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OPTIMUM DESIGN OF A BAND REINFORCED PRESSURISED CYLINDER

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NOTE NO.116

April, 1961.

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Optimum Design of a Band Reinforced Pressurised Cylinder

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SUMMARY

The surface stresses in band reinforced cylindrical pressure vessels are examined, and an equivalent stress determined by using the Mises-Hencky criterion. By comparing the equivalent stress to the band stress, the efficiency of the structural material can be established, and by equating these stresses to their respective yield stresses, the theoretical maximum strength of the structure can be found. Once the material properties of the shell and the reinforcing bands have been specified, the optimum structural layout can be determined.

CONTENTS

NOTATION

A	cross sectional area
D	flexural rigidity of shell
D'	flexural rigidity of shell-stringer combination
E	Young's modulus
F	axial compressive force
k	$\left(\frac{Et'}{4D'R^2}\right)^{\frac{1}{4}}$
ℓ	length of shell, distance between mid-band positions
M	bending moment per unit width
M'	non-dimensional bending moment
N	radial shear force per unit width
p	internal pressure
R	shell radius
t	shell thickness
t', t_e	effective shell thickness
T_1, T_2	resultant longitudinal and circumferential force per unit width
w	radial displacement
w_o	$\frac{\phi\, p\, R^2}{Et'}$
Y	yield stress
α	$\frac{F}{\pi p R^2}$
β	$\frac{2t}{k A_b}$
γ_1 γ_2	defined by equation (8)

Notation (Continued)

ϵ_x, ϵ_θ	longitudinal and circumferential strains
η	$\frac{k\ell}{2}$
ν	Poisson's ratio (taken as 0.3)
σ_1 σ_2	shell surface stresses in longitudinal and circumferential directions
$\bar{\sigma}$	equivalent stress defined by equation (13)
ϕ	$1 + \frac{1}{2}\nu(\alpha - 1)\frac{t}{t_e}$

Subscripts

s	refers to reinforcing stringers
b	refers to reinforcing band

1. Introduction

In the design of thin shell structures, internal pressure can be considered as an effective means of stabilising the body shell against collapse which might otherwise occur due to axial compressive forces. These internal pressure forces can be sufficiently large to cause bursting of the structure and a limiting allowable hoop stress is generally specified, which in turn limits the stabilising pressure which can be used.

It is well known that longitudinal stringers can be introduced in order to assist the shell to carry the axial forces, in much the same way as is used on a conventional aircraft structure. However, unless the axial forces are very high, this form of construction is structurally less efficient than the pressure stabilised shell(1). A combined form of stabilisation may be considered, by using both pressurisation and longitudinal stiffeners, but the effect of the stiffeners on the maximum hoop can be shown to be small, so that a comparatively heavy structure would result.

One form of construction which could be used, might be in the form of radial bands, which are spaced sufficiently close together, so that the hoop stress in the shell is considerably reduced. This could mean that the internal pressure might be increased to sustain a greater axial load, and the bands would help to stabilise the shell during manufacture, and so prevent large shell distortions.

An experimental investigation into the stress distribution in a band reinforced pressure vessel has been made by Mantle, Marshall and Palmer(2), where in this case very closely spaced bands were used. It was found that this form of construction enabled an efficient shell structure to be designed.

The following analysis investigates the stress distribution in a band reinforced cylindrical shell, which is subjected to combined axial load and internal pressure.

By comparing the maximum stress in the shell and reinforcing bands, the optimum geometry can be established.

2. The determination of the stress distribution

If the cylindrical shell of radius R is subjected to a compressive axial force F, and internal pressure p, then the resultant longitudinal force/in T_1 becomes

$$T_1 = \tfrac{1}{2}\, pR - \frac{F}{2\pi R} ,$$

or $$T_1 = \tfrac{1}{2}\, pR\,(1 - \alpha). \tag{1}$$

If the shell has longitudinal stringers, then this force can be expressed in terms of the strain components as

$$T_1 = E_s t_s \epsilon_x + \frac{Et}{1 - \nu^2}\,(\epsilon_x + \nu\epsilon_\theta) , \tag{2}$$

where t_s is the effective thickness of the stringers. The first term on the right hand side will of course disappear if only band reinforcement is used.

