (3.8) has not yet been published, although the author hopes soon to do this. The degenerate problem of the sinusoidally forced, short-period motion has, however, been discussed in Ref. 6. In this paper an airframe was considered whose incremental trim curve was of the same type as Curve 2 of Fig. 2, i.e. initially statically stable and having a 'hard' pitching moment characteristic. It was shown in Ref. 6 that the resulting equation was closely similar to Duffing's equation and the respective solutions also exhibited great similarity. If attention is restricted to the fundamental i.e. $n = 1$ in (3.9), then the graph of the amplitude versus frequency may be looked upon as an approximate nonlinear frequency response diagram, a crude generalization of the well known diagram used in association with linear systems. Curves of this sort have been taken from Ref. 6 and are re-produced in Fig. 6. In Fig. 6(a) it will first be observed that the shape of the curves are dependent on the amplitude of the elevator sinusoid and cannot, as with a linear system, be reduced to one curve by normalization. The vertical arrows indicate 'jumps' of amplitude which occur at the points of vertical tangency of the response curves. Being a dissipative system the fundamental will be out of phase with the forcing sinusoid and the associated phase angle curves shown in Fig. 6(b) also indicate jumps in phase angle.

It has been seen that provided the nonlinear terms in (3.8) or its degenerate forms are relatively small the periodic solution may be readily, if somewhat laboriously, obtained. From Fig. 6(a) it is clear that at some frequencies more than one solution is possible and there arises the problem of determining which of the three amplitudes correspond to solutions which are asymptotically stable; it is these which are physically realizable. To do this use is made of the equation of first variation defined by

$$\dot{\xi} = J_{\bar{w}}\{t,\ p(t)\}.\xi, \qquad (3.10)$$

where

$$J(\bar{w},\ t) = A\bar{w} + G(\bar{w}) + f(t), \qquad (3.11)$$

$J_{\bar{w}}$ is Jacobian matrix of $J(\bar{w},t)$ with respect to $\bar{w}$, and $p(t)$ is the periodic solution whose stability is being investigated. In the present case $J(\bar{w},t)$ will be a 4 x 4 matrix. Equation (3.10) will be a linear differential equation with periodic coefficients and the form of its solution is known from Floquet's theorem (see Ref. 2, p. 78) to be

$$\Phi[\xi(t)] = \underline{p}(t)e^{tR}, \qquad (3.12)$$

where $\Phi[\xi(t)]$ is a 4 x 4 matrix formed by the 4 linearly independent solution vectors ξ, $\underline{p}(t)$ is a periodic matrix and R is a constant matrix. The stability of this solution is determined by the characteristic roots of R, known as the 'characteristic exponents' and these are not readily determined. One method of evaluating the characteristic exponents, to an accuracy consistent with the assumption of small nonlinearity made at the outset, has been developed by Cesari in Ref. 7, Chapter 8 and has proved to be of considerable utility. The details will not be entered into here, but see for example Refs. 8 and 9. Having evaluated the characteristic exponents, then the solutions (3.12) will be asymptotically stable if all the characteristic exponents are negative. If this is so then it follows from Ref. 2, Chapter 13, Theorem 2.1, that the associated periodic solution, $p(t)$, of (3.8) is asymptotically stable.

In Ref. 6 another method, due to Minorsky, has been used to obtain the region, bounded by the locus of vertical tangents, on Fig. 6(a) which corresponds to asymptotically unstable solutions. This result may readily be obtained by the method described in the above paragraph, as may be seen by comparing the results of Ref. 8 with those of Ref. 6.

4. The use of Lyapunov functions

Poincare's theory for determining the stability of equilibrium is effectively restricted to systems of second order. A more general technique for determining the stability of equilibrium of systems of any finite order is that known as Lyapunov's Direct Method. Consider the autonomous real vector system

$$\dot{x} = X(x), \tag{4.1}$$

where X is Lipschitzian or otherwise satisfies a uniqueness condition. Let there exist in a neighbourhood of the origin a scalar function of the solution vector $V(x)$ which is positive definite, i.e. $V(x) > 0$, $V(0) = 0$, and for which the total time derivative $\dot{V}(x)$ is negative definite, i.e. $V(x) < 0$, $\dot{V}(0) = 0$. It then follows that all the solution curves in this neighbourhood move into the origin as $t \to \infty$ and the origin is said to be asymptotically stable. Proofs of this theorem are given in Ref. 13, pp. 37-38 and Ref. 14, p. 15. The existence of $V(x)$, known as a Lyapunov function, is a sufficient, but not necessary, condition for asymptotic stability of the origin.

