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STRESS ANALYSIS OF VEHICLE STRUCTURES

by

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Stress Analysis of Vehicle Structures

- by -

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SUMMARY

A historical review is made of the methods adopted in vehicle design.

This is divided into two sections, vehicles with chassis frames and integral structures.

Complete solutions have been obtained for all types of chassis frames by Erz and the results of this work are given. The simple frame structural analysis has been extended for buses, with semi-integral and integral construction.

Private car structural analysis has been very approximate using pencil and paper methods, various approaches have been tried with little success. The advent of the digital computer has made the analysis of such complex structures possible and the two basic methods of 'displacements' and 'force' are described. The matrix force method due to Argyris is treated in detail with a complete analysis of a simplified box van structure under idealised loading. This method has been chosen for economy in computer space and its application to a more complex structure, an integral Land Rover, is indicated.

With the advent of very large computers it is probable that displacement methods will take precedence as less work in 'choosing' the idealisation and redundant systems is involved.

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Introduction

Vehicle development has grown through many centuries and it has never been necessary for a vehicle with a new power unit to be structurally different from its immediate predecessors. Consequently the first vehicles with internal combustion engines were able to function satisfactorily by using existing horse drawn techniques of coach building - the size of wooden or metal frames required to carry the load safely had been arrived at after generations of experience in coach building. There was no need to drastically re-think the structural problem of the horse-less carriage whereas the first aeroplanes would not fly unless the theoretically lightest structure was used. The necessarily different approach has led to the position today where nearly every advance in structural analysis is made by aircraft engineers and nearly all surface vehicles are designed on a basis of experience.

The considerable advances in vehicle structural efficiency have been largely achieved by intelligent application of the lessons of ever more extensive and rigorous testing. This statement is still basically true in spite of analyses of vehicle structures made in the past.

If there is a movement towards the use of structural analysis in the vehicle industry at the present time it is caused as much by the need to cut down the time spent testing a new design as by the desire for greater structural efficiency. Where there are real gains to be achieved by making the lightest possible vehicle as in racing cars and air transportable vehicles structural analysis is found to be most often used.

In order to base analysis on reasonable loading cases much more information is required than at present available. Only static loading cases are used and these are based on intuition or on the highest acceleration recorded on one vehicle in a series of tests or by assuming that one wheel is resting on a bump of a given size. Faced with this dearth of information a programme of measuring the loads existing between the suspension system and the body structure has been started at the A.S.A.E. References 1 and 2. The tests can only be regarded as preliminary but indicate that the longitudinal forces are likely to be greater than the vertical forces and that the vertical forces are not likely to exceed twice the static load. Extensive tests on military vehicles to establish the acceleration environment for equipment, Ref. 3, giving a maximum acceleration of 4g provide a further guide in establishing a realistic loading case for vehicle structures. It is with this very serious limitation in mind that the work carried out so far on vehicle structural analysis is reviewed and modern methods suggested for future use.

Historical review - vehicles with separate chassis frames

Stressing of the frame for static loading as a simple beam must have been carried out from the early days of vehicle design. The methods in use at the time (1948) are summarised by Dean-Averns, Ref. 4, and the stressing cases considered for a typical public service vehicle were:-

1. Static loading for chassis only.
2. Static loading for body and load.
3. Braking reactions (load transfer due to maximum braking; 0.5 g. at that time).

It is emphasised that these static calculations can only be relevant as comparative stressing cases and Dean-Averns states that if the stress level is kept to 6 tons/sq.ins. the frame will be strong enough for a public service vehicle. Elementary calculations of this type are still the only ones carried out by many design organisations since the continual testing of similar chassis frames allows the maximum stress level to be accurately adjusted to suit the type of frame used.

The torsional stiffness of a chassis assumes great importance when independent front suspension of the wishbone type is used and the stresses involved in frame torsion may well be greater than those due to bending.

Dean-Averns, Ref.4, again supplies a first attempt at solving this problem by comparing the torsional stiffness of ladder frames and those with cruciform bracing. His assumption that a quarter of a rectangular frame may be treated as two cantilevers is incorrect as can be seen from fig. 1. The correct solution to this problem is given in detail in Appendix 2, and also by Cooke in Ref. 5, using strain energy methods.

A comprehensive discussion of the torsional stiffness of chassis frames was published by K. Erz in 1957, Ref. 6. The complete formulae for determining the loads and deflections of a ladder frame are given under the assumption that the side members and cross members are loaded in torsion and bending based on symmetry and Engineers Theory of Bending. He draws attention to the importance of including the effect of warping inhibition at the ends of the cross members when they are of open section and subject to torsion. This effect is expressed as a factor multiplying the torsional stiffness of the section of the cross member. The factor is a function of the torsion stiffness and the torsion bending or warping constant.

i.e. $$J^* = J \frac{\frac{\mu \ell}{2}}{\frac{\mu \ell}{2} - \tanh \frac{\mu \ell}{2}}$$

where J^* = Modified torsional stiffness constant.
J = Normal torsional stiffness constant (free warping)
$\mu = \sqrt{\frac{GJ}{E\Gamma}}$
Γ = Warping constant for section

Table 1 gives formulae originally obtained by Bergmann, Ref. 7, for the constant Γ for various sections commonly used as chassis frame cross members.

A further simplification made by Erz was to ignore the effect of bending deflections of the individual members compared with the torsional deflections for the conventional open section chassis frame.

The deformed shape of the frame is then shown in Fig. 2, and the calculation of loads and the frame stiffness becomes statically determinate. An example is given in the paper showing that this simplification gives a theoretical error of only 4%. Further justification can be found in table 2, which compares the measured torsional stiffness of five chassis frames with the theoretical torsional stiffness calculated by the simplified method (column S_1 assuming free warping of the cross members and column S_2 assuming that warping is completely inhibited).

Frames 2a and 2b were the same except that the thickness of the channel section side members was increased from 6 mm. to 7 mm. (say $^1/_4$" to 9/32") and only a 4% increase in torsional stiffness was obtained. Frames 3a and 3b were again similar except that the flanges of the side members were increased from 70 to 80 mm. (2.75 inches to 3.15 inches). In this case 14% higher torsional rigidity was measured indicating that it is more economical to increase the flange width to obtain torsionally stiff frames.

Erz next compares the torsional stiffness of ladder, trapezoidal, cruciform and backbone type frames. The results shown in Fig. 3, refer to frames of equal weight made up of closed section members and he states that for open double flange section members the order of stiffness of the various types is the same but the difference in magnitude is greater. When discussing the merits of stiff v flexible frames several design points are brought out to minimise stress concentration and it is emphasized that flexible frames must not be too narrow as the stiffening effect of warping inhibition of open section cross members is confined to the ends of the cross members, consequently wide ladder frames are less stiff than narrow ones. An analysis scheme for a space frame chassis is also suggested and formulae for a flat rectangular base type chassis are given but comparative torsional stiffnesses are not given for these alternatives.

The principles used in the complete analysis of the ladder frame are extended by Michelberger, Ref. 8 and 9 to include the effects of the vehicle sides as additional beams, Fig. 4. This author assumes that torsion cannot be transmitted to cross members where they meet the sides of the vehicle. It is interesting to note that the method adopted by these authors to solve the redundant structure problem is basically the same as used in the matrix force method, in fact Michelberger sets out the calculation in matrix form.

A recent contribution to ladder frame design has been made by Seitler Ref. 10, who has shown that since cross members made of circular tubes induce no 'torsion bending' the stresses at the joints are much lower and a very stiff frame can be made with considerable saving in weight.

It is clear that the original frame that he has improved was very stiff compared with a typical rivetted frame and he admits that rivets do, in fact, relieve the very high stresses at the joints of open section frames.

This review of work on frames is by no means comprehensive but indicates the recent emphasis on torsion as the important loading condition.

Vehicles with integral structures

If the integral structure is a box type, e.g. a closed van, the solution can be approached by conventional shell theory. Since torsion is usually the most important case the Bredt-Batho formula may well be sufficient. If substantial corner members run the length of the vehicle the extension of thin wall torsion theory to box beams with corner members has been familar to aircraft stress analysis for a long time. References 11 and 12, give simple and comprehensive treatments respectively.

Car like structures differ radically from the near symmetrical shell and may be approached as a fixed jointed framework. Swallow, Ref. 13, treats one side of a vehicle in this way, as shown in Fig. 5. He also suggested that a torsional calculation can be made treating the body shell as six flat frames. Garrett, Ref. 14, attempted a piecemeal analysis of an integral car structure, in particular treating the sill as the main beam support member for the vehicle and considering in detail sill to inner wing member connections.

The cross member connecting the front spring hanger brackets of the rear spring (the heelboard) is also treated in detail in the first part of the article.

In part 2, Garrett stresses the front end structure, starting with the suspension loads and treating the main front structure as a horizontal portal frame (see Fig. 6), and finally considering the cross member at the base of this portal, the toeboard.

Johnson and Heyl, Ref. 15, discuss the possible load paths in a typical American car body but conclude that measurement of existing structure is the best guide to future design. They produce useful bending moment curves on a typical body side from extensive strain gauge testing.

Erz, Ref. 6, attempts the analysis of an integral bus or coach by approximate methods. He first considers symmetrical bending of the vehicle and assumes the whole body side acts as a beam. The main cut-out is taken to be the rear door, a front door is also present but the loads are expected to be small as a rear engined vehicle only is discussed. The bending moment at the centre of the door opening is replaced by horizontal loads in the cant rail and sill, the horizontal load in the cant rail is then distributed to the window uprights in proportion to their local bending stiffness.

$$\text{i.e.} \qquad Q_{p_r} = \frac{I_r}{\Sigma_1^n I_r} \cdot P \qquad \text{(See Fig. 7).}$$

The critical bending moment is assumed to be at the base of the upright and is given by $M_{p_r} = {}^2/_3\, h_1\, Q_{p_r}$. Where the factor $^2/_3$ is an assumption.

The shear load at the centre of the rear door is assumed to be distributed between the cant rail and the door sill in proportion to their bending stiffness. These separate shear forces produce a bending moment on the cant rail and door sill respectively which are arbitrarily increased by 50% to allow for the 'fixed end' effects on the beams. Referring to Fig. 8, the cant rail shear force

$$Q_u = \frac{I_u}{I_u + I_L} Q,$$

and the maximum bending moment

$$M_u = 0.75\, Q_u b$$

similarly the door sill shear force $Q_L = \frac{I_L}{I_u + I_L} Q$, and the maximum bending moment $M_L = 0.75 Q_L b$.

For torsion the vehicle is treated as a thin tube in torsion about a transverse axis. First the torsion about this axis is calculated from the known couple M_d about the longitudinal axis as shown in Fig. 9.

The shear flow q round the edge of the roof, ends and floor is then calculated by the Bredt-Batho formula:

$$q = \frac{T}{2A}.$$

where A is the area of the side of the vehicle.

This shear flow is resisted by the window uprights in bending as before:

$$M_{t_r} = {}^2/_3 h_1 \frac{I_r}{\Sigma I_r} \cdot q \cdot 1$$

The same shear flow q must also act on the front end of the vehicle and the maximum lateral moment on the windscreen pillar will therefore be:

$$M_{t_p} = \frac{h_1' qb}{3} \quad \text{for the case of a single windscreen.}$$

It is clear that this type of analysis could be used on a passenger car and Brzoska, Ref. 16, indicates this possibility at the end of his paper.

The analysis of redundant structures

Before describing in detail the digital computer method chosen for further study it is necessary to examine briefly some of the methods that have been proposed for general structural analysis and their application to integral vehicle structure.

Basically all analyses must start by idealising the real system since any real structure is infinitely redundant and an approximation must be made to obtain a workable order of redundancy or number of unknowns. The problem is, therefore, to establish an acceptable approximation to the actual structure. This problem can be solved in two ways, either real vehicle structures can be analysed with various idealisations and the results compared with tests on an actual vehicle, or simplified structures embodying certain essential characteristics of present vehicle structures can be analysed and tested. Again the degree of idealisation may be varied and the calculation made more complicated until agreement between theory and practice is achieved. Of these two methods the second has been adopted at the A.S.A.E. as the main streams of student thesis work and two of these theses have been completed (Ref. 17 and 18) analysing essentially the same structure statically and dynamically. Even the highly simplified structures adopted, see fig. 10, proved too ambitious for a one year thesis and a more recent thesis analysed an even simpler structure fig. 11.

To take this work a stage further and to gain experience with a more complicated structure a much simplified Land Rover structure is being analysed under the present contract.

The analysis of a complete car structure by using a digital computer has been successfully carried out in America and is briefly reported in Ref. 19. This programme appears comprehensive but requires a fairly large computer to handle it and is not likely to provide the most economical approach to the analysis.

Displacement and force methods

There are essentially two basic methods of structural analysis, one, choosing displacements as unknowns and finding the value of these displacements to satisfy the conditions of equilibrium of forces at every point in the structure, these are known as 'Displacement Methods'. The second method chooses forces as unknowns and finds the values of these forces to satisfy compatibility conditions in the structure; these are known as 'Force Methods'.

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Since the true structure cannot be analysed the conditions can only apply at points specified by the idealisation adopted.

A useful method of deciding which basic method to use is given by Michelberger (Ref. 8). 'If the possible node displacements of the system exceeds the degree of redundancy it is generally better to use a Force Method and vice-versa'.

Examples are shown in Figures 12 and 13.

This rule must be used with caution as the best method is often dependent on the ease of setting up the equations, since with large computers it is better to set up a large number of simple equations than a few difficult ones. The McKenna paper Ref. 19, previously mentioned is an example of this where a displacement method has been used, involving 6 equations for each node in a structure of about 100 nodes. (Fig. 14). McKenna's method is basically the same as the Livesley Displacement method described in Ref. 20, and summarised later in this report. The A.S.A.E. Thesis of Mann, Ref. 17, used the matrix force method attributed to Argyris, Ref. 2, for stress analysis. The A.S.A.E. Thesis of Cuthill, Ref. 18, used the matrix displacement method given also in Ref. 21, for obtaining the stiffness matrix of the complete structure defined at a large number of points to obtain the dynamic characteristics of the structure.

In the two theses referred to the relevant matrices were set up and computer programmes run but time was not available for sufficient checking of the input data to obtain accurate results. The matrix displacement method of Argyris condenses the matrix to be inverted (or the number of simultaneous equations to be solved) to the number of the external forces or inertia forces defined in the system. By using these condensation techniques the Argyris methods are the ones most suited to small computers of which Pegasus is an example.

The matrix force method

As a simple example consider the 4 pin jointed bars supporting a single load as in Fig. 15a. This system clearly has two indeterminacies or redundancies and could be solved quite simply by writing two simultaneous equations assuming unknown forces in any two of the bars.

It will now be used to show how the Argyris method uses simple concepts to build up these equations. First choose as a basic system sufficient structure to support the external load system (bars 1 and 4) and replace the external load by a unit load.

This is shown in Fig. 15b, and the loads in <u>all</u> the members due to **this** load system can be written as

$$b_o = \begin{bmatrix} b_{01} \\ b_{02} \\ b_{03} = 0 \\ b_{04} \end{bmatrix}$$

where b_o is a matrix having as many rows as there are loads in the system and as many columns as there are different values of the external load R to be considered. Many writers use the term stress resultant for the load in a member due to one particular type of loading, e.g. a separate stress resultant exists for the bending moment and the direct load in a member. Morice, Ref. 22, defines a stress resultant as the integral of a stress over an area or the integral of a stress moment about a chosen axes, the area need not be normal to the axis of the section but is usually taken to be so.

Since linear structures are assumed it seems no disadvantage to refer to a stress resultant as a load, bending moment or shear force existing in a member at any point as long as there is no confusion with the external load system. In the notation of the matrix force method all such internal loads, moments, etc. are denoted by S.

In the present example for a single value of the external load R, S will be a four row single column matrix having values S_1, S_2, etc. as the final (correct) load in bar 1, 2, etc.

Clearly the contribution of the basic system to the S matrix will be

$b_oR = b_{01}R = S_{01}$

$b_{02}R = S_{02} = 0$

$b_{03}R = S_{03} = 0$

$b_{04}R = S_{04}$

In order to include the effect of the redundant members the external force is ignored and two self equilibrating systems are chosen (one for each indeterminacy) from the structure where an unknown load X in any one member will give loads in all the other members of the self equibrating system but zero loads in the remainder of the structure.

Such a system is shown in Fig. 16a, where X_2 is the unknown load in bar 2. Again this can be replaced by a unit load and from simple statics the loads in the other bars can be found and written as:-

b_{112}

b_{122}

b_{132} (= 0)

b_{142}

See Fig. 16b.

Multiplying each of these by the unknown load X_2 we have

$$S_{112} = b_{112} X_2$$
$$S_{122} = b_{122} X_2$$
$$S_{132} = b_{132} X_2 \ (= 0)$$
$$S_{142} = b_{142} X_2$$

A similar result will be obtained by taking an unknown load in bar 3, in the self equilibrating system shown in Fig. 16b.

The loads in all the members due to this load system will be

$$S_{113} = b_{113} X_3$$
$$S_{123} = b_{123} X_3 \ (= 0)$$
$$S_{133} = b_{133} X_3$$
$$S_{143} = b_{143} X_3$$

Since the structure is linear the principle of superposition holds and

$$S_1 = S_{01} + S_{112} + S_{113} = b_{01}R + b_{112} X_2 + b_{113} X_3$$
$$S_2 = S_{02} + S_{122} + S_{123} = b_{02}R + b_{122} X_2 + b_{123} X_3$$
$$S_3 = S_{03} + S_{132} + S_{133} = b_{03}R + b_{132} X_2 + b_{133} X_3$$
$$S_4 = S_{04} + S_{142} + S_{143} = b_{14}R + b_{142}X_2 + b_{143}X_3 \qquad (1)$$

or in matrix form $S = b_0R + b_1X$

Equation (1) expresses the equilibrium condition for the whole structure and to solve for X it is necessary to satisfy compatibility conditions for the structure. In this simple case it can be seen that if cuts are made in bars 2 and 3 the loads X_2 and X_3 must be chosen such that the displacement across the cut is zero.

The displacement across each cut (sometimes called the 'corresponding displacement' for the X_2 and X_3 forces) can be found by the application of the unit load principle which states:-

'The displacement in any direction due to a load system acting on or in a structure is the sum of the deflections in each member of the structure due to that load system multiplied by the load in each member due to a unit load in the direction considered.*

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* The unit load can be supported by any part of the structure able to do so. Thus a statically determinate part of the structure may be used to connect the unit load to the supports. The deflection in each member must be the true deflections when the whole structure is taking the load.

Thus if the final displacements of the members of the 4 bar structure are v_1, v_2, v_3, v_4 and the loads in each member due to a unit load in say the X_2 direction are known the corresponding displacement for X_2 can be found and equated to zero. Now the loads in each member due to a unit load X_2 are b_{112}, b_{122}, b_{132}, b_{142}. The compatibility condition for determining X_2 is, therefore:-

$$b_{112}v_1 + b_{122}v_2 + b_{132}v_3 + b_{142}v_4 = 0$$

There will be a similar condition for X_3

$$b_{113}v_1 + b_{123}v_2 + b_{133}v_3 + b_{143}v_4 = 0.$$

In matrix form the complete compatibility condition will be:-

$$b_1'v = 0 \qquad (2)$$

The relation between the final load in bar 1 say and the final displacement between the ends of the bar can be expressed either in terms of the stiffness or the flexibility of the bar. It is more convenient in this analysis to choose the flexibility f_1 so that:-

$$v_1 = f_1\, S_1$$

Similarly

$$v_2 = f_2\, S_2$$

$$v_3 = f_3\, S_3$$

$$v_4 = f_4\, S_4$$

In matrix form $v = f\, S$ (3)

Where f in this case a diagonal matrix and is called the 'matrix of unassembled flexibilities'.

The compatibility condition now becomes:-

$$b_1'f\, S = 0$$

or $$b_1'f(b_oR + b_1X) = 0$$

i.e. $$b_1'fb_1X = - b_1'fb_oR.$$

$\therefore$ $$X = - (b_1'fb_1)^{-1}\ (b_1'fb_o)R \qquad (4)$$

Equation (4) is often written:

$$X = - D^{-1} D_o R \qquad (4a)$$

where

$$D = b_1' f b_1$$

$$D_o = b_1' f b_o$$

This is a useful notation as D and D_o can be made definite stages in the computer programme.

The stress analysis of the structure is now obtained by substituting equation (4a) in Equation (1).

$$S = (b_o - b_1 D^{-1} D_o) R \qquad (5)$$

The displacement of the structure (r) in the direction of the external load R can also be found from the unit load method since the matrix b_o is a matrix of the loads in all the members due to a unit external load in the R direction and the displacement of all members under the load system R is given by v.

$\therefore \quad b_o' v = r$

i.e. $\quad r = b_o' f S$

$$r = b_o' f (b_o - b_1 D^{-1} D_o) R \qquad (6)$$

In the case of the simple example where there is one external load R, Equation 6 expresses the flexibility of the structure to the external load and the matrix coefficient of R is a scalar. In general where R is a matrix of several external loads each taking several values this coefficient is also a matrix and is usually denoted as F the 'overall flexibility of the assembled structure'.

$\therefore \quad r = FR$

$$F = b_o' f b_o - (b_o' f b_1) D^{-1} D_o.$$

Since f is always a symmetrical matrix $b_o' f b_1$ is the transpose of $b_1' f b_o$

$$\therefore \quad F = b_o' f b_o - D_o' D^{-1} D_o \qquad (7)$$

It can be seen from this introduction to the method that all the static properties of the structure can be determined from matrix manipulation of the three basic matrices b_o, b_1 and f_1. The analyst is now left with the idealisation of the real structures, the choice of the basic system and the choice of the redundant force systems.

The matrix displacement method

This method is a direct parallel with the matrix force method just described and the same 4 bar problem can be used to illustrate the method.

If r is the displacement of the loaded structure in the direction of (and at the point of application of) the external load R the displacements a_{01} etc. of the bars due to a unit value of r can be found and the displacements of each bar (v) written down:-

$$\left.\begin{aligned} v_{01} &= a_{01} \cdot r \\ v_{02} &= a_{02} \cdot r \\ v_{03} &= a_{03} \cdot r \\ v_{04} &= a_{04} \cdot r \end{aligned}\right\} \quad \text{in matrix form } v_o = a_o r \qquad (8)$$

If u is the displacement of the structure at the node point normal to the applied load; the displacements a_{11} etc. of the bars due to a unit displacement in the u direction can be found and the extensions or displacements of the bars again written:-

$$\left.\begin{aligned} v_{11} &= a_{11} \cdot u \\ v_{12} &= a_{12} \cdot u \\ v_{13} &= a_{13} \cdot u \\ v_{14} &= a_{14} \cdot u \end{aligned}\right\} \quad \text{in matrix form } v_1 = a_1 u \qquad (9)$$

$\therefore$ Total displacement of the bars is:-

$$v_1 = v_{01} + v_{11} = a_{01} r + a_{11} u \text{ etc. in matrix form } v = v_o + v_1 \qquad (10)$$

The load in bar 1 due to a displacement v_1 is $S_1 = k_1 v_1$ where k_1 is the stiffness of bar 1.

∴ for all bars: $S = kv$ in matrix form (11)

The equilibrium condition for the structure is that the total force in the u direction is zero.

Now the resolved component of S_1 in the u direction is equal to S_1 multiplied by the same factor as was used to resolve u into the bar 1. direction, i.e. a_{11}:

∴ Total force in the u direction is

$$a_{11} S_1 + a_{12} S_2 + a_{13} S_3 + A_{14} S_4 = a_1' S$$

∴ The equilibrium condition is $a_1' S = 0$ (12)

But $S = kv = kv_o + kv_1$

$$= k\, a_o r + k a_1 u$$

∴ $a_1' S = a_1' k a_o\, r + a_1' k a_1 u = 0$

or $u = - (a_1' k a_1)^{-1}\, a_1' k a_o r$ (13)

or $u = - C^{-1}\, C_o\, r$ (13a)

where $C = a_1' k a_1$

$C_o = a_1' k a_o$

In the particular case of the example chosen u is known to be zero but in general it is given by equation (13a).