The hoop force/in becomes

$$T_2 = \frac{Et}{1 - \nu^2}\,(\epsilon_\theta + \nu\,\epsilon_x) . \tag{3}$$

Equations (1), (2) and (3) can be combined to give

$$T_2 = Et'\,\frac{w}{R} + \tfrac{1}{2}\, pR\,(1 - \alpha)\,\nu\,\frac{t}{t_e} , \tag{4}$$

where $$t_e = t + t_s\,(1 - \nu^2)\,\frac{E_s}{E}$$

and $$t' = \frac{t}{1 - \nu^2}\,(1 - \nu^2\,\frac{t}{t_e})$$

The equilibrium conditions of an element of the shell Fig. 1 are given as

$$\frac{dN}{dx} + p = \frac{T_2}{R} , \quad \text{and } N = -\frac{dM}{dx} , \tag{5}$$

where M and N are the bending moment and radial force/in.

The relationship between the bending moment and the radial displacement w is

$$M = D'\,\frac{d^2w}{dx^2} , \tag{6}$$

where D' is an effective bending rigidity of the stringer-shell combination. Substitution of equations (5) (6) into (4) gives a form of the well known differential equation

$$\frac{d^4 w}{dx^4} + 4k^4 w = \frac{p\phi}{D'}, \tag{7}$$

where $\phi = 1 + \frac{1}{2}\nu(\alpha - 1)\frac{t}{t_e}$, and $4k^4 = \frac{Et'}{D' R^2}$.

If the origin is taken about a point on the shell which is midway between the band reinforcements, then the radial displacement becomes

$$w = w_o + C_1 \cos kx \cosh kx + C_2 \sin kx \sinh kx,$$

where $w_o = \phi\left(\frac{pR^2}{Et'}\right)$

The conditions of zero slope at the frames (i.e. $\frac{dw}{dx} = 0$ at $x = \pm\frac{\ell}{2}$), and equilibrium between the forces in the band, and the shell gives

$$A_b E_b \frac{w}{R} = -2NR, \quad \text{at } x = \pm\frac{\ell}{2},$$

where $N = -D'\frac{d^2 w}{dx^2}$.

There are two conditions which give $C_1 = -w_o \gamma_1$ and $C_2 = -w_o \gamma_2$,

where
$$\gamma_1 = \frac{\sin\eta\cosh\eta + \cos\eta\sinh\eta}{\frac{1}{2}(\sin 2\eta + \sinh 2\eta) + \frac{2t'}{kA_b}\frac{E}{E_b}(\sin^2\eta + \sinh^2\eta)},$$

$$\gamma_2 = \frac{\sin\eta\cosh\eta - \cos\eta\sinh\eta}{\frac{1}{2}(\sin 2\eta + \sinh 2\eta) + \frac{2t'}{kA_b}\frac{E}{E_b}(\sin^2\eta + \sinh^2\eta)}, \quad \ldots\ldots \tag{8}$$

and $\eta = \frac{k\ell}{2}$.

Hence $w = w_o(1 - \gamma_1 \cos kx \cosh kx = \gamma_2 \sin kx \sinh kx)$. (9)

This equation describes the axi-symmetric radial displacements of the shell along its length. In this analysis, an effective stringer skin was used, so that this equation is incapable of recognising the fact that shell quilting between stringers might occur for large values of R/t.

If σ_b is the band stress, then

$$\sigma_b = E_b \frac{w}{R}, \quad \text{when } x = \pm \frac{\ell}{2} \quad .$$

Hence from equation (9),

$$\sigma_b = E_b \frac{w_0}{R} \left[1 - \gamma_1 \cos \eta \cosh \eta - \gamma_2 \sin \eta \sinh \eta \right] \tag{10}$$

Having established the radial displacement of the shell, it is a simple matter to establish the distribution of the shear force and bending moment from equations (5) and (6).

These stresses together with the membrane stresses arising from equations (1) and (4) constitute the stress distribution in the shell.

