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The Dry-Bed Problem in Shallow-Water Flows

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CONTENTS

ABSTRACT

The shallow water equations are a useful model for free-surface gravity water flows and some flows in the atmosphere. A common occurrence in water flows is the propagation of water fronts into a dry bed. In this paper we present the exact solution of the special Riemann problem in which one of the data states consists of a dry horizontal bed. The solution is then used locally in conjunction with the weighted average flux method (WAF) to solve the general initial boundary value problem for the unsteady one-dimensional shallow water equations.

1. INTRODUCTION

The shallow-water equations are a useful mathematical model for a good variety of fluid dynamics problems. Common examples are tidal waves in oceans, waves in shallow beaches and flood waves in rivers. If suitably reinterpreted, the same equations can be used to model atmospheric flows.

The shallow-water equations are an approximation to the full free-surface gravity flow problem with viscosity and surface tension neglected. They result from the additional assumption that the pressure is given as in hydrostatics (Stoker, 1957). The shallow-water equations are a set of non-linear hyperbolic equations. Their non-linear character rules out analytical techniques for most problems of practical interest. Their hyperbolic character rules out most obvious numerical methods, for discontinuous solutions are admissible.

Especially difficult problems to solve numerically are those involving bores or hydraulic jumps. Conservative, high resolution methods based on local Riemann problems have proved very successful in gas dynamics flows involving shock waves and contact surfaces. Their performance in solving the shallow-water equations is also very satisfactory (Glaister, 1987); Toro, 1990).

The special difficulty we are interested in in this paper is that in which water flows into a dry bed. This may be interpreted as yet another moving boundary problem. Our solution to the problem is based on the exact solution to the special Riemann problem in which one of the data states is a dry horizontal bed. The local solution is then utilised in conjunction with the Weighted Average Flux (WAF) Method (Toro, 1989, 1990). In this way, the moving interface between water and no-water is captured automatically by the numerical method.

Computed results are presented for a one-dimensional test problem. The paper is organised as follows: section 2 contains the formulation of the problem. In section 3 we solve the dry-bed Riemann problem. In section 4 we outline the WAF numerical method and the way in which the dry-bed Riemann problem can be incorporated into the method. A test with exact solution is presented. Conclusions are drawn in section 5.

2. FORMULATION OF THE PROBLEM

The unsteady one-dimensional shallow-water equations in pseudo-conservation form are

$$\begin{bmatrix} \phi \\ \phi u \end{bmatrix}_t + \begin{bmatrix} \phi u \\ \phi u^2 + \frac{1}{2}\phi^2 \end{bmatrix}_x = \begin{bmatrix} 0 \\ g\phi h_x \end{bmatrix} \qquad (1)$$

where

$$\phi = g(\eta + h) \qquad (2)$$

Here u is the particle velocity, h is the undisturbed depth, η is the elevation of the free surface and g is the acceleration due to gravity. Fig. 1 illustrates more clearly the meaning of h and η. Note that $\eta = \eta(x, t)$, $u = u(x, t)$ and $h = h(x)$. We consider the unknowns of the problem to be u and ϕ, which depend on space and time. For the purpose of this paper we shall consider the case of a horizontal bed. Equations (1) reduce to the homogeneous conservation laws

$$U_t + F(U)_x = 0 \qquad (3)$$

with

$$U = \begin{bmatrix} \phi \\ \phi u \end{bmatrix} \text{ and } F(U) = \begin{bmatrix} \phi u \\ \phi u^2 + \frac{1}{2}\phi^2 \end{bmatrix} \qquad (4)$$

U is the vector of conserved variables and F is the vector of respective fluxes.

Instead of expressing the conservation laws in differential form, as in (3), one can also write them in integral form as

$$\oint (Udx - Fdt) = 0 \tag{5}$$

This is more general, for it admits discontinuous solutions.

Equations (3) are hyperbolic with real eigenvalues

$$\lambda_1 = u - a \quad , \quad \lambda_2 = u + a \tag{6}$$

where the wave propagation speed a (celerity, or "sound speed") is given as

$$a = \sqrt{\phi} = \sqrt{g(\eta + h)} \tag{7}$$

The usual assumption is that $\eta > -h$, i.e. the free surface does not touch the bed. Then the eigenvalues λ_1 and λ_2 are distinct and the equations (3) are strictly hyperbolic.

In this paper we study the limiting case $\eta = -h$ in which $a = 0$ in some part of the domain.

3. THE DRY-BED RIEMANN PROBLEM

The Riemann problem for (3) is the initial-value problem (IVP)

$$\left.\begin{aligned} & U_t + F_x = 0 \;, \quad -\infty < x < \infty \;, \; t > 0 \\ & U(x,\,0) = \begin{cases} U_L \;, & x < 0 \\ U_R \;, & x > 0 \end{cases} \end{aligned}\right\} \quad (8)$$

where the states U_L and U_R are constant.