From equations (10) and (13a)

$$v = [a_o - a_1 C^{-1} C_o] r \qquad (14)$$

and the matrix of the loads in the members is given by

$$S = kv = k[a_o - a_1 C^{-1}\, C_o] r \qquad (15)$$

in terms of the deflection r, but the external load R is the resolved sum of the loads S in the r direction, i.e.

$$R = a_o'\, S = a_o' k [a_o - a_1 C^{-1}\, C_o] r \qquad (16)$$

But $R = Kr$

Where K is 'overall stiffness of the assembled structure' for the external load system R and is the inverse of the matrix F defined in equation (7).

This matrix K can now be defined in terms of the resolution coefficients a_o and a_1 as

$$K = a_o' k \, a_o - C_o' C^{-1} C_o \tag{17}$$

In order to determine the loads in the members it is necessary to first calculate K, then use equation (16) to calculate r, i.e. $r = K^{-1}R$ and then equation (15) to calculate S.

The advantage of this method lies in its use for frameworks where the a_o and a_1 matrices are simply obtained from geometry. Further the fact that all possible displacements are taken into account in one or other of the matrices reduces the chances of error compared with the force method where the order of redundancy must be estimated and the redundant systems are left to the choice of the analyst. The method can be extended to fixed jointed frameworks and to panelled structures but the possible displacements are large in these cases and the size of computer required increases rapidly.

The Livesley displacement method

In this method an alternative method is used to obtain the overall stiffness matrix of the structure in the directions of the external loads (the K matrix).

It applies mainly to frameworks and although it may be extendable to panelled structures panels are normally replaced by a diagonal rod. As in the Argyris formulation all orthogonal displacements of the members in the structure are considered, the axes being defined for the whole structure. Rotation matrices are used to resolve the displacements along the directions of the members of the structure and these rotation matrices serve also to transform the known stiffness in the direction of the member into an effective stiffness in the direction of the co-ordinate axis. (See Fig. 18). It is clear that the rotation matrices, taken for all members, are equivalent to the a_o and a_1 matrices of the Argyris method. The product of the stiffnesses and deflections in the main co-ordinate directions give the loads exerted by a member at the node point in question. The external load, if any, at the point can now be equated to the sum of these loads:-

$$W_a = T^{-1} k_{AB} T (\Delta_A - \Delta_B)$$

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where:

W_a = Matrix of external loads at point A (in main co-ordinate directions)

T = Rotation matrix.

k_{AB} = Stiffness of member along its own direction

Δ_A = Matrix of displacements of point A. (in main co-ordinate directions).

Δ_B = Matrix of displacements of point B. (in main co-ordinate directions).

The sum is taken over all members converging on point A.

Equations of this type can be built up for all points in the structure and by inserting the known conditions at each point (e.g. zero external load where appropriate, zero displacement at supports), the simultaneous equations can be condensed to form a set equal to the number of external loads. The coefficient matrix of this set of simultaneous equations is the K matrix of the system.

The advantages of the method would appear to be that it builds up the coefficients for each equation using simple small rotation matrices compared with the large a_o and a_1 matrices that have to be prepared for the Argyris method. The disadvantage lies in the condensation of the simultaneous equations which is not automatic in the references seen by the writer. Some combination of the two approaches would be possible by combining the rotation matrix idea into the Argyris method so that the a_o and a_1 matrices can be built up in the computer.

Idealisation of a structure

The Matrix Force method was originally devised for highly redundant continuous stressed skin structures which could be broken down into a rectangular grid of stress carrying panels with stringers at their boundaries. The major assumption was made that the panels only carry shear and that the shear flow in each panel is constant. The stringers are assumed to take only the end load and this end load varies linearly along the stringer. This idealisation offends the principles of shear diffusion and compatibility of the local displacement between the stringer and the panel. These errors are small and can always be made negligible by taking a fine enough grid.

The idealisation has been shown to give very good results for typical aircraft stressed skin construction. It is clear that van bodies and bus bodies can be idealised in this way and the worked example in Appendix 1 is of a simple van body.

For fuselage analysis Argyris and Kelsey, Ref. 23, propose to assume that the frames take bending while the remainder of the structure is limited to end load and shear.

The combining of bending and end loads in a member bordering a shear panel further offends compatibility conditions but it is shown in Ref. 24 that it gives good agreement with measured loads in a structure containing a large cutout or doorway.

The basic matrix force method is not confined to this type of structure and the analysis of a rectangular frame in torsion is given in Appendix 2.

When the idealisation of the structure has been made it is necessary to determine its indeterminacy or order of redundancy.

Very general formulae have been obtained for doing this for frameworks by Henderson and Bickley, Ref. 25, and explained more simply by Morice, Ref. 21. The frameworks may be either fixed or pin jointed or with mixed joints. The principle on which the formulae are based is that any two dimensional ring with rigid joints has three indeterminacies and a three dimensional ring has six indeterminacies.

i.e. $n_s = 3R = 3\,(M - N + 1)$ For plane frames (18a)

and $n_s = 6R = 6\,(M - N + 1)$ For 3-Dimensional frames (18b)

where n_s = No. of indeterminacies for a stiff jointed frame.

R = No. of complete rings in the structure.

M = No. of members in the structure

N = No. of nodes (joints) in the structure.

The supports for the structure must be included in a special way and examples of such stiff jointed frames are shown in Fig. 19. Departures from completely stiff frameworks are dealt with by subtracting a number of 'releases' e.g.: a hinge removing a bending moment across a joint.

Indeterminacy $n = n_s - r$ (19)

where r is No. of releases.

The difficulty in using the method arises in assessing the number of releases in any framework and in guarding against mechanisms ($n < 0$) which may arise in part of a structure while the overall formulae gives a statically determinate ($n = 0$) or redundant result ($n > 0$).

This last point can be checked by treating the structure piecemeal or by ensuring that the equilibrium equations (one for each redundancy) are linearly independent. The matrix force method requires that the D matrix (see equation 4a) represents linearly independent equations so that this condition must be fulfilled.

Equation (18a) can be simply modified for a structure made up of shear carrying panels set in rigid jointed rings of framework

$$n_s = 3\,(M - N + 1) + P \tag{20a}$$

where P is the number of Panels.

Releases are applied only to the surrounding framework.

For the original Agyris assumption of the framework carrying end loads only the number of releases are the same as for a pin-jointed structure.

In a plane structure $r = 2M - N$ (21)

If a bending stiffness is assumed at any joint between the ends of two members this reduces the releases by one.

An example of this is given in Figures 20a and 20b.

The same argument can be applied to a three dimensional structure containing shear carrying panels. In structures like the van body of Appendix 1 each surface acts as a plane redundant structure and to obtain the three dimensional result it is necessary to replace each plane redundant structure by a single panel before applying the formulae.

$$n = 6\,(M - N + 1) + P - r \tag{22}$$

where $r = 5M - 3N$ (23)

It can be seen from Figure 21a that for cube like structures the indeterminacy is zero. An example of a two bay structure is given in Fig. 21b showing an indeterminacy of one.

It is necessary to carry out this procedure before using the matrix force method because although the condition of non-singularity of the D matrix ensures that the redundant systems chosen are independent it does not ensure that the correct number has been chosen to correspond with the idealisation assumed.

To assist in visualising the shear panel end load carrying bar type of structure a form of model has been devised that reproduces most of the characteristics of the idealised structure. The indeterminacy of simple sections of the structure can be demonstrated with this model technique and mechanisms can be avoided as they show up clearly in the model. Basically the model consists of flat bars that can rotate about separate pins at each end, and diamond shaped panels that are slotted where they meet the bars at the apex of the diamond. For a single plane structure the bars clearly constitute a pin jointed system. Since the panels are slotted normal to the edge member bar only a shear force can be transmitted to the panel. The only difference between the model and the idealisation is that in the model the shear force is transferred from the bar to the panel at the mid point of the bar instead of along the length of the bar as a constant sheaf flow. Since the idealisation offends compatibility in this respect it is clearly impossible to make a model to reproduce the theoretical idealisation exactly. Where two planes meet spiders of pins are provided which connect the planes at the bar intersections. The idealisation calls for ball joints of zero radius and the model does not represent this condition accurately. The essential components of the model technique are shown in Fig. 22, and the structure analysed in Appendix 1 is shown built up from model components in Fig. 23.

The Land Rover project

The application of the formulae for determining the order of redundancy of the structure shows that the number 52 assumed in the work reported in Ref. 26, is incorrect.

Most of the work during the period since Ref. 26, was issued has been devoted to an endeavour to find a method of combining the two dimensional and the three-dimensional formulae into a comprehensive formulae for this type of structure. This attempt has failed and while the simple formulae deal adequately with the main box structure aft of the scuttle the front end structure is complicated by having modified egg-box characteristics. The application of the formulae to this section has been checked by an analysis of this structure making various assumptions. The fact that the work reported in Ref. 26, is incorrect has given the opportunity to revise the idealisation of the structure in view of a detailed analysis of the structure in the plane of the bonnet top. The results of the two detailed analyses will be given first followed by an estimation of the order of redundancy of the structure as at present idealised. A diagram of the structure to be analysed in its modified form is given in Fig. 24, showing the node numbering system used.

Simplified analysis of front end structure

The simplified structure was made statically determinate in all possible planes so that the analysis concentrates on the three dimensional egg box effect, it has been shown that indeterminacies in individual planes can be incorporated simply. For this reason the sections of rudimentary chassis which take bending moment only are left out, further, the outside face of the wing is assumed to be a rectangular panel with no cutout for the wheel.

Three cross members are retained across the bonnet top but the bending stiffness of the ring previously suggested is omitted.

The structure appears as in Fig. 25, with an imaginary panel c_o continuing the vehicle floor across the wheel arch. The order of redundancy of this structure for symmetrical loads is shown to be 2.

The panel c_o was removed in the computer analysis by the use of the Argyris cut out procedure given in Ref. 21 and 27 and briefly in Appendix 1. Since the order of redundancy is so low this analysis could be done by pencil and paper methods but it was found that the condensing of the data from the 31 loads defined in the structure to the four coefficients of the two simultaneous equations was time consuming and led to errors. It was, therefore, decided to use Pegasus and to gain experience in using the condensation techniques proposed in Ref. 23, and also the methods of reading in data using zero sub-matrices were developed. Since the matrix division was only concerned with 2 x 2 matrices the computer time for this programme was only a few minutes. The results are shown in Fig. 26.

It was proposed that the basic systems for the final calculation should be made up of a set of systems transferring the loads at the extremities of the cross members to the longitudinals of the rudimentary chassis and each concentrated load on the chassis would then be supported by the axle loads. This method was adopted for the first programme of the simplified front end structure but to ensure its accuracy another programme was run using basic systems that linked each down load with the support points. The results of this programme were identical within rounding off limits and the equivalence of the two systems established. It can be seen from Figs. 27a and 27b that the first method gives much more simple basic systems.

Since the simplified analysis assumed two redundancies and then effectively removed one of them by making a cut out it seemed reasonable that a programme could be written with just one redundancy for the whole front end. Advantage was also taken of the fact that the loading assumed gave no loads in the outside panels b_1 and b_6 or in the remaining floor panel c_1. The structure to be analysed could then be represented as in Fig. 28.

This analysis also gave the same result showing that the front end structure has only one redundancy due to the three cross members in the bonnet top plane.

Structure in plane of the bonnet top

It was originally proposed to idealise the structure in this plane with a stiff ring round the bonnet opening.

A completely stiff ring in one plane has three redundancies and the computation would be simplified if these could be omitted. The structure was analysed in the two forms shown in Figs. 29a and 29b, with 4 redundancies for the first case and 6 in the second (Fig. 30). (The indeterminacy formulae indicate 8 redundancies, but for symmetrical loadings 2 on each side will be identical). A symmetrical load was imposed on each system by assuming unit tension in the member joining points 69 and 69'. This method of loading is not satisfactory in that the unit external load is divided between direct tension in the member (approx. 0.98 of the load) and the load taken by the remainder of the structure. Since the relative flexibilities of these local paths are different the remainder of the structure is only dealing with approx. .02 of a unit load and all loads in the members are small.

Redundant system No. 2 used in the analysis represents the structure in the plane of the bonnet top with no bending stiffness included, and for symmetrical loads it is a sufficient structure. By reducing the load at point 69 in the member 69-44 to the mean value computed for the two redundant structures the comparative table of loads table 3 can be drawn up.

From the table it can be seen that the end loads are within 10% for the three idealisations except for F_{43X} and F_{68X} where the actual loads are small. The errors due to ignoring the corner fixations on the shear loads in panels c_8 and c_9 are approximately 14% but again the actual shear loads are small. The conclusion can therefore be drawn that for symmetrical loading the bending stiffnesses may be ignored for this plane structure.

This simplification allows a reduction of 6 redundant systems and 7 loads to be defined in the structure.

It is clear that for future work on torsion some of these bending stiffnesses will have to be included since the structure is a mechanism for antisymmetrical loading.

Order of redundancy of the structure

It would be instructive to carry out similar detailed analyses of other separate parts of the structure since further simplifications as have been found above may become evident. However, such local analyses are time consuming and it is proposed to revert to the original programme of drawing up an analysis for the complete structure under symmetrical loading.

It has already been stated in the section dealing with idealisation that good agreement with experiment has been obtained by assuming the members bordering a cut out constitute a stiff ring. The bending stiffness of the joints is assumed infinite and the Engineers theory of bending stiffness assumptions applied to the members ignoring the effect of the shear panel. This idea is extensively used in the side view of the proposed Land Rover. It provides a simple way of dividing up a structure and allowing for local changes in geometry, e.g. wheel arches. While the essential repetitive simplicity of the matrix force method will never be used on irregular structures like vehicle bodywork it is hoped that this device will go some way to making the writing of programmes standardised.

The order of redundancy of each of the plane structures making up the integral vehicle is given in Figs. 31 to 34. The floor structure consists of members in bending under loads normal to the plane; for this case additional redundancies can be readily seen by making 'cuts' in a simplified diagram, Fig. 35.

The order of redundancy of the complete structure can now be estimated from the sum of the individual sections and including one extra for the three dimensional front end structure. This gives a total of 48 which is effectively the number of simultaneous equations to be solved by the computer.

Conclusions

The literature on the use of digital computers for structural analysis is growing fast. Improvements are continually being made in the methods of analysis used in aircraft design. Some of these later methods are outlined in Ref. 28, where the emphasis is on the use of displacement methods. The larger computers now available make these methods with their more simple programming possible in spite of the large increase in the amount of information to be processed. The swing to displacement methods is not complete and the force method is used by Denke in the same reference to analyse non-linear structures. The editor of Ref. 28, advances the ideas first suggested by Argyris of using force and displacement methods on the same structure and ensuring that the true solution lies between the bounds set by the two methods. He shows that it is necessary to be consistent in the idealisation and the formulation of the equations to ensure the desired result.

No literature has yet been found dealing with the vehicle problem which may be summarised as a combination of fixed jointed frameworks and paralleled structures. The results obtained by Marsden, Ref. 24, indicated that the force method can be simply adapted to give satisfactory results for this type of structure in one case. It is hoped that the simplified Land Rover analysis will confirm this.

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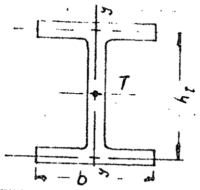
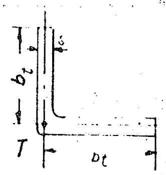
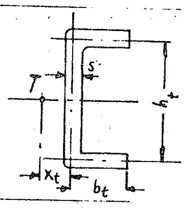


$$\Gamma = \left(\frac{h_t}{2}\right)^2 \cdot I_y$$

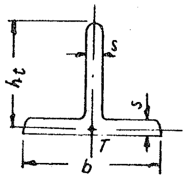


$$\Gamma = \frac{1}{18}\, b_t^3 \cdot s^3$$

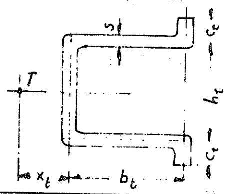


$$x_t = \frac{b_t}{2 + \dfrac{h_t}{3b_t}}$$

$$\Gamma = \frac{s}{3}\left(\frac{h_t}{2}\right)^2 \cdot b_t^2\,(2b_t - 3x_t)$$

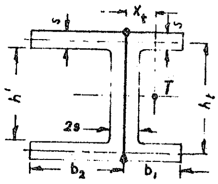


$$\Gamma = \frac{1}{18}\, s^3\left(\frac{b}{2}\right)^3 + \frac{1}{36}\, s^3 h_t^3$$



$$x_t = \frac{3h_t^2 b_t^2 - 8b_t c_t^3 + 6\, b_t h_t^2 c_t}{h_t^3 + 6h_t^2 b_t + 6h_t^2 c_t + 12 h_t c_t^2 + 8c_t^3}$$

$$\Gamma = \frac{sb_t}{12} \cdot \left[2h_t^2 b_t^2 - 3h_t^2 b_t x_t + 6h_t^2 b_t c_t - 6h_t^2 x_t c_t - 12 h_t b_t c_t^2 + 8c_t^3 (x_t + b_t)\right]$$



$$x_t = \frac{b_2^2 - b_1^2}{2(b_1 + b_2) + \dfrac{2}{3} \cdot \dfrac{h'^3}{h_t^2}}$$

$$\Gamma = \frac{s}{12} h_t^2 \cdot \left[2(b_1^3 + b_2^3) - 3x_t\,(b_2^2 - b_1^2)\right]$$

TABLE 1

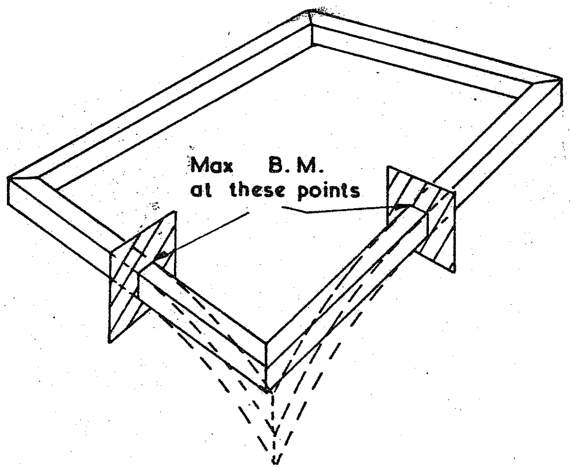
FRAME	TORSIONAL RIGIDITY in ft/lbs/rad			DIFFERENCE $S_2 - S_3$	
	CALCULATION		TEST		
	Uninhibited warping. S1	Totally inhibited warping S2	S3	in ft lbs	in % of calculation
1	1512	9661	9406	255	2·6
2 ⓐ	1432	9973	8828	145	1·6
2 ⓑ	1975	10530	9154	976	9·6
3 ⓐ	1751	13530	12910	620	4·5
3 ⓑ	1817	16350	14690	1660	10·2

TABLE 2.

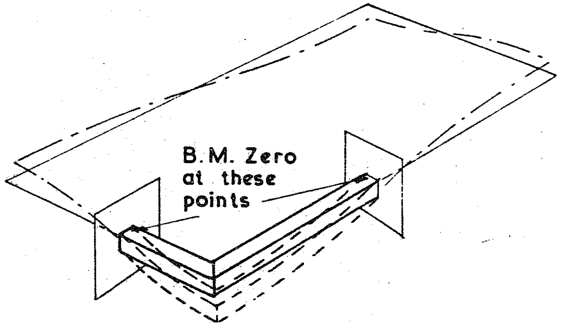
TABLE 3

Comparison of loads in bonnet top for three idealisations

Loads Defined in the Structure	Non-redundant Basic System 2	Bonnet top 1	Bonnet top 2
$F_{69-69'y}$	+ 1	+ .983	+ .983
M_{69z}	+ 0	+ .000199	+ .000208
M_{68z}	+ 0	+ 0	- .0000745
M_{67z}	+ 0	+ 0	+ .0000490
M_{66z}	+ 0	- .000897	- .0000923
F_{65y}	- .0278	- .0291	- .0293
F_{66-41y}	+ .0445	+ .0460	+ .0459
F_{67y}	+ 0	+ .0000089	+ .0000264
F_{68y}	+ 0	- .0000199	- .0000405
F_{69-44y}	- .0167	- .0169	- .0166
F_{41x}	- .0295	- .0308	- .0310
F_{42x}	- .0197	- .0210	- .0215
F_{43x}	- .0098	- .00924	- .00859
F_{66x}	+ .0295	+ .0308	+ .0310
F_{67x}	+ .0197	+ .0211	+ .0215
F_{68x}	+ .0098	+ .00924	+ .00859
C_6	- .00164	- .00171	- .00172
C_7	+ .00098	+ .000975	+ .000945
C_8	+ .00098	+ .000925	+ .000861
C_9	+ .00098	+ .000924	+ .000858
$F_{66-66'y}$	+ .0445	+ .0460	+ .0459



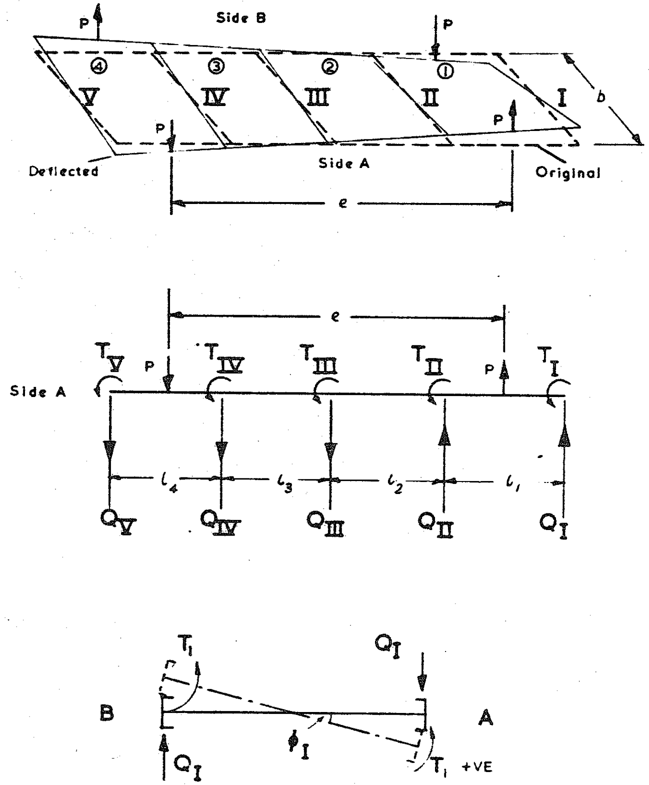
DEAN-AVERNS ASSUMPTION



CORRECT ASSUMPTION

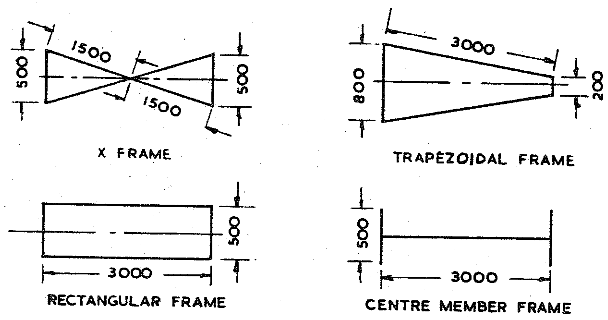
Inflection points at centre of sides

FIG. I



FRAME ANALYSIS - INFINITE BENDING STIFFNESS

FIG. 2



RELATIVE TORSIONAL RIGIDITY

EQUAL WIDTH FRAMES

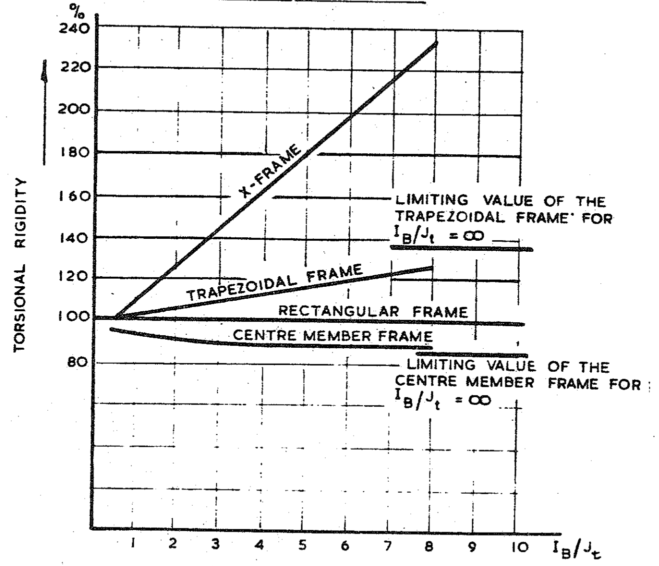
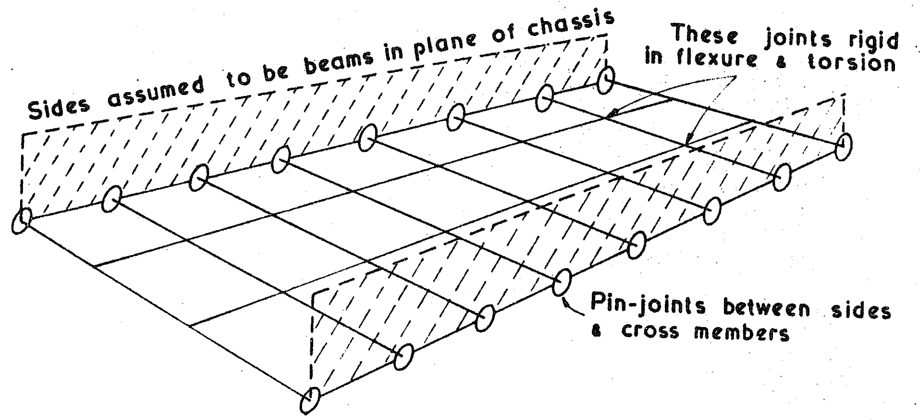
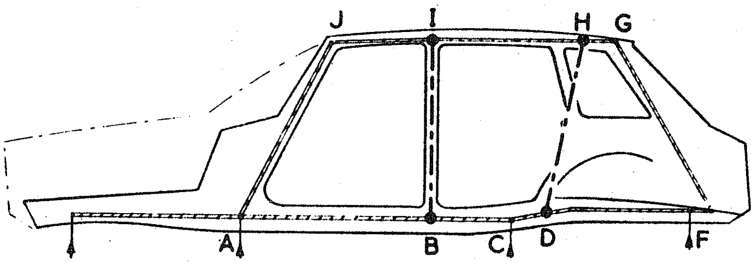


FIG 3



MICHELBERGER'S REPRESENTATION OF A SEMI-INTEGRAL VECHILE

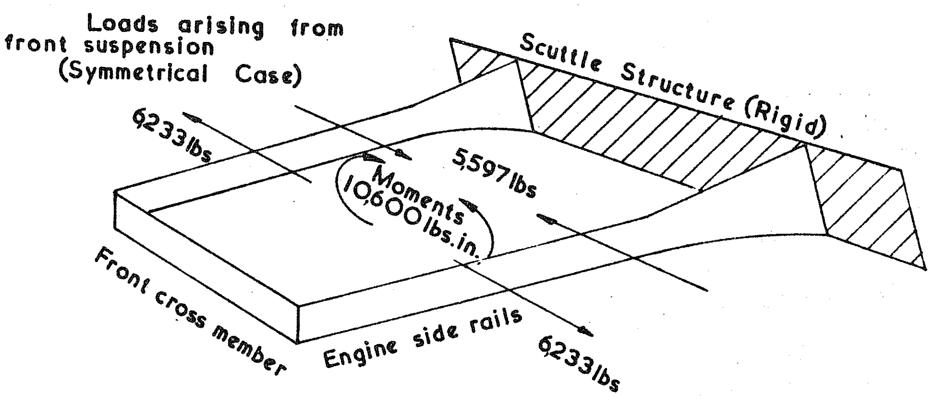
FIG. 4



MEMBERS BI & DH PIN JOINTED, ALL OTHERS RIGID.

