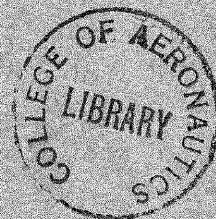THE COLLEGE OF AERONAUTICS

CRANFIELD

SIMULATION OF RATIONAL TRANSFER FUNCTIONS

WITH ADJUSTABLE COEFFICIENTS

by

R. J. A. Paul

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Simulation of Rational Transfer Functions

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SUMMARY

Design techniques are described for the simulation of quadratic functions with independent control of each coefficient. Rational function approximations, for the simulation of dead time, are considered. Other typical examples are described and include the simulation of Butterworth functions, Chebyshev functions and orthonormal functions, which have application in self optimizing control systems.

CONTENTS

1. Introduction ·

In the equlization of control systems there is often a requirement for networks having rational transfer functions with the facility of independent control of one or more coefficients. Again, in many forms of self optimizing control systems these forms of networks are required, where adjustment of the relevant coefficient is automatically controlled to meet a certain performance index.

This note is concerned with the design of such networks and their application to specific problems.

The methods described are an extension of the work by Mathews and Seifert[1]

2. The Generalized Two Port Active Network Arrangement

The derivation of all rational transfer functions given in this note is based on the arrangement shown in Fig. 2.1.
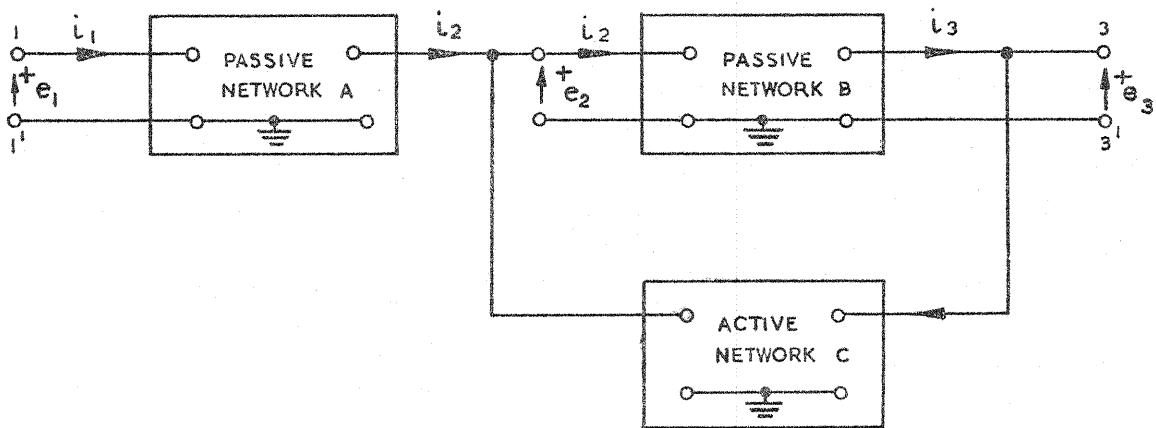

Fig. 2.1. The Basic Arrangement

Input terminals 1 1' ; Output terminals 3 3' .

Networks A and B are both passive and comprise combinations of linear resistors and capacitors.

Network C represents a computing or operational amplifier having an infinite driving point input impedance and a zero driving point output impedance. The forward open circuit voltage transfer function is of the form $- K \cdot G_1(s)$ where K is a positive constant, and s is the complex frequency $(\sigma + jw)$.

A detailed analysis of the above arrangement is given in Appendix A where the following result is derived: -

i.e.
$$\frac{E_3(s)}{E_1(s)} = \frac{y_{21}^{A}(s)}{y_{12}^{B}(s)} \quad , \qquad (2.1)$$

where $\quad y_{21}{}^{A}(s) = \dfrac{I_2(s)}{E_1(s)}\Bigg|_{E_2(s)=0}$ $\equiv$ forward short circuit transfer admittance of network A,

and $\quad y_{12}{}^{B}(s) = \dfrac{I_2(s)}{E_3(s)}\Bigg|_{E_2(s)=0}$ $\equiv$ reverse short circuit transfer admittance of network B.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (2.2)

The derivation of equation 2.1 is based on the following assumptions: -

(a) Networks A and B are initially relaxed (i.e. zero charge on all capacitors).

(b) $K\ G_1(s) \to \infty$ over the frequency range of interest.

(c) $y_{11}{}^{C}(s) = 0$.

(d) $y_{22}{}^{C}(s) = -\infty$

(e) $y_{12}{}^{B}(s) \geqslant \dfrac{y_{22}{}^{A}(s) - y_{11}{}^{B}(s)}{K}$ for the frequency range of interest.

(f) $y_{21}{}^{B}(s)$ and $y_{22}{}^{B}(s)$ are both finite over the frequency range of interest.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (2.3)

It should be noted that these assumptions are implied in conventional analogue computing circuit applications and, in practice, may be achieved without undue difficulty.

Mathews and Seifert[1] describe synthesis procedures, for rational transfer functions, based on equation (2.1). They also outline a further synthesis procedure, based on use of an operational amplifier and two sign reversing amplifiers, for the simulation of a wider range of rational transfer functions than that possible in the former method. A further advantage of the latter method is that the number and range of values of components is reduced, compared with the one amplfier configuration.

As stated in the introduction, this note describes modifications of these procedures, which result in rational transfer functions with coefficients which may be adjusted independently, within prescribed limits. The methods outlined involve the use of RC tee networks and some standard forms are described in the next section.

3. Standard Forms of T.Network

TABLE 3.1.

NETWORK	$y_{21}(s) = -y_{12}(s)$
	$\dfrac{s^2 C_A^2 R_A}{1 + s2C_A R_A} = \dfrac{1}{2R_C}\left(\dfrac{R_C}{2R_A} \cdot \dfrac{s^2 T^2}{1+sT}\right) = \dfrac{1}{2R_C}\left(\dfrac{N_1 s^2 T^2}{1+sT}\right)$
	$\dfrac{sC_B}{1+sC_B R_B} = \dfrac{1}{2R_C}\left(\dfrac{2R_C}{R_B} \cdot \dfrac{sT}{1+sT}\right) = \dfrac{1}{2R_C}\left(\dfrac{N_2 sT}{1+sT}\right)$
	$-\dfrac{1}{2R_C\left(1 + \dfrac{sC_C R_C}{2}\right)} = \dfrac{1}{2R_C}\left(\dfrac{1}{1+sT}\right)$

where

$$T = 2C_A R_A = C_B R_B = \frac{C_C R_C}{2} = CR$$

Note:- denominator of each function contains $(1 + sT)$.

3.1.1. Special Case of (3.1)

3.1.1.1. $N_1 = N_2 = 1$

$$N_1 = \frac{R_C}{2R_A} = 1 \quad ; \quad N_2 = \frac{2R_C}{R_B} = 1 \, ,$$

$$T = C_A R_C = 2 C_B R_C = \frac{C_C R_C}{2} \quad .$$

i.e. $\quad C_A = 2 C_B = \frac{C_C}{2} = C \quad .$ $\hspace{3cm}$ (3.1.)

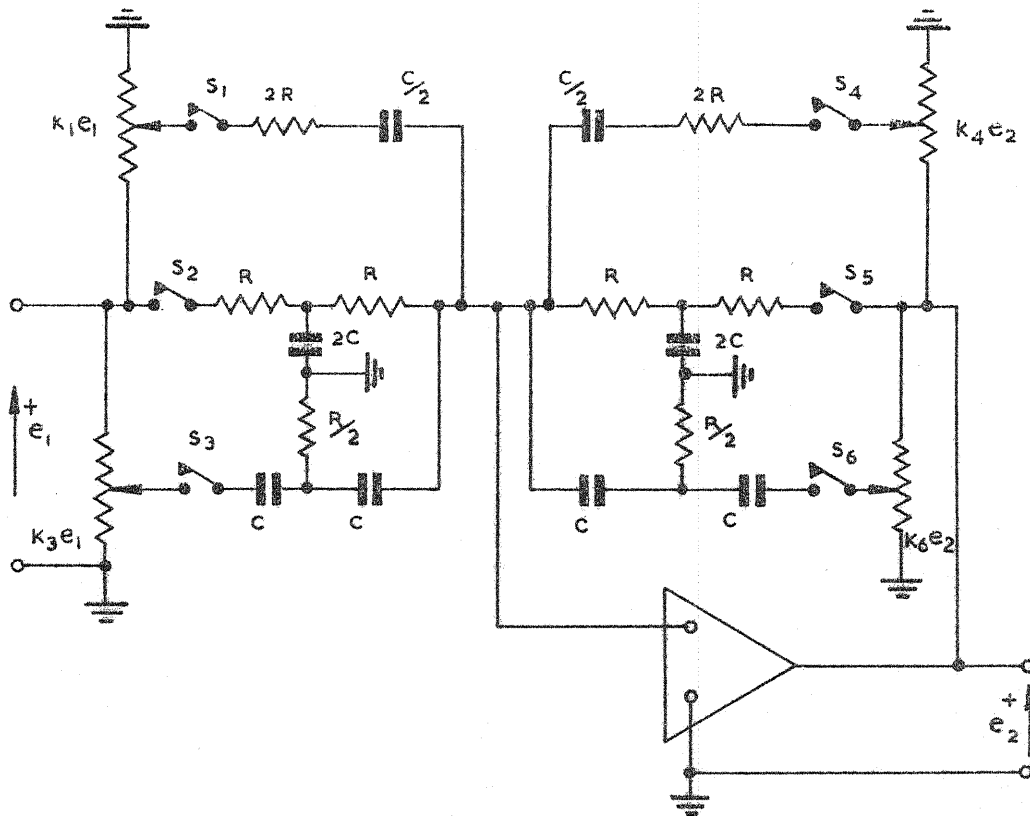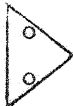Let $\quad R_C = R = \frac{R_B}{2} = 2 R_A$

$$C = C_A = 2 C_B = \frac{C_C}{2} \quad .$$

Some possible circuit arrangements with their respective transfer functions, based on the use of these forms of network are given in Table 3.2.

TABLE 3.2.

NETWORK

---

denotes an operational amplifier

$T = RC$; k's ≤ 1

Assumption:- output resistance of each potentiometer is negligible.

NETWORK CONDITIONS		$\dfrac{E_2(s)}{E_1(s)}$
All switches closed		$-\dfrac{1 + k_1 sT + k_3 s^2 T^2}{1 + k_4 sT + k_6 s^2 T^2}$
$S_1, S_2, S_4, S_5, S_6,$	closed	$-\dfrac{1 + k_1 sT}{1 + k_4 sT + k_6 s^2 T^2}$
S_3	open	
S_1, S_2, S_4, S_5	closed	$-\dfrac{1 + k_1 sT}{1 + k_4 sT}$
S_3, S_6	open	
S_3, S_5, S_6	closed	$-\dfrac{k_3 s^2 T^2}{1 + k_6 s^2 T^2}$ *
S_1, S_2, S_4	open	
S_1, S_5, S_6	closed	$-\dfrac{k_1 sT}{1 + k_6 s^2 T^2}$ *
S_2, S_3, S_4	open	
S_2, S_6	closed	$-\dfrac{1}{k_6 s^2 T^2}$
S_1, S_3, S_4, S_5	open	

* An additional amplitude limiting circuit is required for these cases.

3.1.1.2. $\underline{N_1 = 1 ; N_2 = 2}$

$$N_1 = 1 = \frac{R_C}{2R_A} \quad , \quad \text{i.e. } R_C = 2R_A . \qquad)$$

$$N_2 = 2 = \frac{2R_C}{R_B} \quad , \quad \text{i.e. } R_C = R_B . \qquad)$$

$$T = 2 C_A R_A = C_C R_B = \frac{C_C R_C}{2} = CR . \qquad) \qquad (3.2.)$$

$$R = R_C = R_B = 2R_A , \qquad)$$

$$C = \frac{C_C}{2} = C_B = C_A . \qquad)$$

    With combinations of the above forms of networks the transfer functions,
which may be obtained, are the same as those tabulated in Table 3.2 except
that the sT terms in the numerator and denominator is replaced by 2 sT.

    Permutations of the forms in 3.2.1, and 3.2.2 are obviously possible.

