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Preventive Maintenance for Systems with Repairable Minor Failures

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Abstract. Reliability modelling of repairable systems deals mostly with two types of repair. Perfect repair brings a system to 'as good as new' state. Minimal repair, on the contrary, returns a system to the state immediately prior to failure. In this paper, we consider perfect and imperfect preventive maintenance actions for a system subject to minor and major failures. Minor failures are minimally repaired, whereas a major failure terminates the operational function of the system and can be considered as an end-of-life event. The preventive maintenance strategies that we propose and analyse increase mission success probability and extend the expected lifetime of the system. The modeling is illustrated with numerical examples.

Keywords: Preventive maintenance; perfect repair; minimal repair; imperfect maintenance; nonhomogeneous Poisson process

1. Introduction

Traditionally, reliability modeling of repairable systems deals mostly with two types of repair. *Perfect* or ideal repair returns a system to 'as good as new' state. Therefore, the sequence of operating times forms in this case a renewal process. The most common realization of perfect repair in practice is the replacement of the failed system with a new identical one. Minimal repair, on the contrary, returns a system to a state (defined in statistical terms) immediately prior to failure (see, e.g., references [1-2]]). It is well known that in the latter case the corresponding sequence of lifetimes is described by the nonhomogeneous Poisson process (NHPP) with the rate equal to the failure rate defined by the baseline lifetime distribution of a system. A natural example of minimal repair is when the failed system is replaced by the identical one that was operating for the same time in the same conditions but did not fail and, therefore, could be considered as statistically identical.

In order to compare different maintenance actions, the basis for comparison should be chosen. A manufacturer (if he, for instance, provides a warranty) or the user are obviously interested in minimizing the operational costs of repairable systems. This characteristic for perfectly repaired systems is often defined as a stationary one via the concept of the renewal reward theory [3] as the long-run expected cost per unit of time (cost rate), i.e., the *mean cost incurred at the renewal cycle/ duration of the renewal cycle.* Numerous optimal maintenance policies minimizing this metric were discussed in the literature. The most popular strategy considers the setting when a system is perfectly repaired either upon failure or on attaining age T, whichever comes first. Then the optimal T minimizing the expected cost rate can be obtained (see the 'classical' paper by Barlow and Hunter [4]). The other standard strategy considers replacements at periodic instants of time $T, 2T, 3T, ...$ and minimal repairs in-between. Then again, optimal period minimizing the cost rate is obtained. Those are well-known cost driven important optimal decisions focused on perfect and imperfect preventive maintenance strategies [5-15].

There are numerous modifications of the basic models. However, we feel that some of the essential notions and approaches were overlooked in this 'endless' flow of literature. Some of the criticism of the renewal reward reasoning is based on the fact that in reality, we do not achieve asymptotic values of costs per unit of time as prescribed by the renewal-rewards theorems. This is true, but still these values are usually a very good estimate of the real quantities in practice and, furthermore, a 'one cycle solution' can be also [16]. On the other hand, it should be noted that in practice, we have a lot of applications when a system is subject to minor failures (which can be repaired minimally, perfectly or imperfectly) and a critical (disastrous) failures that terminate the operation of a system and cannot be repaired (e.g., failure of a mission or a death of an organism in biological applications). The possible optimal PM actions in these cases should be of interest as a possibility for, e.g., a *life extension* or for *increasing the probability of a mission success.* Both of these applications can be very important in practice. For instance, when a mission is very important (e.g., space or combat mission) or when the unique complex system is very expensive and the extension of its lifetime becomes vital.

Our setting somehow resembles the basic Brown-Proschan model [2] (or its time-dependent generalization [17] when each failure is minor (minimally repaired) with probability q and is major (perfectly repaired) with probability p, however, in contrast to this basic model, we consider the process only to the first major failure. To the best of our knowledge, this PM model in the current setting was not considered in the literature so far. Note that, preventive maintenance for the classical Brown-Proschan model with a random, time-independent p in a different from our approach context was reported recently in Lim et al. [18].

The paper is organized as follows. In Section 2, we describe our general setting. In Section 3, we consider increasing of the probability of a mission success via the PM actions is considered, whereas in Section 4, we deal with the cost-effective optimal PM schedules for extending the time to the major failure of partially repairable systems. Finally, concluding remarks are given in the last section.