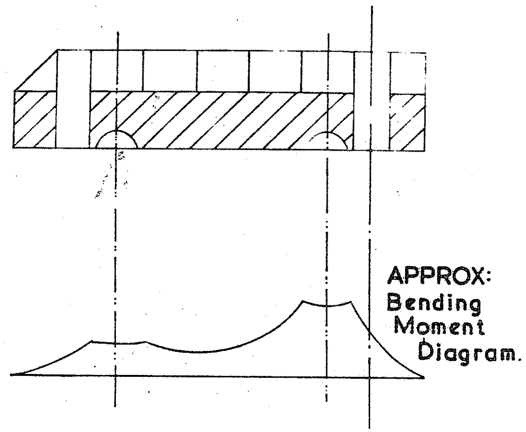
SYSTEM HAS 5 INDETERMINACIES AS A PLANE FRAME.

FIG 5



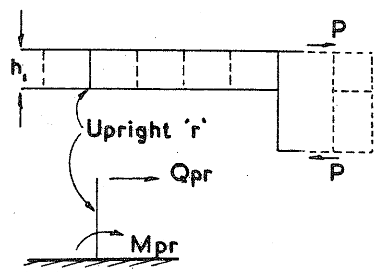
GARRETT'S SCHEME FOR ANALYSIS OF FRONT END STRUCTURE HORIZONTAL LOADS ONLY

FIG. 6

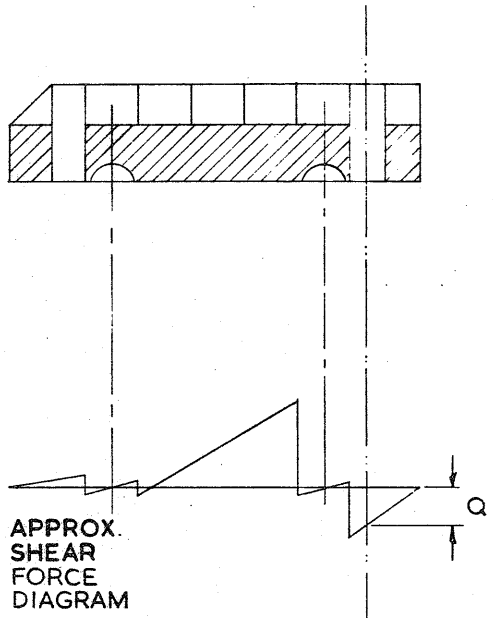


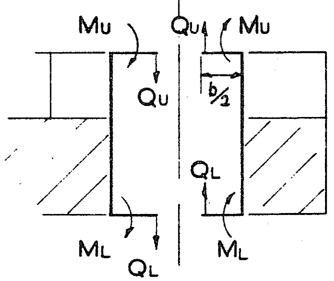
$$Q_{pr} = \frac{I_r}{\Sigma I_r} P$$

$$M_{pr} = \frac{2}{3} Q_{pr} h_1$$



COACH STRESS ANALYSIS ERZ.
LOADS DUE TO BENDING
FIG 7



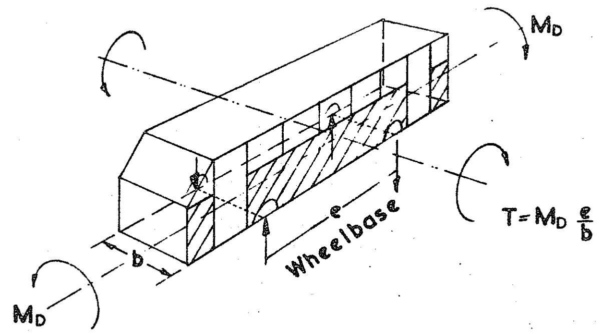


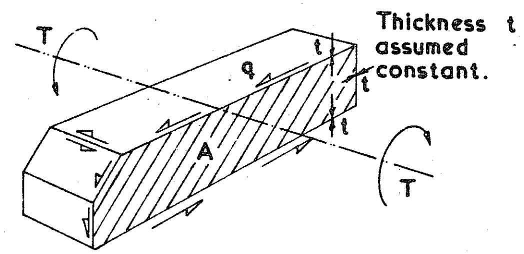
$$Q_U = \frac{I_U}{I_U + I_L} Q$$

$$M_U = 0 \cdot 75\, Q_U b$$

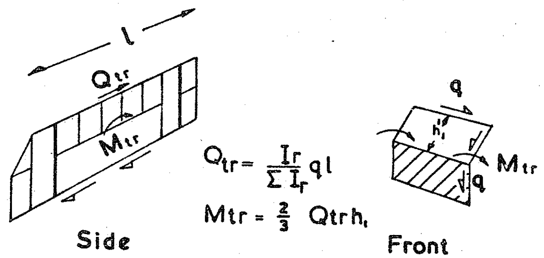
FIG 8

COACH STRESS ANALYSIS - ERZ
LOADS DUE TO SHEAR





Resisting Shear Flow $q = \frac{T}{2A}$

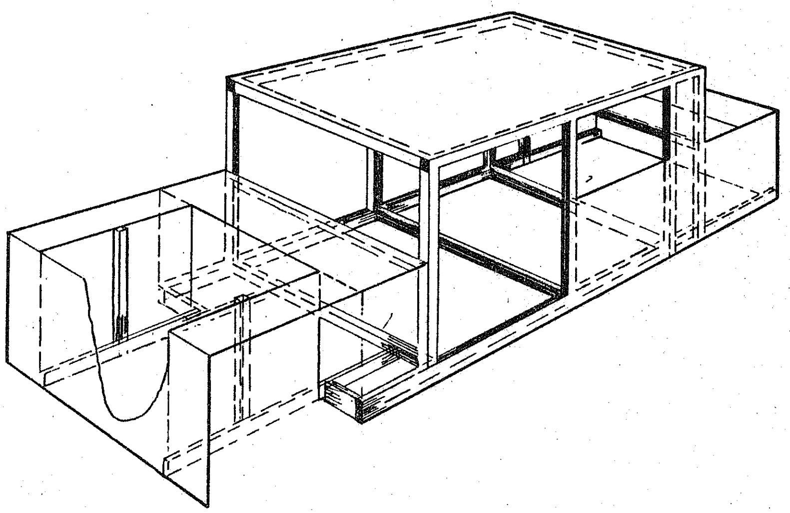


$$Q_{tr} = \frac{I_r}{\Sigma I_r} q l$$

$$M_{tr} = \frac{2}{3} Q_{tr} h_1$$

Side
Front

COACH STRESS ANALYSIS - ERZ
LOADS DUE TO TORSION - FIG 9



MODEL BODY STRUCTURE – MATRIX FORCE ANALYSIS

FIG 10

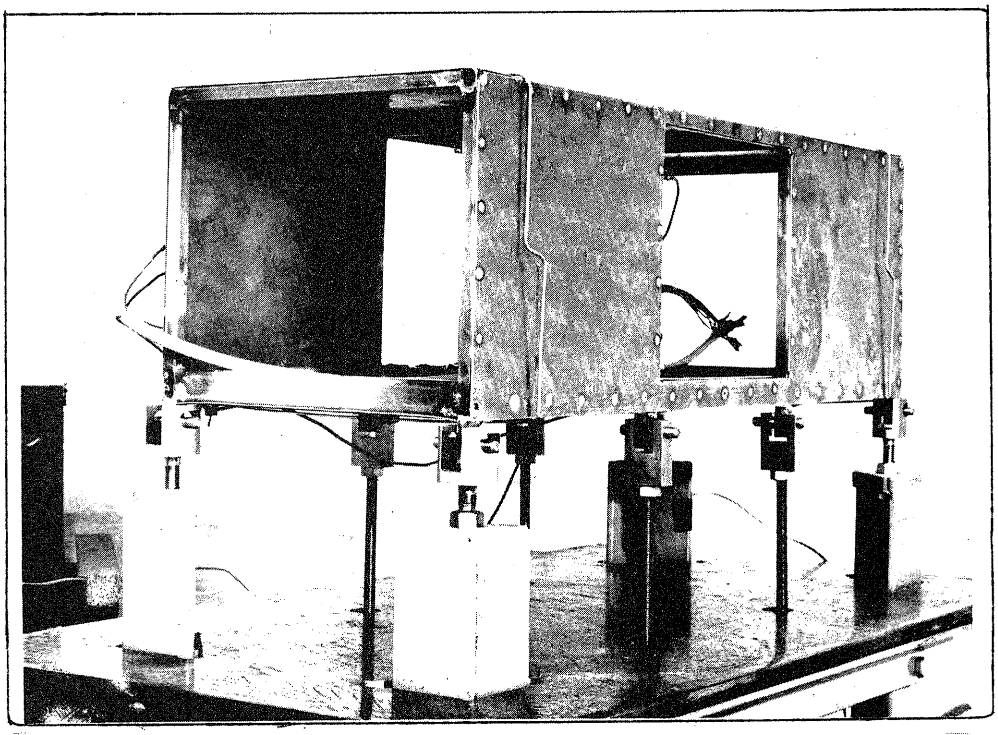


FIG. 11 STRUCTURE WITH A LARGE CUT-OUT
MATRIX FORCE ANALYSIS

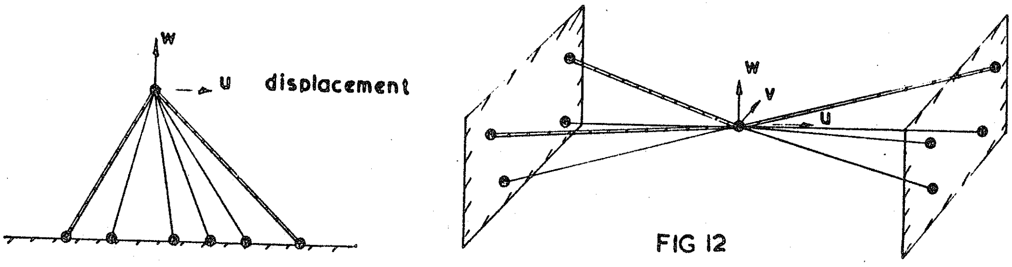


FIG 12

TWO DIMENSIONS

Possible node displacements 2

Degree of redundancy 4

THREE DIMENSIONS

Possible node displacements 3

Degree of redundancy 5

EXAMPLES FOR DISPLACEMENT METHOD

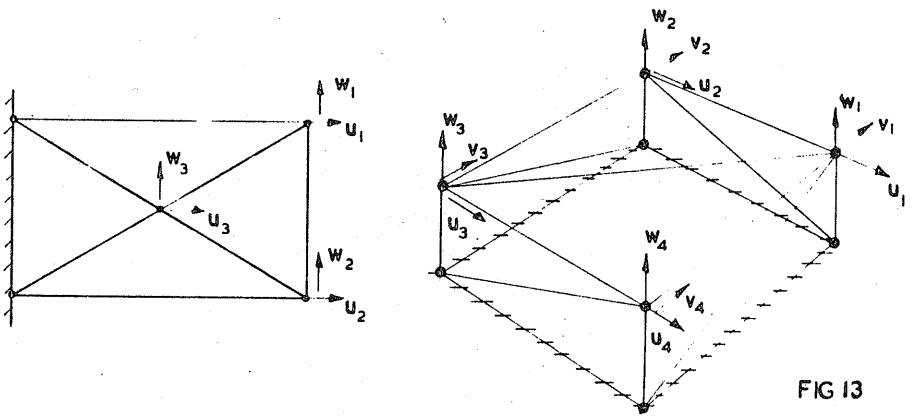


FIG 13

TWO DIMENSIONS

Possible node displacement 6

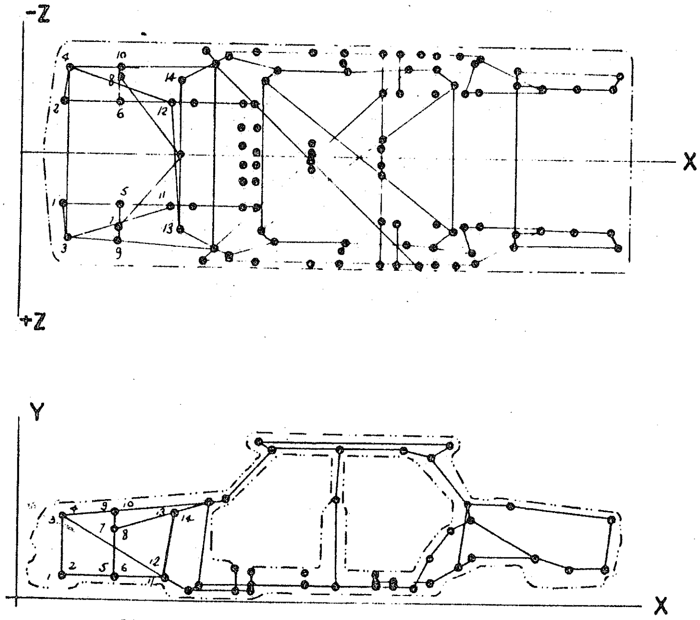
Degree of redundancy 1

THREE DIMENSIONS

Possible node displacement 12

Degree of redundancy 1

EXAMPLES FOR FORCE METHOD



Total number of node points = 99

McKENNA DISPLACEMENT METHOD

FIG 14

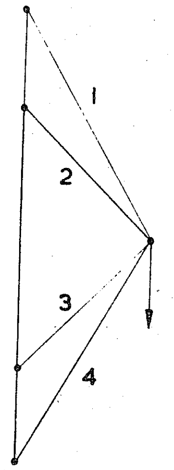
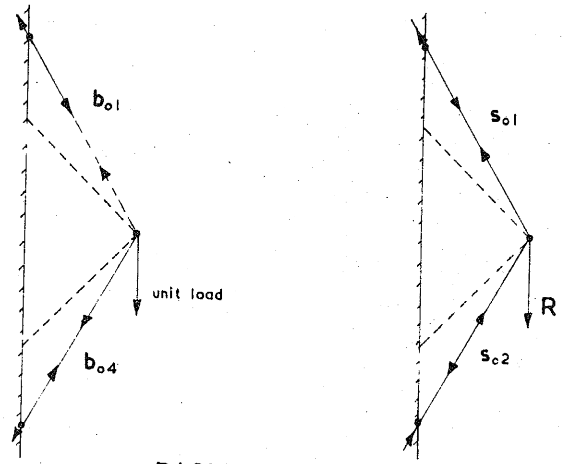
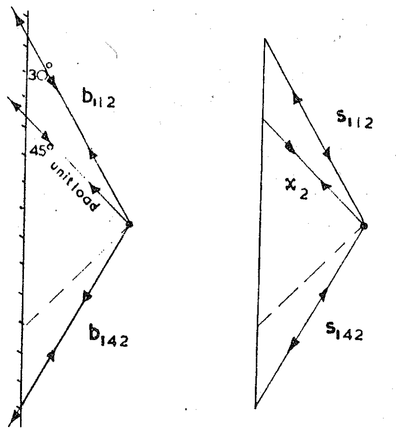


FIG 15 a

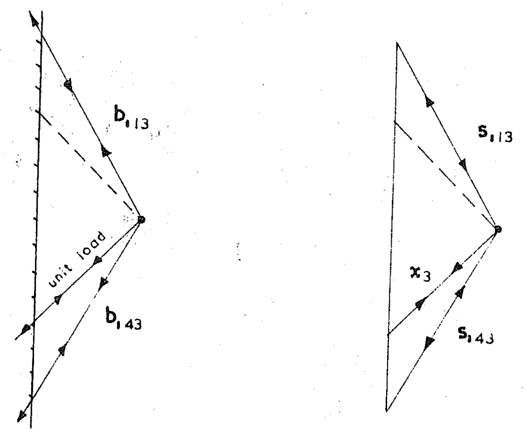


BASIC SYSTEM

FIG 15 b

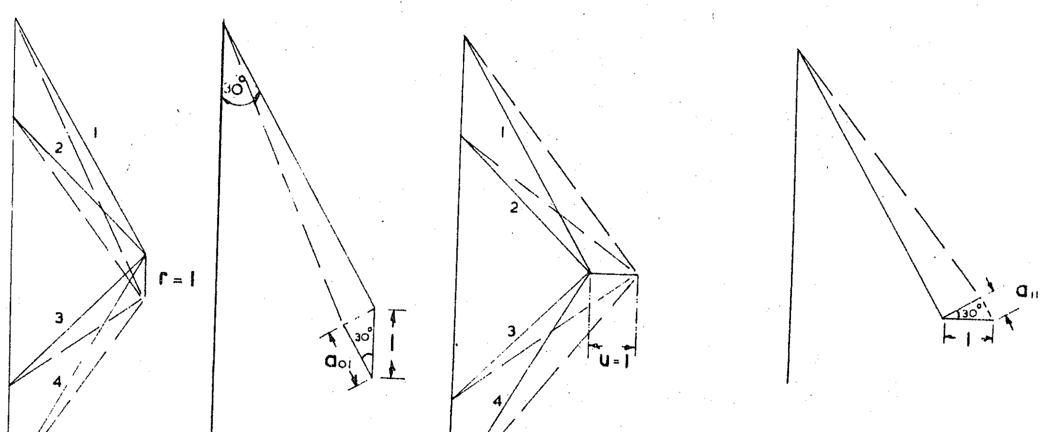


FIRST REDUNDANT SYSTEM



SECOND REDUNDANT SYSTEM

FIG 16

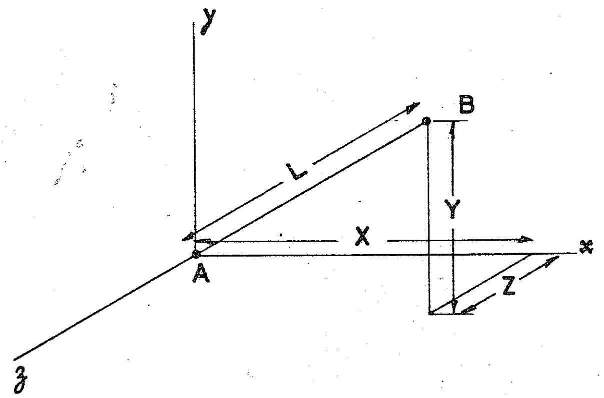


$a_{11} = \sin 30°$

$a_{01} = \cos 30°$

DISPLACEMENT METHOD

FIG 17



x, y, z — Main Co-ordinate Directions

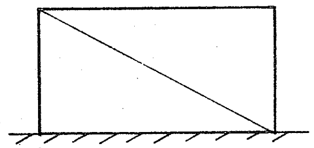
For a pin jointed member AB.

$$T^{-1} k\, T = \frac{EA}{L^3}\begin{bmatrix} X^2 & XY & XZ \\ XY & Y^2 & YZ \\ XZ & YZ & Z^2 \end{bmatrix}$$

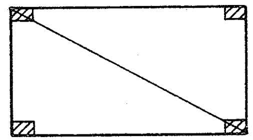
E = Youngs Modulus
A = Cross Section Area
} of member AB

LIVESLEY DISPLACEMENT METHOD

FIG 18



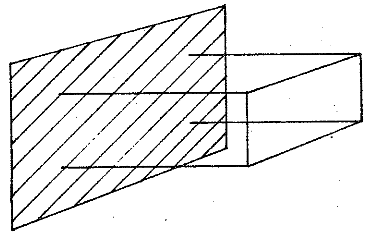
Actual Structure



Representation

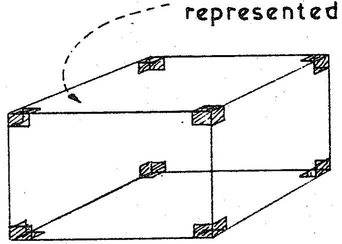
$M = 5$
$N = 4$
$n_s = 6$

Example of a Plane Frame



Actual Structure

The built-in wall must be represented by singly-connected stiff members.



Representation

$M = 11$
$N = 8$
$n_s = 24$

Example of a 3-Dimensional Frame

ORDER OF REDUNDANCY OF STIFF JOINTED FRAMEWORKS.

FIG. 19

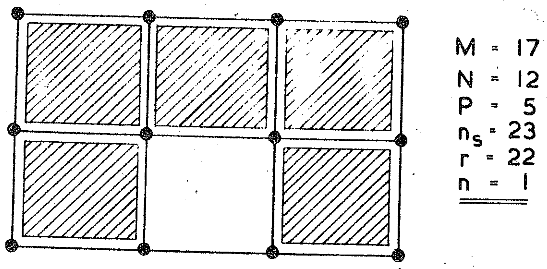


FIG 20a

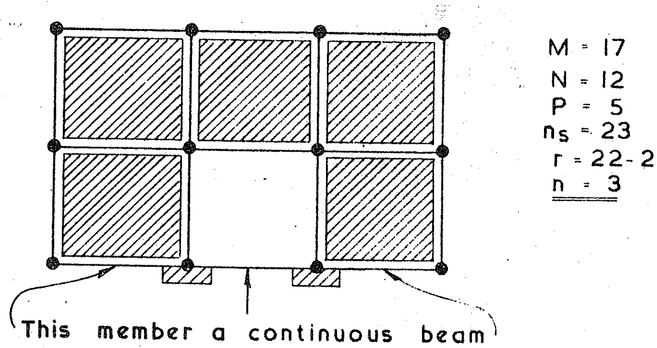


FIG 20b

Order of Redundancy of Plane Panelled Structures.