If no longitudinal stringers exist, then the surface stresses in the shell become

$$\sigma_1 = \frac{6D}{t^2} \frac{d^2w}{dx^2} + \frac{1}{2} \frac{pR}{t} (1 - \alpha) \tag{11}$$

and

$$\sigma_2 = \nu\sigma_1 + \frac{Ew}{R} \, . \tag{12}$$

The other stress components are all zero.

The design condition for the shell, is that the equivalent stress[3] is given by the equation

$$\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 = \bar{\sigma}^2 \tag{13}$$

3. The derivation of σ_1 and σ_2

The bending moment in the shell M, can be found from equations (6) and (9) as

$$\frac{2\,k^2M}{\phi p} = M' = \gamma_1 \sin kx \sinh kx - \gamma_2 \cos kx \cosh kx \, ,$$

where γ_1 and γ_2 are given in equation (8).

Since $\phi = 1 + \frac{1}{2}\nu\,(\alpha - 1)$, when no stringers are present,
then

$$\frac{2}{\nu}(\phi - 1) = \alpha - 1,$$

and equation (11) can be written as

$$\sigma_1 = \frac{3\,M'\,\phi\,p}{k^2\,t^2} + \frac{pR}{\nu t}(\phi - 1).$$

For an unreinforced shell, $k^2 = \frac{\sqrt{3(1-\nu^2)}}{Rt}$, and σ_1 becomes

$$\frac{\sigma_1}{\frac{pR}{t}} = \sqrt{\frac{3}{1-\nu^2}}\;\phi\,(y_1 \sin kx \sinh kx - y_2 \cos kx \cosh kx) - \frac{(1-\phi)}{\nu}\,. \qquad \ldots\ldots \quad (14)$$

Similarly from equation (12) σ_2 becomes

$$\frac{\sigma_2}{\frac{pR}{t}} = \nu\sqrt{\frac{3}{1-\nu^2}}\;M'\phi - \phi\,(y_1 \cos kx \cosh kx + y_2 \sin kx \sinh kx) + 1,$$

or

$$\frac{\sigma_2}{\frac{pR}{t}} = \nu\sqrt{\frac{3}{1-\nu^2}}\;\phi(y_1 \sin kx \sinh kx - y_2 \cos kx \cosh kx)$$
$$- \phi\,(y_1 \cos kx \cosh kx + y_2 \sin kx \sinh kx) + 1\,. \qquad (15)$$

4. An examination of the parameter ϕ

For a shell having no longitudinal stiffeners,

$$\phi = 1 + \frac{\nu}{2}(\alpha - 1)\,,$$

where $\alpha = \dfrac{F}{\pi p R^2}$.

If no axial compressive force F exists, and the shell is subjected to the internal pressure only, then

$$\phi = 0.85 \quad \text{if} \quad \nu = 0.3 \qquad (16)$$

However, for the case of a pressurised shell having an axial compressive force, then α and hence ϕ, is a function of that force, in accordance with the above equations.

For the ballistic missile application, if one introduces the stabilising pressure in order to produce zero longitudinal stress in the shell,[4] then

$$\alpha = 1 \quad \text{and} \quad \phi = 1 \tag{17}$$

Hence the limiting conditions for ϕ, are that it must be between the values given in equations (16) and (17)

i.e. $.85 < \phi < 1.0$.

In the following work, and the figures which are presented, only the lower value of ϕ is considered.

5. Discussion of Results

The band stress given by equation (10) is presented in Fig. 2 for various shell geometries. The analysis assumes that the band depth is small compared with the radius, and if this is not the case, it would be better to replace the shell radius with the centroidal radius of the band. The only stress which is considered in the band is the hoop stress, as this will be the only significant stress, unless the band width is large compared with its thickness.

The Fig. 2 suggests that a maximum value of the band stress is developed for a shell parameter $\eta = 1.5$, but for most missile applications the value of η will generally exceed this value.

For the commercial application referred to earlier[2], where comparatively large shell thicknesses are used, the value of η will be small, and hence the stress developed in the bands will be small compared with the nominal hoop stress in the shell.