2. The setting

Consider a system subject to minor failures that are instantaneously minimally repaired and to a major failure that is unrepairable and terminates the operational function of our system. Fatal failures often can result in large economic loss and probabilities of these events should be minimized as in the case of, e.g., space or combat missions. Another example is the deteriorating systems with a relatively long lifetime when it is not already cost-wise reasonable to perform a repair after a major failure. Furthermore, we can think also about an organism, whose death can be considered as a major failure, whereas 'minimal repairs' are executed throughout its lifetime. Thus, we can qualify the described type of systems as partially repairable. It is quite natural to implement some measures extending the lifetime and/or increasing the probability of a mission success of these critical systems. One of these measures is preventive maintenance (PM) that is widely used especially for repairable systems. One can find numerous papers that deal with various modifications of the basic PM models (see, e.g., [4, 6, 7] and references therein). However, we believe, that the described partially repairable case that seems simpler from the first sight, did not attract the deserved attention. Therefore, in this note, we will try to fill this gap to some extent considering some basic simple PM models for the partially repairable systems. Note that classical PM strategies with infinite time are formally non-applicable as the time till the major failure is finite. However, obviously, we can consider a single cycle (till the major failure) and describe optimal strategies of preventive maintenance in this case that will minimize overall or per unit time operational costs.

Let T be the time to any failure (minor or major) of our system with a finite expectation $\mu \equiv E[T] < \infty$ and absolutely continuous cumulative distribution function (Cdf) $F(t) = P[T \le t]$. Denote the corresponding survival function by $\bar{F}(t) = 1 - F(t)$, the probability density function (pdf) by $f(t)$ and the failure rate by $\lambda(t)$. Assume that each failure is minor with probability $q(t)$ and major (terminating) with probability $p(t) = 1 - q(t)$.

Minor failures are instantaneously minimally repaired. Denote the time to the fatal failure by T_P. It is well known that the time to the fatal failure has the following distribution [17,19]:

$$F_P(t) = \Pr[T_P \le t] = \exp\left\{ -\int_0^t p(u)\lambda(u)du \right\}. \quad (1)$$

with the corresponding failure rate

$$\lambda_P(t) = p(t)\lambda(t) \quad (2)$$

and the density function $f_P(t)$. It also follows from e.g., Finkelstein and Cha [20] that the process of minimal repairs (before the major failure) in this case follows the NHPP with rate

$$\lambda_q(t) = q(t)\lambda(t). \quad (3)$$

In what follows we will assume that a system is deteriorating, which is manifested by the *increasing* $\lambda_P(t)$. By implementing the corresponding PM actions we want to extend the useful life of deteriorating systems or to increase a mission success probability. However, as usual, it should be cost-effective (optimal in a suitable sense), otherwise we can perform the instantaneous PM as often as technically possible and will achieve the maximal extension of the initial lifetime T_P. We will first consider how to increase mission success probability by implementing the corresponding PM actions, where optimality is understood as the minimal number of PMs that achieve the required value of probability.

3. Maximizing mission success probability

As it was stated in the Introduction, increasing the probability of a mission success can be crucial in practice, e.g., for missions with high importance (e.g., space or combat missions). In this section, for the setting to be described, we will obtain the optimal (minimal) number of PMs that achieve the required probability of a mission success. We start first with perfect PM, that, according to its definition, decreases the failure rate to its initial value at $t = 0$.

Let t_m be the mission duration and $P_r(t_m)$ be the required mission success probability and let

$$\bar{F}_P(t_m) < P_r(t_m) \quad (4)$$

meaning that existing mission success probability does not meet these requirements. Therefore, we want to implement the preventive maintenance. As perfect maintenance brings the failure rate to its initial $\lambda_P(0)$, and the failure rate $\lambda_P(t)$ is increasing, we must first check that this ideal case meets our requirement, i.e.,

$$\bar{F}_{id}(t) \equiv \exp\{-\lambda_P(0)t\} > P_r(t_m)\,. \qquad (5)$$

If this is the case, then we can proceed with PMs. Note that, when $\lambda_P(0) = 0$, as for many distributions used in reliability modeling (e.g., the Weibull distribution in the forthcoming examples), inequality (5) holds automatically ($\exp\{-\lambda_P(0)t\} \equiv 1$) and this formal check is not required. Assume first, that only one PM can be scheduled at time a and let us find a maximizing the corresponding probability. Thus, we must obtain

$$\min_a \left\{ \int_0^a \lambda_P(u)du + \int_0^{t_m - a} \lambda_P(u)du \right\}, \quad a \in [0, t_m)\,, \qquad (6)$$