A principal shortcoming of this method is that there are, in existence, no generally applicable techniques for constructing $V(x)$. In the case of linear systems with constant coefficients a very elegant method exists (see Ref. 14, pp. 26-29) for constructing $V(x)$ as a positive definite quadratic form. The resulting stability criteria are precisely those of

Routh. In the case of nonlinear systems the form of V(x) is based largely on inspired guesswork and a fund of appropriate forms for V(x) has been established. Consider for example the equation

$$\ddot{x} + B(x).\dot{x} + C(x) = 0, \tag{4.2}$$

which may be expressed as the equivalent system

$$\left.\begin{aligned} \dot{x} &= y - F(x) \\ \dot{y} &= - C(x) \end{aligned}\right\}, \tag{4.3}$$

where

$$F(x) = \int_0^x B(x)dx \tag{4.4}$$

Write

$$G(x) = \int_0^x C(x)dx \tag{4.5}$$

and assume that there exists a Lyapunov function of the form

$$V(x,y) = G(x) + y^2/2. \tag{4.6}$$

If (4.2) were representative of a nonlinear mechanical system, then V(x,y) would describe the contours of total energy in the x,y plane. Differentiating V(x,y) totally with respect to t gives

$$\begin{aligned} \dot{V}(x,y) &= C(x).\dot{x} + y\dot{y} \\ &= C(x)[y - F(x)] - yg(x) \end{aligned}$$

or

$$\dot{V}(x,y) = - C(x).F(x). \tag{4.7}$$

In order that V(x,y) be positive definite and $\dot{V}(x,y)$ sign definite, certain restrictions have to be placed on the form of B(x) and C(x). For this purpose it is assumed that

$$\left.\begin{aligned} &\text{(a)}\quad F(0) = C(0) = 0, \\ &\text{(b)}\quad C(x)/x > 0,\ F(x)/x > 0,\ x \neq 0, \\ &\text{(c)}\quad G(x) \to \infty \text{ as } ||x|| \to \infty \end{aligned}\right\} \tag{4.8}$$

With these conditions it is clear that V(x,y) is positive definite and $\dot{V}(x,y)$

is negative definite for all x,y, and it follows that the origin is asymptotically stable for arbitrary initial x,y.

It is of interest to apply these results to the equation

$$\ddot{\xi} + B(\xi).\dot{\xi} + C(\xi) = 0, \qquad (4.9)$$

where

$$B(\xi) = - B_1$$

$$C(\xi) = - [(A_1 + 3A_3w_s{}^2)\xi + 3A_3w_s\xi^2 + A_3\xi^3],$$

which is a slightly simplified form of (2.14) with z_η and B_3 both zero. Thus

$$F(\xi) = - B_1\xi$$

and

$$G(\xi) = - [\tfrac{1}{2}(A_1 + 3A_3w_s{}^2)\xi^2 + A_3w_s\xi^3 + \tfrac{1}{4}A_3\xi^4].$$

The conditions (4.8) (a) and (c) are clearly satisfied and so is the condition $F(\xi)/\xi > 0$ of (4.8) (b) provided $B_1 < 0$, which it is, normally. The breakdown in meeting the conditions of (4.8) arises in the condition $C(\xi)/\xi > 0$ and as a result gives rise to certain asymptotic stability boundaries. Two cases arise.

Case 1. $A_1 < 0$, $A_3 < 0$

The ξ v $C(\xi)$ curves for this case are shown in Fig. 7(a). The zeros of $C(\xi)$ correspond to the roots of equation

$$\xi\{A_3\xi^2 + 3A_3w_s\xi + (A_1 + 3A_3w_s{}^2)\} = 0 \qquad (4.10)$$

which are

$$\xi = 0,\ \tfrac{1}{2}\{- 3w_s \pm (- 4A_1/A_3 - 3w_s{}^2)^{\frac{1}{2}}\} \qquad (4.11)$$

Since A_1 and A_3 have the same sign then the discriminant is negative and the only real root is $\xi = 0$. The curves of Fig. 7(a), therefore, cut the ξ - axis once only. Clearly $C(\xi)/\xi > 0$ for all w_s and ξ, and, thereby, the equilibrium point is always asymptotically stable. This agrees with the conclusion in Section 2 and corresponds to the response curves of Fig. 5 (b), (c).