Examination of equations (14) and (15) together with equation (13) suggests that the stress conditions at only two points in the shell need be examined. These are at the ends of the shell (when $x = \pm \frac{\ell}{2}$) and at the mid-shell position ($x = 0$). The equivalent shell stress $\bar{\sigma}$, is shown as a ratio of the nominal hoop stress at these two positions in Figs. 3 and 4, for various shell geometries, and the equivalent shell stress to frame stress ratio $\frac{\bar{\sigma}}{\sigma_b}$ is presented in Figs. 5 and 6.

The theoretical maximum strength of the band reinforced shell will be obtained when simultaneous yielding occurs in both the band and the cylinder. Hence for any shell having a known geometry, and hence η and β, simultaneous yielding will occur when $\bar{\sigma} = Y$ and $\sigma_f = Y_b$, and the ratio $\frac{Y}{Y_b}$ will be given by Figs. 5 and 6. Clearly $Y_b < Y$ so that a lower grade of material can be efficiently used for the bands.

Once the properties Y and Y_b are known, and the values of R, t and ℓ given, then the value of η is known. Hence the value of β, can be found, and the optimum value of A_b determined.

6. References

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3. Hill, R. — The mathematical theory of plasticity. Clarendon Press, Oxford. 1950.

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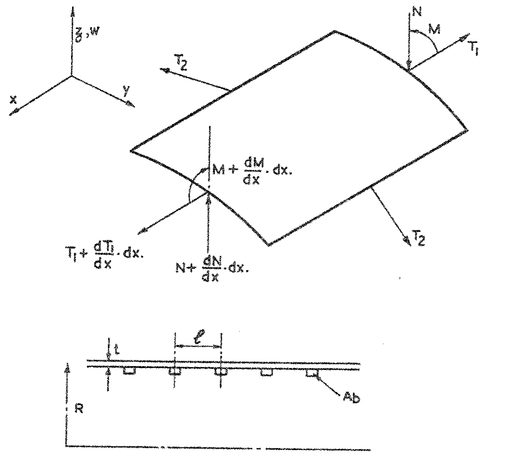

FIG. 1. Forces and moments on an element of the shell and geometry of reinforced cylinder

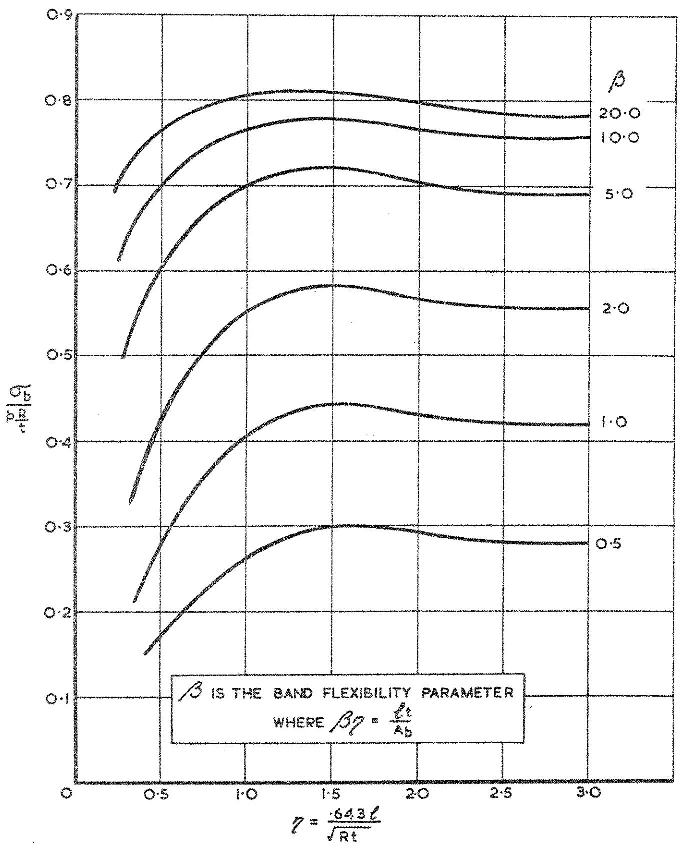

FIG. 2. The stress in reinforcing bands for various shell geometries and band flexibilities