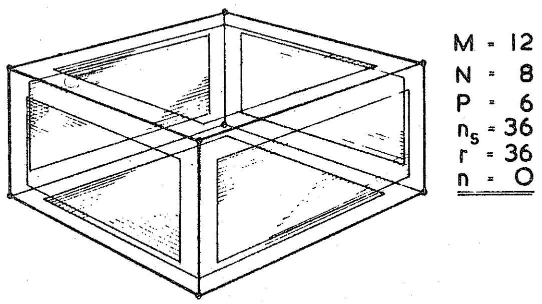


FIG 21a.

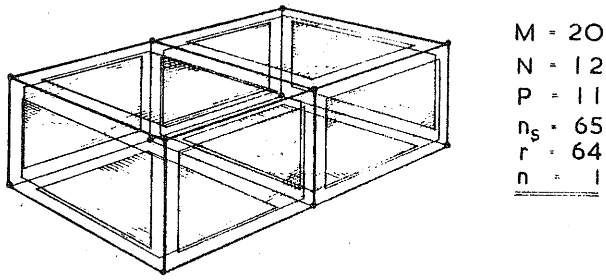


FIG 21b.

Order of Redundancy of 3-Dimensional Panelled Stuctures.

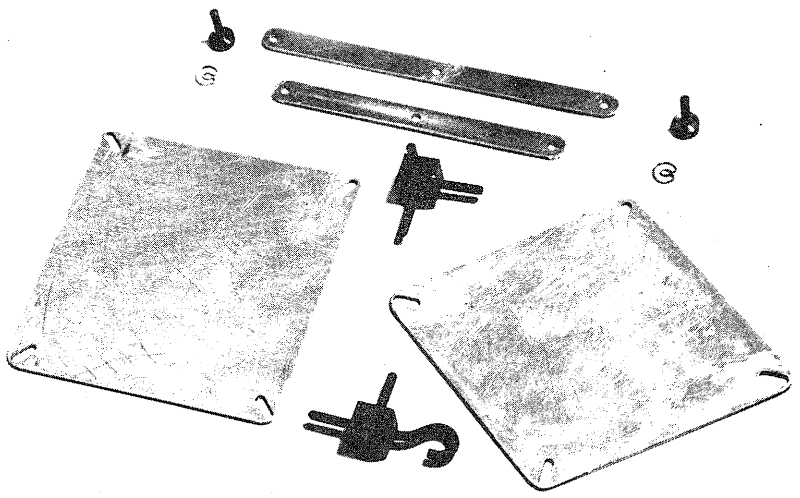


FIG. 22 COMPONENTS FOR ASSEMBLING MODELS OF SHEAR PANEL-END LOAD MEMBER STRUCTURES

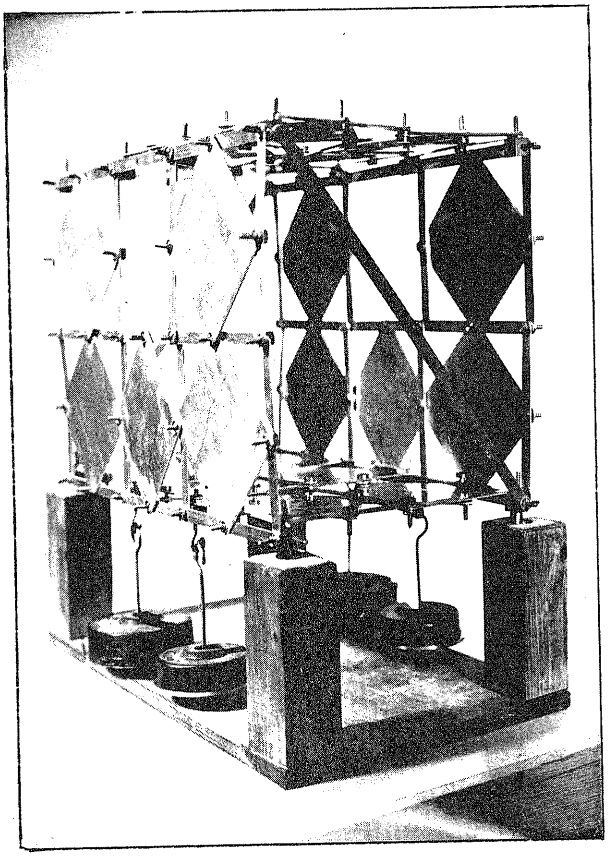
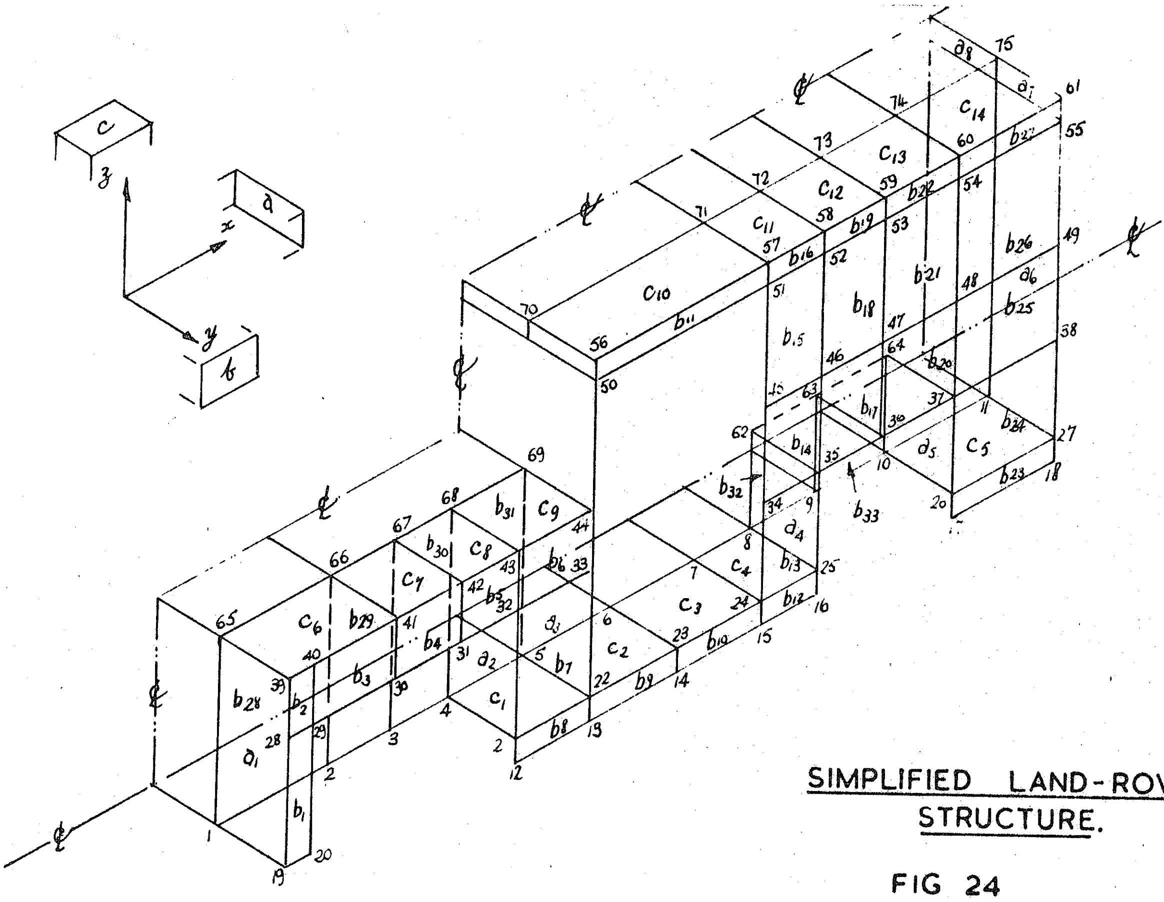
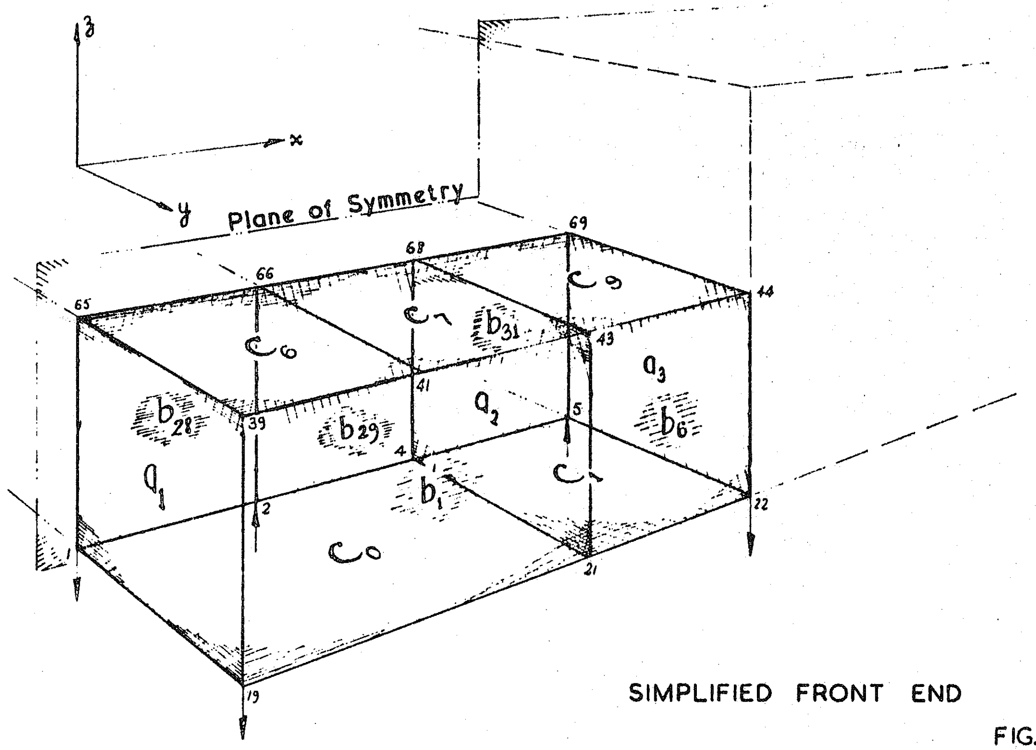


FIG. 23 BOX VAN WITH WINDOW STRUCTURAL MODEL OF IDEALISATION.



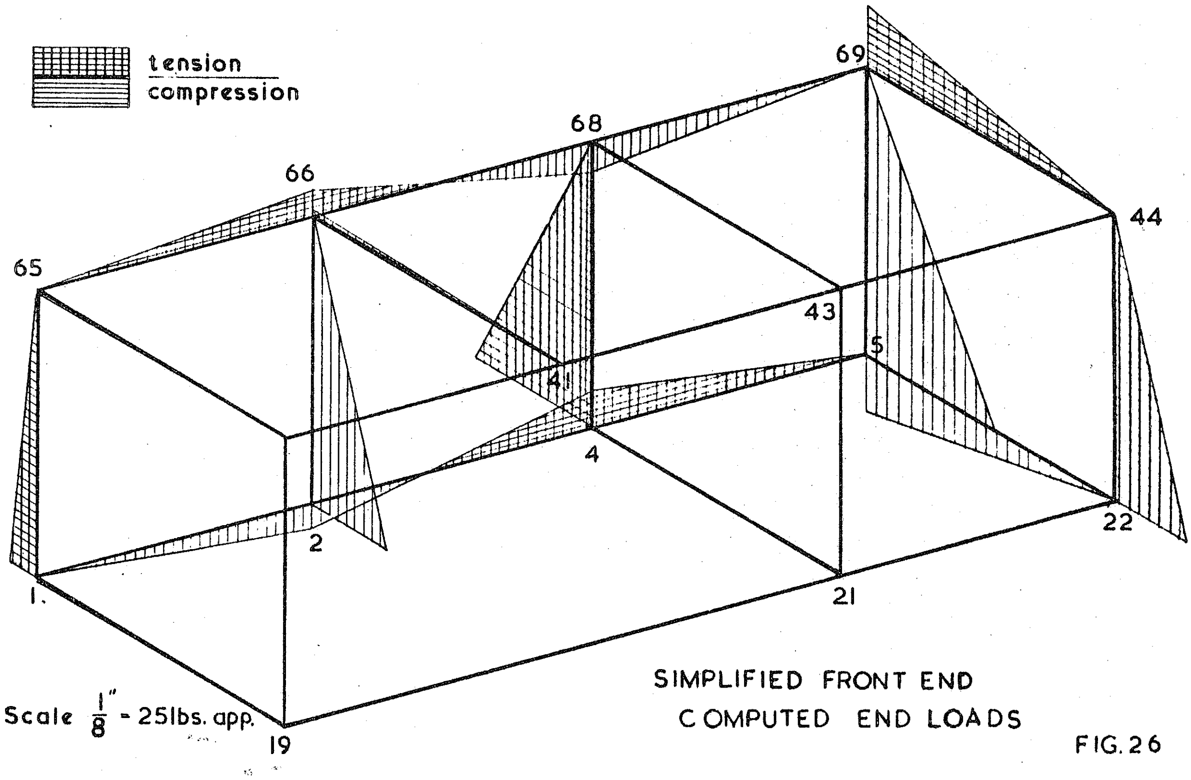
SIMPLIFIED LAND-ROVER STRUCTURE.

FIG 24



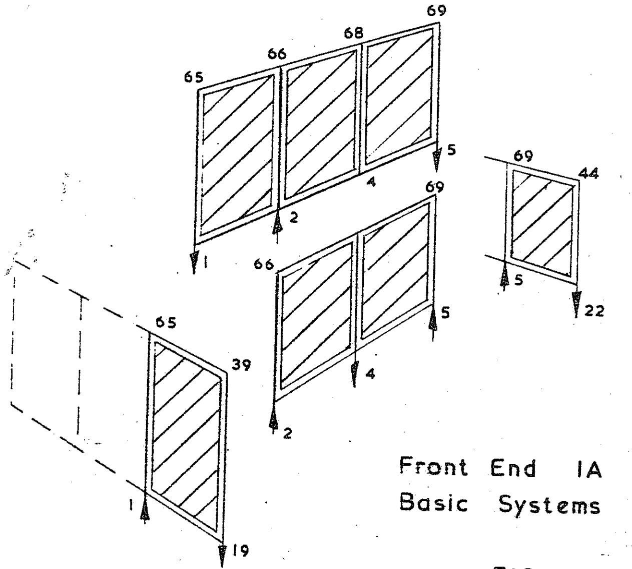
SIMPLIFIED FRONT END

FIG. 25



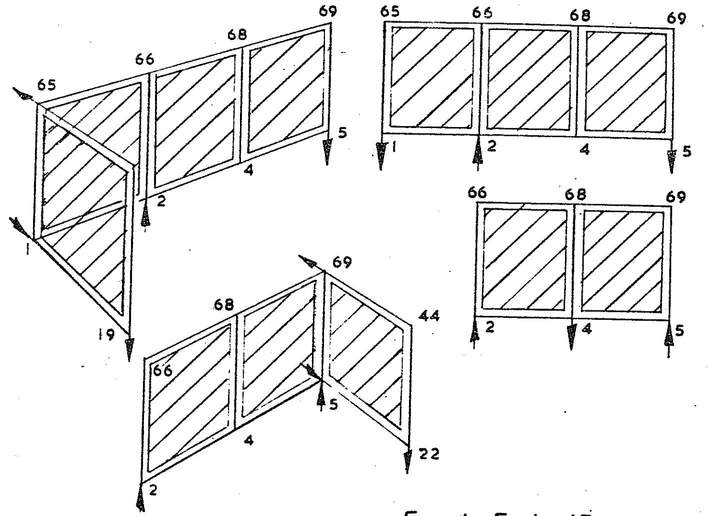
SIMPLIFIED FRONT END
COMPUTED END LOADS

FIG. 26



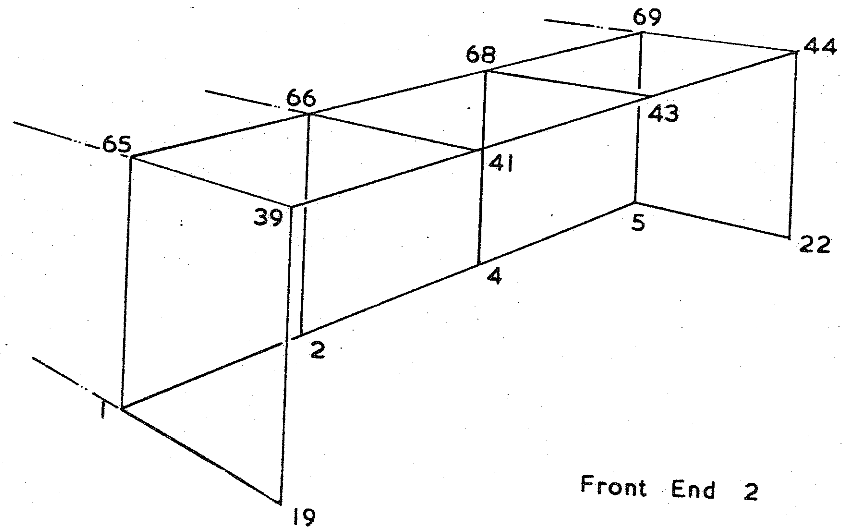
Front End 1A
Basic Systems

FIG 27a



Front End 1B
Basic Systems

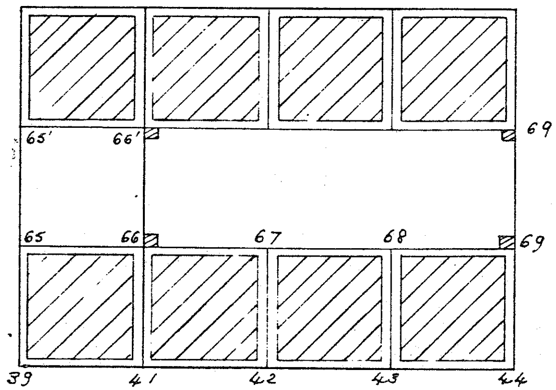
FIG.27b



Front End 2

FIG 28

Bonnet Top 1



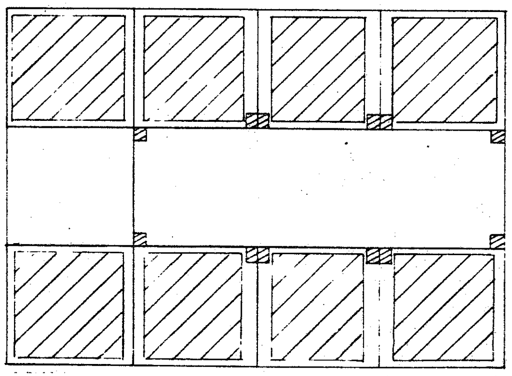
M = 29 N = 20 P = 8

$n_s = 3(M - N + 1) + P = 38$

$r = 2M - N - 4$ Bending Fixations = 34

$n = 4$

FIG. 29a



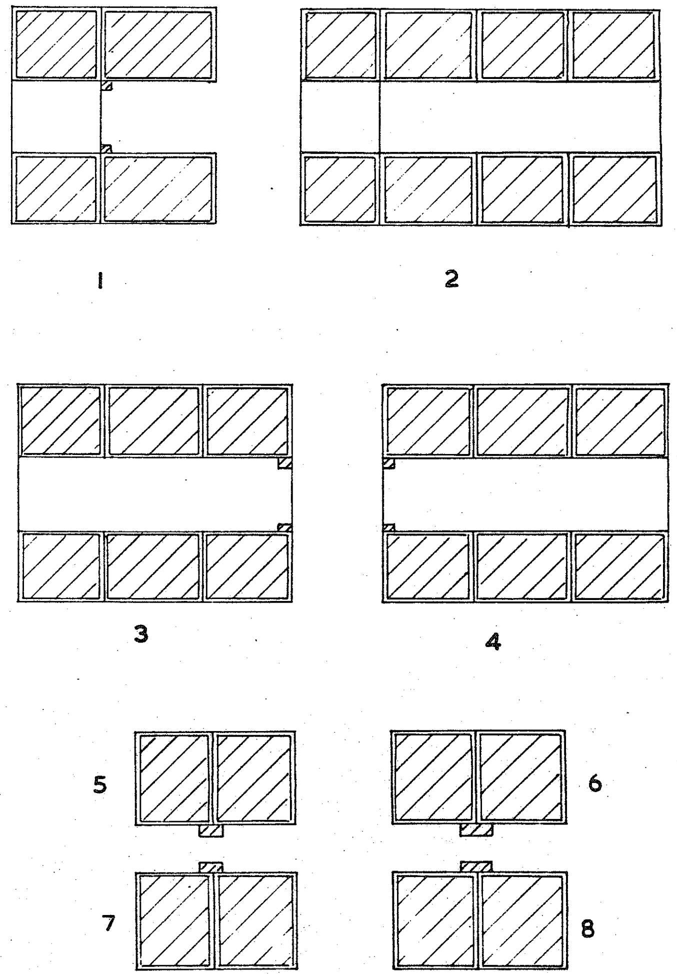
$n_s = 38.$

$r = 30.$

$n = 8.$

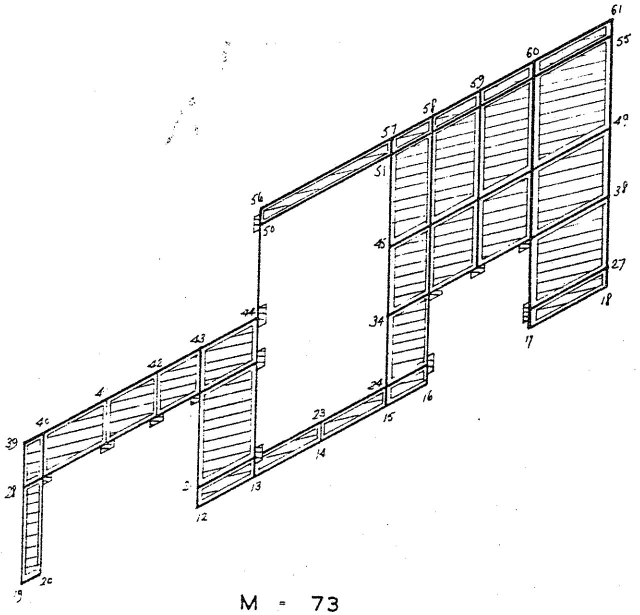
FIG. 29b

Bonnet Top 2



BONNET TOP REDUNDANT SYSTEMS

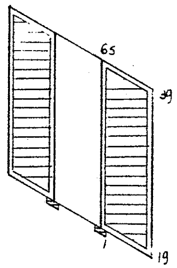
FIG. 30



M = 73
N = 48
P = 26
n_s = 104
r = 85
n = 19 (Order of Redundancy)

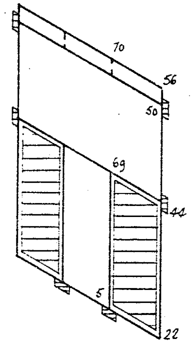
BODY SIDE STRUCTURE - IDEALISATION

FIG.31



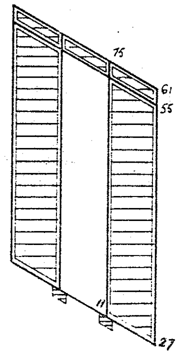
M = 10
N = 8
P = 2
n_s = 11
r = 10
n = 1

Front Cross Section



M = 16
N = 12
P = 2
n_s = 17
r = 14
n = 3

Scuttle Cross Section

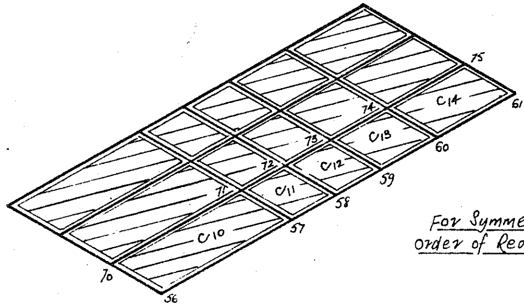


M = 17
N = 12
P = 5
n_s = 23
r = 20
n = 3

Rear End Cross Section

ORDER of REDUNDANCY of TRANSVERSE PLANE STRUCTURES

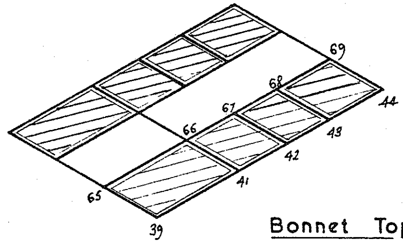
FIG. 32



M = 38
N = 24
P = 15
n_s = 60
r = 52
n = 8

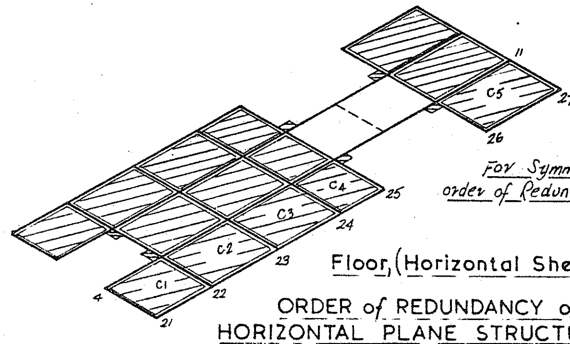
For Symmetrical Loads order of Redundancy = 4.