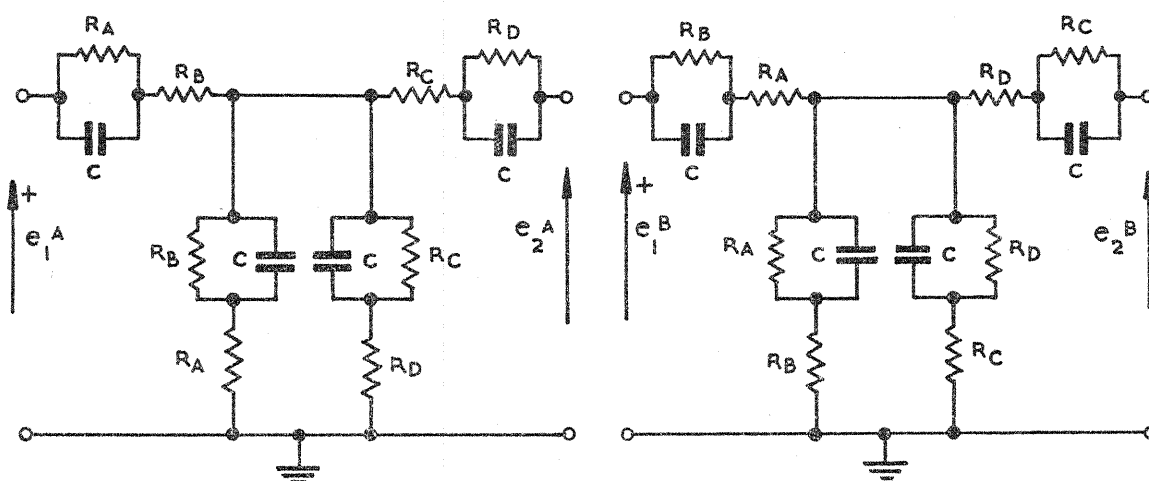   A simplified network having many practical applications is shown in
Fig. 3.1.
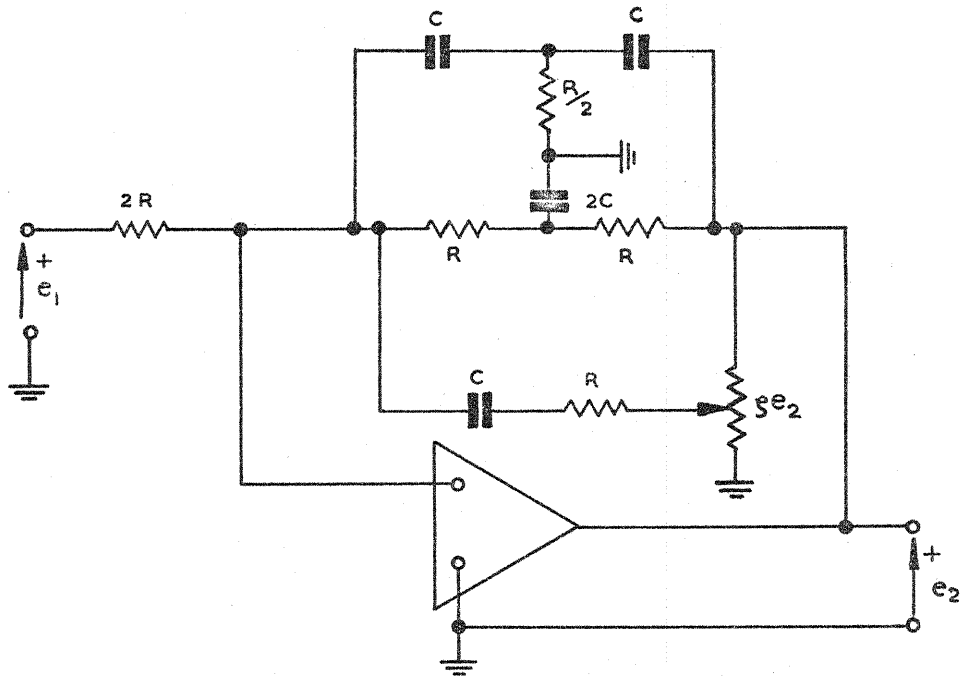
$$\frac{E_2(s)}{E_1(s)} = - \frac{1 + sT}{1 + 2\,\mathcal{J}sT + s^2 T^2} ,$$

$$(T = RC).$$

Fig. 3.1.1.

3.2. Type 2 Form

A further form of T network, which when used in the general active network gives a factored form of rational transfer function is shown in Fig. 3.2.1.

(a) Network A   (b) Network B

Fig. 3.2.1.

For Network A,

$$y_{21}^{A} = \frac{(1 + sCR_A)\ (1 + sCR_D)}{[2+sC(R_A+R_B)]\ [R_C+R_D+sCR_CR_D] + [2+sC(R_C+R_D)]\ [R_A+R_B+sCR_AR_B]} \tag{3.2.1}$$
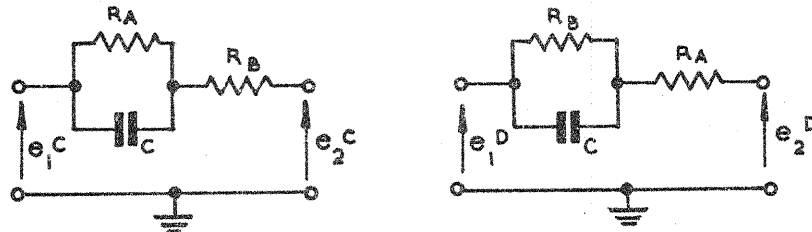
For Network B,

$$y_{21}^{B} = \frac{(1 + sCR_B)\ (1 + sCR_C)}{[2+sC(R_A+R_B)]\ [R_C+R_D+sCR_CR_D] + [2+sC(R_C+R_D)]\ [R_A+R_B+sCR_AR_B]} \tag{3.2.2}$$

The main feature of these two networks is that the denominator of the forward short circuit admittance, of each network, is identical.

3.2.1. Reduced Form of Network

The networks shown in Fig. 3.2.1. may be simplified to the form shown in Fig. 3.2.2.

(a) Network C

(b) Network D

Fig. 3.2.2.

For Network C,

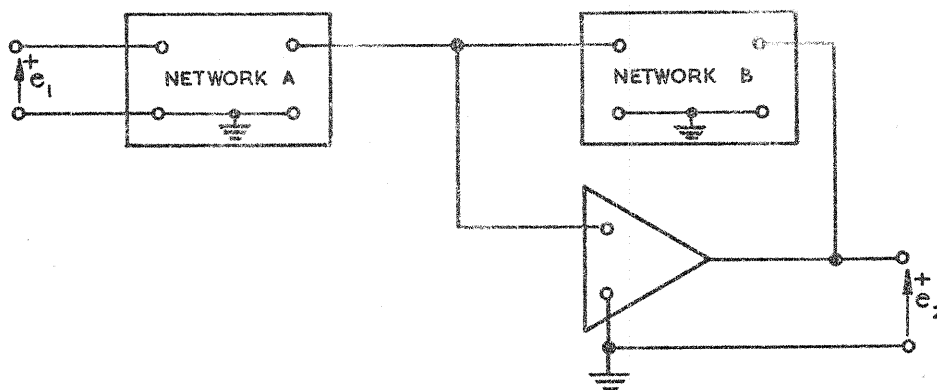$$y_{21}^{C} = \frac{(1 + sCR_A)}{R_A + R_B + sCR_A R_B} \cdot \qquad (3.2.3)$$

For Network D,

$$y_{21}^{D} = \frac{(1 + sCR_B)}{R_A + R_B + sCR_A R_B} \cdot \qquad (3.2.4)$$

3.2.2. Active Networks

The active network arrangement using networks A and B is shown in Fig. 3.2.3.

$$\frac{E_2(s)}{E_1(s)} = - \frac{(1 + sCR_A)(1 + sCR_D)}{(1 + sCR_B)(1 + sCR_C)} \cdot \qquad (3.2.5)$$

Fig. 3.2.3.
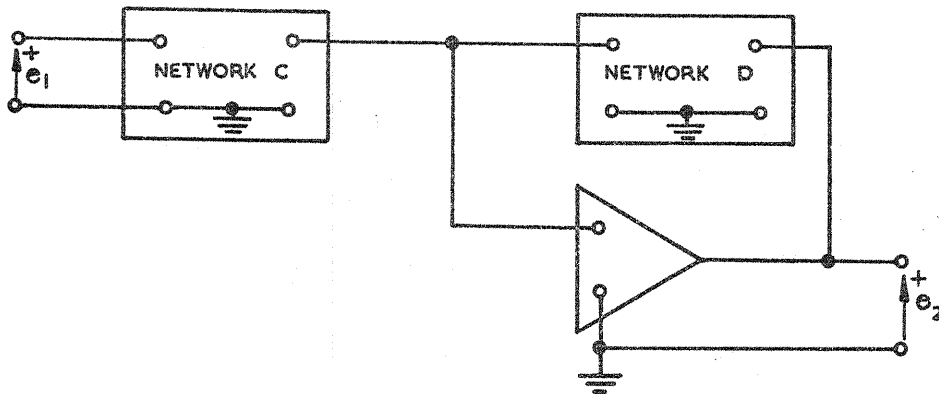
$$\frac{E_2(s)}{E_1(s)} = - \frac{(1 + sCR_A)}{(1 + sCR_B)} \cdot \qquad\qquad (3.2.6)$$

Fig. 3.2.4.

The arrangement shown in Fig. 3.2.3 permits independent adjustment of each factor. This facility requires the use of four matched sections R_A, R_B, R_C, and R_D.

Independent adjustment of each factor in the arrangement shown in Fig. 3.2.4 requires the use of two matched sections of each of the resistors R_A and R_B.

A practical version of this latter arrangement has been built in the laboratory and has been used extensively in the simulation of servo-systems. The practical circuit details are shown in Fig. 3.2.5.
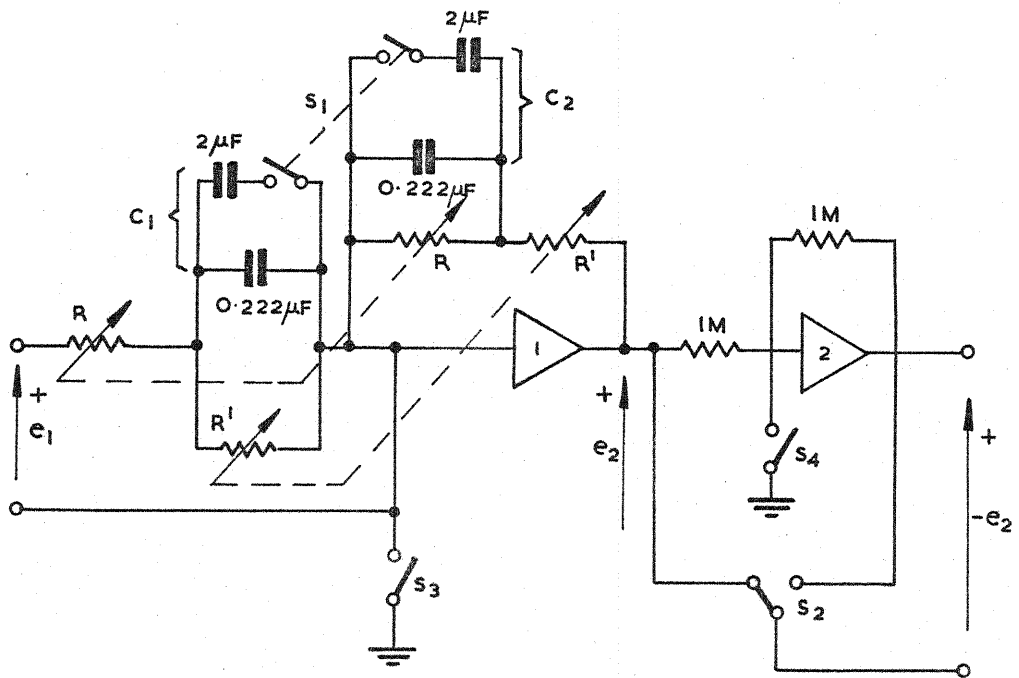
R ≡ 2 ganged Helical-Potentiometer
(each matched section 0.47 MΩ)

Switches S_2 and S_3 are used for
the purpose of zero adjustment.

R' ≡ "   "   "   "

$$\frac{E_2(s)}{E_1(s)} = - \frac{1 + sT_1}{1 + sT_2} \left\{ \begin{array}{l} \text{with } s_1 \text{ open, max. value of } T_1 \text{ and } T_2 \text{ respectively} \\ \hspace{4cm} \text{is 0.1 sec.} \\ \text{with } s_1 \text{ closed, max. value is increased to 1.0 sec.} \end{array} \right\}.$$

Fig. 3.2.5.

3.3. An Oscillator based on the use of T Networks

From Table 3.2 the relevant transfer function is,

$$G(s) = - \frac{k_1 sT}{1 + k_6 s^2 T^2} , \qquad (3.3.1)$$

If $k_1 = 1$; $k_6 = 1$, then

$$G(s) = - \frac{sT}{1 + s^2 T^2} = \text{say } \frac{E_2(s)}{E_1(s)} . \qquad (3.3.2)$$

If $E_1(s)$ is the transform of a step function having a magnitude E, then the inverse transform is given by :

$$E_2(t) = - E \sin wt. \qquad (3.3.3)$$

$$\text{where} \qquad w = \frac{1}{T} \qquad (3.3.4)$$

This is the equation of an oscillator.

In practice, slight changes in operating conditions and values of components result in spurious phase errors and an amplitude control circuit is necessary.