Case 2. $A_1 < 0, A_3 > 0.$

Here three real roots of (4.10) are possible, as shown in Fig. 7(b). When $w_s = 0$, the roots are

$$\xi = 0, \pm (- A_1/A_3)^{\frac{1}{2}}.$$

In the open interval

$$- (- A_1/A_3)^{\frac{1}{2}} < \xi < + (- A_1/A_3)^{\frac{1}{2}} \qquad (4.12)$$

the condition $C(\xi)/\xi > 0$ is satisfied and the origin is asymptotically stable. This means that if the initial value of ξ lies in the open interval (4.12) then the solution curves will spiral into the origin, i.e. the equilibrium point $w = w_s = 0$.

When $w_s \neq 0$ the picture changes, the positive real root becoming smaller and the negative real root becoming more negative, as in Fig. 7(b). Again if the initial value of $\xi = - w_s$ lies in the open interval between these roots then the solution curves will spiral into the origin. For modest w_s the situation corresponds to that of Fig. 5(e), the point 1 now being the origin of Fig. 7(b) and the origin of Fig. 5(e) being a point $\xi = - w_s$ on Fig. 7(b). With increasing w_s a situation is finally reached where the positive root of (4.11) becomes zero. Thus

$$- 3w_s + (- 4A_1/A_3 - 3w_s{}^2)^{\frac{1}{2}} = 0$$

or

$$w_s = + [- A_1/(3A_3)]^{\frac{1}{2}} \qquad (4.13)$$

This corresponds to the merging of the points 1 and 2 on Fig. 5(e) and represents the upper limit to the stability boundary.

Systems of higher order

The previous example has shown that if it is manifest that there exists about a final equilibrium point an interval of asymptotic stability on the w axis which contains the initial point $w = 0$ (or $\xi = - w_s$), then the response curve will move into $w = w_s$ as $t \to \infty$. This idea can be extended to systems of higher order and the conditions, similar to equation (4.8), which ensure sign-definiteness of the appropriate V and $\dot{V}$ will, in general, define a region of asymptotic stability in the space of $w, \dot{w}, \ldots, \overset{(n)}{w}$.

It is unfortunate that appropriate Lyapunov functions for equation (1.15) have not, as yet, been discovered and much more work needs to be done before the application of the technique to equations of this type is possible.

5. Conclusion

The preceding sections have demonstrated three of the principal techniques available for determining the solutions, or their stability, of nonlinear ordinary differential equations. By considering the application of these techniques to a particular type of equation, describing the response of an airframe to certain elevator motions, it is clear that these methods are far from complete. This situation arises as a result of the very limited existing knowledge of the nature of the solutions of nonlinear equations, a problem which is proving to be very formidable indeed. However, some progress is being made. In particular, the method proposed by Cesari in Ref. 10, and the development of methods for constructing Lyapunov functions do offer reasonable prospects for the future.

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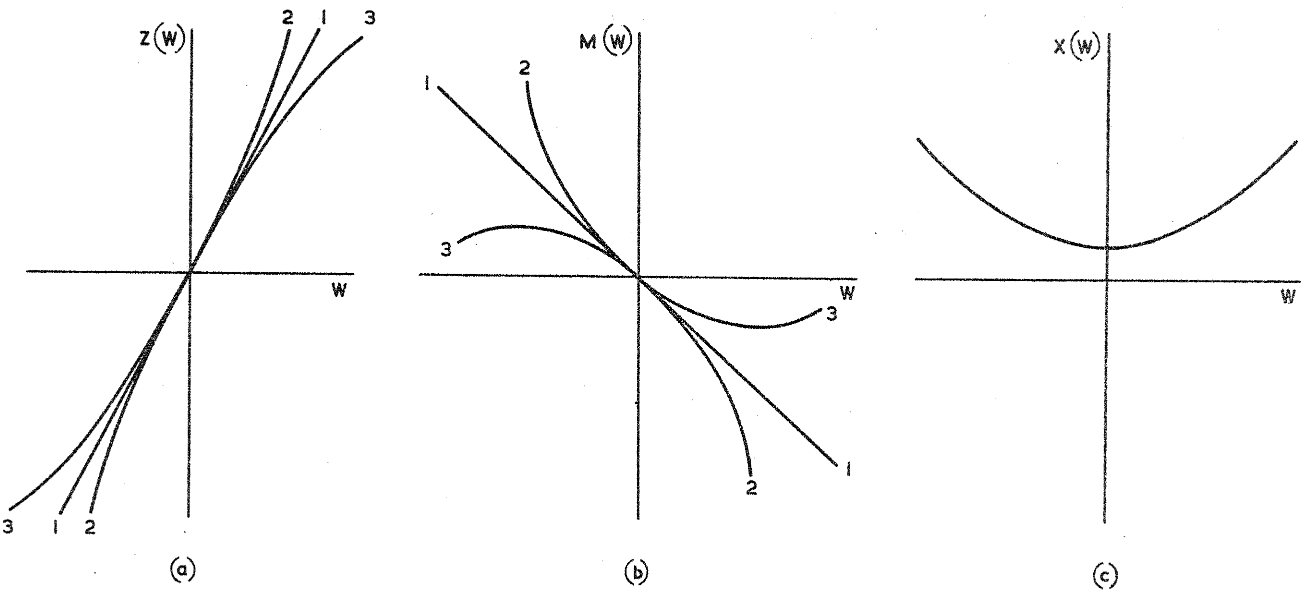