Roof Panel & Stiffners



M = 29
N = 20
P = 8
n_s = 38
r = 38
n = 0 (Symmetrical Loads only)

Bonnet Top.



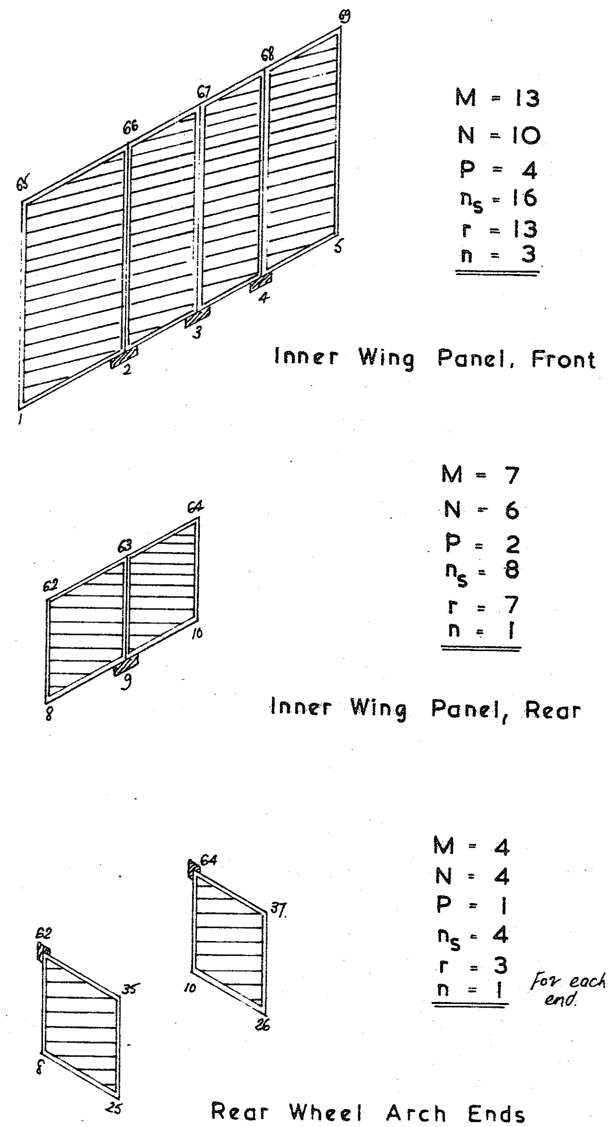
M = 42
N = 28
P = 15
n_s = 60
r = 50
n = 10

For Symmetrical Loads order of Redundancy = 5.

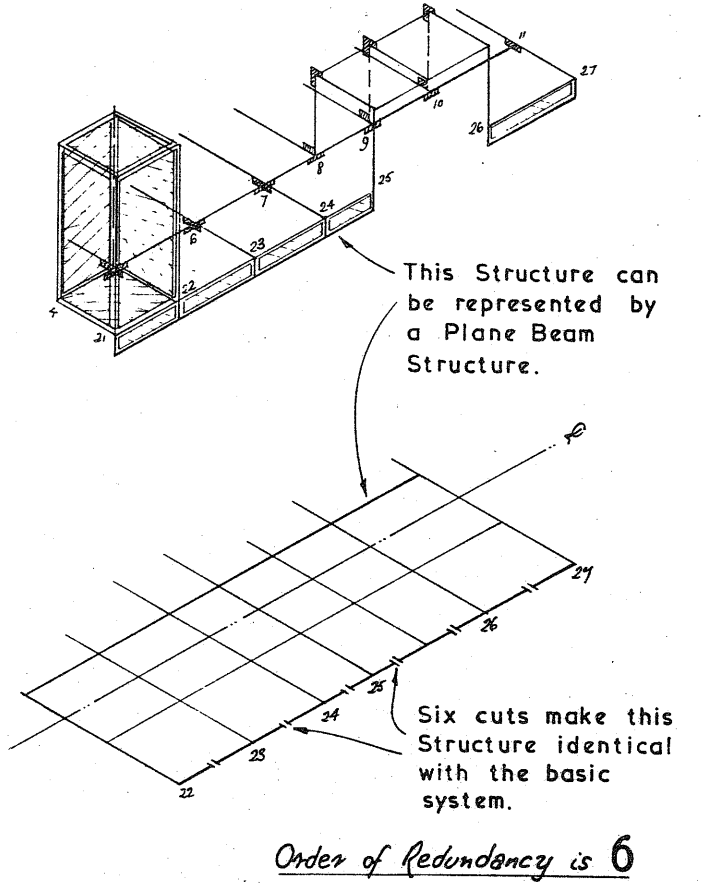
Floor, (Horizontal Shear Only)

ORDER of REDUNDANCY of HORIZONTAL PLANE STRUCTURES.

FIG. 33



ORDER of REDUNDANCY of WHEEL ARCH PLANE STRUCTURES. FIG. 34



ORDER of REDUNDANCY of FLOOR BENDING MEMBERS

FIG.35

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-XIV.1 2 2

Appendix 1

Matrix force analysis of an idealised van body

The structure to be analysed is shown in Fig. 1. It will be seen that the structure is symmetrical and is symmetrically loaded about a vertical plane through the longitudinal centre line. The structure itself is doubly symmetrical and it would be possible to analyse one quarter. The loading is not symmetrical about the lateral centre line and while this could be allowed for by combining symmetrical and unsymmetrical longitudinal loading it is physically more simple to understand the analysis of half the structure with two separate loads applied R_1 and R_2. The structure is an example of the end load carrying edge member, shear carrying panel type referred to in the report and the window will be allowed for by applying the Argyris cut-out technique.

This assumes that a panel of the same gauge and material as the surrounding panels is assumed to be in place of the window for the main analysis, a modification is then made such that the final load in the panel is zero.

Numbering system and sign convention

No attempt is to be made in this example analysis to use condensation techniques which avoid specifying a given load more than once in the matrices, it is, therefore, necessary to number the members in some arbitrary sequence. The chosen system is shown in Figs. 2 and 3. Since there may be a different load at each end of the edge members, the ends are numbered as shown, i.e. each load in the structure or row in the S matrix will have a bar number and an end number. The shear panels only have one load and consequently occupy only one row each in the S matrix.

The sign convention is equally arbitrary and is shown in Fig. 4. The essential points about this convention are (a) the loads acting on a member are always drawn and appear in the matrices, (b) the loads are always drawn $+^{ve}$ and if they carry a negative sign act in the opposite direction to the arrow. The convention can be expressed in words as follows:-

1. A load applying tension to one end of a bar is $+^{ve}$

2. A shear flow is $+^{ve}$ when the force acting on the edge furthest from an axis is acting in the $+^{ve}$ direction of that axis, e.g. for a panel in the x,y plane in positive shear the edge of the panel furthest from the axis (most $+^{ve}$ value of x) will be acted on by a force in the +y direction.

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Basic system

Assuming that it is stable the simplest section of the structure that can support the external load is the lower half of the body side.

This is shown in Fig. 5 with the loads necessary for equilibrium with the external load system equivalent to $R_1 = k\ (2\ell_1 + \ell_2)$.

This value of R_1 is chosen rather than unity to avoid writing fractions at every point in the diagram. When numbers are substituted the division has to be made in each case or the external load R_1 may be divided by $h\ (2\ell_1 + \ell_2)$. The basic system is statically determinate and in order to obtain the load values shown in the figure it is only necessary to start at one corner say rod c_1 end 1, where the support load $h(\ell_1 + \ell_2)$ is inducing a compression of that amount. It is evident that the load end 2 of c_1 must be zero as there is no reacting structure and the shear flow in panel e_1 is thus determined.

If q is shear flow in e_1

$$q\ h = -\ h(\ell_1 + \ell_2) + 0$$

$$\therefore \quad q = -\ (\ell_1 + \ell_2) \text{ etc.}$$

The other basic system is similar and is shown in detail in Fig. 6. The b_o matrix can now be written assuming the rows are allocated in consecutive order, although only 15 of the 74 load defining rows have entries the b_o matrix is a 74 row by 2 column matrix and zeros must be put in where necessary, see table 1.

Order of redundancy

The structure as idealised is a mechanism for pin joints, the ends are therefore assumed to be stiff in shear to stabilise the structure.

It has been shown that a cuboid of this nature is simply stiff and the order of redundancy will be the sum of the orders for the three plane structures, (roof, floor and side).

The opposite side should also be included but will clearly behave the same as the side being analysed.

For each face, M = No. of bars = 17

N = No. of nodes = 12

P = No. of panels = 6

$\therefore$ For a stiff structure $n_s = 3\ (M - N + 1) + P = 24$

For pin joints, releases, $r = 2M - N = 22$

$\therefore$ Order of redundancy $= n = 2$

The total redundancies in the analysis will therefore be 6 and the b_1 matrix will have 74 rows by 6 columns.

Redundant systems

The simplest single plane self equilibrating system whereby all members in the system have forces induced but have zero reactions on the remainder of the structure is the four panel X system of Argyris. The whole structure is divided into the six overlapping systems shown in Fig. 7, the floor and roof systems are not standard in that only two of the four panels exist in the analysis but the presence of the missing structure can be allowed for. The load distribution in system X_3 is shown in Fig. 8, again the load in the 'starting' panel e_1 is chosen as $+\ell_2$ instead of + 1, to avoid fractions. The equations will solve correctly for the unknown X_1 .... X_6 for any base value of the load system so that it is not necessary to correct for the change from + 1 to $+ \ell_2$, but to preserve the simplicity of the example this has in fact been done when substituting figures.

The loads in redundant system X_4 will be similar to X_3. Remembering that X_5 is the same as X_1 and X_6 the same as X_2 the b_1 matrix can be compiled. (The loads in X are given in Fig. 9). It will be noted from table 2 that this is done in partitioned form, a separate matrix is written for all 'a' members, etc.

Partitioning is necessary as the final matrix to be compiled the 'flexibility matrix of the unassembled members' is square and would require 74 rows and 74 columns if used in full.

Such a matrix requires 74 x 74 = 5,476 storage spaces in the computer out of the 6,142 spaces available so that little else could be stored at the same time. The partitioned f matrix is made up as follows:-

$f_a = 30 \times 30 = 900$
$f_b = 16 \times 16 = 256$
$f_c = 16 \times 16 = 256$
$f_d = 6 \times 6 = 36$
$f_e = 6 \times 6 = 36$
Total 1484 spaces

Since the five sub matrices are not required in the computer store at the same time, further savings are possible and the whole calculation can be performed with less than 1500 spaces.

The Flexibility Matrix

As referred to above this is strictly a matrix of the flexibilities of the unassembled members since the D and D_o matrices referred to in the theory are strictly flexibility matrices as is the F matrix or the overall flexibility of the structure to the external load.

The displacement of end 1 (δ_1) of a bar having an end load P_1 at end 1 and P_2 at end 2 is

$$\delta_1 = \frac{1}{6AE}(2P_1 + P_2) = f_{11}\,P_1 + f_{12}\,P_2$$

$\therefore$ The flexibility matrix for the bar a_1, say, is

$$fa_1 = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}_{a_1}$$

where $f_{12} = f_{21}$ by Maxwells reciprocal theorem.

The flexibility matrix for the bars consists therefore of a main diagonal f_{11}, f_{22}, ... etc. and two, equal, sub-diagonals with every other space zero. This is only true as long as the rows corresponding to each end of one bar are kept next to one another and it is normally advisable to do this. A device exists for reading in this type of square matrix into the computer without punching all the zeros in the data type, this device is indicated in the programme.

Some of the panel adjacent to each bar will contribute to the end load stiffness of the bar and this is normally taken as $^1/_6$ of the cross section area, see Fig. 10. Where there is no actual bar this panel area is taken to be an equivalent bar. Where the panel is to be cut out for the window no area allowance is made for that panel.

The flexibility of a shear carrying panels is only one term

$$f_{d1} = \frac{\text{Area}}{\text{G x thickness}} = \frac{l_1 n}{G\ t}.$$

Vehicle Data

As the analysis is to be performed on the computer it is necessary to substitute numbers for the formulae so far used. The following values have been assumed.

Dimensions

Total length	= 14 ft.	ℓ_1	= 60 inches
		ℓ_2	= 48 inches
Width	= 7 ft. 6 ins.	n	= 45 inches
Height	= 10 ft.	h	= 60 inches

Material. Aluminium Alloy of a quality used in commercial vehicle body building is assumed.

2% Proof Stress = 19 - 20 tons/sq.in.	
Side and Roof Panels, 16 S.W.G.	t = .064 inches
Floor Panels $^1/_8$" Plate	t = .125 inches
Floor sides, centre rail and cross bearers	A′ = 1.44 sq. ins.
Waist rail and roof centre rail, top of window frame, intermediate verticals and roof cross members	A′ = 0.295 sq.ins.
End verticals and roof crossmembers	A′ = 0.73 sq.ins.

There are no roof contrails except where the window frame occurs, end load carrying members are made solely from the adjoining panels.

Loading Cases. Total Load (including structure) 5 tons with a load factor of 3g.

(a) Load evenly distributed over the whole body:-

$$R_1 = R_2 = 2.41 \text{ tons.}$$

(b) 2/3 of load evenly distributed in the front half of the body, the remainder evenly distributed in the rear half.

$$R_1 = 3.21 \text{ tons} \quad R_2 = 1.61 \text{ tons.}$$

Programme

The programme is written using the Pegasus Matrix Interpretive Scheme and is given in full. The partitioning scheme mentioned earlier can be written as:-

$$S = \begin{bmatrix} S_a \\ S_b \\ S_c \\ S_d \\ S_e \end{bmatrix} = \begin{bmatrix} b_a \\ b_b \\ b_c \\ b_d \\ b_e \end{bmatrix} R$$

Now $\qquad b = b_o - b_1 D^{-1} D_o$

$$\therefore \begin{bmatrix} b_a \\ b_b \\ b_c \\ b_d \\ b_e \end{bmatrix} = \begin{bmatrix} b_{oa} \\ b_{ob} \\ b_{oc} \\ b_{od} \\ b_{oe} \end{bmatrix} - \begin{bmatrix} b_{1a} \\ b_{1b} \\ b_{1c} \\ b_{1d} \\ b_{1e} \end{bmatrix} D^{-1} D_o$$

or

$$b_a = b_{oa} - b_{1a} D^{-1} D_o \text{ etc.}$$

Also it can be shown that:

$$D = b_1' f b_1 = b_{1_a}' f_a b_{1_a} + b_{1_b}' f_b b_{1_b} + b_{1_c}' f_c b_{1_c}$$

$$+ b_{1_d}' f_d b_{1_d} + b_{1_e}' f_e b_{1_e}$$

Similarly $D_o = b_{1_a}' f_a b_{oa} + \dots + b_{1_e}' f_e b_{oe}$

The remainder of the programme is self explanatory except for the device for reading in the f matrices. This method was added after the programme was first drafted which explains the high store numbers used. The programme includes examples of overwriting or re-using a store space after one part of the calculation is complete, by extreme use of this and other refinements not included in this simple programme it is possible to accommodate fairly large problems on Pegasus even with its limited store space.

Cut Out Procedure

In principle it is required to introduce an initial displacement H in the panel e_5 such that the final load corresponding to e_5 is zero.

Partition all matrices to separate this row (in general there will be more than one row).

i.e. $$S = \begin{bmatrix} S_g \\ S_h \end{bmatrix} = \begin{bmatrix} b_{og} \\ b_{oh} \end{bmatrix} R + \begin{bmatrix} b_{1g} \\ b_{1h} \end{bmatrix} X$$

The initial displacement H can be written as a column matrix of the displacements in all the members:

$$\begin{bmatrix} 0 \\ H \end{bmatrix}$$

these displacements will be added to the original displacements v due to the external load system. The compatibility condition for the structure will now be that the total displacements across the 'cut' redundancies are zero, i.e.

$$b_1' \left(\begin{bmatrix} v_g \\ v_h \end{bmatrix} + \begin{bmatrix} 0 \\ H \end{bmatrix} \right) = 0$$

where $v = \begin{bmatrix} v_g \\ v_h \end{bmatrix}$

v is defined in equation (3) of the main text

$$\therefore \quad b_1' f \begin{bmatrix} S_g \\ S_h \end{bmatrix} + b_1' \begin{bmatrix} 0 \\ H \end{bmatrix} = 0$$

and $b_1' f S + b_1' \begin{bmatrix} 0 \\ H \end{bmatrix} = 0$

or $D_o R + DX + \begin{bmatrix} b_1'{}_g, & b_1'{}_h \end{bmatrix} \begin{bmatrix} 0 \\ H \end{bmatrix} = 0$

i.e. $\quad D_o R + DX + b_1'{}_h H = 0$

$$\therefore \quad X = - D^{-1} D_o R - D^{-1} b_{1_h}' H \qquad (1.4)$$

To find H, the final load in the cut out member S_h is zero and

$$\begin{bmatrix} S_g \\ 0 \end{bmatrix} = \begin{bmatrix} b_{og} \\ b_{oh} \end{bmatrix} R + \begin{bmatrix} b_{1_g} \\ b_{1_h} \end{bmatrix} [- D^{-1} D_o R - D^{-1} b_1'{}_h H]$$

or

$$0 = b_{o_h} R - b_{1_h} D^{-1} D_o R - b_{1_h} D^{-1} b_1'{}_h H$$

$$\therefore \quad H = \begin{bmatrix} b_{1_h} D^{-1} b_1'{}_h \end{bmatrix}^{-1} \begin{bmatrix} b_{oh} - b_{1_h} D^{-1} D_o \end{bmatrix} R \qquad (1.5)$$

Then

$$S_g = \begin{bmatrix} b_{og} - b_{1g} D^{-1} D_o - D^{-1} D_o - D^{-1} b_{1h}' \begin{bmatrix} b_{1_h} D^{-1} b_{1_h} \end{bmatrix}^{-1} \begin{bmatrix} b_{oh} - b_{1_h} D^{-1} D_o \end{bmatrix} \end{bmatrix} R \qquad (1.6)$$

This is a simple calculation as b_{1_h} and b_{o_h} are the rows of the b_1 and b_o matrices corresponding to the cut-out, in this case single row matrices. The additional matrix to be inverted is $[b_{1_h}' D^{-1} b_{1_h}]$ and in this case is only a scalar. The values of D and D_o have been calculated and can be retained in the computer. The programme for carrying out this additional calculation is given as 'BOX VAN WITH WINDOW' Table assuming that computer sorter is arranged as at the end of the 'BOX VAN', computation. The calculated values are given in Figs. 11 to 16.

In the Programme as tabulated, the additional symbol E is used where:

$$E = D^{-1} b_{1h}' H$$

$$\therefore \quad S_g = b_g - b_{1g} E.$$

In partitioned form:

$$S_g = \begin{bmatrix} S_a \\ S_b \\ S_c \\ S_d \\ S_{e(1-4)} \\ S_{e6} \end{bmatrix} = \begin{bmatrix} b_a \\ b_b \\ b_c \\ b_d \\ b_{e(1-4)} \\ b_{e6} \end{bmatrix} - \begin{bmatrix} b_{1a} \\ b_{1b} \\ b_{1c} \\ b_{1d} \\ b_{1e(1-4)} \\ b_{1e6} \end{bmatrix} E$$

where $S_{e(1-4)}$ refers to row numbers 1 to 4 of the e partitioned matrix. S_{e6} refers to row 6 since S_{e5} refers to row 5 (or panel e_5 the window) and $S_{e5} = 0 = S_h$.

APPENDIX 1 TABLE 1

The b_o Matrix

$b_o = \dfrac{h}{h(2\ell_1+\ell_2)}$ (Load system)

Load system	R_1	R_2	Member	End
b_{oa}	$+0$	$+0$	a_1 END	1
	$\ell_1(\ell_1+\ell_2)$	ℓ_1^2	a_1	2
	$\ell_1(\ell_1+\ell_2)$	ℓ_1^2	a_2	1
	ℓ_1^2	$\ell_1(\ell_1+\ell_2)$	a_2	2
	ℓ_1^2	$\ell_1(\ell_1+\ell_2)$	a_3	1
	$+0$	$+0$	a_3	2
	6 rows of zeros		For a_4, a_5 and a_6	
	$+0$	$+0$	a_7	1
	$-\ell_1(\ell_1+\ell_2)$	$-\ell_1^2$	a_7	2
	$-\ell_1(\ell_1+\ell_2)$	$-\ell_1^2$	a_8	1
	$-\ell_1^2$	$-\ell_1(\ell_1+\ell_2)$	a_8	2
	$-\ell_1^2$	$-\ell_1(\ell_1+\ell_2)$	a_9	1
	$+0$	$+0$	a_9	2
	12 rows of zeros		For a_{10} to a_{15}	
b_{ob}	16 rows of zeros		For b_1 to b_8	
b_{oc}	$-h(\ell_1+\ell_2)$	$-h\ell_1$	c_1	1
	$+0$	$+0$	c_1	2
	$h(2\ell_1+\ell_2)$	$+0$	c_2	1
	$+0$	$+0$	c_2	2
	$+0$	$h(2\ell_1+\ell_2)$	c_3	1
	$+0$	$+0$	c_3	2
	$-h\ell_1$	$-h(\ell_1+\ell_2)$	c_4	1
	$+0$	$+0$	c_4	2
	8 rows of zeros		For c_5 to c_8	
b_{od}	6 rows of zeros		For d_1 to d_6	
b_{oe}	$-(\ell_1+\ell_2)$	$-\ell_1$	Shear flow in e_1	
	$+\ell_1$	$-\ell_1$	Shear flow in e_2	
	$+\ell_1$	$+(\ell_1+\ell_2)$	Shear flow in e_3	
	3 rows of zeros		For e_4 to e_6.	