A practical arrangement which has been investigated is shown in Fig. 3.3.1.
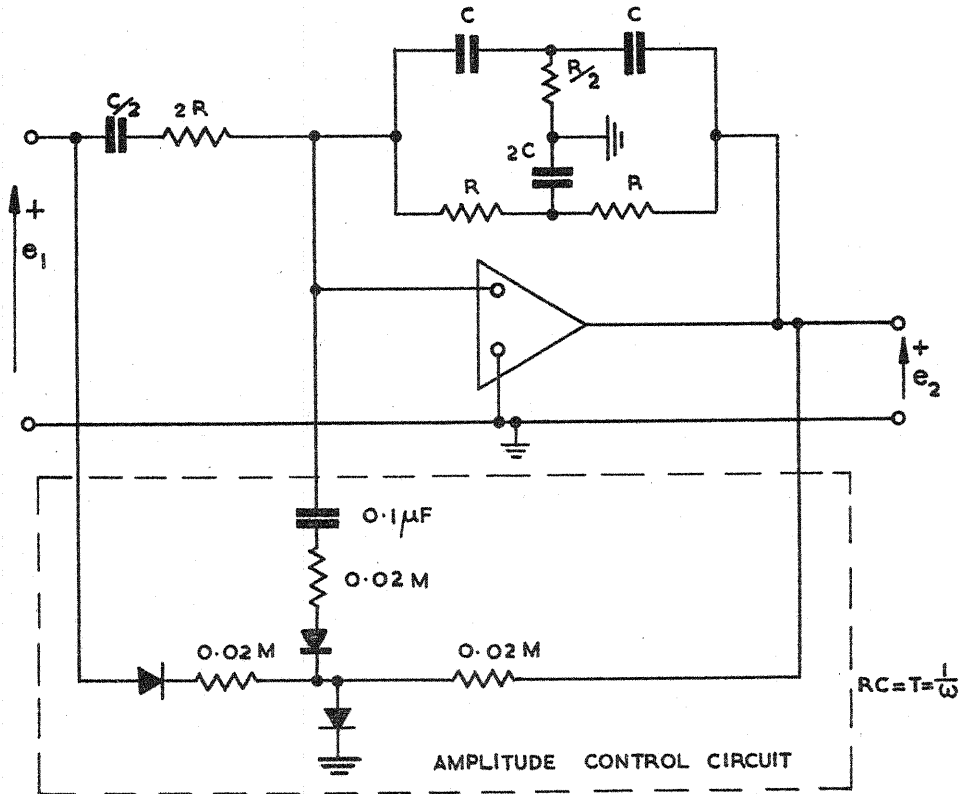
Fig. 3.3.1. Amplitude Controlled Oscillator

A detailed analysis of the optimum amplitude control circuit has not been completed but results which have been obtained from the arrangement shown in Fig. 3.3.1. are as follows:-

E_1 volts	E_2 volts
0.25	0.275
2.50	3.0
25.0	30.0

These results indicate that the presence of the diodes has a greater effect on low amplitude signals than that with high amplitude values, as might be expected.

4. Networks Using the Minimum Number of Components

4.1. General Remarks

The network described in Section 3 may be simplified if the transfer function to be simulated is one of the following forms :-

$$\text{(i)} \qquad G(s) = - \frac{ksT}{1 + asT + bs^2T^2} , \qquad\qquad (4.1.1)$$

$$\text{or} \qquad \text{(ii)} \qquad G(s) = - \frac{1}{1 + asT + bs^2T^2} . \qquad\qquad (4.1.2)$$

where k, a and b are positive constants which do not require to be altered from their initial value. The time constant T may be required to be adjustable.

The reduction in the number of components is obtained by the incorporation of extra feedback paths.

$$\text{4.2.} \qquad G(s) = - \frac{ksT}{1 + asT + bs^2T^2} . \qquad\qquad (4.2.1)$$

Consider Fig. 4.2.1.

Fig. 4.2.1

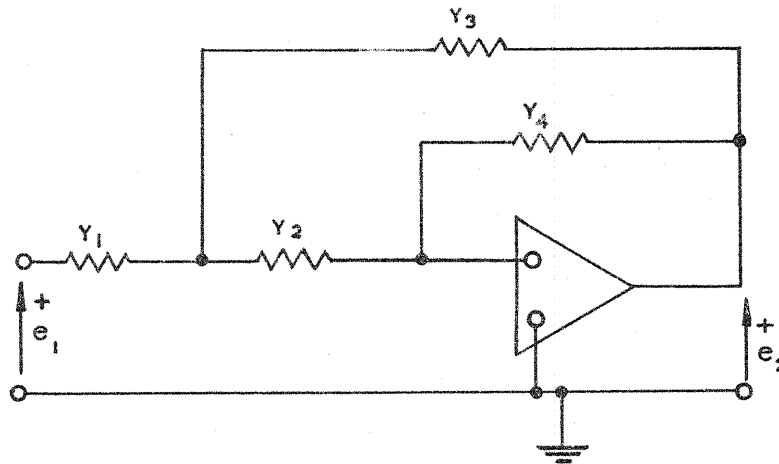$$\frac{E_1(s)Y_1(s)+E_2(s)Y_3(s)}{(Y_1(s)+Y_3(s))} \cdot \frac{(Y_1(s)+Y_3(s))Y_2(s)}{(Y_1(s)+Y_2(s)+Y_3(s))} = -E_2(s)Y_4(s) \qquad (4.2.2)$$

therefore $\quad \dfrac{E_2(s)}{E_1(s)} = -\dfrac{Y_1(s)\,Y_2(s)}{Y_4(s)\,(Y_1(s)+Y_2(s)+Y_3(s))+Y_2(s)Y_3(s)}$, $\qquad (4.2.3)$

with $\quad Y_1(s) = sC_1 \; ; \; Y_2(s) = \dfrac{1}{R_2} \; ; \; Y_3(s) = \dfrac{1}{R_3} \; ; \; Y_4(s) = sC_4$. $\qquad (4.2.4)$

$$\frac{E_2(s)}{E_1(s)} = -\frac{sC_1 R_3}{1+sC_4(R_2+R_3)+s^2 C_1 C_4 R_2 R_3}, \qquad (4.2.5)$$

$\therefore \quad G(s) = \dfrac{E_2(s)}{E_1(s)} = -\dfrac{sC_1 R_3}{1+sC_4(R_2+R_3)+s^2 C_1 C_4 R_2 R_2} \equiv \dfrac{ksT}{1+asT+bs^2 T^2}$. $\qquad (4.2.6)$

Equating terms:

$$C_1 R_3 = kT, \qquad)$$
$$C_4(R_2+R_3) = aT, \qquad) \qquad (4.2.7)$$
$$C_1 C_4 R_2 R_3 = bT^2, \qquad)$$

$\therefore \quad C_4 R_2 = \dfrac{bT}{k}$, $\qquad (4.2.8)$

and $\quad \dfrac{R_2+R_3}{R_2} = \dfrac{ka}{b}$. $\qquad (4.2.9)$

i.e. $\quad \dfrac{R_3}{R_2} = \dfrac{ka-b}{b}$, $\qquad (4.2.10)$

$$\frac{C_1}{C_4} \cdot \frac{R_3}{R_2} = \frac{k^2}{b}, \qquad (4.2.11)$$

$$\frac{C_1}{C_4} = \frac{k^2}{b} \times \frac{b}{ka-b} = \frac{k^2}{ka-b}. \qquad (4.2.12)$$

A typical example which has application in Section 5 has values as follows:

$$k = 2 \; ; \; a = 1 \; ; \; b = \frac{1}{3}, \qquad (4.2.13)$$

i.e. $\quad G(s) = -\dfrac{2sT}{1+sT+\dfrac{s^2 T^2}{3}}$. $\qquad (4.2.14)$

Hence $\quad \dfrac{R_3}{R_2} = \dfrac{2-\dfrac{1}{3}}{\dfrac{1}{3}} = 5$. $\qquad (4.2.15)$

$$\frac{C_1}{C_4} = \frac{4}{2-\dfrac{1}{3}} = \frac{12}{5}. \qquad (4.2.16)$$

$$C_1 R_3 \;=\; 2T \;=\; 2RC . \qquad\qquad (4.2.17)$$

If
$$R_3 \;=\; 2R \qquad\qquad)$$
$$\qquad\qquad)\qquad\qquad (4.2.18)$$
$$C_1 \;=\; C . \qquad\qquad) ,$$

then the network to satisfy these conditions is shown in Fig. 4.2.2.
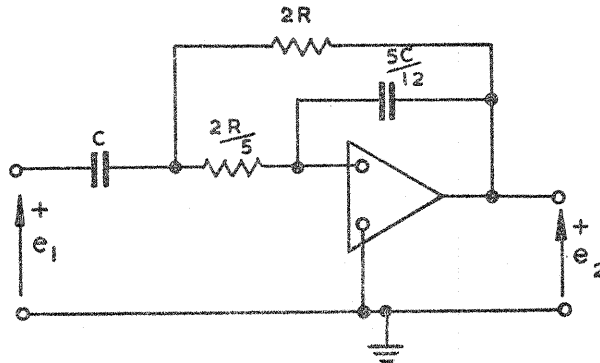
$$G(s) \;=\; \frac{E_2(s)}{E_1(s)} \;=\; \frac{-\,2sT}{1 + sT + \dfrac{s^2 T^2}{3}} \;.$$

$$T \;=\; RC ,$$

Fig. 4.2.2.

4.3.
$$G(s) \;=\; -\,\frac{1}{1 + asT + bs^2 T^2} \;. \qquad\qquad (4.3.1)$$

Consider Fig. 4.3.1.
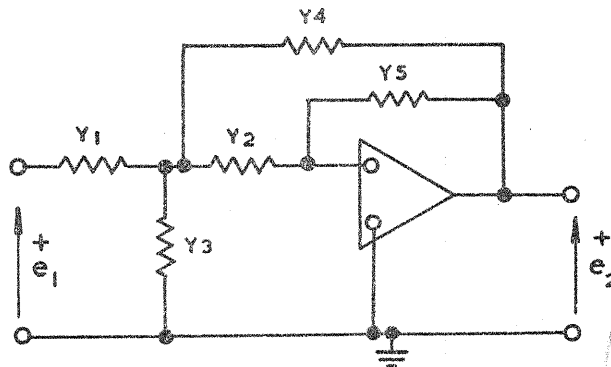
Fig. 4.3.1.

$$\frac{E_1(s)\,Y_1(s)+E_2(s)Y_4(s)}{Y_1(s)+Y_3(s)+Y_4(s)} \cdot \frac{Y_2(s)(Y_1(s)+Y_3(s)+Y_4(s))}{Y_1(s)+Y_2(s)+Y_3(s)+Y_4(s)} = -E_2(s)Y_5(s), \qquad (4.3.2)$$

$$\therefore \quad \frac{E_2(s)}{E_1(s)} = -\frac{Y_1(s)\,Y_2(s)}{Y_5(s)(Y_1(s)+Y_2(s)+Y_3(s)+Y_4(s))+Y_2(s)Y_4(s)} \qquad (4.3.3)$$

$$\equiv -\frac{1}{1 + asT + bs^2 T^2} \qquad (4.3.4)$$

If $\quad Y_1(s) = \dfrac{1}{R_1}$; $Y_2(s) = \dfrac{1}{R_2}$; $Y_4(s) = \dfrac{1}{R_1}$; $Y_5(s) = sC_5$; $Y_3(s) = sC_3$, $\qquad (4.3.5)$

then $\quad \dfrac{E_2(s)}{E_1(s)} = -\dfrac{1}{1 + sC_5(R_1 + 2R_2) + s^2 C_3 C_5 R_1 R_2} \qquad (4.3.6)$

$$\equiv -\frac{1}{1 + asT + bs^2 T^2} \qquad (4.3.7)$$

Equating terms,

$$\left.\begin{array}{l} C_5(R_1 + 2R_2) = aT \\[4pt] C_3 C_5 R_1 R_2 = bT^2 . \end{array}\right\} \qquad (4.3.8)$$

If $\qquad 2R_2 = R_1$, $\qquad (4.3.9)$

then $\qquad \left.\begin{array}{l} 2C_5 R_1 = aT . \\[8pt] C_3 \dfrac{C_5 R_1^2}{2} = bT^2 . \end{array}\right\} . \qquad (4.3.10)$

$$\therefore \qquad C_3 R_1 = \frac{4bT}{a} , \qquad (4.3.11)$$

and $\qquad \dfrac{C_5}{C_3} = \dfrac{aT}{2} \cdot \dfrac{a}{4bT} = \dfrac{a^2}{8b} . \qquad (4.3.12)$

As a typical example, if $a = 1$; $b = 1$,

then $\qquad \left.\begin{array}{l} \dfrac{C_5}{C_3} = \dfrac{1}{8} \\[10pt] \dfrac{R_1}{R_2} = 2 \\[10pt] C_5 R_1 = \dfrac{T}{2} . \end{array}\right\} \qquad (4.3.13)$

If $T = RC$, make $R_2 = R$ and $C_5 = \dfrac{C}{4}$. $\quad$)

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$) $\qquad\qquad$ (4.3.14)

Hence $R_1 = 2R$ and $C_3 = 2C$. $\qquad\qquad\qquad$)

The network which satisfies these conditions is shown in Fig. 4.3.2.
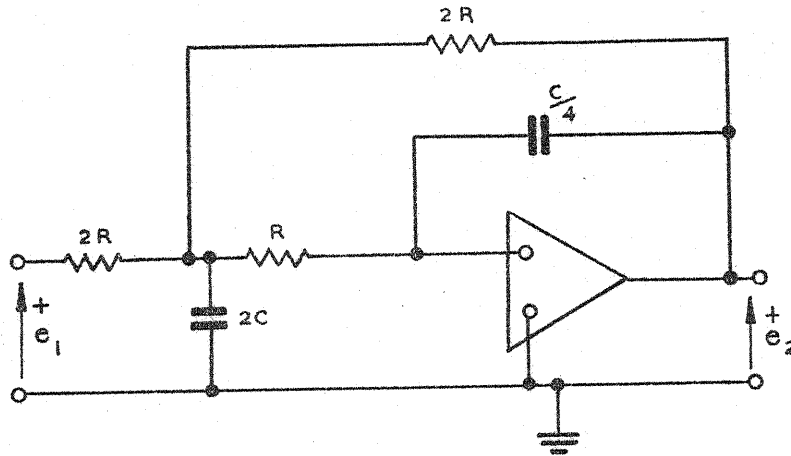
$$\frac{E_2(s)}{E_1(s)} = G(s) = -\frac{1}{1 + sT + s^2T^2} \quad .$$

$$T = RC$$

Fig. 4.3.2.