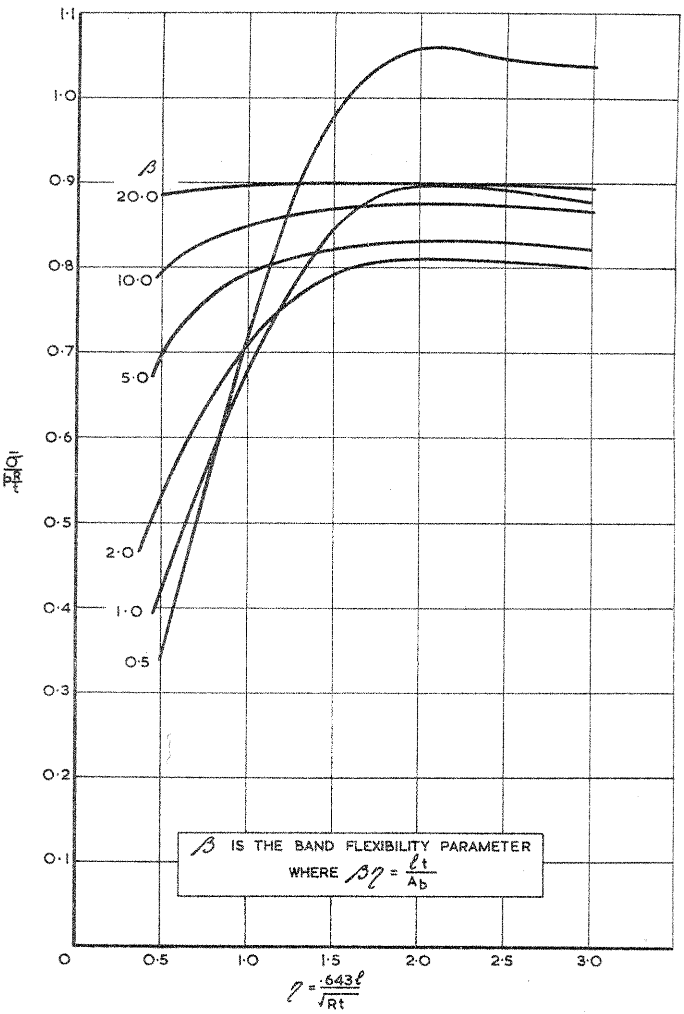

FIG. 3. The equivalent surface stress in the shell at its end ($x = \frac{\ell}{2}$)

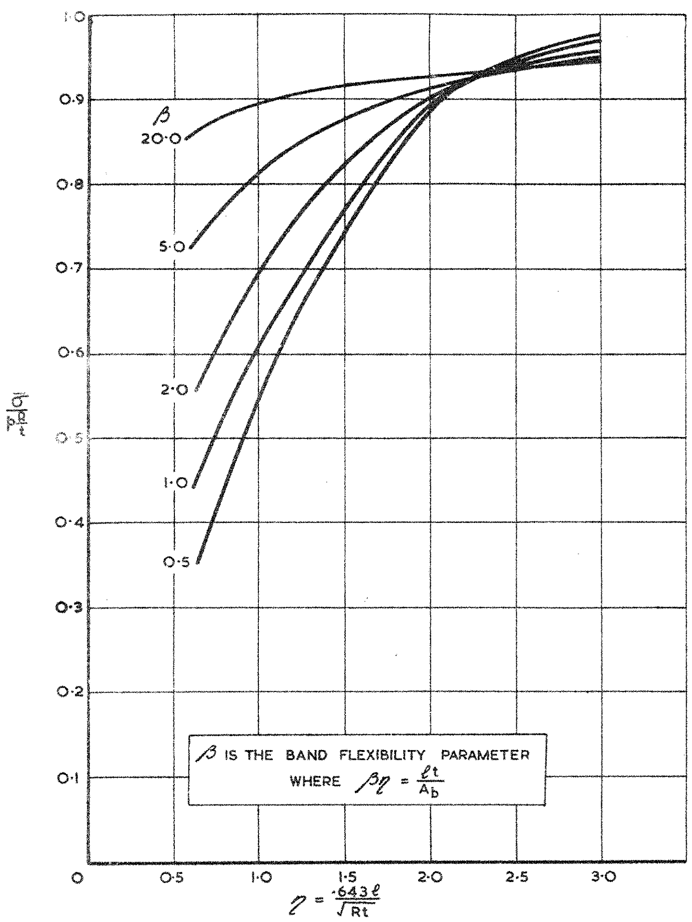

FIG. 4. The equivalent surface stress in the shell at its centre (x = 0)