APPENDIX 1 TABLE 2

The b_1 Matrix

Load System:	X_1	X_2	X_3	X_4	X_5	X_6	Member and End
$b_{1a} = \frac{1}{\ell_2}$	0	0	0	0	0	0	a_1 end 1
	$-\ell_1\ell_2$	0	$-\ell_1\ell_2$	0	0	0	a_1 end 2
	$-\ell_1\ell_2$	0	$-\ell_1\ell_2$	0	0	0	a_2 end 1
	0	$-\ell_1\ell_2$	0	$-\ell_1\ell_2$	0	0	a_2 end 2
	0	$-\ell_1\ell_2$	0	$-\ell_1\ell_2$	0	0	a_3 end 1
	0	0	0	0	0	0	a_3 end 2
	0	0	0	0	0	0	a_4 end 1
	$+2\ell_1\ell_2$	0	0	0	0	0	a_4 end 2
	$+2\ell_1\ell_2$	0	0	0	0	0	a_5 end 1
	0	$+2\ell_1\ell_2$	0	0	0	0	a_5 end 2
	0	$+2\ell_1\ell_2$	0	0	0	0	a_6 end 1
	0	0	0	0	0	0	a_6 end 2
	0	0	0	0	0	0	a_7 end 1
	0	0	$+2\ell_1\ell_2$	0	0	0	a_7 end 2
	0	0	$+2\ell_1\ell_2$	0	0	0	a_8 end 1
	0	0	0	$+2\ell_1\ell_2$	0	0	a_8 end 2
	0	0	0	$+2\ell_1\ell_2$	0	0	a_9 end 1
	0	0	0	0	0	0	a_9 end 2
	0	0	0	0	0	0	a_{10} end 1
	0	0	$-\ell_1\ell_2$	0	$-\ell_1\ell_2$	0	a_{10} end 2
	0	0	$-\ell_1\ell_2$	0	$-\ell_1\ell_2$	0	a_{11} end 1
	0	0	0	$-\ell_1\ell_2$	0	$-\ell_1\ell_2$	a_{11} end 2
	0	0	0	$-\ell_1\ell_2$	0	$-\ell_1\ell_2$	a_{12} end 1
	0	0	0	0	0	0	a_{12} end 2
	0	0	0	0	0	0	a_{13} end 1
	0	0	0	0	$+2\ell_1\ell_2$	0	a_{13} end 2
	0	0	0	0	$+2\ell_1\ell_2$	0	a_{14} end 1
	0	0	0	0	0	$+2\ell_1\ell_2$	a_{14} end 2
	0	0	0	0	0	$+2\ell_1\ell_2$	a_{15} end 1
	0	0	0	0	0	0	a_{15} end 2

APPENDIX 1 TABLE 2

The b_1 Matrix (Cont.)

$b_{1b} = \frac{1}{\ell_2}$

Load System:	X_1	X_2	X_3	X_4	X_5	X_6	Member and End
	0	0	0	0	0	0	b_1 end 1
	$-n\ell_2$	0			0	0	b_1 " 2
	0	0			0	0	b_2 " 1
	$+n(\ell_1+\ell_2)$	$-n\ell_1$			0	0	b_2 " 2
	0	0			0	0	b_3 " 1
	$-n\ell_1$	$+n(\ell_1+\ell_2)$			0	0	b_3 " 2
	0	0			0	0	b_4 " 1
	0	$-n\ell_2$			0	0	b_4 " 2
	0	0			0	0	b_5 " 1
	0	0			$-n\ell_2$	0	b_5 " 2
	0	0			0	0	b_6 " 1
	0	0			$+n(\ell_1+\ell_2)$	$-n\ell_1$	b_6 " 2
	0	0			0	0	b_7 " 1
	0	0			$-n\ell_1$	$+n(\ell_1+\ell_2)$	b_7 " 2
	0	0			0	0	b_8 " 1
	0	0	0	0	0	$-n\ell_2$	b_8 " 2

$b_{1c} = \frac{1}{\ell_2}$

Load System:	X_1	X_2	X_3	X_4	X_5	X_6	Member and End
	0	0	0	0	0	0	c_1 end 1
			$-h\ell_2$	0			c_1 " 2
			0	0			c_2 " 1
			$+h(\ell_1+\ell_2)$	$-h\ell_1$			c_2 " 2
			0	0			c_3 " 1
			$-h\ell_1$	$+h(\ell_1+\ell_2)$			c_3 " 2
			0	0			c_4 " 1
			0	$-h\ell_2$			c_4 " 2
			$-h\ell_2$	0			c_5 " 1
			0	0			c_5 " 2
			$+h(\ell_1+\ell_2)$	$-h\ell_1$			c_6 " 1
			0	0			c_6 " 2
			$-h\ell_1$	$+h(\ell_1+\ell_2)$			c_7 " 1
			0	0			c_7 " 2
			0	$-h\ell_2$			c_8 " 1
	0	0	0	0	0	0	c_8 " 2

APPENDIX 1 TABLE 2

The b_1 Matrix (Cont.)

Load System:		X_1	X_2	X_3	X_4	X_5	X_6	Panel
b_{1d} =	$\frac{1}{\ell_2}$	$+\ell_2$	0	0	0	0	0	d_1
		$-\ell_1$	$+\ell_1$	⋮	⋮	0	0	d_2
		0	$-\ell_2$	⋮	⋮	0	0	d_3
		0	0	⋮	⋮	$+\ell_2$	0	d_4
		0	0	⋮	⋮	$-\ell_1$	$+\ell_1$	d_5
		0	0	0	0	0	$-\ell_2$	d_6
b_{1e} =	$\frac{1}{\ell_2}$	0	0	$+\ell_2$	0	0	0	e_1
		⋮	⋮	$-\ell_1$	$+\ell_1$	⋮	⋮	e_2
		⋮	⋮	0	$-\ell_2$	⋮	⋮	e_3
		⋮	⋮	$-\ell_2$	0	⋮	⋮	e_4
		⋮	⋮	$+\ell_1$	$-\ell_1$	⋮	⋮	e_5
		0	0	0	$+\ell_2$	0	0	e_6

PROGRAMME TAPE

INST. NO.	INSTRUCTION AS PUNCHED	INSTRUCTION	END SPACE IN STORE
	D		
	N		
	BOX VAN		
	BLANK TAPE OR 2 F. SHIFTS		
	J64.0		
0*	(0,30X6) → 1	Read b_{1a} into store	180
1	(1,30X6)* → 181	Transpose b_{1a} to b_{1a}'	360
2	(0,61/) → 5499	Read zero and diagonals of f_a into store	5559
3	(5499)X(3999,100X15) → 3999	Put zeros in spaces 3999 to 5498	5498
4	(5500,30/)+(4000,30X30) → 4000	Main diagonal of f_a } f_a starts 4000	4899
5	(5530,30/)+(4001,30X30) → 4001	} Sec. diagonals of f_a	
6	(5530,30/)+(4030,30X30) → 4030		
7	(181,6X30)X(4000,30X30) → 1261	Form $b_{1a}' f_a$	1440
8*	(1261,6X30)X(1,30X6) → 181	Form $(b_1' f b_1)a$	216
9*	(0,30X2) → 217	Read b_{oa} into store	276
10*	(1261,6X30)X(217,30X2) → 277	Form $(b_1' f b_o)a$	288
11*	(0,16X6) → 289	Read b_{1b} into store	384
12	(289,16X6)* → 385	Transpose b_{1b} to b_{1b}'	480
13	(0,32/) → 5560	Read diagonals of f_b into store	5591
14	(5560,16/)+(4900,16X16) → 4900	Main diagonal of f_b } f_b starts 4900	5155
15	(5576,16/)+(4901,16X16) → 4901	} Sec diagonals of f_b	
16	(5576,16/)+(4916,16X16) → 4916		
17	(385,6X16)X(4900,16X16) → 737	Form $b_{1b}' f_b$	832
18*	(737,6X16)X(289,16X6) → 385	Form $(b_1' f b_1)_b$	420
19*	(0,16X6) → 421	Read b_{1c} into store	516
20	(421,16X6)* → 517	Transpose b_{1c} to b_{1c}'	612
21	(0,32/) → 5592	Read diagonals of f_c into store	5623
22	(5592,16/)+(5156,16X16) → 5156	Main diagonal of f_c } f_c starts 5156	5411
23	(5608,16/)+(5157,16X16) → 5157	} Sec. diagonals of f_c	
24	(5608,16/)+(5172,16X16) → 5172		
25	(517,6X16)X(5156,16X16) → 869	Form $b_{1c}' f_c$	964
26*	(869,6X16)X(421,16X6) → 517	Form $(b_1' f b_1)_c$	552

INST. NO.	INSTRUCTION AS PUNCHED	INSTRUCTION	END SPACE IN STORE
27*	(0,16X2) → 553	Read b_{oc} into store	584
28*	(869,6X16)X(553,16X2) → 585	Form $(b_o' f b_o)_c$	596
29*	(0,6X6) → 597	Read b_{1d} into store	632
30	(597,6X6)* → 633	Transpose b_{1d} to b_{1d}'	668
31	(0,6/) → 669	Read f_d into store	674
32	(633,6X6)X(669,6/) → 675	Form $b_{1d}' f_d$	710
33*	(675,6X6)X(597,6X6) → 633	Form $(b_1' f b_1)_d$	668
34*	(0,6X6) → 669	Read b_{1e} into store	704
35*	(669,6X6)* → 3000	Transpose b_{1e} to b_{1e}'(Put in separate part of store for cutout	3035
36	(0,6/) → 741	Read f_e into store	746
37	(3000,6X6)X(741,6/) → 747	Form $b_{1e}' f_e$	782
38*	(747,6X6)X(669,6X6) → 705	Form $(b_1' f b_1)_e$	740
39*	(0,6X2) → 783	Read b_{oe} into store	794
40	(747,6X6)X(783,6X2) → 795	Form $(b_1' f b_o)_e$	806
41*	(783,6X2) → 741	Replace b_{oe} in correct position	752
42*	(795,6X2) → 753	Replace $(b_1' f b_o)_e$ in correct position	764
43	(181,6X6)+(385,6X6) → 181	Add $(b_1' f b_1)_a$ to $(b_1 f b_1)_b$	216
44	(181,6X6)+(517,6X6) → 181	Add $(b_1' f b_1)_c$ to sum	216
45	(181,6X6)+(633,6X6) → 181	Add $(b_1' f b_1)_d$ to sum	216
46	(181,6X6)+(705,6X6) → 181	Add $(b_1' f b_1)_e$ to sum to form D	216
47*	(181,6X6) → 3036	Replace total $(b_1' f b_1) = D$ for cut out use	3071
48	(277,6X2)+(585,6X2) → 277	Add $(b_1' f b_o)_a$ to $(b_1' f b_o)_b$	288
49	(277,6X2)+(753,6X2) → 277	Add $(b_1 f b_o)_e$ to sum to form D_o	288
50*	(181,6X6),(277,6X2) → 385	Form $D^{-1} D_o = D_1$	396
51	(1,30X6)X(385,6X2) → 765	Form $b_{1a}D_1$	824
52*	(217,30X2)-(765,30X2) → 765	Form $b_a = b_{oa} - b_{1a}D_1$	824
53	(289,16X6)X(385,6X2) → 825	Form b_{1b} D_1	856
54	(421,16X6)X(385,6X2) → 857	Form $b_{1c}D_1$	888
55*	(553,16X2)-(857,16X2) → 857	Form $b_c = b_{oc} - b_{1c}D_1$	888
56	(597,6X6)X(385,6X2) → 889	Form b_{1d} D_1	900
57	(669,6X6)X(385,6X2) → 901	Form b_{1e} D_1	912
58*	(741,6X2)-(901,6X2) → 901	Form $b_e = b_{oe} - b_{1e}$ D_1	912

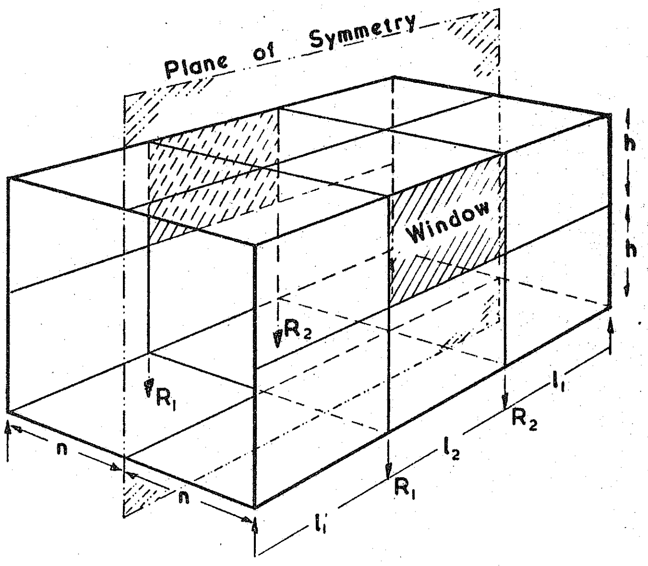
INST. NO.	INSTRUCTION AS PUNCHED	INSTRUCTION	END SPACE IN STORE
59	(0) → 913	Read (- 1) into store	913
60*	(913)X(825,16X2) → 825	Form $b_b = - b_{1b}D_1$	856
61*	(913)X(889,6X2) → 889	Form $b_d = - b_{1d}D_1$	900
62	(0,2X2) → 914	Read R into store	917
63	(765,30X2)X(914,2X2) → 918	Form $S_a = b_a R$	977
64	(825,16X2)X(914,2X2) → 978	Form $S_b = b_b R$	1009
65	(857,16X2)X(914,2X2) → 1010	Form $S_c = b_c R$	1041
66	(889,6X2)X(914,2X2) → 1042	Form $S_d = b_d R$	1053
67	(901,6X2)X(914,2X2) → 1054	Form $S_e = b_e R$	1065
68	(918,30X2)(6,2) → 0	Read out S_a	
69	(978,16X2)(6,2) → 0	Read out S_b	
70	(1010,16X2)(6,2) → 0	Read out S_c	
71	(1042,6X2)(6,2) → 0	Read out S_d	
72	(1054,6X2)(6,2) → 0	Read out S_e	
	* S		

N.B. Asterisk after instruction number indicates result is required in store for future use.

D
N
BOX VAN WITH WINDOW
BLANK TAPE OR 2 FIGURE SHIFTS
J64.0

INST. NO.	INSTRUCTION AS PUNCHED	INSTRUCTION	END SPACE IN STORE
0	(905) → 1100	First term of b_h	1100
1	(911) → 1101	Second term of b_h	1101
2	(1100,1X2)X(914,2X2) → 1102	Form b_hR	1103
3	(3024,6X1) → 5000	Copy 5th column of $b_{1e}'(=b_1'h)$ to vacant part of store	5005
4	(3036,6X6),(3024,6X1) → 1104	Form $D^{-1}b_{1h}'$	1109
5	(5000,1X6)X(1104,6X1) → 1116	Form $b_{1h}D^{-1}b_{1h}'$	1116
6	(1116),(1102,1X2) → 1117	Form H	1118
7	(1104,6X1)X(1117,1X2) → 1119	Form E	1130
8	(1,30X6)X(1119,6X2) → 1131	Form $b_{1a}E$	1190
9	(289,16X6)X(1119,6X2) → 1191	Form $b_{1b}E$	1222
10	(421,16X6)X(1119,6X2) → 1223	Form $b_{1c}E$	1254
11	(597,6X6)X(1119,6X2) → 1255	Form $b_{1d}E$	1266
12	(3000,6X4)* → 1267	Transpose $b_{1e}'(1-4)$	1290
13	(1267,4X6)X(1119,6X2) → 1291	Form $b_{1e(1-4)}E$	1298
14	(3030,1X6)X(1119,6X2) → 1305	Form $b_{1e6}E$	1310
15	(1054,6X2)* → 3100	Transpose $S_e = b_eR$ to form $R'b_e'$	3111
16	(3100,2X4)* → 3120	Transpose first 4 columns to form $b_{e(1-4)}R$	3127
17	(918,30X2)-(1131,30X2) → 1310	Form new $S_a = b_aR - b_{1a}E$	1369
18	(978,16X2)-(1191,16X2) → 1370	Form new S_b	1401
19	(1010,16X2)-(1223,16X2) → 1402	Form new S_c	1433
20	(1042,6X2)-(1255,6X2) → 1434	Form new S_d	1445
21	(3120,4X2)-(1291,4X2) → 1446	Form $S_{e(1-4)}$	1453
22	(3110,1X2)-(1305,1X2) → 1454	Form S_{e6}	1455
23	(1310,30X2)(6,2) → 0	Read out new S load matrices	
24	(1370,16X2)(6,2) → 0	Read out new S load matrices	
25	(1402,16X2)(6,2) → 0	Read out new S load matrices	
26	(1434,6X2)(6,2) → 0	Read out new S load matrices	
27	(1446,4X2)(6,2) → 0	Read out new S load matrices	
28	(1454,1X2)(6,2) → 0	Read out new S load matrices	

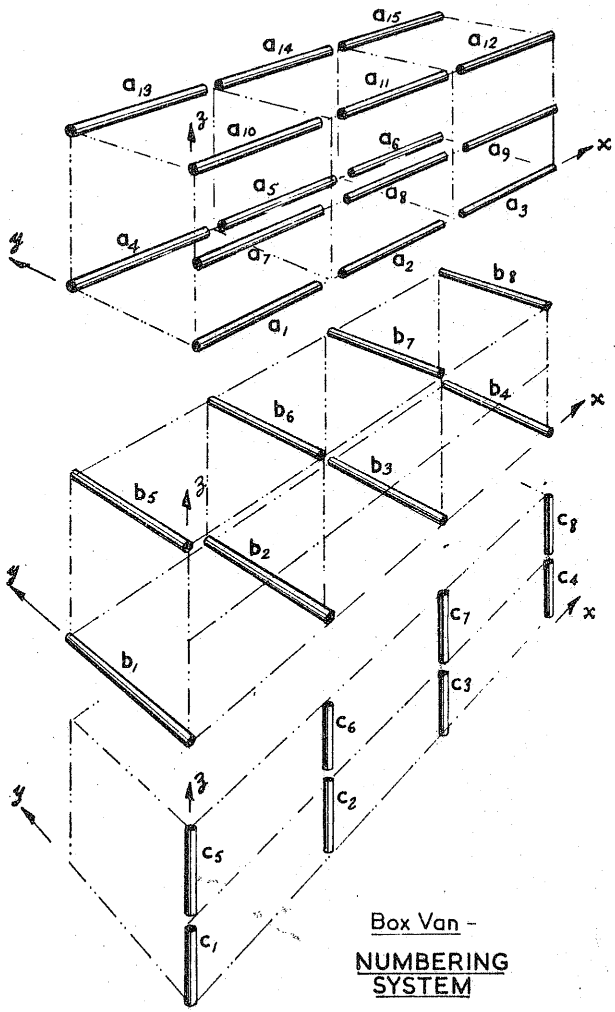
*
S



Box Van

IDEALISED MODEL OF STRUCTURE

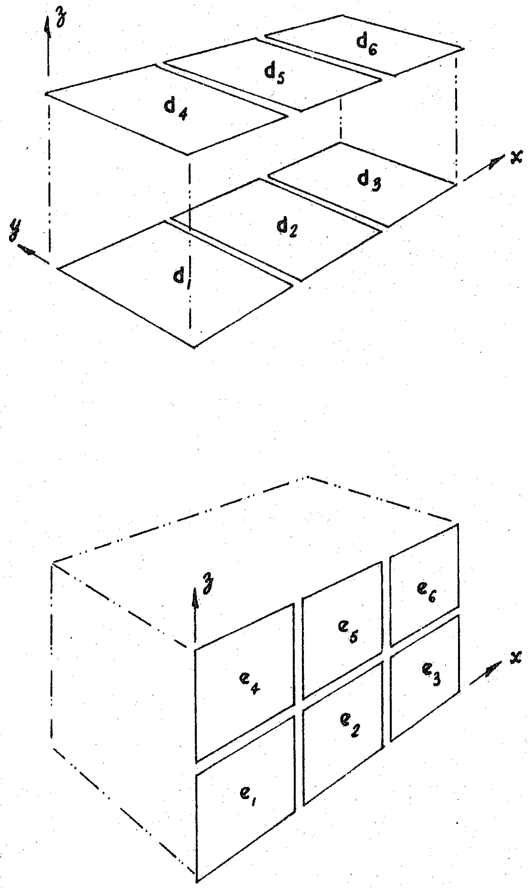
APP.I. FIG I.



Box Van -

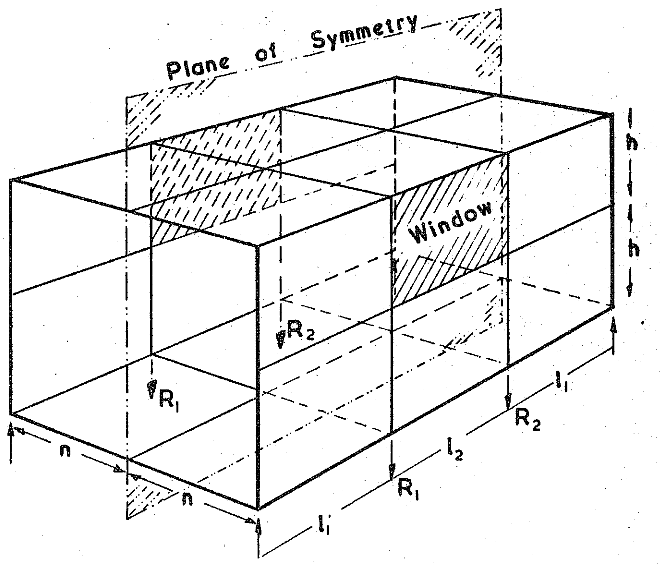
NUMBERING SYSTEM

APP.I FIG.2.

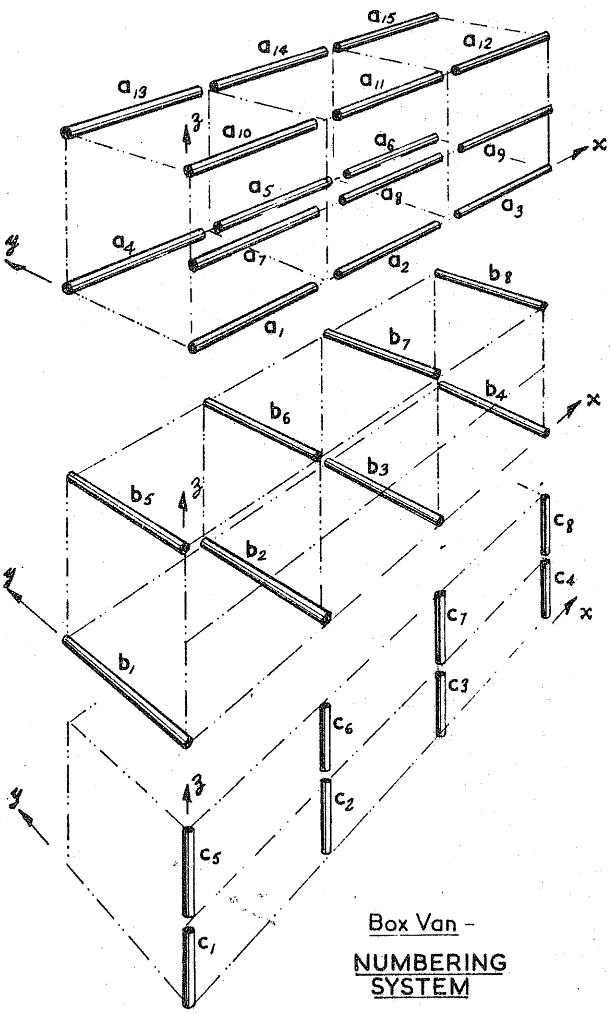


Box Van - Panel Numbering System

APP.I FIG.3

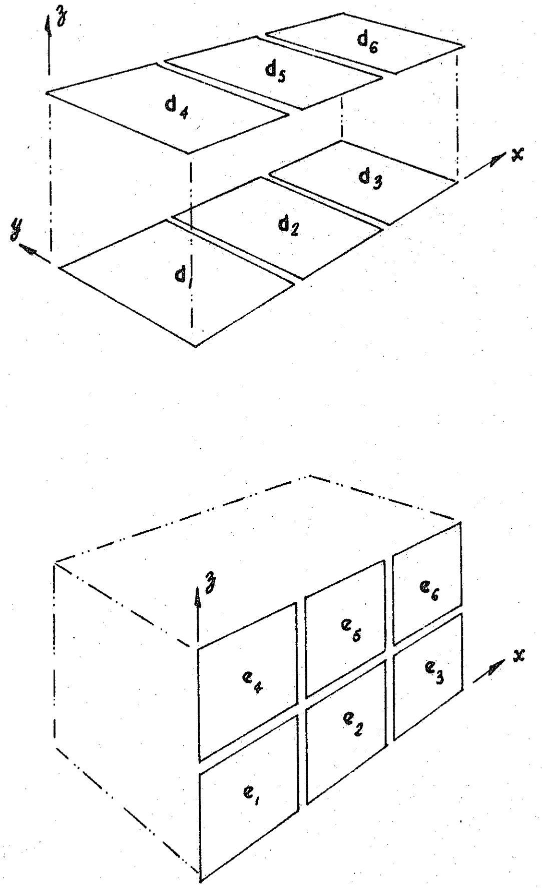


Box Van
IDEALISED MODEL OF STRUCTURE
APP. I. FIG I.