which simply follows from maximizing

$$\bar{F}_m(a, t_m) = \exp\left\{ -\int_0^a \lambda_P(u)du \right\} \exp\left\{ -\int_0^{t_m - a} \lambda_P(u)du \right\}.$$

where $\bar{F}_m(a, t_m)$ is the corresponding survival function for the case with one PM at a. Indeed, similar to the series system, the first multiplier in the r.h.s. gives probability of survival before the PM, and the second one, after the PM, defines survival probability in the rest of the interval. It also takes into account that after the PM, the failure rate is set to its initial value. After differentiating the sum of integrals in (6) with respect to a and equating the result to zero, we arrive at the following equation with respect to an optimal a:

$$\lambda_P(a) = \lambda_P(t_m - a)\,. \qquad (7)$$

Equation (7) has a trivial unique solution $a^* = t_m/2$ for increasing functions [21]. It is also clear that it is a minimum for (6) as the maximum is achieved for $a = 0$ and $a = t_m$.

Obviously, the same reasoning can be applied to the case of n PMs at times $a_1 < a_2 < \ldots a_n$. Thus we must find

$$\min_{\{a_i\}} \left\{ \int_0^{a_1} \lambda_P(u)du + \int_0^{a_2 - a_1} \lambda_P(u)du + \ldots + \int_0^{t_m - a_n} \lambda_P(u)du \right\}.$$

After differentiating and equating the result to zero, and using the same argument as while discussing (7), we arrive at the simultaneous equations with respect to optimal $\{a_i^*\}, i = 1, 2, \ldots n$ with a solution:

$$a_i^* = i\frac{t_m}{n+1}, i = 1, 2, \ldots, n\,.$$

Thus under given assumptions the PMs should be performed equidistantly. When there are n PMs, the mission success probability, $\bar{F}_m(n, t_m)$ is

$$\bar{F}_m(n, t_m) = (\bar{F}_P(t_m / n))^n\,, \qquad (8)$$

where $F_P(t)$ is defined in (1). Note that $\overline{F}_m(n,t_m)$ is increasing with *n*, as each additional PM obviously increases the corresponding survival function, and

$$\lim_{n\to\infty}(\overline{F}_P(t_m/n))^n = \exp\{-p(0)\lambda(0)t_m\}. \qquad (9)$$

Equation (9) is obtained for the case $p(0)\lambda(0)\neq 0$. When $p(0)\lambda(0)=0$, this limit is equal to 1.

The minimal number of PMs to meet the requirement $P_r(t_m)$ (see also (4)) can be obtained as

$$n^* = \min_n\{(\overline{F}_P(t_m/n))^n \geq P_r(t_m)\}. \qquad (10)$$

Example 1. Let $\overline{F}_P(t)=\exp\{-(\lambda t)^2\},\ \lambda(t)=2\lambda^2 t,\ \lambda>0$ (where, for simplicity, $p(x)\equiv 1$), which corresponds to the Weibull distribution with linear failure rate. Then (10) turns to

$$n^* = \min_n\{(\exp\{-(\lambda t_m)^2/n\}\geq P_r(t_m)\}. \qquad (11)$$

It is obvious that inequality in (11) can be easily achieved by the sufficiently large *n*.

In practice, most of the preventive maintenance actions are imperfect. Even the replacement of a system by a 'new one, strictly speaking, is not ideal as a system could be subject to different tests at the production phase (e.g., burn-in) and can be also stored for some time. There are numerous models of imperfect repair/preventive maintenance (see, e.g., references [8],[10-14], [20]). We will suggest here a simple model that to the best of our knowledge was not considered in the literature. Furthermore, it seems to be quite realistic from the practical point of view.

After each PM, the failure rate in the considered above perfect PM model was set at its initial level $\lambda_P(0)=p(0)\lambda(0)$. We will assume now that the failure rate after the PM at calendar time x and at time t after the last maintenance has the following form

$$\lambda_{im}(x,t)=\lambda_0(x)+\lambda_P(t),\ x>0, t\geq 0, \qquad (11)$$

where the function $\lambda_0(x), \lambda_0(0)=0$ is assumed to be increasing showing the value of 'additional' failure rate that is added to $\lambda_P(t)$ after each imperfect PM. The fact that $\lambda_0(x)$ is increasing means that the quality of imperfect PM is also deteriorating with each repair. Thus, in accordance with (11), the survival function that describes time to the next failure after the PM at calendar time x is $\overline{F}_P(t)\exp\{-\lambda_0(x)t\}$, where the second multiplier shows the effect of imperfect maintenance on the baseline survival probability.