FIG.1. TYPICAL NONLINEAR AERODYNAMIC CHARACTERISTICS.

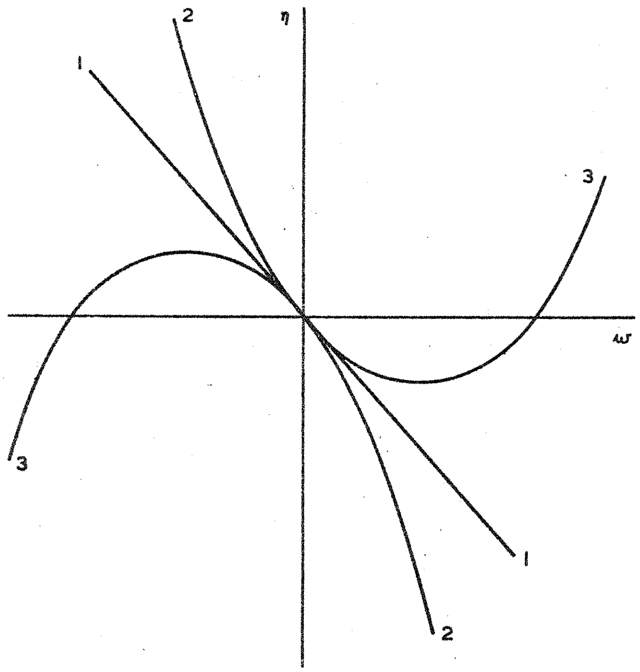


FIG.2. INCREMENTAL TRIM CURVES. $(W_0 = 0)$

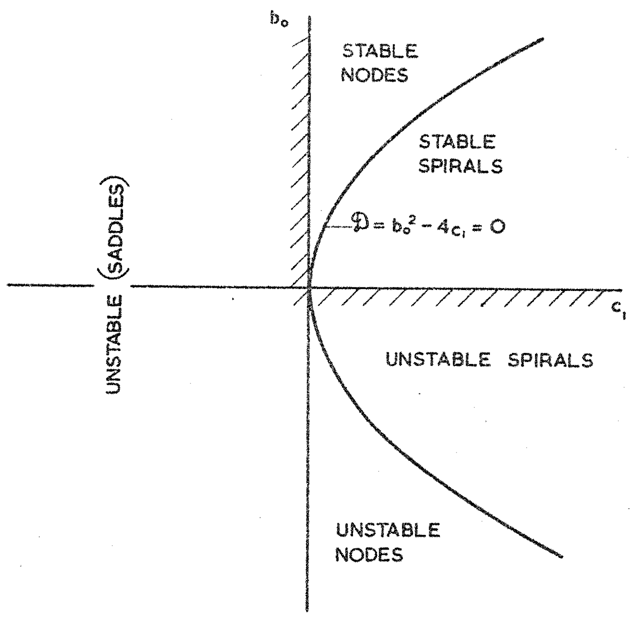


FIG.3. CLASSIFICATION OF THE SINGULARITIES OF EQUATION 2.3

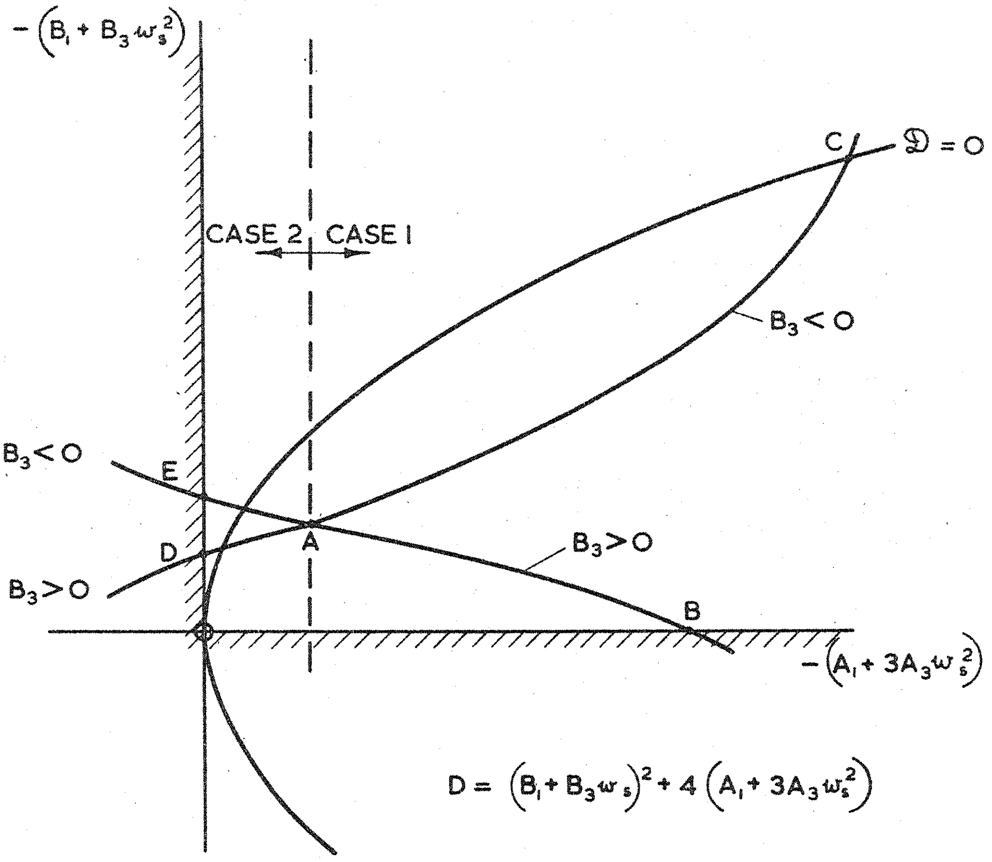


FIG.4. CLASSIFICATION OF THE STABILITY OF THE EQUILIBRIUM POINTS OF EQUATION 2.14.