5. Simulation of Dead Time (Transportation Lag)

5.1. Second Order Rational Function Approximation

If a function $f(t)$ has a Laplace Transform $f(s)$,

i.e. $\qquad\qquad f(t) \supset f(s)$, $\qquad\qquad\qquad\qquad\qquad$ (5.1.1)

then the delayed function

$$f(t - T_d) \supset e^{-sT_d} f(s) .$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (5.1.2)

The factor e^{-sT_d} may be approximated by a rational function over a limited range of frequency. One form of approximation, which is considered in this section, is the cascade connection of networks having forward transfer functions of the form $G(s)$,

where
$$G(s) = \frac{1 - asT + bs^2 T^2}{1 + asT + bs^2 T^2} \,, \tag{5.1.3}$$

and
$$G(jw) = \frac{(1 - bw^2 T^2) - jawT}{(1 - bw^2 T^2) + jawT} \,. \tag{5.1.4}$$

If
$$v = wT \,, \tag{5.1.5}$$

then
$$G(jv) = \frac{(1 - bv^2) - jav}{(1 - bv^2) + jav} \,, \tag{5.1.6.}$$

i.e.
$$G(vj) = 1 \cdot e^{-j2\phi} \,, \tag{5.1.7}$$

where
$$\phi = \arctan \frac{av}{1 - bv^2} \,. \tag{5.1.8}$$

i.e.
$$\tan \phi = \frac{av}{1 - bv^2} \,, \tag{5.1.9}$$

and
$$\sec^2 \phi \, \frac{d\phi}{dv} = \frac{a(1 + bv^2)}{(1 - bv^2)^2} \tag{5.1.10}$$

$$\therefore \quad \frac{d\phi}{dv} = \frac{a(1 + bv^2)}{(1 - bv^2)^2 \left(1 + \dfrac{a^2 v^2}{(1 - bv^2)^2}\right)} \,, \tag{5.1.11}$$

i.e.
$$\frac{d\phi}{dv} = \frac{a(1 + bv^2)}{(1 - bv^2)^2 + a^2 v^2} \,. \tag{5.1.12}$$

$$\therefore \quad \frac{1}{a} \, \frac{d^2\phi}{dv^2} = \frac{((1 - bv^2)^2 + a^2 v^2)2bv - (1 + bv^2)(-4bv(1 - bv^2) + 2a^2 v)}{\left[(1 - bv^2)^2 + a^2 v^2\right]^2} \,. \tag{5.1.13}$$

Now consider the function $G_1(jw) = e^{-jwT_d}$. $\tag{5.1.14}$

$$\left| G_1(jw) \right| = 1 \,.$$

$$\angle G_1(jw) = \theta = -wT_d \,. \tag{5.1.15}$$

If
$$T = T_d \,, \tag{5.1.16}$$

then
$$v = wT_d \,, \tag{5.1.17}$$

and
$$\frac{d\theta}{dv} = -1 \,. \tag{5.1.18}$$

The latter equation is valid for all values of v. The approximate function cannot meet this requirement and the question arises, how to choose the best approximating function. For example, by making $v = kwT_d$ it is possible to have a negative and

positive phase error over a certain frequency range with a zero error between the two limits. However, if a number of the approximate functions are to be connected in tandem, it would seem desirable to make the error zero at $v = 0$. Thus with reference to equation (5.1.12), to meet this condition

$$\frac{d\phi}{dv}\bigg|_{v=0} = 0.5 , \tag{5.1.19}$$

since

$$\theta\bigg|_{v=0} = 2\phi\bigg|_{v=0} \tag{5.1.20}$$

Hence

$$a = 0.5 \tag{5.1.21}$$

and

$$\frac{d^2\phi}{dv^2}\bigg|_{v=0} = 0 . \tag{5.1.22}$$

$\therefore$

$$b + 2b - a^2 = 0 ,$$

i.e.

$$b = \frac{a^2}{3} = \frac{1}{12} . \tag{5.1.23}$$

With these values of a and b,

$$\frac{d\phi}{dv} = \frac{0.5\left(1 + \frac{v^2}{12}\right)}{\left(1 - \frac{v^2}{12}\right)^2 + \frac{v^2}{4}} , \tag{5.1.24}$$

$$= \frac{0.5\left(1 + \frac{v^2}{12}\right)}{\left(1 + \frac{v^2}{12} + \frac{v^4}{144}\right)} , \tag{5.1.25}$$

$$= 0.5\left[1 - \frac{v^4}{v^4 + 12v^2 + 144}\right] \tag{5.1.26}$$

Thus the term $\left(\dfrac{v^4}{v^4 + 12v^2 + 144}\right)$ may be regarded as an error factor.

For this error to be $\leqslant 0.01$,

then

$$\frac{v^4}{v^4 + 12v^2 + 144} \leqslant 0.01 . \tag{5.1.27}$$

Hence

$$0.99v^4 - 0.12v^2 - 1.44 \leqslant 0 \tag{5.1.28}$$

put

$$u = v^2 , \tag{5.1.29}$$

then

$$0.99u^2 - 0.12u - 1.44 \leqslant 0 \tag{5.1.30}$$

i.e. $\qquad$ $u \leqslant 1.26,$ $\hfill$ (5.1.31)

and $\qquad$ $v \leqslant 1.12 .$ $\hfill$ (5.1.32)

i.e. the error will exceed 1% if $v = wT$ exceeds 1.12.

The transfer function of the approximate network, with $a = 0.5$ and $b = \dfrac{1}{12}$, is

$$G(s) = \frac{1 - \dfrac{sT}{2} + \dfrac{s^2 T^2}{12}}{1 + \dfrac{sT}{2} + \dfrac{s^2 T^2}{12}} , \hspace{2cm} (5.1.33)$$

i.e.

$$G(s) = 1 - \frac{sT}{1 + \dfrac{sT}{2} + \dfrac{s^2 T^2}{12}} . \hspace{2cm} (5.1.34)$$

Since $\qquad$ $T = T_d ,$

$$G(s) = 1 - \frac{sT_d}{1 + \dfrac{sT_d}{2} + \dfrac{s^2 T_d^2}{12}} . \hspace{2cm} (5.1.35)$$

If $\qquad$ $T_d = 2T_1 ,$

then $\qquad$
$$G(s) = 1 - \frac{2sT_1}{1 + sT_1 + \dfrac{s^2 T_1^2}{3}} , \hspace{2cm} (5.1.36)$$

Equation 5.1.33 is known as the fourth order Padé rational polynomial fraction[2].

A good review of time delay approximation and relevant networks is given by King and Rideout[3].

5.2. <u>Simulation of approximate Function</u> $G(s) = -\left[1 - \dfrac{2sT_1}{1 + sT_1 + \dfrac{s^2 T_1^2}{3}} \right]$

If $\qquad$
$$-\frac{2sT_1}{1 + sT_1 + \dfrac{s^2 T_1^2}{3}} = \frac{E_2(s)}{E_1(s)} , \hspace{2cm} (5.2.1)$$

the network which satisfies this function has been discussed in section 4, and is shown in Fig. 4.2.2, where in this case $T_1 = RC$.

The complete simulation of $G(s) = -\left[1 - \dfrac{2sT_1}{1 + sT_1 + \dfrac{s^2 T_1^2}{3}} \right]$,

is shown in Fig. 5.2.1.
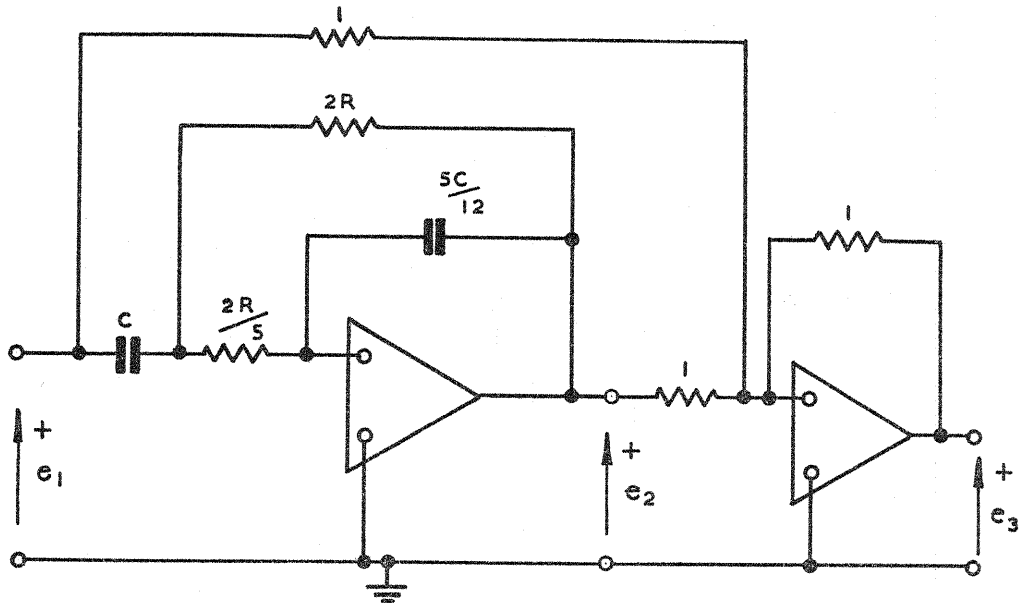
$$\frac{E_2(s)}{E_1(s)} = - \frac{2sT_1}{1 + sT_1 + \frac{s^2 T_1^2}{3}} \quad , \quad \frac{E_3(s)}{E_1(s)} = - \left[1 - \frac{2sT_1}{1 + sT_1 + \frac{s^2 T_1^2}{3}} \right]$$

$$T_1 = RC = \frac{T_d}{2} \quad , \quad \text{(where } T_d \text{ is the dead time)}.$$

Fig. 5.2.1. Simulation of Network Approximating Dead Time

A similar network has been suggested by King and Rideout[3] and is shown in Fig. 5.2.2.
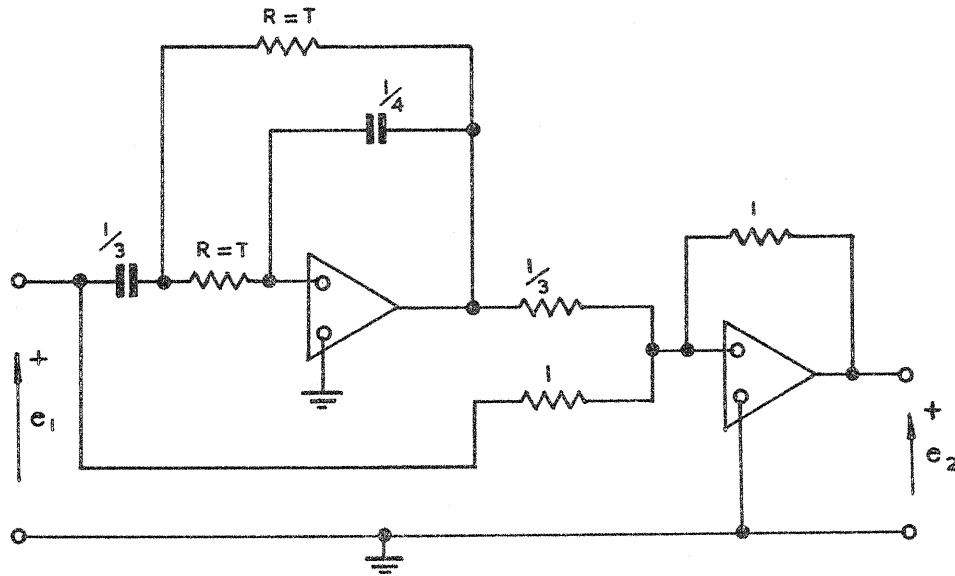
$$\frac{E_2(s)}{E_1(s)} = - \left[1 - \frac{sT}{1 + \dfrac{sT}{2} + \dfrac{s^2 T^2}{12}} \right]$$

$T_d = T$ where T_d is dead time.

Fig. 5.2.2.

5.3. Tandem Arrangement of Approximate Functions to Simulate Dead Time

If $G(s) = - \dfrac{1 - sT_1 + \dfrac{s^2 T_1^2}{3}}{1 + sT_1 + \dfrac{s^2 T_1^2}{3}}$, (5.3.1)

then with n such functions connected in tandem as shown in Fig. 5.3.1.,

$$\left[G(s) \right]^n = \frac{E_{n+1}(s)}{E_1(s)} .$$ (5.3.2)

Fig. 5.3.1

The choice of T_1 is now governed by the number of networks used.