Similar to (6) consider first one PM in $[0,t_m)$

$$\min_a\left\{\int_0^a \lambda_P(u)du + \int_0^{t_m-a}(\lambda_0(a)+\lambda_P(u))du\right\}. \qquad (12)$$

After differentiating the sum of integrals and equating the result to zero, we arrive at the following equation with respect to a:

$$\lambda_P(a)-\lambda_P(t_m-a)+(t_m-a)\lambda_0'(a)-\lambda_0(a)=0. \qquad (13)$$

Let, for simplicity, $\lambda_P(0) = \lambda_0(0) = 0$. Then rearranging (13) as

$$\lambda_P(a) - \lambda_0(a) = \lambda_P(t_m - a) - (t_m - a)\lambda_0'(a), \tag{14}$$

it can be shown (e.g., graphically) that it has a unique solution under our assumptions. Indeed, both symmetrical curves $\lambda_P(a)$ and $\lambda_P(t_m - a)$ that cross at $a = t_m/2$ are shifted lower but maintain the zero values at $a = 0$ and $a = t_m$, respectively, therefore they have to cross as well. For instance, for the specific case, $\lambda_P(t) = k_1 t,\ \lambda_0(t) = k_2 t;\ k_1 > k_2$ (see the next example), it is easy to see that $a = t_m / 2$ is still a solution to (14). A similar reasoning can be applied to the case of n PMs at times $a_1 < a_2 < \ldots a_n$. Thus we must find

$$\min_{\{a_i\}} \left\{ \int_0^{a_1} \lambda_P(u)du + \int_0^{a_2 - a_1} (\lambda_0(a_1) + \lambda_P(u))du + \ldots + \int_0^{t_m - a_n} (\lambda_0(a_n) + \lambda_P(u))du \right\}.$$

that corresponds to optimal $\overline{F}_P * (t_m, n)$ for the fixed n, i.e.,

$$\overline{F}_P * (t_m, n) = \exp\left\{ -\min_{\{a_i\}} \left\{ \int_0^{a_1} \lambda_P(u)du + \int_0^{a_2 - a_1} (\lambda_0(a_1) + \lambda_P(u))du + \ldots + \int_0^{t_m - a_n} (\lambda_0(a_n) + \lambda_P(u))du \right\} \right\}.$$

As derivatives of the sum of integrals with respect to $a_i, i = 1,2,\ldots,n$, similar to (13), involve only two terms, the simultaneous equations have an optimal solution $a_1 * < a_2 * < \ldots. < a_n *$ that can be obtained numerically. As $\overline{F}_P * (t_m, n)$ is increasing in n, the optimal $n*$ can be obtained.

Example 2. Let $\lambda_P(t) = k_1 t,\ \lambda_0(t) = k_2 t;\ k_1 > k_2$. Then it can be shown by simple derivations that, similar to the perfect PM case, $a_i * = i\dfrac{t_m}{n+1}, i = 1,2,\ldots,n$. The optimal number of PMs can be obtained from the relation:

$$n* = \min_n \{\overline{F}_P * (t_m, n) \geq P_r(t_m)\}.$$

Assume that the mission success probability requirement is given as $P_r(t_m) = 0.9$ and let $k_1 = 0.005$, $k_2 = 0.0018$, $t_m = 10$. Then $\overline{F}_P(t_m, 0) \approx 0.81$ and we need preventive maintenance to improve this probability. Table 1 and Fig. 1 show how the mission success probability approaches the required value with the number of PMs increasing from 1 to n. Thus $\overline{F}_P * (t_m, n^*) = 0.9007$, $n^* = 10$.

n	1	2	3	4	5	6	7	8	9	10
$\overline{F}_P * (t_m, n)$	0.8437	0.8665	0.8781	0.8851	0.8899	0.8933	0.8958	0.8978	0.8994	0.9007