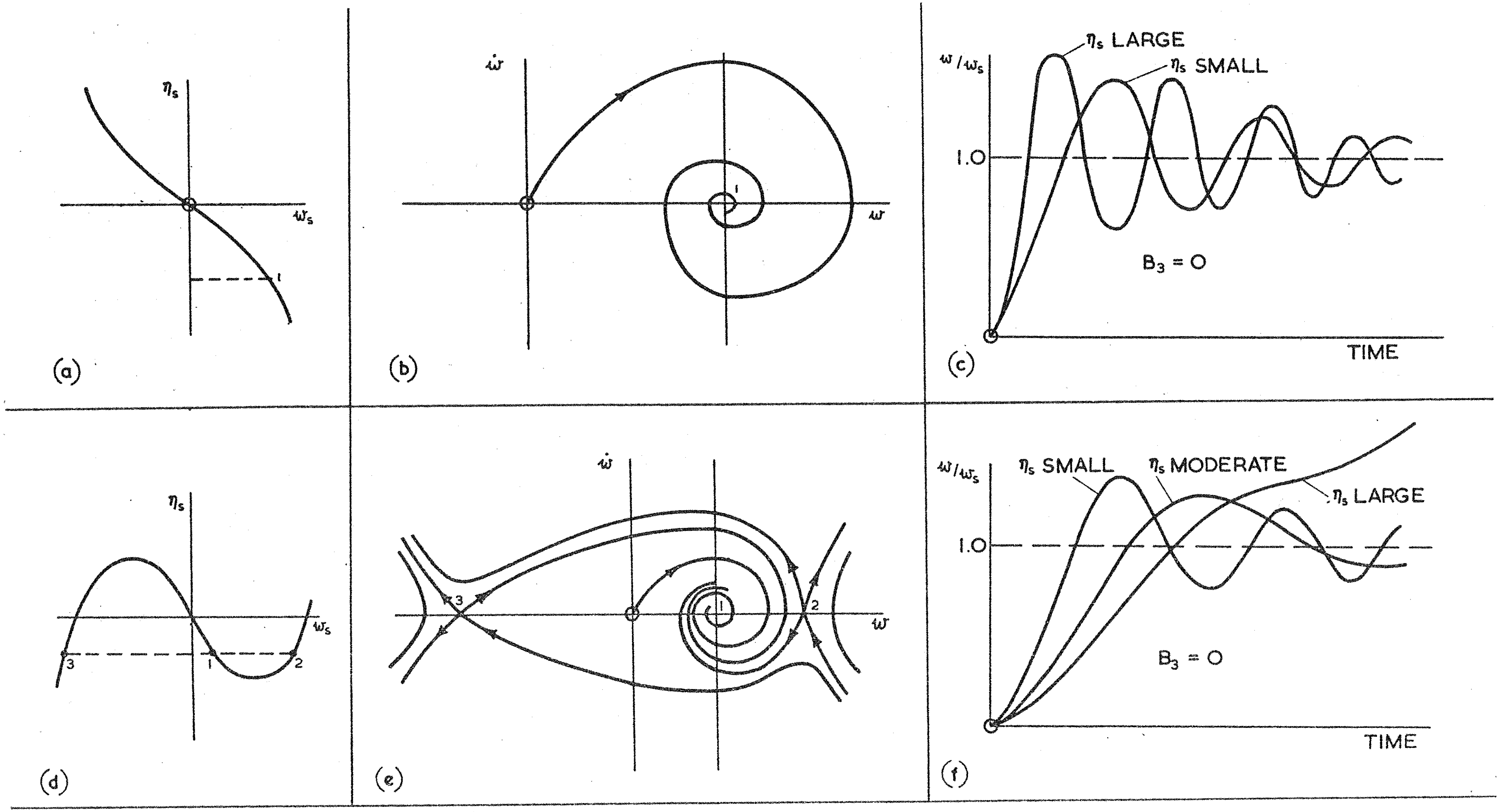