For one network,
$$T_1 = \frac{T_d}{2} \quad , \tag{5.3.3}$$

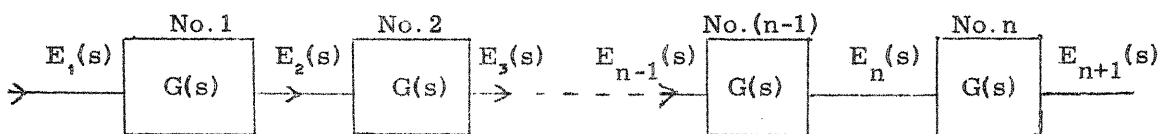where T_d is the dead time.

Hence, for n networks in tandem,
$$T_1 = \frac{T_d}{2n} \quad , \tag{5.3.4}$$

Under these conditions the error will be less than 1%, provided
$$v \leqslant 1.12 \, n, \tag{5.3.5}$$

where
$$v = wT_d \quad , \tag{5.3.6}$$

An alternative arrangement for cascading sections is given by King and Rideout[3], in which n sections connected in tandem only require $n + 1$ amplifiers.

6. Butterworth and Chebyshev Approximations of Low Pass Filter

6.1. Introductory remarks[4,5]

If $F(jw)$ represents the square of the magnitude of a network function, then,
$$F(jw) = \left| G(jw) \right|^2 = G(jw) \cdot G(-jw) \quad , \tag{6.1.1}$$

where $G(jw)$ is the given frequency response network function.

Also,
$$F(s) = G(s) \cdot G(-s) \quad , \tag{6.1.2}$$

which is valid for all values of s, but $F(s)$ is not equal to the square of the magnitude of $G(s)$, except for $s = jw$.

Now if, $\quad v = wT$, (T is a positive constant) $\tag{6.1.3}$

and $\quad F(jv) = G(jv) \cdot G(-jv)$, $\tag{6.1.4}$

then $\quad F(sT) = G(sT) \cdot G(-sT)$. $\tag{6.1.5}$

If now, consideration is given to the ideal lowpass filter characteristics shown in Fig. 6.1.1(a), there are two well known approximations to this ideal function. The characteristic for the first is shown in Fig. 6.1.1.(b) and is known as a "maximally-flat" or Butterworth approximation, with

$$F(jv) = \left| G(jv) \right|^2 = \frac{1}{1 + v^{2n}} \quad , \tag{6.1.6}$$

The second characteristic is shown in Fig. 6.1.1.(c) and is known as "an equal-ripple" or Chebyshev approximation with

$$F(jv) = \left| G(jv) \right|^2 = \frac{1}{1 + a^2 C_n^2(v)} \qquad (6.1.7)$$

in which $a < 1$ is a real number which controls the ripple amplitude, and where $v = 1$, corresponds to the edge of the pass band.

$C_n(v)$ is a Chebyshev polynomial defined by

$$C_n(v) = \text{Cos. } n(\text{arc. cosine } v) \text{ for } |v| < 1 \cdot \qquad (6.1.8)$$
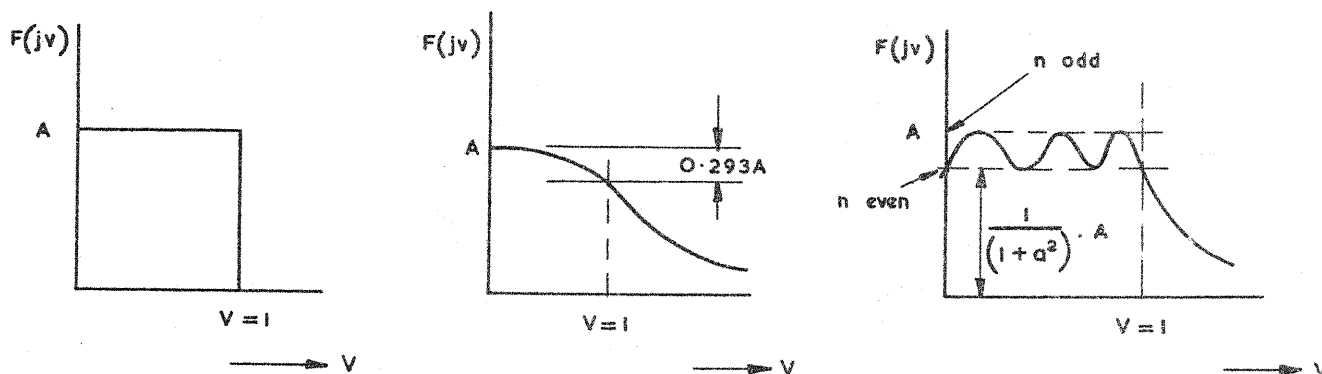
(a) Ideal low pass function

(b) Butterworth approximation

(c) Chebyshev approximation

Note A is a scale factor.

Fig. 6.1.1.

6.2. Butterworth Functions

$$F(sT) = G(sT) \cdot G(-sT) = \frac{1}{1 + (-1)^n (sT)^{2n}} \qquad (6.2.1.)$$

The poles of $F(sT)$ are

$$p_k = e^{j(2k-1+n)\frac{\pi}{2n}}, \quad k = 1, 2 \ldots 2n , \qquad (6.2.2)$$

6.2.1. n = 4

For n = 4, the left hand poles are :

$$p_1 = e^{j\frac{5\pi}{8}} \; ; \; p_2 = e^{j\frac{7\pi}{8}} \; ; \; p_3 = e^{j\frac{9\pi}{8}} \; ; \; p_4 = e^{j\frac{11\pi}{8}} . \tag{6.2.3}$$

Thus for n = 4,

$$G(sT) = \frac{1}{(sT - p_1)(sT - p_2)(sT - p_3)(sT - p_4)} \tag{6.2.4}$$

$$= \frac{1}{\left(sT - e^{j\frac{7\pi}{8}}\right)\left(sT - e^{j\frac{9\pi}{8}}\right)} \cdot \frac{1}{\left(sT - e^{j\frac{5\pi}{8}}\right)\left(sT - e^{j\frac{11\pi}{8}}\right)} \tag{6.2.5}$$

$$= \frac{1}{1 + sT\left(e^{j\frac{\pi}{8}} + e^{-j\frac{\pi}{8}}\right) + s^2T^2} \cdot \frac{1}{1 + sT\left(e^{j\frac{3\pi}{8}} + e^{-j\frac{3\pi}{8}}\right) + s^2T^2} \tag{6.2.6}$$

$$= \frac{1}{(1 + 1.8478sT + s^2T^2)} \cdot \frac{1}{1 + 0.7654sT + s^2T^2} \tag{6.2.7}$$

$$= \frac{1}{1 + 2.6131sT + 3.4142s^2T^2 + 2.6131s^3T^3 + s^4T^4} . \tag{6.2.8}$$

6.2.2. n = 5

In a similar manner to that used in 6.2.1., the factors of G(sT) may be derived. These factors are found to be :-

$$G(sT) = \frac{1}{(1 + sT)(1 + 0.61804sT + s^2T^2)(1 + 1.61804sT + s^2T^2)} \tag{6.2.9}$$

$$= \frac{1}{1 + 3.2361sT + 5.2361s^2T^2 + 5.2361s^3T^3 + 3.2361s^4T^4 + s^5T^5} . \tag{6.2.10}$$

6.2.3. n = 6

$$G(sT) = \frac{1}{(1 + 1.93186sT + s^2T^2)(1 + 1.41422sT + s^2T^2)(1 + 0.51764sT + s^2T^2)}$$

$$\ldots \ldots \tag{6.2.11}$$

$$= \frac{1}{1 + 3.8637sT + 7.4641s^2T^2 + 9.1416s^3T^3 + 7.4641s^4T^4 + 3.8637s^5T^5 + s^6T^6} .$$

$$\ldots \ldots \tag{6.2.12}$$

6.2.4. <u>n = 3</u>

$$G(sT) = \frac{1}{(1 + sT)(1 + sT + s^2T^2)} \qquad\qquad (6.2.13)$$

$$= \frac{1}{1 + 2sT + 2s^2T^2 + s^3T^3} \,. \qquad\qquad (6.2.14)$$

6.2.5. <u>n = 2</u>

$$G(sT) = \frac{1}{1 + 1.4142sT + s^2T^2} \,, \qquad\qquad (6.2.15)$$

6.3. <u>Simulation of Butterworth Functions</u>

The method of simulating Butterworth functions is apparent from the preceding analysis.

Thus for n even, the function may be synthesized by the cascade connection of quadratic factors.

For n odd, the cascade connection of $\frac{1}{(1 + sT)}$ and quadratic factors results.

The terms $-\frac{1}{1 + sT}$ may be obtained as shown in Fig. 6.3.1.
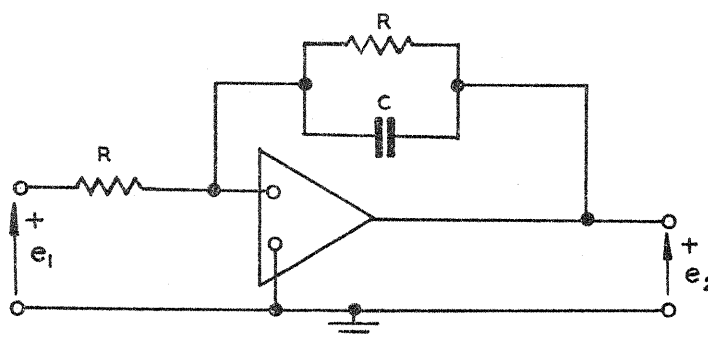
$$RC = T$$
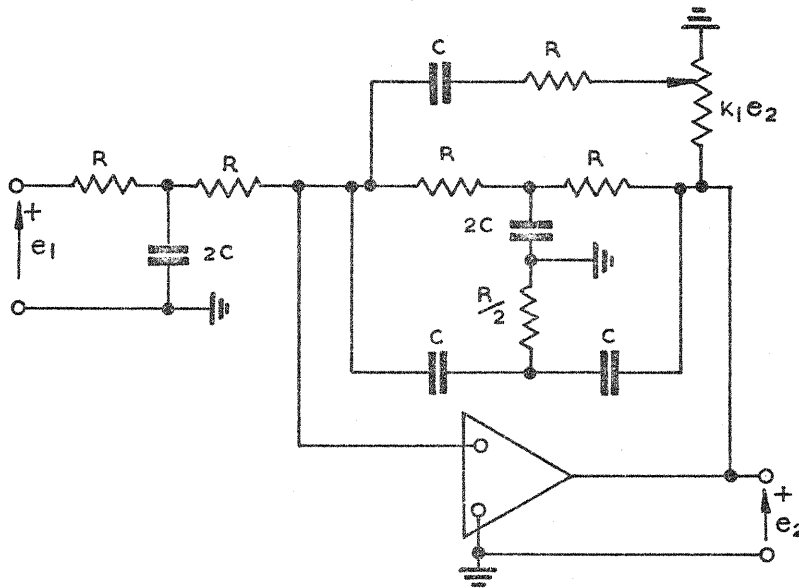
$$\frac{E(s)}{E(s)} = -\frac{1}{1 + sT}$$

<u>Fig. 6.3.1.</u>

Each of the quadratic factors are of the form:-

$$G(sT) = \frac{1}{1 + ksT + s^2T^2} \quad , \qquad\qquad (6.3.2)$$

where k is a constant.

The simulation of $-\dfrac{1}{1 + k_1 2sT + s^2T^2}$ is shown in Fig. 6.3.2.

Note: Output impedance of potentiometer assumed to be negligible.

$$\frac{E_2(s)}{E_1(s)} = -\frac{1}{1 + k_1 2sT + s^2T^2} \; ; \; T = RC \; ; \; k_1 < 1 \; .$$

Fig. 6.3.2.

Adjustment of T is obtained by the simultaneous adjustment of all resistors or all capacitors or both. A practical arrangement would result if all resistors were in the form of precision ten turn potentiometers, used as variable resistors, and fixed values of capacitances. This arrangement would require the use of a six gang potentiometer for each quadratic factor. If n such factors were connected in cascade, then a $6n$ gang potentiometer would be required or alternatively arrangements would have to be made to gang individual six gang potentiometers.

In view of these difficulties an arrangement of switched resistors on a decade scale may be preferable.

The cascade connection of quadratic factors is shown in Fig. 6.3.3. which illustrates as a typical example the simulation of a Butterworth Function with n=4.
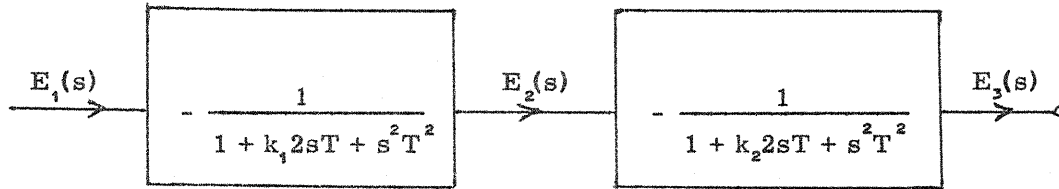
$$\frac{E_2(s)}{E_1(s)} = - \frac{1}{1 + k_1 2sT + s^2 T^2} \quad ; \quad \frac{E_3(s)}{E_2(s)} = - \frac{1}{1 + k_2 2sT + s^2 T^2} \quad .$$

$$\frac{E_3(s)}{E_2(s)} = \frac{1}{(1 + k_1 2sT + s^2 T^2)(1 + k_2 2sT + s^2 T^2)} \quad .$$

Fig. 6.3.3. Simulation of Butterworth Function (n = 4)

6.4. Chebyshev Functions

$$F(jv) = \left| G(jv) \right|^2 = \frac{1}{1 + a^2 C_n^2(v)} \left. \begin{array}{c}) \\ ) \\ ) \\ ) \end{array} \right. \quad (6.4.1)$$

and $\quad C_n(v) = \cos. n(\text{arc cosine } v) \text{ for } |v| \le 1$).