Table 1. Values of the mission success probability for different n

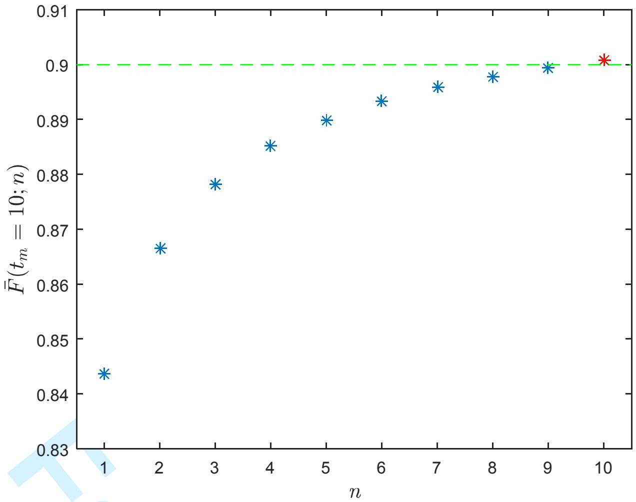


Fig 1. Values of the mission success probability for different number of PMs, *n*

4. Extending the lifetime by PMs

The reasoning in the previous section was aimed at achieving the required reliability characteristics for a given mission time without direct consideration of the corresponding costs. In this section, for the same basic setting with minimal repairs and a major (fatal) failure (end-of-life event), we will deal with a problem of minimizing expected costs on a life cycle of a system until its major failure and will obtain an optimal strategy for the corresponding PMs. Note that for the non-reparable and partially repairable systems, the classical PM strategies described in the Introduction do not work, as we do not have a stationary regime in this case, however, we can consider a single cycle and look at the optimal strategy of PMs in this case that will minimize overall or per unit time operational costs.

First, we must define the corresponding cost structure. Let $C_m(x), C_{pm}$ be the costs of the minimal repair and of the preventive maintenance, respectively. Let the latter, for simplicity, does not depend on the calendar time x, however the time-dependent case can be also considered. It is reasonable to assume also that as a system is wearing out, its minimal repair cost $C_m(x)$ is increasing. The expected operational costs before the major failure, in accordance with (1)-(3) and Boland [22] are

$$C_p = \int_0^{\infty} f_P(y) \int_0^{y} C_m(x)\lambda_q(x)dxdy, \tag{15}$$

Thus the average cost rate on a lifecycle is

$$c_p = \frac{\int_0^{\infty} f_P(y) \int_0^{y} C_m(x)\lambda_q(x)dxdy}{\mu_p}, \tag{16}$$

where $\mu_p = \int_0^{\infty} \overline{F}_P(x)dx$ is the mean time to the major failure.

Let $C_m(x)$ be also a constant, i.e., $C_m(x) \equiv C_m$ Then (16) simplifies to

$$c_p = \frac{C_m \int_0^\infty f_P(y) \int_0^y \lambda_q(x) dx dy}{\mu_p}. \tag{17}$$

We want to improve now the performance characteristics of our system by implementing PM. Let us perform periodic perfect PMs at $na, n = 1,2,...; a > 0$. As the perfect maintenance brings the failure rate to its initial $\lambda_P(0)$, and $\lambda_P(t)$ is increasing, the PMs will increase the expected time to a major failure. For instance, when $a \to 0$, in the limit, the time to a major failure tends to the exponentially distributed random variable with parameter $p(0)\lambda(0)$). Therefore, without considering costs of PMs:

$$C_p(a) \to_{a\to 0} C_m \frac{q(0)}{p(0)},$$

$$c_p(a) \to_{a\to 0} C_m q(0)\lambda(0).$$

This relationships show the potential minimal costs of minimal repairs, and similar to (5) can be used for preliminary analysis of the problem. Assume, in what follows that $\lambda_q(x) = q\lambda(x), \lambda_p(x) = p\lambda(x)$, i.e., $q(x) \equiv q, p(x) \equiv p$. Implementing the PM actions will increase the expected costs as the cost of each PM is C_{pm}. Therefore, we must find an optimal period a that minimizes the expected costs. Denote by $S(a)$ the probability of survival (without a major failure, but with possible minor failures that are instantaneously minimally repaired) of our system between the two perfect PMs. Then, in accordance with (1),

$$S(a) = \exp\left\{-\int_0^a \lambda_p(x) dx\right\}.$$

Then the expected number of PMs before the major failure, in accordance with the corresponding geometric random variable, is

$$S(a)(1-S(a)) + 2S^2(a)(1-S(a)) + 3S^3(a)(1-S(a)) + ... = \frac{S(a)}{1-S(a)}. \tag{18}$$