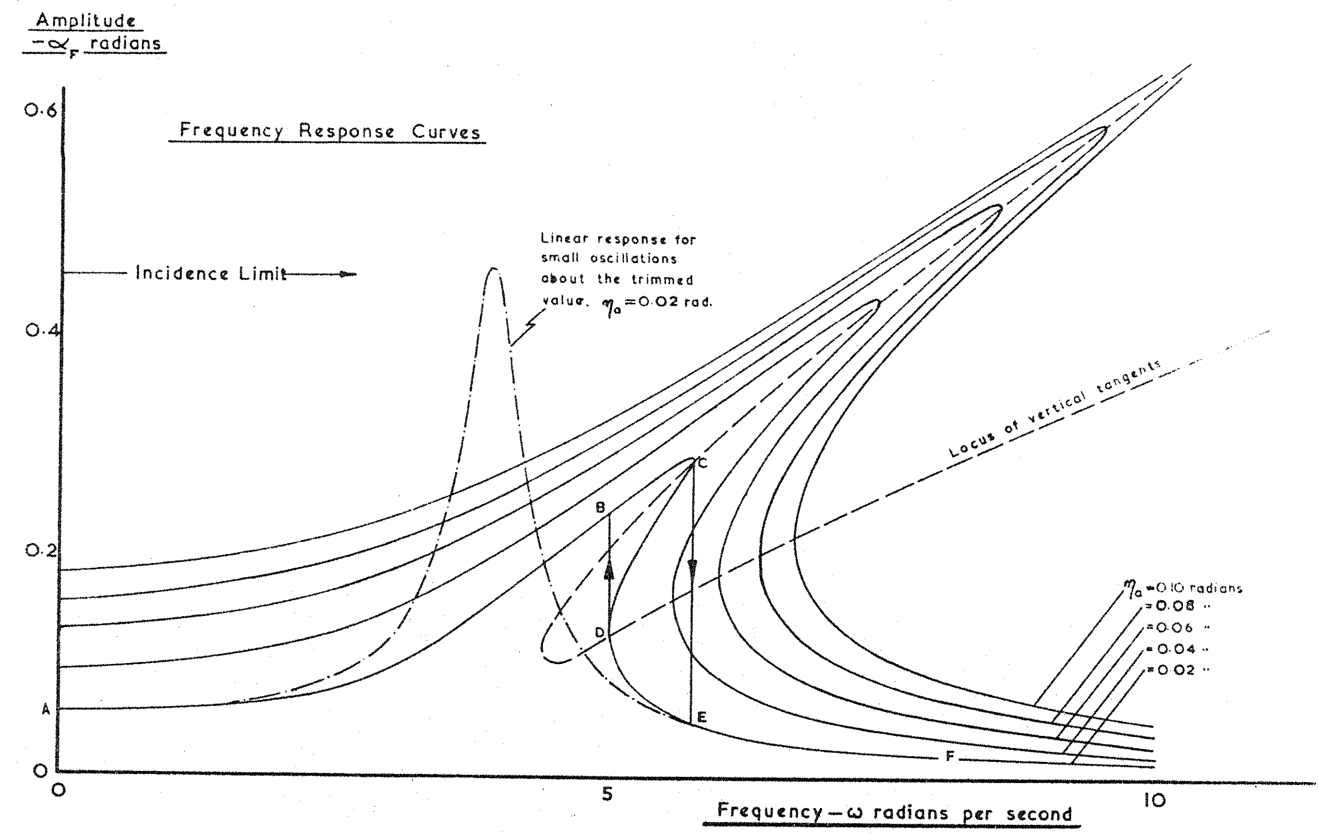