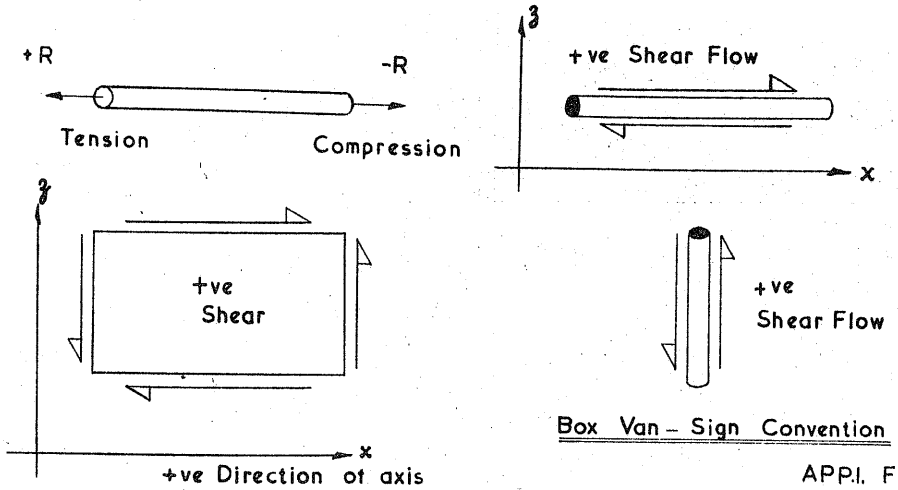
Box Van -
NUMBERING SYSTEM

APP. I FIG. 2.



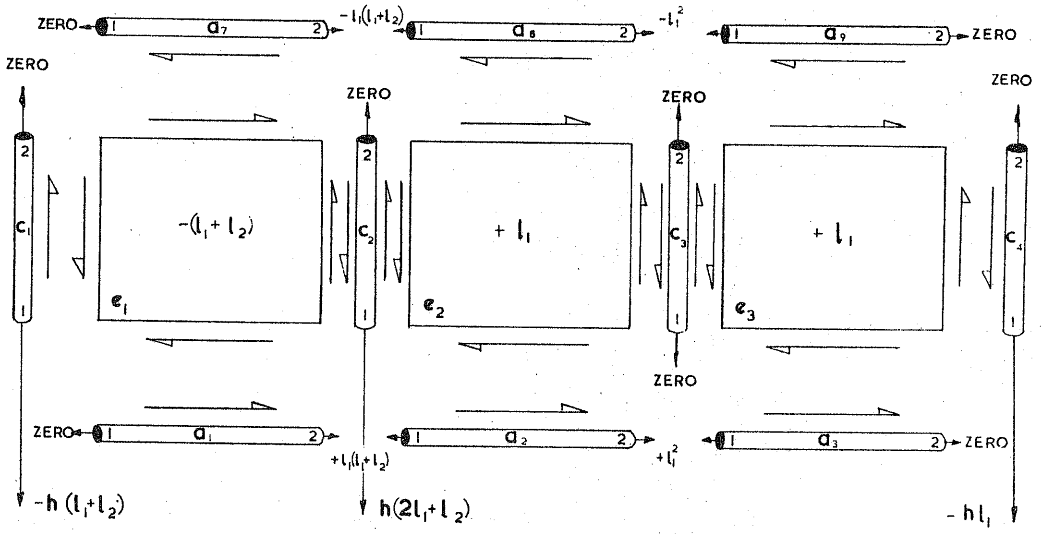
Box Van - Panel Numbering System

APP. I FIG.3

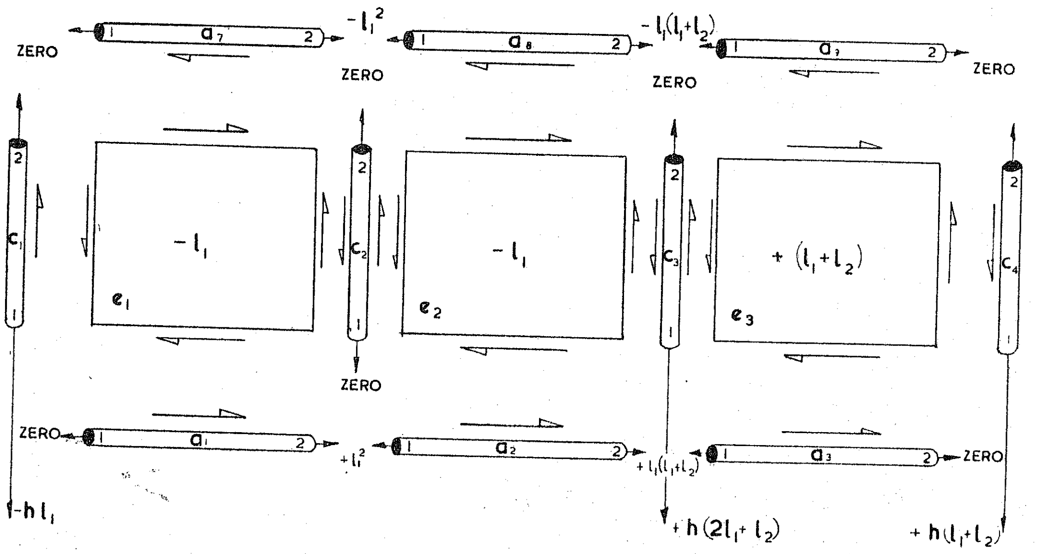


Box Van – Sign Convention

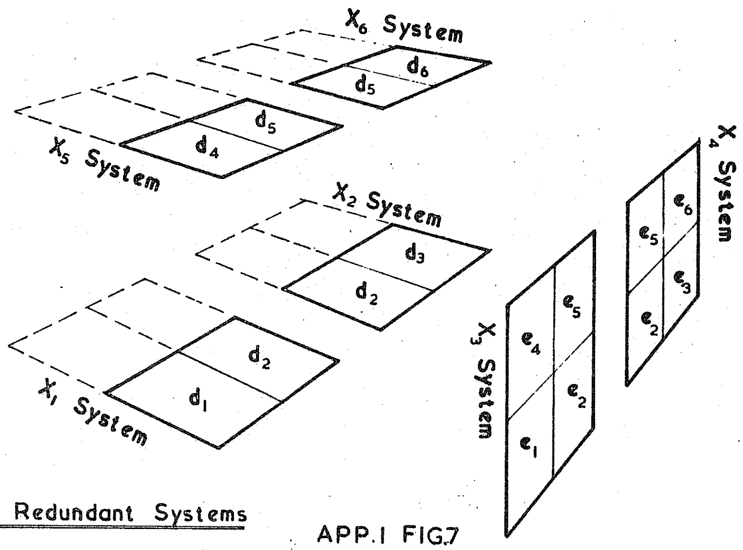
APP.I. FIG.4.



Box Van – R_1 Basic System APP.I. FIG.5

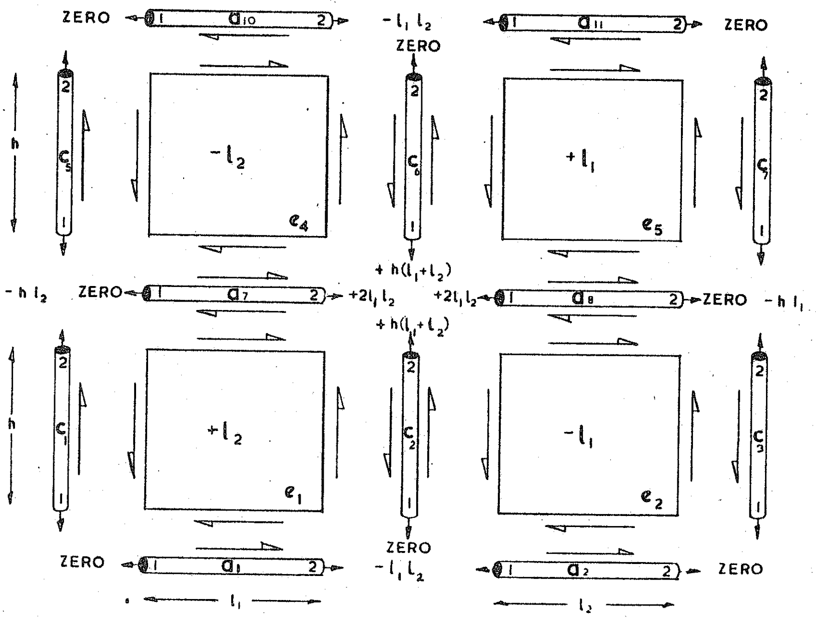


Box Van – R_2 Basic System APP.I FIG.6

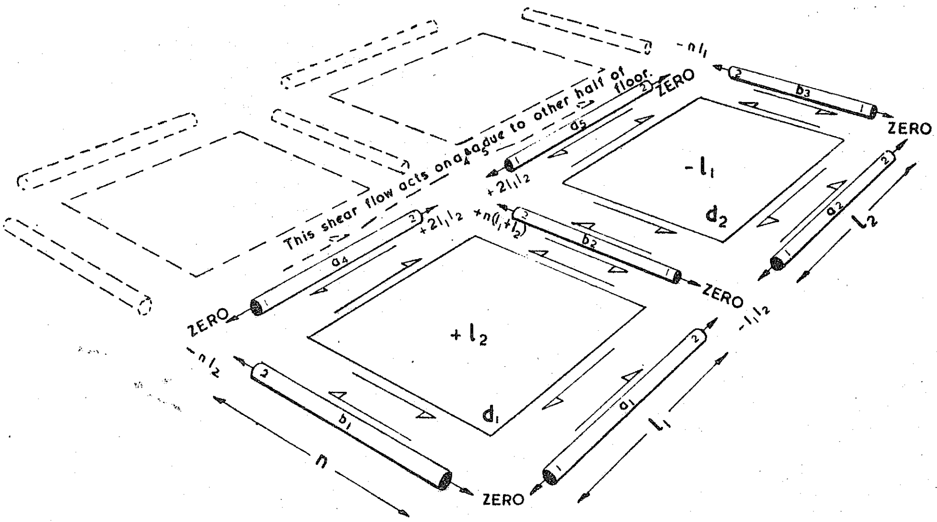


Box Van – Redundant Systems

APP. I FIG.7

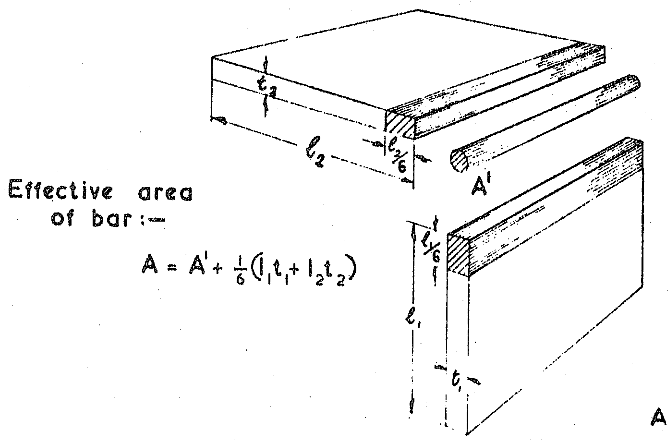


Box Van — Redundant System X_3 APP. I. FIG.8

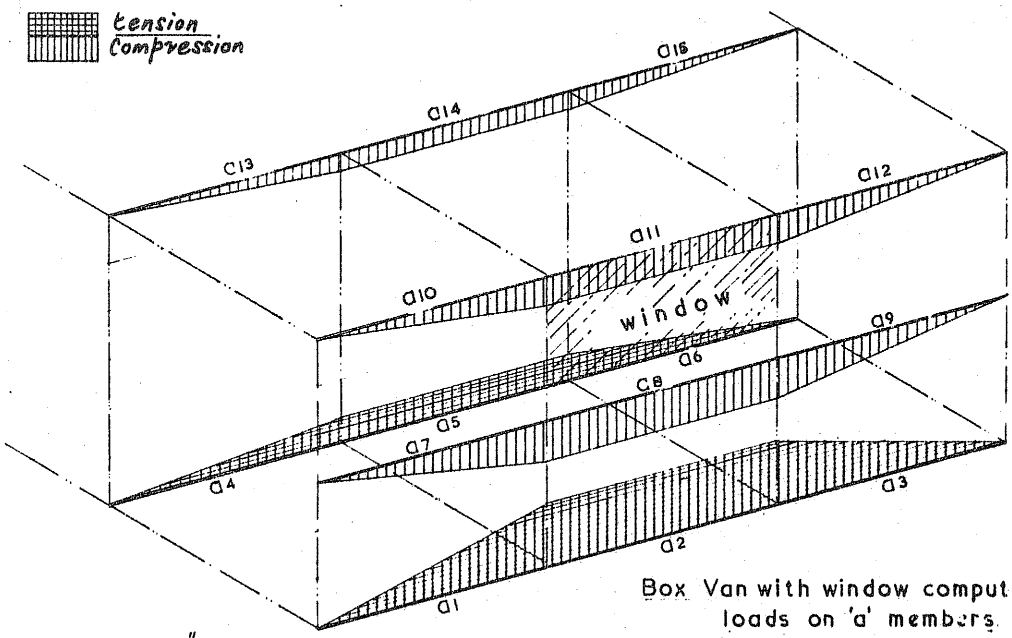


Box Van – Redundant System X_1

APP. I FIG.9



APP.I FIG.10

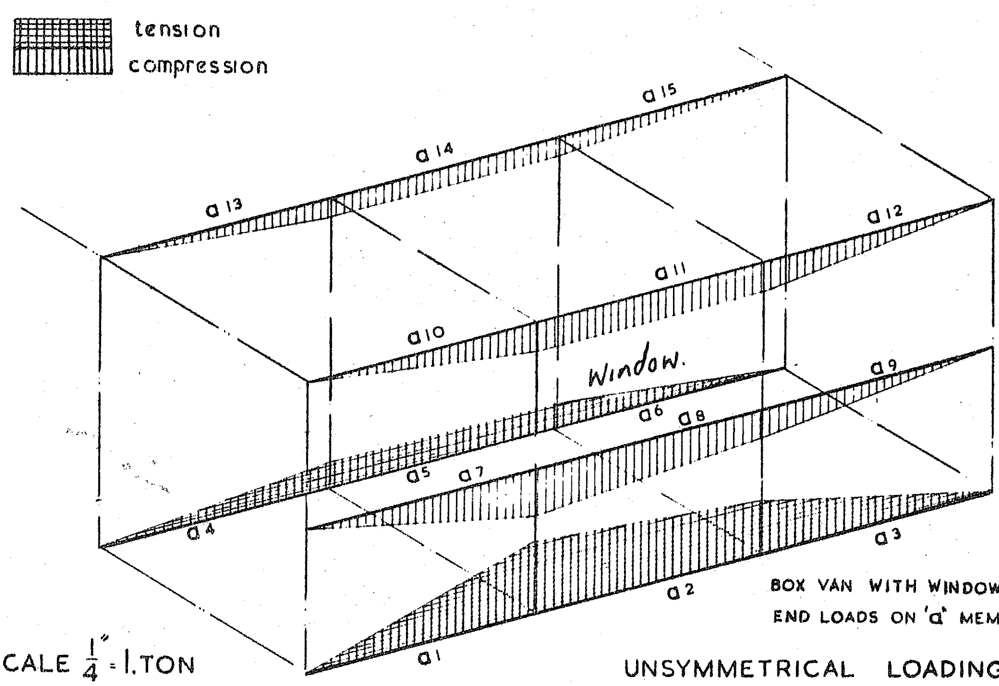


Box Van with window computed end loads on 'a' members.

SCALE $\frac{1}{4}''$ = 1.TON

SYMMETRICAL LOADING

APP I. FIG II.

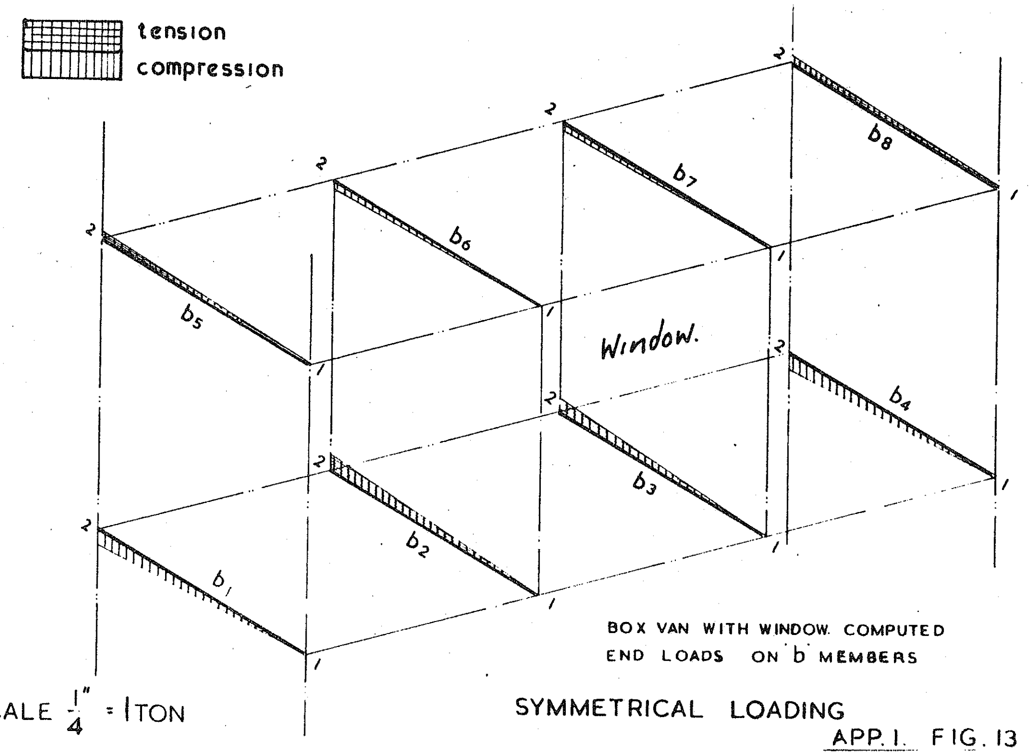


BOX VAN WITH WINDOW COMPUTED END LOADS ON 'a' MEMBERS

SCALE $\frac{1}{4}''$ = 1.TON

UNSYMMETRICAL LOADING

APP.I. FIG 12.

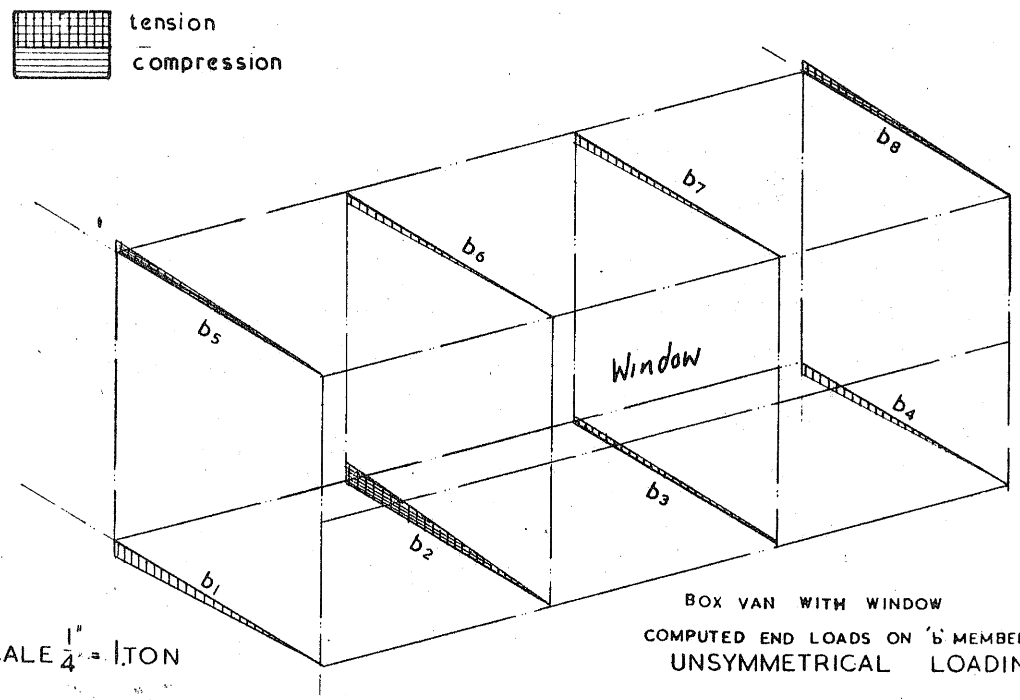


BOX VAN WITH WINDOW. COMPUTED END LOADS ON 'b' MEMBERS

SCALE $\frac{1''}{4}$ = 1 TON

SYMMETRICAL LOADING

APP. I. FIG. 13



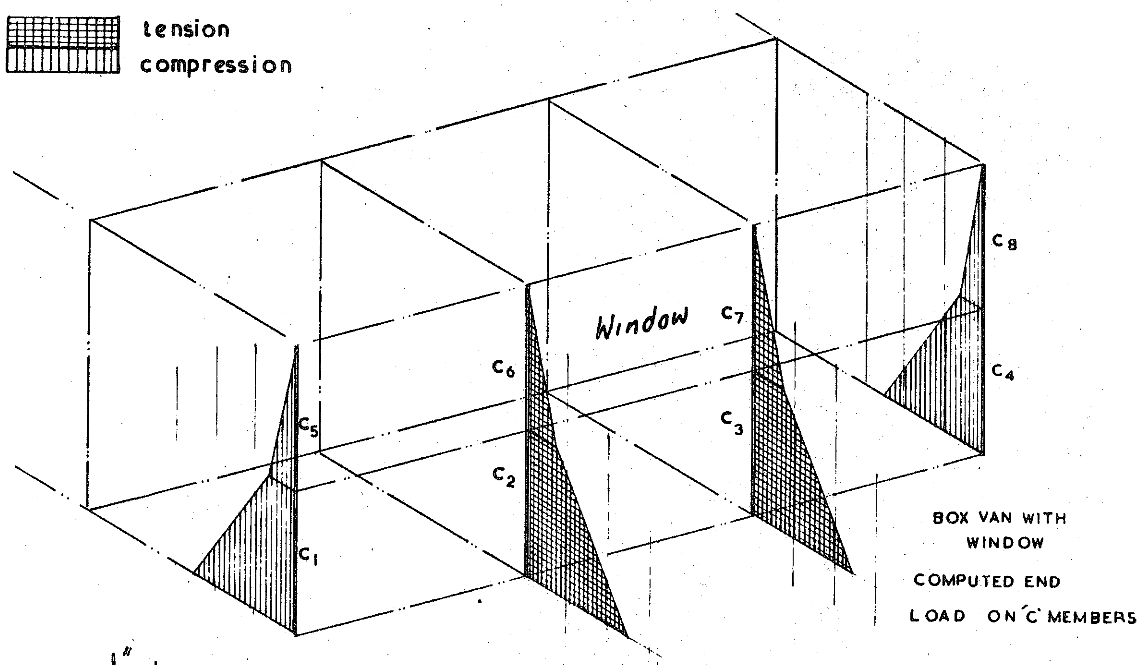
BOX VAN WITH WINDOW

COMPUTED END LOADS ON 'b' MEMBERS

SCALE $\frac{1''}{4}$ = 1 TON

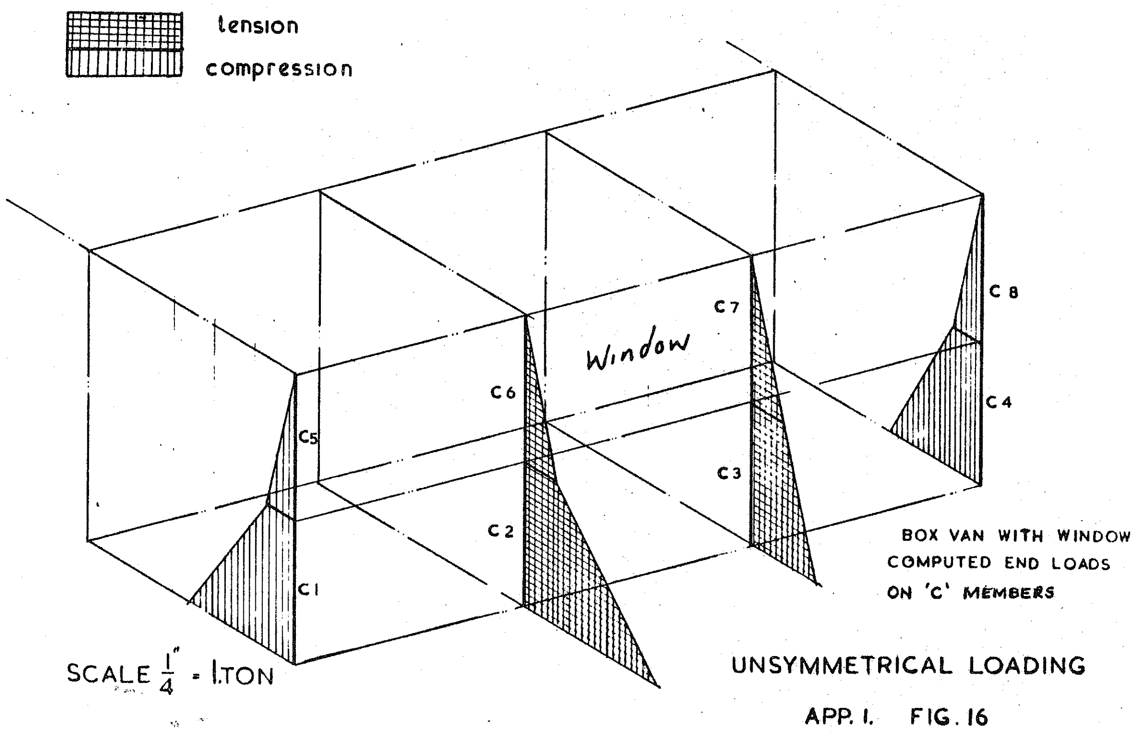
UNSYMMETRICAL LOADING

APP. I. FIG 14



SCALE $\frac{1''}{4}$ = 1 TON

SYMMETRICAL LOADING
APP. I. FIG. 15.