If jv is replaced by λ , where $\lambda = sT$, then

$$C_n(\frac{\lambda}{j}) = \cosh(n \cosh^{-1} \frac{\lambda}{j}) = \pm \frac{j}{a} \quad . \quad (6.4.2)$$

If now a new variable $z = x + jy$ is defined

and $\qquad \lambda = j \cosh z = j \cosh(x + jy)$, $\qquad\qquad (6.4.3.)$

then $\qquad C_n(\frac{\lambda}{j}) = \cosh nz = \cosh n(x + jy) = \pm \frac{j}{a}$. $\qquad (6.4.4)$

By expanding $\cosh nz$ and equating real and imaginary parts, the values of x and y may be determined. If these values of x and y are substituted into equation 6.4.3., the corresponding values of λ are given, which are the pole locations, i.e. $\lambda_k = \sigma_k + jv_k$

The result of these operations will be :-

$$\sigma_k = \sinh \alpha \ . \ \sin \frac{(2k-1)\pi}{2n} \qquad (6.4.5)$$

and

$$v_k = \cosh \alpha \ . \ \cos \frac{(2k-1)\pi}{2n} \quad , \qquad (6.4.6)$$

where

$$\alpha = \frac{1}{n} \sinh^{-1} \left(\frac{1}{a}\right) \qquad)$$
$$\qquad\qquad\qquad\qquad\qquad) \qquad (6.4.7)$$
$$k = 1, \ 2, \ 3 \ \ldots \ldots 2n \qquad) \quad .$$

This result leads to the expression

$$\frac{\sigma_k^2}{\sinh^2 \alpha} + \frac{v_k^2}{\cosh^2 \alpha} = 1 \quad . \qquad (6.4.8)$$

This is the equation of an ellipse in the λ-plane, with the major axis coincident with the jv axis.

Van Valkenburg[5] has described a procedure where the poles of the Chebyshev function may be derived from those of the Butterworth function of the same order. This procedure is as follows:-

Locate poles for the Butterworth case and reduce the real part by multiplying by $\tanh \alpha$ to give the real part of the equivalent pole of the Chebyshev case. The imaginary part in both cases remains the same.

If necessary the frequency may be inversely scaled by the $\cosh \alpha$ factor to give the ellipse equation 6.4.8.

To illustrate this procedure, the case for $n = 4$ is considered.

The Butterworth function for $n = 4$, is

$$G_B(sT) =$$
$$\frac{1}{(sT+0.9239+j0.3827)(sT+0.9239-j0.3827)(sT+0.3827+j0.9239)(sT+0.3827-j0.9239)}$$

$$\ldots \ldots \quad (6.4.10)$$

For the Chebyshev Function,

Take $n = 4$, $a = 0.1$

$$\therefore \quad \alpha = \frac{1}{4} \sinh^{-1} 10 = 0.75 \ . \qquad)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad)$$
$$\tanh \alpha = 0.6352 \ , \qquad\qquad\qquad) \qquad (6.4.11)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad)$$
$$\cosh \alpha = 1.295 \ . \qquad\qquad\qquad)$$

Multiplying the real parts of the roots by 0.6352 then

$$G'(sT) = \frac{1}{(sT+0.587+j0.3827)(sT+0.587-j0.3827)(sT+0.243+j0.9239)(sT+0.243-j0.3827)}$$

$$\ldots\ldots \quad (6.4.12)$$

If the frequency scaling is now incorporated, then neglecting the amplitude scale factor

$$G_C(sT) = \frac{1}{(s^2T^2 + 0.63sT + 1.46)(s^2T^2 + 1.51sT + 0.76)} . \quad (6.4.13)$$

(slide rule accuracy) .

Seshu and Balabanian[4] give the expression

$$G_C(sT) = \frac{1}{(s^2T^2 + 0.644sT + 1.534)(s^2T^2 + 1.519sT + 0.823)} . \quad (6.4.14)$$

From 6.4.14.

$$G_C(sT) = \frac{0.652}{1 + 0.42sT + 0.652 s^2T^2} \cdot \frac{1.21}{1 + 1.84sT + 1.21s^2T^2} . \quad (6.4.15)$$

If the numerator terms are made unity in each case, then,

$$G'_C(sT) = \frac{1}{1 + 0.42sT + 0.65s^2T^2} \cdot \frac{1}{1 + 1.84sT + 1.21s^2T^2} , \quad (6.4.16)$$

Thus for n even, as in the Chebyshev use, the function comprises quadratic factors of the form $\dfrac{1}{1 + k_1 sT + k_2 s^2 T^2}$.

6.5. Simulation of Chebyshev Functions

The functions to be simulated are of the form

$$G(sT) = - \frac{1}{1 + k_1 sT + k_2 s^2 T^2} . \quad (6.5.1)$$

This transfer function may be realized as one of the special cases of those shown in Table 3.2.

Alternatively the arrangement discussed in 4.3. might be preferable if the minimum number of components is to be used.

7. Orthonormal Functions

Orthonormal Approximation Functions are widely used in the synthesis of linear systems.

For example,[6] the impulse response of a linear system may be approximated by a sum of pre-determined orthonormal functions $f_r(t)$ so that

$$w^*(t) = \sum_{r=1}^{n} a_r f_r(t) \tag{7.1.1}$$

where a_r is a constant and $w^*(t)$ is the approximate impulse response.

The functions are chosen by varying the a_r to make the integral square error a minimum.

Thus if $w(t)$ is the true impulse response, then the error $e(t)$ is

$$e(t) = \int_{-\infty}^{\infty} \left[w(t) - w^*(t) \right]^2 dt = \int_{-\infty}^{\infty} w(t)^2 dt - 2 \sum_{r=1}^{n} a_r \int_{-\infty}^{\infty} w(t) f_r(t) dt$$

$$+ \sum_{r=1}^{n} \sum_{p=1}^{n} a_r a_p \int_{-\infty}^{\infty} f_r(t) f_p(t) dt \ .$$

$$\dots \dots \tag{7.1.2}$$

When the functions $f_r(t)$ are orthonormal they satisfy the relation,

$$\int_{-\infty}^{\infty} f_r(t) f_p(t) = 0 \quad \text{for } r \neq p \qquad)$$
$$= 1 \quad \text{for } r = p \qquad) \ . \tag{7.1.3}$$

Thus (7.1.2) becomes,

$$e(t) = \int_{-\infty}^{\infty} w(t)^2 dt - 2 \sum_{r=1}^{n} a_r \int_{-\infty}^{\infty} w(t) . f_r(t) dt + \sum_{r=1}^{n} a_r^2 \ . \tag{7.1.4}$$

Making $\dfrac{\partial e(t)}{\partial a_r} = 0$ for all r to minimize $e(t)$ in 7.1.4 gives

$$a_r = \int_{-\infty}^{\infty} w(t) f_r(t) dt \ , \tag{7.1.5}$$

and the minimum integral square error is given by

$$e(t)_{min} = \int_{-\infty}^{\infty} w(t)^2 dt - \sum_{r=1}^{n} a_r^2 \ . \tag{7.1.6}$$

The transfer function of the approximate function is given by

$$w^*(s) = \sum_{r=1}^{n} a_r f_r(s) \tag{7.1.7}$$

where $\qquad\qquad w^*(t) \supset w^*(s)$ \hfill (7.1.8)

and s is the complex variable $\sigma + jw$.

For discrete parameter systems $w^*(s)$ and $f_r(s)$ will be rational functions of s. Now by Parseval's Theorem

$$\int_{-\infty}^{\infty} f_r(t)\, f_p(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_r(jw).\ f_p(-jw)\, dw = \begin{array}{l} 0 \text{ for } r \neq p \\ 1 \text{ for } r = p \end{array} \Big) \tag{7.1.9}$$

Since $f_r(jw)\, f_p(-jw)$ goes to zero more rapidly then $\frac{1}{jw}$ as w $\to \infty$ equation 7.1.9 may be expressed as a contour integral where the contour encircles the entire right half s-plane.

Thus

$$\frac{1}{2\pi j} \int_c f_r(s)\, f_p(-s)\, ds = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} f_r(s)\, f_p(-s)\, ds \qquad)$$
$$= \sum \text{residues of } f_r(s)\ .\ f_p(-s) \text{ in right half plane} \qquad)$$
$$= 0 \text{ for } r \neq p \qquad)$$
$$= 1 \text{ for } r = p\ . \qquad) \tag{7.1.10}$$

Various forms of orthonormal functions are used and typical examples are given in Refs. 6 and 7. The functions considered in this note have particular application in model approximations of control systems and this topic will be the subject of a further note.

7.2. Simulation of certain types of Orthonormal Functions

The functions considered, as stated in the last paragraph, have particular application in the synthesis of approximate mathematical models of control systems. Such models form the basis of some types of self-optimizing systems.

It should be noted that all ganged potentiometers used in the networks are adjusted, manually or automatically, to minimize a certain error or performance index.

7.2.1. $\quad \dfrac{E_2(s)}{E_1(s)} = G(s) = -\dfrac{\sqrt{2a}}{s+a}$. \hfill 7.2.1.1)

where a is a positive constant which should be capable of adjustment.

For simulation purposes, if $a > 1$, it is convenient to express a in the form

$$a = \frac{1}{kT} \tag{7.2.1.2}$$

where T is a pre-determined positive integer and k is a positive variable coefficient with

$$k \leqslant 1. \tag{7.2.1.3}$$

T is chosen so that it exceeds the largest time constant in the system under consideration.

Thus
$$\frac{E_2(s)}{E_1(s)} = -\frac{\frac{\sqrt{2}}{\sqrt{a}}}{1+\frac{s}{a}} = -\frac{\sqrt{2kT}}{1+skT} . \tag{7.2.1.4}$$

The instrumentation of equation 7.2.1.4 is shown in Fig. 7.2.1.1.
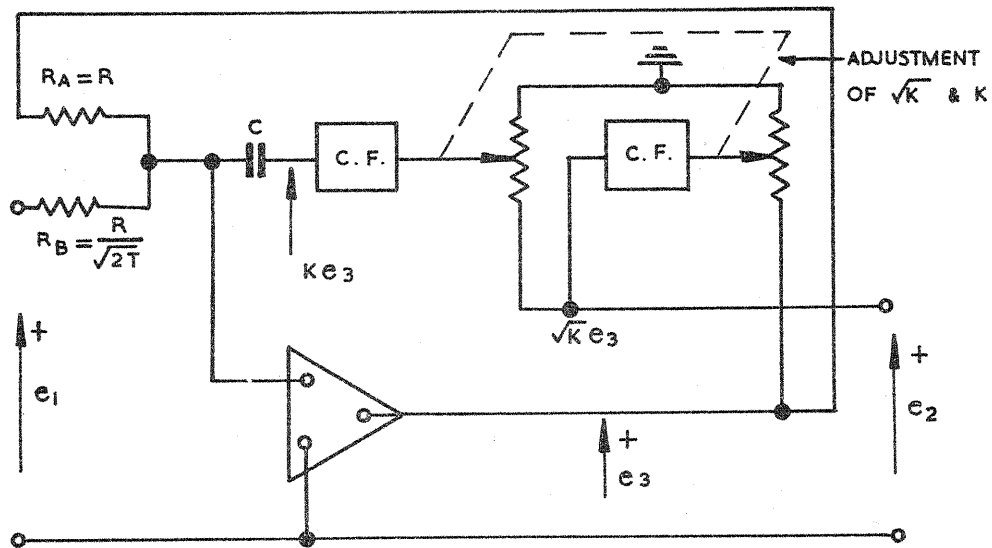
C.F. denotes cathode follower.

$$T = RC \; ; \quad \frac{E_2(s)}{E_1(s)} = -\frac{\sqrt{2kT}}{1+skT} \equiv -\frac{\sqrt{2a}}{s+a} . \tag{7.2.1.5}$$

Fig. 7.2.1.1.

A special case of this network is given when $C = 1 \times 10^{-6}$ farad and $R = T \times 10^6$ ohm. This results in $R_A = T \times 10^6$ ohms and $R_B = \sqrt{.5T} \times 10^6$ ohms, if the time constant T is expressed in seconds.

7.2.2.
$$\frac{E_2(s)}{E_1(s)} = G(s) = - \frac{s-a}{s+a}. \tag{7.2.2.1}$$

a is to be capable of variation and may be > or < 1.