FIG.5. INCREMENTAL TRIM CURVES AND THEIR ASSOCIATED RESPONSE CURVES.



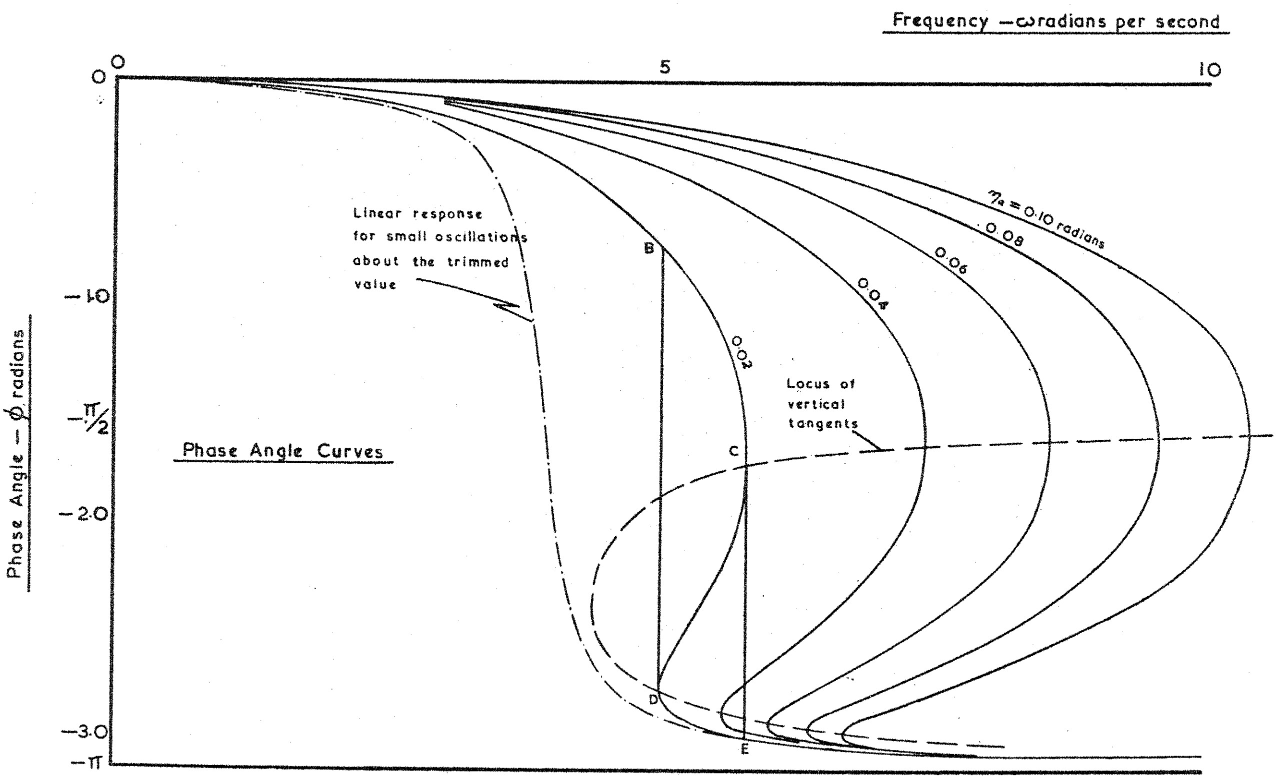


FIG.6. AMPLITUDE AND PHASE ANGLE VERSUS FREQUENCY CURVES FOR AIRFRAME WITH NONLINEAR AERODYNAMIC CHARACTERISTICS (FROM REF.6.)