Therefore, the expected cost till the major failure can be obtained as

$$C_p(a) = \left(C_{pm} + C_m \int_0^a \lambda_q(x) dx\right) \frac{S(a)}{1-S(a)} + C_m \int_0^a f_P(y) \int_0^y \lambda_q(x) dx dy, \tag{19}$$

where the first term is just the product of the expected number of PMs and the expected cost for one PM cycle and the second term defines the expected cost on the last terminated by the major failure PM cycle. On the other hand, the expected time to the major failure is

$$\mu_p(a) = a \frac{S(a)}{1-S(a)} + \mu_{pa}(a), \tag{20}$$

where μ_{pa} is now a conditional expectation of a mean time to a major failure in [0,a) (the time since the last PM and to the failure) given the failure had occurred in this interval, i.e.,

$$\mu_{pa}(a) = \frac{\int_0^a (S(x) - S(a))dx}{1 - S(a)} < a .$$

As mentioned, the second terms in (19) and (20) correspond to the period of length a where the major failure had occurred. Due to the Wald's inequality, the first term in (19) is the product of expected costs between two PMs and the expected number of PMs, whereas the first term in (20) is the product of the length of the PM period a and the expected number of full periods before the major failure. Thus

$$c_p(a) = \frac{(C_{pm} + C_m \int_0^a \lambda_q(x)dx)\frac{S(a)}{1 - S(a)} + C_m \int_0^a f_P(y) \int_0^y \lambda_q(x)dxdy}{a\frac{S(a)}{1 - S(a)} + \mu_{pa}(a)} . \qquad (21)$$

and our optimization problem of obtaining the optimal PM period a^* is formulated as

$$c_p(a^*) = \min_{a>0} c_p(a) .$$

It is not so simple to analyze the shape of $c_p(a)$ analytically and we will consider the corresponding numerical examples further. However, some simple intuitive reasoning can be sufficient for the general considerations on existence of the optimal a that minimizes (21). It can be easily seen that

$$\lim_{a \to 0} c_p(a) = \lim_{a \to 0} \frac{C_{pm}}{a} = \infty ,$$

whereas by applying the L'Hopital's rule:

$$\lim_{a \to \infty} c_p(a) = \lim_{a \to \infty} \frac{C_m \int_0^a f_P(y) \int_0^y \lambda_q(x)dxdy}{\mu_{pa}} = \infty ,$$

which means that $c_p(a)$ has, at least, one minimum in $[0, \infty)$.

On the other hand, we can also approximate (19) by

$$\widetilde{C}_p(a) = (C_{pm} + C_m \int_0^a \lambda_q(x)dx)[\frac{S(a)}{1 - S(a)} + 1]$$
$$= (C_{pm} + C_m \int_0^a \lambda_q(x)dx)\frac{1}{1 - S(a)}$$

and (20) by $a/(1 - S(a))$. The meaning of this approximation is in substitution of the last terminated PM cycle by the full one of length a. Therefore, the accuracy of this approximation increases with the number of periods before the major failure (i.e. as a decreases). Thus

$$\tilde{c}_P(a) = \frac{(C_{pm} + C_m \int_0^a \lambda_q(x)dx)}{a} \,. \tag{22}$$

Expression (22) can be easily analyzed now. Let $\lambda(x) \to \infty$ as $x \to \infty$. Note that our reasoning can be easily adjusted to the case when the failure rate is increasing to a constant. By similar reasoning as above, $\tilde{c}_P(a) \to \infty$ when $a \to 0$ and $a \to \infty$. Thus, $\tilde{c}_P(a)$ has, at least, one minimum. As (22) is much simpler than (21), we can go now further in our analysis. Equating $\tilde{c}'_P(a)$ to 0, the condition for minimum can be expressed as

$$a\lambda_q(a) - \int_0^a \lambda_q(x)dx = \frac{C_{pm}}{C_m} \,. \tag{23}$$

It can be seen that under our assumptions (assume for simplicity additionally that $\lambda_q(x)$ is a convex function, e.g. as for Weibull distribution with increasing failure rate), the l.h.s. of (23) is increasing from 0 to ∞ and, therefore, there is a single minimum for the function $\tilde{c}_P(a)$ that approximates $c_p(a)$. However, the accuracy of this approximation is not always sufficient that can be seen from the example below. Indeed for various values of parameters $\tilde{c}_P(a)$ can provide a very good approximation for $c_p(a)$ when a is relatively small. However as a increases $c_p(a) - \tilde{c}_P(a)$ also increases and the value of the approximate and the 'exact' optimal period can differ substantially. Therefore, the suggested approximation is useful for a general analysis, however, in practice one should rather use the exact relationship (21).