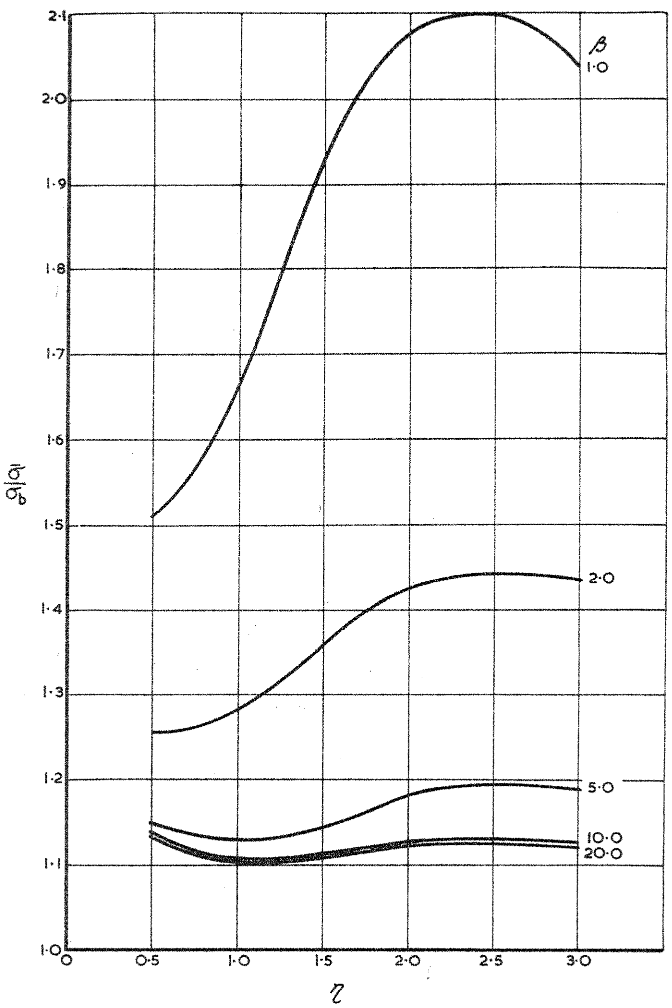

FIG. 5. The ratio of the equivalent stress to band hoop stress at the end of the shell (x = $\pm \frac{\ell}{2}$)

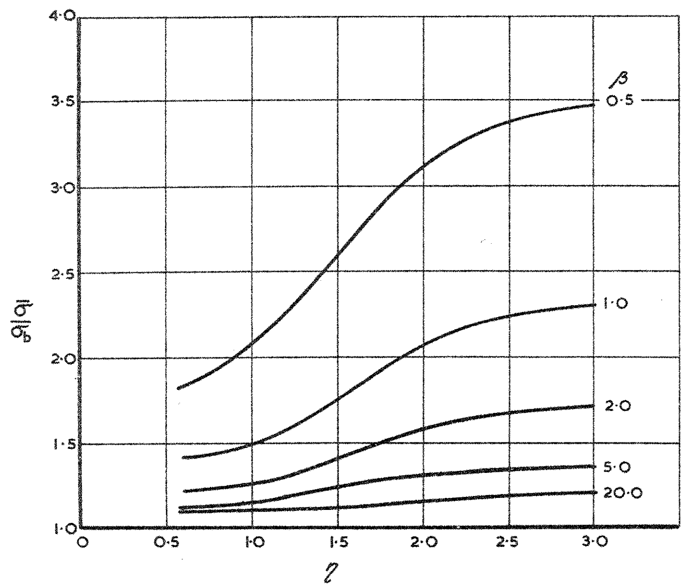

FIG. 6. The ratio of the equivalent stress to band hoop stress at the centre of the shell (x = 0)