If
$$a = 1/kT \tag{7.2.2.2}$$

where T is a positive integer and a pre-determined value and k is a variable coefficient,

then
$$\frac{E_2(s)}{E_1(s)} = - \frac{\frac{s}{a} - 1}{\frac{s}{a} + 1} \tag{7.2.2.3}$$

$$= - \left(1 - \frac{2}{\frac{s}{a} + 1} \right) \tag{7.2.2.4}$$

$$= - \left(1 - \frac{2}{1 + skT} \right). \tag{7.2.2.5}$$

The instrumentation of equation 7.2.2.5 is shown in Fig. 7.2.2.1.
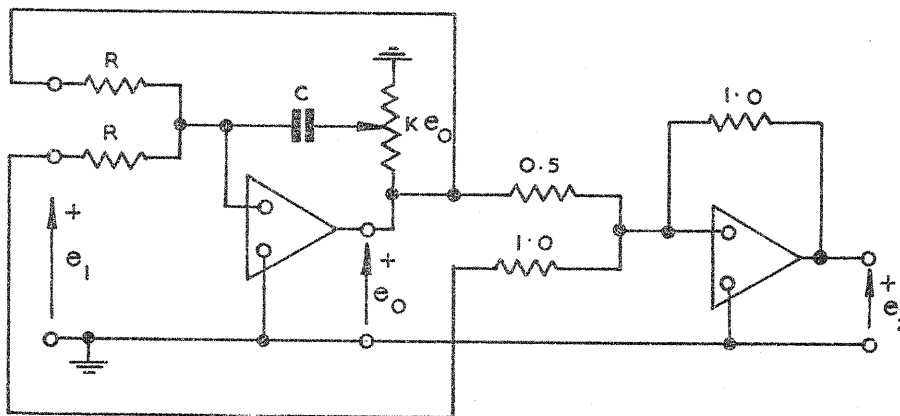
$$RC = T; \quad \frac{E_2(s)}{E_1(s)} = - \left(1 - \frac{2}{1 + skT} \right) \equiv - \frac{s-a}{s+a}. \tag{7.2.2.6}$$

Fig. 7.2.2.1.

An alternative arrangement is shown in Fig. **7.2.2.2.** in which the function $-\frac{\sqrt{2a}}{s+a}\ E_1(s)$ is also available as an output.

$$T = RC \ ; \quad \frac{E_4(s)}{E_1(s)} = -\frac{2kT}{1+skT} = -\frac{\sqrt{2a}}{s+a} \ ; \quad \frac{E_2(s)}{E_1(s)} = -\left(1 - \frac{2\sqrt{2T}}{\sqrt{2T}(1+skT)}\right)$$

$$= -\frac{s-a}{s+a} \ . \qquad (7.2.2.7)$$

Fig. 7.2.2.2.

In Fig. 7.2.2.2. a two-ganged potentiometer is required to give the variable coefficients $\sqrt{k}$ and k respectively.

7.2.3. $\qquad \dfrac{E_2(s)}{E_1(s)} = G(s) = -\dfrac{2\sqrt{ba}(s+b)}{s^2 + 2bas + b^2}$, $\qquad (7.2.3.1)$

where b and a are to be adjusted.

b can be $>$ or $<$ 1 but $a \leqslant$ 1. $\qquad (7.2.3.2)$

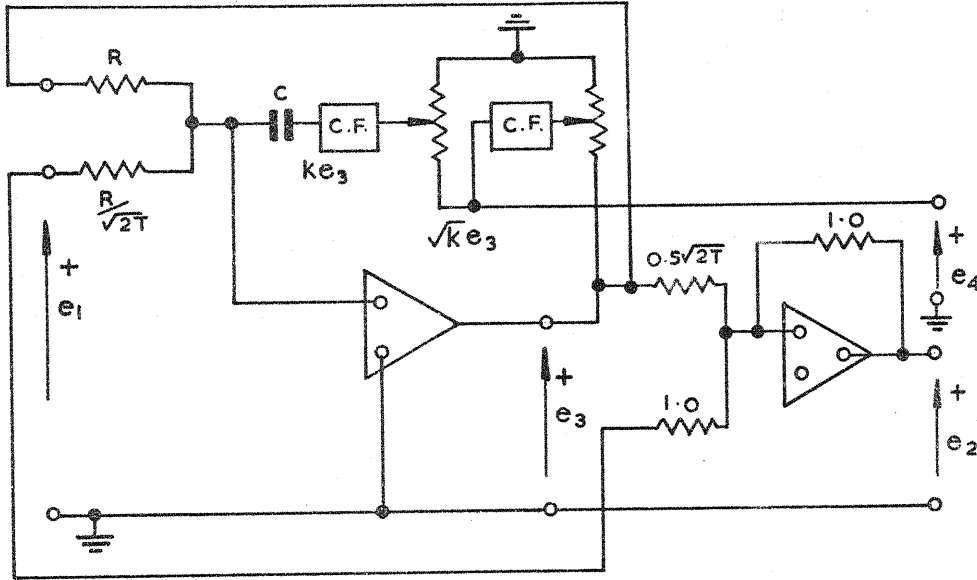Let $b = \dfrac{1}{kT}$ $\qquad (7.2.3.3)$

where T is a positive integer $\geqslant$ 1 and fixed and k is a variable coefficient where $k \leqslant$ 1.

Then, $\qquad \dfrac{E_2(s)}{E_1(s)} = -\dfrac{2\sqrt{\dfrac{a}{b}}\ (1+\dfrac{s}{b})}{1+\dfrac{2a}{b}s+\dfrac{s^2}{b^2}}$ $\qquad (7.2.3.4)$

$$= -\frac{2\sqrt{akT}\ (1+skT)}{1+2sakT+s^2k^2T^2} \ , \qquad (7.2.3.5)$$

The instrumentation of equation 7.2.3.5. is shown in Fig. 7.2.3.1.
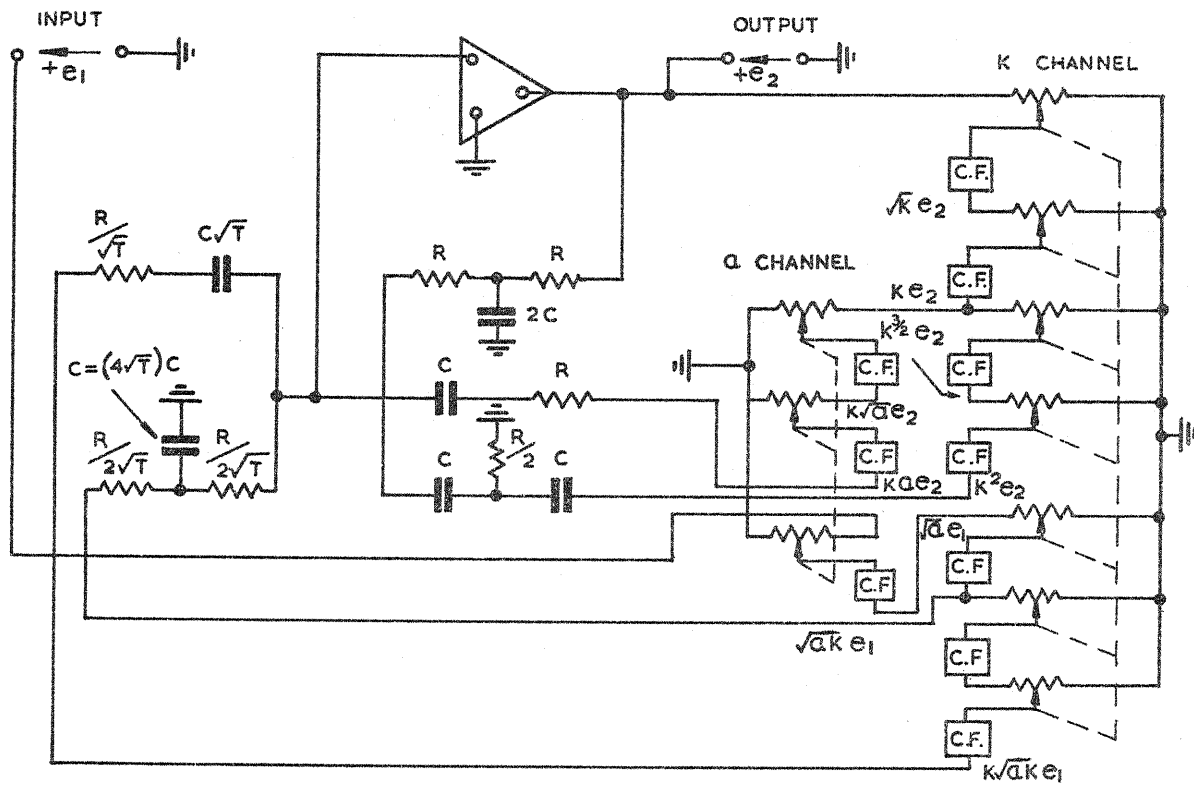
$$T = RC \; ; \quad \frac{E_2(s)}{E_1(s)} = - \frac{2\sqrt{akT}\,(1 + skT)}{1 + 2sakT + s^2 k^2 T^2} \equiv - \frac{2\sqrt{ab}\,(s + b)}{s^2 + 2bas + b^2} \quad (7.2.3.6)$$

Fig. 7.2.3.1.

Referring to Fig. 7.2.3.1 it will be noted, that to achieve the appropriate transfer function with variation of k, a seven ganged potentiometer is required. The variation due to a is obtained with a three ganged potentiometer.

The time constant $T = RC$ would be chosen to exceed that of the largest time constant of the system under investigation.

7.2.4. Tandem Connection of Networks

In many applications there is a requirement for the following types of orthonormal transfer functions

$$(i) \quad \frac{\sqrt{2a}}{s+a} \cdot \frac{s-b}{s+b} \cdot \frac{s-c}{s+c} \dots \dots \dots \dots \frac{s-g}{s+g} \, , \qquad (7.2.4.1)$$

$$(ii) \quad \frac{\sqrt{2a}}{s+a} \cdot \frac{s-b}{s+b} \dots \dots \dots \dots \frac{2\sqrt{ef} \,(s+f)}{s^2 + 2efs + f^2} \, , \qquad (7.2.4.2)$$

It will be apparent that these transfer functions may be obtained by the tandem connection of the networks discussed, taking due account of signs.

8. Conclusions

Rational second order transfer functions, with coefficients capable of independent adjustment, may be synthesized with relative ease by the use of computing amplifiers associated with passive RC networks. The methods described in the note result in the use of the minimum number of amplifiers but, in general, the number of passive components used is increased compared with the number resulting from design methods in which there is no restriction on the number of amplifiers used.

9. Acknowledgements

Thanks are due to Mr. J. Fisher, of the Department of Electrical and Control Engineering, who was responsible for the construction and testing of experimental models of some of the networks described.

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APPENDIX A

Analysis of the Generalized Two Port Active Network[1,4,8]

Description

Use is made of a computing or operational amplifier associated with passive linear two port networks. The use of one-port networks may be considered as a special case.

The general arrangement is shown in Fig. 11.1.
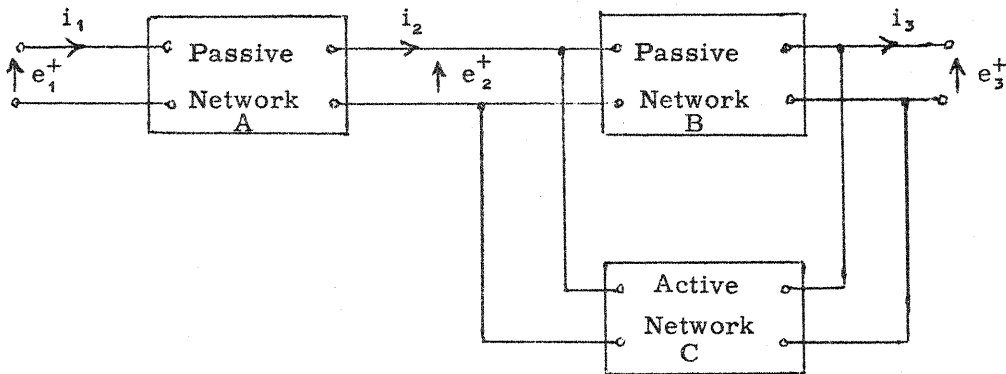

Fig. 11.1

Network B is a two port network placed in parallel with the computing amplifier, i.e. Active network C.