Example 3. Let $C_{pm} = 100$, $C_m = 10$, $\lambda(t) = 0.005\, t$.

a. Let p=0.005. Then

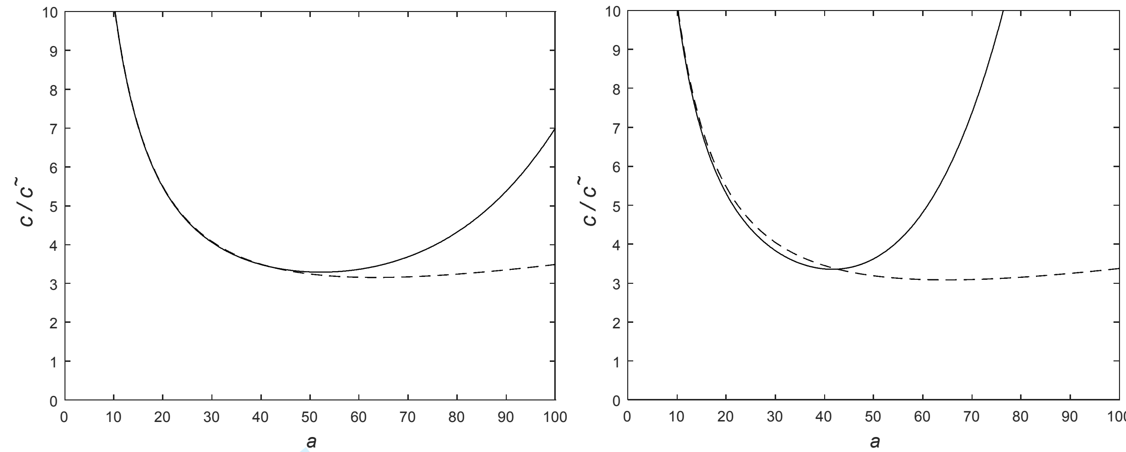
$$a^* = 52.2 \quad ; \quad c(a^*) = 3.290896$$

$$\tilde{a}^* = 63.4 \quad ; \quad \tilde{c}(\tilde{a}^*) = 3.348055$$

b. Let p=0.05. Then

$$a^* = 41.6 \quad ; \quad c(a^*) = 3.353932$$

$$\tilde{a}^* = 64.9 \quad ; \quad \tilde{c}(\tilde{a}^*) = 3.082207$$



(i) (ii)

Fig 2. Approximate (dash) and ‘exact’ (solid) cost rate function for (i) p=0.005 (ii) p=0.05

Remark 1. The imperfect PM model (similar to (11)) can be also considered for systems with minimally repaired minor failures. However, its presentation is much more cumbersome and will be reported elsewhere, whereas in the current paper our aim was to introduce this new approach and to illustrate it via simple practical examples.

5. Concluding remarks

In this paper, we consider perfect and imperfect preventive maintenance actions for systems with minor and major failures. Minor failures are minimally repaired, thus forming the corresponding non-homogeneous Poisson process with rate $\lambda_q(t)$ defined in (3), whereas the major failure terminates the operational function of a system and, therefore, can be considered as an end-of-life event. Traditionally, we assume that repair and PM are instantaneous, as usually in practice the corresponding durations are negligible in comparison with times to failures.

The PM considered in Section 3 increases the mission success probability. In the simplest case, it is optimal in the defined sense when planned equidistantly. The imperfect repair of the specific form is also discussed, however, in this case, computational methods should be used for obtaining the sequence of optimal PM times.

The PM in Section 4 increases the time to a major failure of a system. As the corresponding costs are involved in this case, the PM schedule should be cost-optimal. The suggested approach defines the cost rate and deals with its optimization. Usually, this setting characterises deteriorating complex systems with relatively long lifetimes. As an example, we can think about automobiles or road machines when probability of a major failure (non-repairable) increases with time. Another important example is a biological organism, whose death can be considered as a major failure. PM for the latter setting is an interesting and important novel application and we plan to report the relevant results elsewhere. It seems also reasonable to consider the generalization of the suggested model to the case when performance of a system is characterized by the output function [23].