SCALE $\frac{1''}{4}$ = 1 TON

UNSYMMETRICAL LOADING

APP. I. FIG. 16

Nᵒ. 1 3.9

Appendix 2

Calculation of frame stiffness by the matrix force method

Each member of the frame is assumed to be loaded in longitudinal bending and torsion only. The external torsion is applied to the frame by equal forces at the corners and these can be taken for convenience as unit forces, i.e. forces of magnitude one pound giving a torque on the frame of 2b as in Fig. 1.

These forces will give each corner of the frame a vertical displacement r, in the direction of the local external force. The frame will be twisted by an angle $\theta = \frac{2r}{b}$.

$$\text{The torsional stiffness} = \frac{2b}{2r/_b} = \frac{b^2}{r}$$

Now the matrix force method will give the flexibility of the frame in the direction of the applied force, i.e. the displacement per unit force; this is normally defined as the F matrix, but in this case it is equal to 4r. As there is only one external force system (four unit forces constituting a torque) and consequently only one term in the F matrix.

The general formula for the overall flexibility of the structure F is given by equation (7) as:

$$F = b_o{}' f b_o - D_o{}' D^{-1} D_o$$

An 'element' of the structure is defined as one discrete member under one type of loading, e.g. each side member (length 2a) is made up of two elements, one solely in bending and the other in torsion.

It is assumed that the structure is always below the yield point when both types of loading are applied together, therefore, the effects of each type of load may be calculated separately and then added.

Notation and Sign Convention

For this example the loads are defined as in Fig. 2.

There are three loads defined for each member, a torsion and a bending moment at each end. The bending moments at each end will be different but will be varying linearly along the member (constant shear in the element).

Basic System

The external loads can be supported by a system of three elements, the side members will act as cantilevers supported by the cross member at one end, which will be only in torsion as in Fig. 3.

The matrix of the loads b_o will be a column with only three terms other than zero, M_{1y}, M_{4y} and T_{14}.

If b_o is partitioned into the column matrix:

$$b_o = \left\{ b_{oy}, b_{ox}, b_{ot} \right\}$$

b_{oy} refers to moments about the y axis

b_{ox} refers to moments about the x axis

b_{ot} refers to torsion.

Where b_{oy} is a column matrix of the four moments M_y taken in numerical order, i.e.

	M_{1y}	M_{2y}	M_{3y}	M_{4y}	Moments
$b_{oy} =$	{+2a,	0,	0,	-2a}	Values

Similarly

	M_{1x}	M_{4x}	M_{2x}	M_{3x}	Moments
$b_{ox} =$	{ 0,	0,	0,	0 }	Values

It is desirable to change the numerical order for the columns of b_{ox} since M_{1x} and M_{4x} refer to the bending moments at each end of the element 1 - 4. It is possible to carry out the analysis if this rule is not adhered to but it will result in non-zero terms in the flexibility matrix away from the main and subsidiary diagonals.

Also:

	T_{12}	T_{23}	T_{34}	T_{41}	Torsions
$b_{ot} =$	{0,	0,	0,	-2a}	Values.

Redundant systems

Two redundant systems are required, each being self equilibrating:

System 1

Unit torque on cross member 1 - 4 and the opposite torque on cross member 2 - 3. These torques will induce constant bending moments in the side members, 1 - 2 and 3 - 4; see Fig. 4.

This system gives the first column of b_1 as $\{b_{11y}, b_{11x}, b_{11t}\}$

Where each sub-matrix has the same significance as for b_o and each row in the sub matrices refers to the same load as for the b_o case.

Then:

$$b_{11y} = \{-1, \quad -1, \quad +1, \quad +1\}$$

$$b_{11x} = \{0, \quad 0, \quad 0, \quad 0\}$$

$$b_{11t} = \{0, \quad -1, \quad 0, \quad +1\}$$

<u>System 2</u>

Unit torque is now assumed on side members 1 - 2 and 3 - 4. <u>But</u> instead of being opposite torques they are in the same direction. This type of system is suggested by Erz, Ref. 6, and other writers but they have assumed hinges in the system and the redundant unit load then loads the members in an unequal way. It is a feature of the Argyris treatment that self-equilibrating force systems may be taken as part of the structure without the need for actual 'hinge' or 'cuts' in the system. (In this case all the members are used in the redundant system, but if the frame had intermediate cross members the same type of redundant system could be used involving one bay at a time).

This torsion clearly imposes bending moments in the end cross members Fig. 5a.

The moment system shown is <u>not</u> in equilibrium. Consider the cross member 1 - 4 and draw the moments in the directions that they are acting on the element. Fig. 5b.

The cross member can only be in equilibrium if loads are applied at the ends to produce a total clockwise couple of +2. Fig. 5c.

And similar forces must exist (in the opposite direction) on cross member 2 - 3.

Since there are no external forces these forces must be balanced by forces acting on the side members, i.e. for member 1 - 2. See Fig. 5d.

Since there is only a torque on this member the anticlockwise couple of $\frac{2a}{b}$ induced by these forces must be equilibrated by M_y moments at the ends as shown in Fig. 5e.

A similar argument applies to the member 3 - 4 which is in equilibrium under the forces and moments shown in Fig. 5f.

The moments about the y axis at the ends of the side members must now be balanced by torques in the cross members, these torques will be equal to + a/b on 1 - 4 and + a/b on 2 - 3.

The complete equilibrium load diagram is shown in Fig. 5g.

It can be seen from Fig. 5g that this redundant system allows a variation of bending moment along the cross members which is necessary in the final analysis since the centre is a point of contra-flexure.

The second column of b_1 say $b_{12} = \{b_{12}y, b_{12}x, b_{12}t\}$

		M_{1y}	M_{2y}	M_{3y}	M_{4y}	Moment
where $b_{12}y$	=	{ $-a/b$,	$+a/b$,	$-a/b$,	$+a/b$ }	Value
		M_{1x}	M_{4x}	M_{2x}	M_{3x}	Moment
$b_{12}x$	=	{ +1	-1	-1	+1 }	Value
		T_{12}	T_{23}	T_{34}	T_{41}	Torque
$b_{12}t$	=	{ +1	$+a/b$	+1	$+a/b$ }	Value

Flexibility matrix

The flexibility sub-matrix for an element with linearly varying bending moment is:

$$\frac{\text{Length}}{6\,E\,I}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

where each row refers to the load (bending moment) at one end of the element.

If the flexibility matrix is partitioned as follows:

$$f = \begin{bmatrix} f_y & 0 & 0 \\ 0 & f_x & 0 \\ 0 & 0 & f_t \end{bmatrix}$$

where the four rows of f_y correspond to the rows of b_{oy} and b_{1y} etc. then:

$$f_y = \frac{2a}{6\,E\,I_a}\begin{bmatrix}2 & 1 & 0 & 0\\ 1 & 2 & 0 & 0\\ 0 & 0 & 2 & 1\\ 0 & 0 & 1 & 2\end{bmatrix}$$

and

$$f_x = \frac{2b}{6\,E\,I_b}\begin{bmatrix}2 & 1 & 0 & 0\\ 1 & 2 & 0 & 0\\ 0 & 0 & 2 & 1\\ 0 & 0 & 1 & 2\end{bmatrix}$$

This assumes that all members are of the same material and that the two side members have the same second moment of area I_a as do the cross members I_b.

The torsion flexibility matrix has one row for each member and takes the form

$$f_t = \begin{bmatrix}\frac{2a}{G\,J_a} & 0 & 0 & 0\\ 0 & \frac{2b}{G\,J_b} & 0 & 0\\ 0 & 0 & \frac{2a}{G\,J_a} & 0\\ 0 & 0 & 0 & \frac{2b}{G\,J_b}\end{bmatrix}\quad\begin{matrix}\text{Torsion in } 1-2\\ \text{Torsion in } 1-3\\ \text{Torsion in } 3-4\\ \text{Torsion in } 4-1\end{matrix}$$

The matrix manipulation can now be carried out with the partitioned matrices as follows:

$$b_o' f b_o = b_{oy}' \; f_y b_{oy} + b_{ox}' \; f_x \, b_{ox} + b_{ot}' \; f_t \, b_{ot}$$

$$b_1' f b_1 = b_{1y}' \; f_y b_{1y} + b_{1x}' f_x \, b_{1x} + b_{1t}' \; f_t \, b_{ot}$$

and $b_o' f b_1 = (b_1' \; f b_o) = b_{oy}' \; f_y \, b_{1y} + b_{ox}' \; f_x \, b_{1x} + b_{ot}' f_t \, b_{1t}$

Each of the right hand side terms may be simply evaluated e.g.:

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$$b_{oy}' \, f_y = \frac{2a}{6EI_a} [+2a, 0, 0, -2a] \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$= \frac{a}{3EI_a} [+4a, +2a, -2a, -4a]$$

$$b_{oy}' \, f_y b_{oy} = \frac{a}{3EI_a} [+4a, +2a, -2a, -4a] \begin{bmatrix} +2a \\ 0 \\ 0 \\ -2a \end{bmatrix}$$

and

$$b_{oy}' \, f_y b_{1y} = \frac{a}{3EI_a} [+4a, +2a, -2a, -4a] \begin{bmatrix} -1 & -{}^a/_b \\ -1 & +{}^a/_b \\ +1 & -{}^a/_b \\ +1 & +{}^a/_b \end{bmatrix}$$

$$= \frac{a}{3EI_a} [-12a, \; - \frac{4a^2}{b}]$$

The matrix which requires inverting is:

$$D = b_1' \, fb_1 = \begin{bmatrix} \left(\frac{4a}{E\ I_a} + \frac{4b}{GJ_b}\right) & 0 \\ 0 & \left(\frac{4a^3}{3b^2EI_a} + \frac{4b}{3EI_b} + \frac{4a}{GJ_a} + \frac{4a^2}{bGJ_b}\right) \end{bmatrix}$$

The inverse of a 2 x 2 diagonal matrix is simply the reciprocal of each term on the diagonal, i.e.:

$$D^{-1} = \begin{bmatrix} \dfrac{1}{\frac{4a}{EI_a} + \frac{4b}{GJ_b}} & 0 \\ 0 & \dfrac{1}{\frac{4}{b^2}\left(\frac{a^3}{3EI_a} + \frac{b^3}{3EI_b} + \frac{ab^2}{GJ_a} + \frac{a^2b}{GJ_b}\right)} \end{bmatrix}$$

The other terms are:

$$b_o{}' f b_o = \frac{16a^3}{3EI_a} + \frac{8a^2b}{GJ_b}$$

$$D_o = \left[\left(-\frac{4a^2}{EI_a} - \frac{4ab}{GJ_b} \right), \left(-\frac{4a^3}{3b\ EI_a} - \frac{4a^2}{GJ_b} \right) \right]$$

The matrix multiplication $D_o\ D^{-1}$ gives simply:

$$\left[\frac{-\frac{4a^2}{EI_a} - \frac{4ab}{GJ_b}}{\frac{4a}{EI_a} + \frac{4b}{GJ_b}}, \quad \frac{-\frac{4a^3}{3b(EI_a)} - \frac{4a^2}{GJ_b}}{\frac{4}{b^2}k'} \right]$$

where $k' = \frac{a^3}{3EI_b} + \frac{b^3}{3EI_b} + \frac{ab^2}{GJ_a} + \frac{a^2b}{GJ_b}$

This matrix can be further simplified to give:-

$$D_o\ D^{-1} = \left[-a, -\frac{b}{k'}\left(\frac{a^3}{3EI_a} + \frac{a^2b}{GJ_b} \right) \right]$$

$$\therefore \quad D_o{}'D^{-1}D_o = \left[-a_1, -\frac{b}{k'}\left(\frac{a^3}{3EI_a} + \frac{a^2b}{GJ_b} \right) \right] \times$$

$$\begin{bmatrix} -4a\left(\frac{a}{EI_a} + \frac{b}{GJ_b} \right) \\ -\frac{4}{b}\left(\frac{a^3}{3EI_a} + \frac{a^2b}{GJ_b} \right) \end{bmatrix}$$

$$= 4a^2\left(\frac{a}{EI_a} + \frac{b}{GJ_b} \right) + \frac{4}{k'}\left(\frac{a^3}{3EI_a} + \frac{a^2b}{GJ_b} \right)^2$$

The flexibility matrix of the assembled structure is now:

$$F = b_o{}' f b_o - D_o{}'D^{-1}D_o$$

$$\therefore \quad F = 4a^2\left(\frac{4a}{3EI_a} + \frac{2b}{GJ_b} \right) - 4a^2\left(\frac{a}{EI_a} + \frac{b}{GJ_b} \right) - \frac{4}{k'}\left(\frac{a^3}{3EI_a} + \frac{a^2b}{GJ_b} \right)^2$$

This expression can be simplified to give

$$F = \frac{4}{k'}\left(\frac{a^3}{3EI_a} + \frac{a^2b}{GJ_b}\right)\left(\frac{b^3}{3EI_b} + \frac{ab^2}{GJ_a}\right)$$

Since torsional stiffness $= \dfrac{b^2}{r} = \dfrac{4b^2}{F}$

The Frame Torsional Stiffness is:-

$$\frac{b^2k'}{\left(\frac{a^3}{3EI_a} + \frac{a^2b}{GJ_b}\right)\left(\frac{b^3}{3EI_b} + \frac{ab^2}{GJ_a}\right)}$$

This result has been obtained by Cooke in Ref. 5, using strain energy methods.

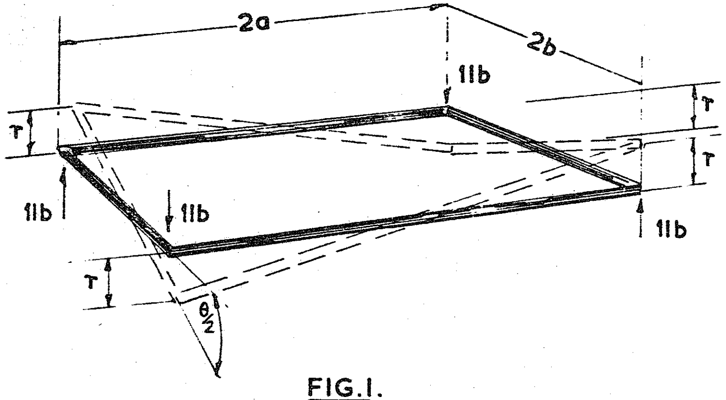
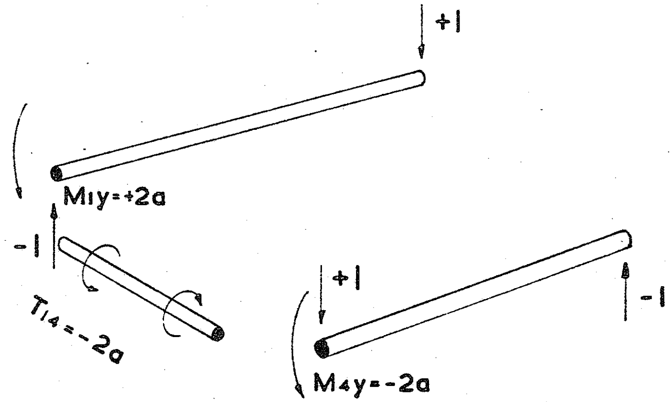
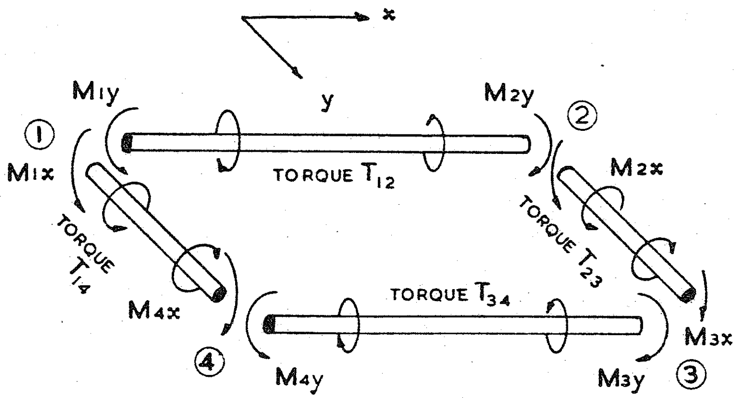


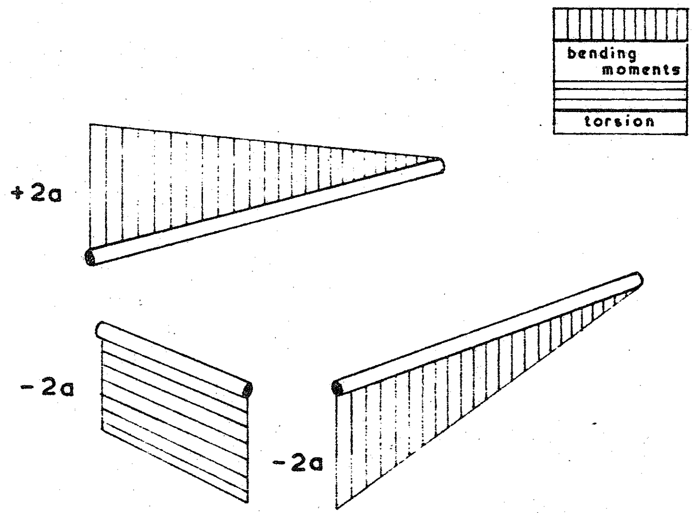
FIG.1.





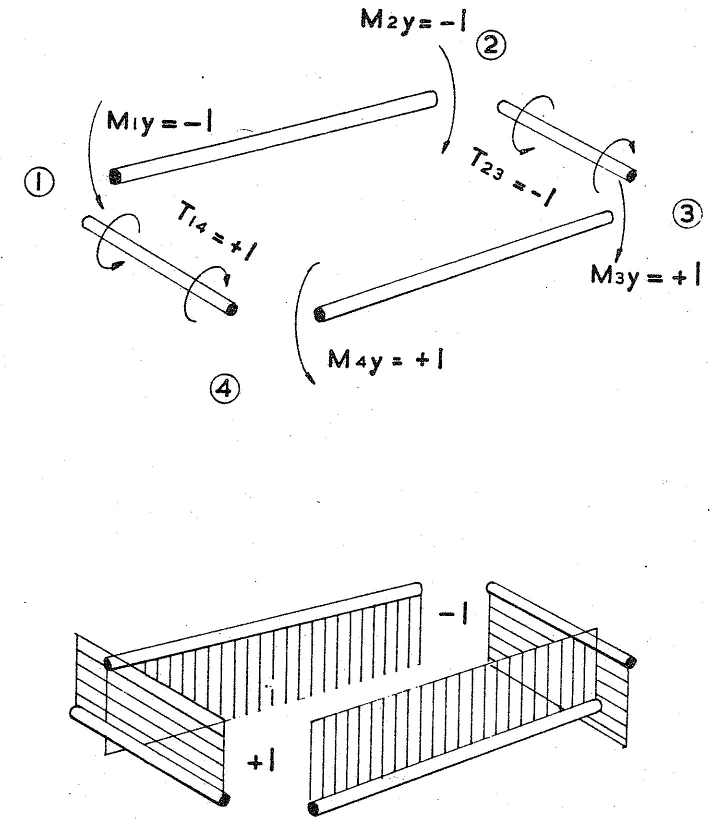
ALL MOMENTS & TORQUES DRAWN POSITIVE.

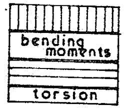
APP.2. FIG. 2



BASIC SYSTEM

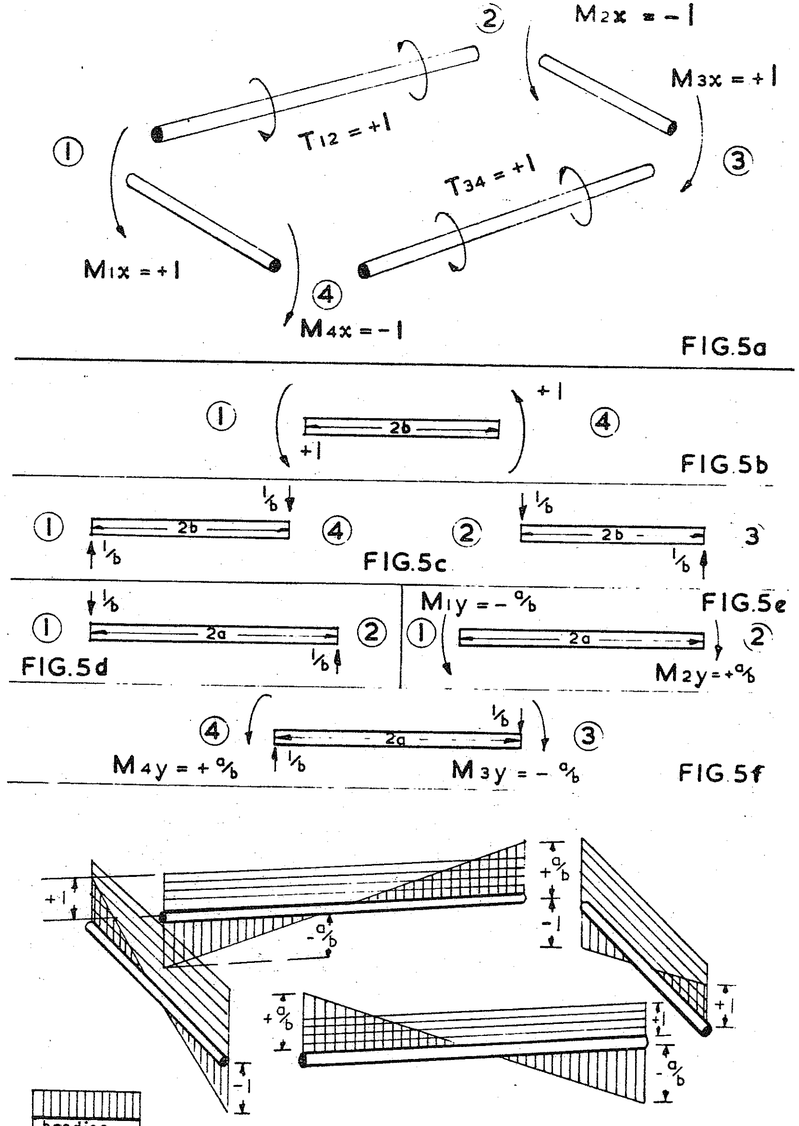
APP.2. FIG.3.

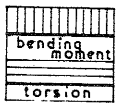




APP.2. FIG.4.

REDUNDANT SYSTEM 1





APP.2. FIG.5g.

REDUNDANT SYSTEM 2