For the specific purpose considered in this note the above arrangement may be simplified since the input and output voltages of each network are referred to a common reference line (earth potential). This results in the simplified diagram shown in Fig. 11.2.
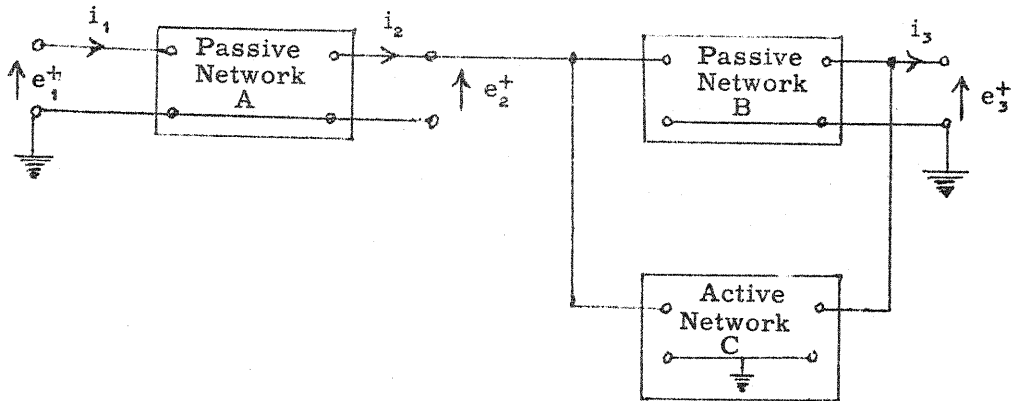
Fig. 11.2

Analysis

For any linear two port network we have,

$$
\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{bmatrix} \begin{bmatrix} E_1(s) \\ E_2(s) \end{bmatrix} \quad , \tag{11.1}
$$

where

$E_1(s)$ is the transformed input voltage,

$E_2(s)$ is the transformed output voltage,

$I_1(s)$ is the transformed input current,

$I_2(s)$ is the transformed output current,

and, $y_{11}(s) = \dfrac{I_1(s)}{E_1(s)}\bigg|_{E_2(s)=0}$ ≡ short circuit admittance of input port)

$y_{12}(s) = \dfrac{I_1(s)}{E_2(s)}\bigg|_{E_1(s)=0}$ ≡ - reverse short circuit transfer admittance)

$y_{21}(s) = \dfrac{I_2(s)}{E_1(s)}\bigg|_{E_2(s)=0}$ ≡ forward short circuit transfer admittance)

$y_{22}(s) = \dfrac{I_2(s)}{E_2(s)}\bigg|_{E_1(s)=0}$ ≡ - short circuit admittance of output port)

(11.2)

For the parallel arrangements of networks B and C in Fig. 11.2,

$$
\begin{bmatrix} I_2(s) \\ I_3(s) \end{bmatrix} = \begin{bmatrix} y_{11}^{B+C}(s) & y_{12}^{B+C}(s) \\ y_{21}^{B+C}(s) & y_{22}^{B+C}(s) \end{bmatrix} \begin{bmatrix} E_2(s) \\ E_3(s) \end{bmatrix} , \qquad (11.3)
$$

where

$$
\begin{bmatrix} y_{11}^{B+C}(s) & y_{12}^{B+C}(s) \\ y_{21}^{B+C}(s) & y_{22}^{B+C}(s) \end{bmatrix} = \begin{bmatrix} y_{11}^{B}(s) & y_{12}^{B}(s) \\ y_{21}^{B}(s) & y_{22}^{B}(s) \end{bmatrix} + \begin{bmatrix} y_{11}^{C}(s) & y_{12}^{C}(s) \\ y_{21}^{C}(s) & y_{22}^{C}(s) \end{bmatrix} . \qquad (11.4)
$$

Since network C represents a computing amplifier the following assumption is made:

i.e. $\qquad y_{11}^{C}(s) = 0$. $\qquad (11.5)$

This is a reasonable assumption for a linear thermionic value amplifier.

If $\quad y_{11}^{C}(s) = 0$, then $y_{12}^{C}(s) = 0$. $\qquad (11.6)$

If $a_{11}(s)$ represents the reciprocal of the forward voltage transfer function, then,

$$
a_{11}^{BC}(s) = \frac{E_2(s)}{E_3(s)} \bigg|_{I_3(s)=0} = - \frac{y_{21}^{B+C}(s)}{y_{22}^{B+C}(s)} . \qquad (11.7)
$$

If $\qquad \dfrac{1}{a_{11}^{C}(s)} = - K \, G_1(s)$ $\qquad (11.8)$

where K is a positive constant and G(s) is a rational function of s, resulting from the poles and zeros of the amplifier,

then

$$
y_{21}^{C}(s) = K \, G_1(s) \, y_{22}^{C}(s) . \qquad (11.9)
$$

This results in,

$$
\begin{bmatrix} y_{11}^{B+C}(s) & y_{12}^{B+C}(s) \\ y_{21}^{B+C}(s) & y_{22}^{B+C}(s) \end{bmatrix} = \begin{bmatrix} y_{11}^{B}(s) & y_{12}^{B}(s) \\ y_{21}^{B}(s) + KG_1(s) \, y_{22}^{C}(s) & y_{22}^{B}(s) + y_{22}^{C}(s) \end{bmatrix} . \qquad (11.10)
$$

Thus,

$$\frac{1}{a_{11}^{BC}(s)} = \frac{E_3(s)}{E_2(s)}\bigg|_{I_3(s)=0} = -\frac{y_{21}^{B}(s) + K \cdot G_1(s) \cdot y_{22}^{B}(s)}{y_{22}^{B}(s) + y_{22}^{C}(s)} \, . \qquad (11.11)$$

The transmission matrix $\mathscr{A}^{BC}$ for the parallel connection of networks B and C is given by:-

$$\mathscr{A}^{BC} = \begin{bmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{bmatrix} . \qquad (11.12)$$

i.e.

$$\begin{bmatrix} E_1(s) \\ I_1(s) \end{bmatrix} = \begin{bmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{bmatrix} \begin{bmatrix} E_2(s) \\ I_2(s) \end{bmatrix} , \qquad (11.13)$$

where

$$\mathscr{A}^{BC} = \frac{1}{y_{21}^{B+C}(s)} \begin{bmatrix} -y_{22}^{B+C}(s) & 1 \\ -\det y^{B+C}(s) & y_{11}^{B+C}(s) \end{bmatrix} ; \qquad (11.14)$$

$$= \frac{1}{y_{21}^{B}(s) + K \cdot G_1(s) \, y_{22}^{C}(s)} \begin{bmatrix} -(y_{22}^{B}(s) + y_{22}^{C}(s)) & 1 \\ -\det y^{B+C}(s) & y_{11}^{B}(s) \end{bmatrix} .$$

$$\dots\dots \quad (11.15)$$

The transmission matrix $\mathscr{A}$ of the complete arrangement is given by

$$\mathscr{A} = \mathscr{A}^{A} \quad \mathscr{A}^{BC} , \qquad (11.16)$$

i.e.

$$\begin{bmatrix} E_1(s) \\ I_1(s) \end{bmatrix} = \begin{bmatrix} \mathscr{A}^{A} \end{bmatrix} \begin{bmatrix} \mathscr{A}^{BC} \end{bmatrix} \begin{bmatrix} E_2(s) \\ I_2(s) \end{bmatrix} . \qquad (11.17)$$

$$\mathscr{A}^{A} = \begin{bmatrix} a_{11}^{A}(s) & a_{12}^{A}(s) \\ a_{21}^{A}(s) & a_{22}^{A}(s) \end{bmatrix} . \qquad (11.18)$$

$$\mathscr{A} = \begin{bmatrix} a_{11}^{A}(s) & a_{12}^{A}(s) \\ a_{21}^{A}(s) & a_{22}^{A}(s) \end{bmatrix} \begin{bmatrix} -(y_{22}^{B}(s) + y_{22}^{C}(s)) & 1 \\ -\det y^{B+C}(s) & y_{11}^{B}(s) \end{bmatrix} \cdot \frac{1}{y_{21}^{B}(s) + KG_1(s) \, y_{22}^{C}(s)} ,$$

$$\dots\dots \quad (11.19)$$

$$= \begin{bmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{bmatrix} . \qquad (11.20)$$

Hence,

$$a_{11}(s) = \frac{1}{y_{21}^{B}(s) + KG_{1}(s)\, y_{22}^{C}(s)} \cdot \left\{ -a_{11}^{A}(s)\,(y_{22}^{B}(s) + y_{22}^{C}(s)) - a_{12}^{A}(s)\cdot \det\!\left[y^{B+C}\right](s) \right\} \cdot$$

$$\ldots\ldots \quad (11.21)$$

Thus the forward open circuit voltage transfer function of the complete arrangement is given by,

$$G(s) = \frac{1}{a_{11}(s)} = \frac{y_{21}^{B}(s) + KG_{1}(s)\, y_{22}^{C}(s)}{\dfrac{y_{22}^{A}(s)}{y_{21}^{A}(s)} \cdot (y_{22}^{B}(s) + y_{22}^{C}(s)) - \dfrac{\det. y^{B+C}(s)}{y_{21}^{A}(s)}} \cdot \qquad (11.22)$$

Since $\quad a_{12}^{A}(s) = \dfrac{1}{y_{21}^{A}(s)} \cdot$ $\qquad (11.23)$

Now $\quad \det y^{B+C}(s) = y_{11}^{B+C}(s)\, y_{22}^{B+C}(s) - y_{12}^{B+C}(s)\, y_{21}^{B+C}(s) \qquad (11.24)$

$$= y_{11}^{B}(s)\,(y_{22}^{B}(s) + y_{22}^{C}(s)) - y_{12}^{B}(s)\,(y_{21}^{B}(s) + KG_{1}(s)\, y_{22}^{C}(s)) \cdot$$

$$\ldots\ldots \quad (11.25)$$

Since from equation 11.6,

$$y_{11}^{C}(s) = 0 \qquad \left.\begin{array}{c} \\ \\ \end{array}\right) \cdot$$

and $\quad y_{12}^{C}(s) = 0 \qquad \left.\begin{array}{c}\\\end{array}\right)$ $\qquad (11.26)$

Equation 11.22 may be simplified if it is assumed that the output impedance of the amplifier is zero,

i.e. $\quad y_{22}^{C}(s) = -\infty$ $\qquad (11.27)$

This condition may be achieved, to a high degree of approximation, in a high quality computing amplifier by employing a cathode follower output stage with internal voltage feedback.

Assuming equation 11.27 is valid, then

$$G(s) = \frac{K\, G_{1}(s)}{\dfrac{y_{22}^{A}(s)}{y_{21}^{A}(s)} - \dfrac{y_{11}^{B}(s) - y_{12}^{B}(s)\, KG_{1}(s)}{y_{21}^{A}(s)}} \qquad (11.28)$$

$$G(s) = \frac{E_{3}(s)}{E_{1}(s)} = \frac{K\, G_{1}(s)\, y_{21}^{A}(s)}{y_{22}^{A}(s) - y_{11}^{B}(s) + y_{12}^{B}(s)\, KG_{1}(s)} \qquad (11.29)$$

Equation 11.29 is valid over a finite frequency range, from zero frequency upwards, provided that, $y_{21}^{B}(s)$ and $y_{22}^{B}(s)$ are both finite over this frequency range.

This condition is valid for RC passive networks, and attention is confined to these networks for the remainder of the analysis.

If now, the poles and zeros of the amplifier network C are far removed from those of networks A and B then over the frequency range of interest,

$$G_{1}(s) \triangleq 1 \quad ,$$
(11.30)

This means that the computing amplifier must have a frequency response function which extends over a much wider frequency range than the frequency range of interest.

Using assumption 11.30., equation 11.29 is simplified to

$$\frac{E_{3}(s)}{E_{1}(s)} = G(s) = \frac{y_{21}^{A}(s)}{\dfrac{y_{22}^{A}(s) - y_{11}^{B}(s)}{K} + y_{12}^{B}(s)} \quad .$$
(11.31)

Now for an RC passive network $y_{22}^{A}(s)$ or $y_{11}^{B}(s)$ may approach infinity as s approaches infinity.

The desired form of G(s) is given by,

$$G(s) = \frac{y_{21}^{A}(s)}{y_{12}^{B}(s)} \quad .$$
(11.32)

The factor $\dfrac{y_{22}^{A}(s) - y_{11}^{B}(s)}{K}$ may, therefore, be considered as an error function.

In order to keep this function small,

$$y_{12}^{B}(s) \gg \frac{y_{22}^{A}(s) - y_{11}^{B}(s)}{K} \quad ,$$
(11.33)

for all values of s.

Thus, if $y_{22}^{A}(s)$ or $y_{11}^{B}(s)$ approach infinity at infinite value of s, then y_{12}^{B} should also approach infinity as s approaches infinity.

Over the frequency range of interest $y_{22}^{A}(s)$ and $y_{11}^{B}(s)$ will be finite and thus, to achieve a small error, K must be large.

We may therefore write,

$$\text{Lt}_{K \longrightarrow \infty} \; G(s) = \frac{y_{21}^{A}(s)}{y_{12}^{B}(s)} \qquad (11.34)$$

i.e. $\quad \dfrac{E_3(s)}{E_1(s)} = G(s) = \dfrac{y_{21}^{A}(s)}{y_{12}^{B}(s)} \; , \qquad (11.35)$

over a finite frequency range as $K \longrightarrow \infty$.

The effect of a finite value of K may be calculated from equation 11.31.

The assumptions which have been made in the derivation of equation 11.35 may be summarised as follows:-

(i) $\quad y_{11}^{C}(s) = 0$.

(ii) $\quad y_{22}^{C}(s) = -\infty$.

(iii) $\quad y_{21}^{B}(s)$ and $y_{22}^{B}(s)$ are both finite over the frequency range of interest. (This condition is satisfied by RC passive networks).

(iv) $\quad G_1(s) \triangleq 1$ over the frequency range of interest.

(v) $\quad y_{12}^{B}(s) \gg \dfrac{y_{22}^{A}(s) - y_{11}^{B}(s)}{K}$ for the frequency range of interest.

(vi) If $y_{22}^{A}(s)$ or $y_{11}^{B}(s) \longrightarrow \infty$ as $s \longrightarrow \infty$, then $y_{12}^{B}(s)$ should $\longrightarrow \infty$ as $s \longrightarrow \infty$.

(vii) $\quad K \longrightarrow \infty$ over the frequency range of interest.

(11.36)

Note:- These assumptions are implied in conventional analogue computer studies and, as already stated, may be approximated to a high degree of accuracy with modern computing circuits.

Similar results, taking due account of the different sign convention used, are given in Ref. 1.