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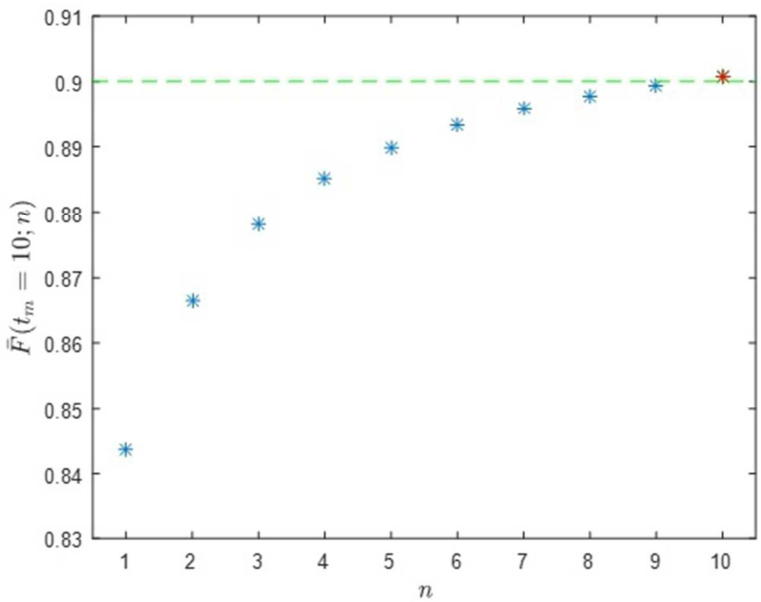

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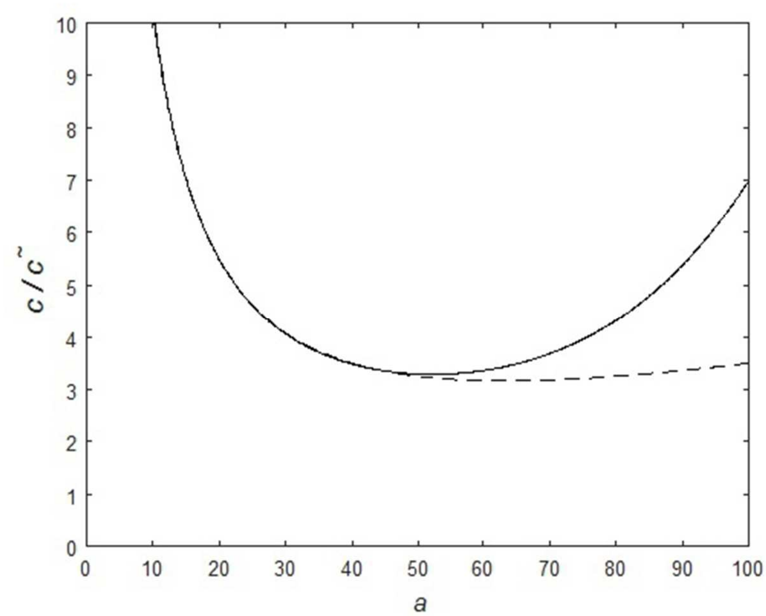

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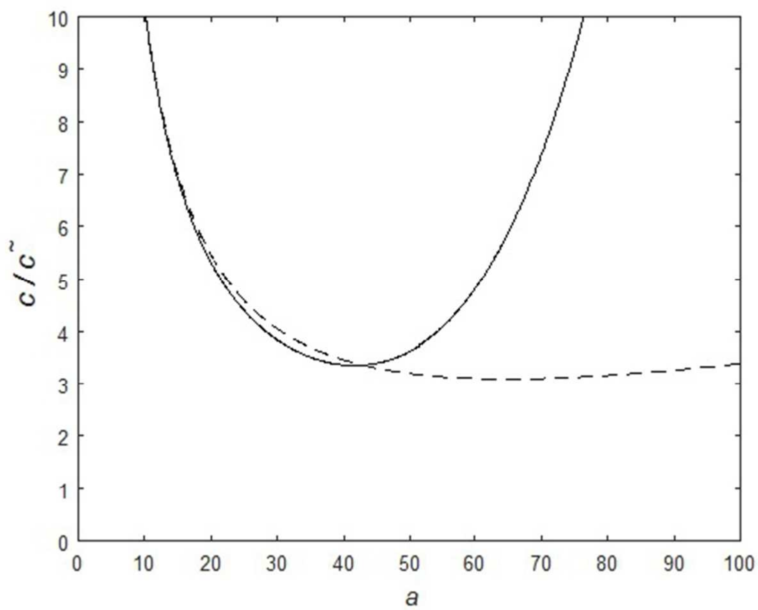

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