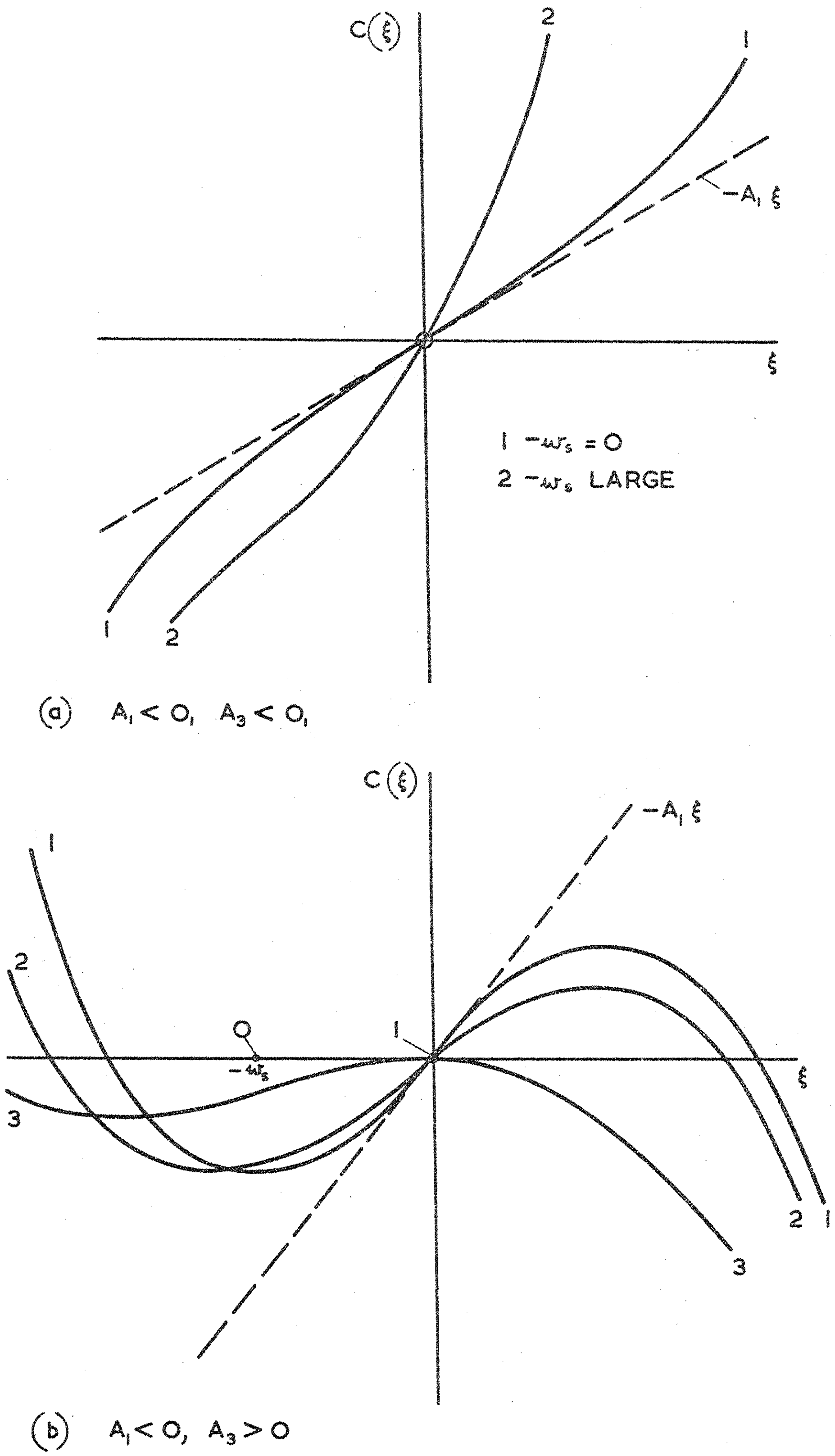


FIG.7. THE ξ v $C(\xi)$ CURVES ASSOCIATED WITH EQUATION 4.9