The exact solution to the IVP (8), along with several approximate Riemann solvers, were presented by this author in Ref. 4. (Toro, 1990) for the strictly hyperbolic case. The structure of the solution in the x - t plane is shown in Fig. 2. There is a left wave associated with the eigenvalue $\lambda_1 = u - a$ and a right wave associated with the eigenvalue λ_2. These simple waves separate three constant states U_L (left data), U_* and U_R (right data) and can be either bores or depression waves. The respective gas dynamics analogues are shocks and rarefactions.

For the case of smooth simple waves the solution at any point (x, t) can be determined completely by using two well known results, namely, (a) a depression fan is formed of characteristics of constant slope and (b) the Riemann invariants $I(\lambda_1)$ and $I(\lambda_2)$ are constant along characteristics associated with the wave speeds λ_1 and λ_2 respectively. The latter statement can more formally be expressed as

$$u - 2a = I(\lambda_1) \text{ along } \frac{dx}{dt} = u - a \quad (9)$$

$$u + 2a = I(\lambda_2) \text{ along } \frac{dx}{dt} = u + a \quad (10)$$

We now consider the Riemann problem for the limiting case in which one of the constant-data states is a dry bed. There are two cases.

CASE 1:

$$u(x,\ 0) = \begin{cases} u_L\ , & x < 0 \\ 0\ , & x > 0 \end{cases} \quad , \ \phi\ (x,\ 0) = \begin{cases} \phi_L > 0\ , & x < 0 \\ 0 & , \ x > 0 \end{cases}$$

Here the speed u_L is a constant that can be different from zero. The special case $u_L = 0$ is identical to the idealised dam-break problem in which the water flows into a dry bed. See Stoker (1957) for details.

The star constant state in the conventional Riemann problem (see Fig. 2) is not present here, neither is the right wave. There is only one wave present and that is a depression wave on the left side. This is consistent with the loss of strict hyperbolicity. The structure of the solution is shown in Fig. 3. The bounding characteristics on the left and right are given by

$$\frac{dx}{dt} = u_L - a_L = S_L \tag{11}$$

$$\frac{dx}{dt} = S_R \tag{12}$$

These correspond to the head and tail of the depression respectively. The terminal characteristic $\frac{dx}{dt} = S_R$ represents the water front and may be viewed as the coalescence of the head of the depression, the missing right wave (trivial) and the missing constant state star.

The solution U(x, t) outside the depression fan is trivial. See Fig. 3. To find the solution at a point $(\hat{x}, \hat{t})$ inside the fan we proceed as follows:

The straight characteristic through the origin (0, 0) and $(\hat{x}, \hat{t})$ satisfy

$$\frac{\hat{x}}{\hat{t}} = u - a \qquad (13)$$

and the right Riemann invariant (equation 10) gives

$$u_L + 2a_L = u + 2a \qquad (14)$$

These two equations in the two unknowns u and a have solutions

$$u = \frac{1}{3} (u_L + 2a_L + 2 \hat{x}/ \hat{t}) \qquad (15)$$

$$a = \frac{1}{3} (u_L + 2a_L - \hat{x}/ \hat{t}) \qquad (16)$$

The limiting characteristic $\frac{dx}{dt} = S_R = u^* - a^*$ must have $a^* = 0$, since $\phi_R = 0$ and so equation (14) gives immediately

$$S_R = u_L + 2a_L \qquad (17)$$

CASE 2:

$$u(x, 0) = \begin{cases} 0 \; , & x < 0 \\ u_R \; , & x > 0 \end{cases} \; ; \; \phi(x, 0) = \begin{cases} 0 \; , & x < 0 \\ \phi_R > 0 \; , & x > 0 \end{cases}$$

In this case the left wave is missing and the right wave is a depression wave bounded by the characteristics

$$\frac{dx}{dt} = S_L = u_R - 2a_R \tag{18}$$

and

$$\frac{dx}{dt} = S_R = u_R + a_R \tag{19}$$

The solution for u and a inside the depression fan is found to be

$$u = \frac{1}{3}(u_R - 2a_R + 2\,\hat{x}/\hat{t}) \tag{20}$$

$$a = \frac{1}{3}(-u_R + 2a_R + \hat{x}/\hat{t}) \tag{21}$$

The exact solution to the dry-bed Riemann problem is now complete for both cases 1 and 2. This solution can be used directly to solve exactly dam-break problems into a dry bed; they are a special case in which all velocities are zero at time zero.

Fig. 4 shows the exact solution to the example

$$\left.\begin{aligned} \phi(x,0) &= \begin{cases} 1.0 , & 0 \le x < \frac{1}{2} \\ 0.0 , & \frac{1}{2} < x \le 1 \end{cases} \\ u(x,0) &= 0 , \quad 0 \le x \le 1 \end{aligned}\right\} \qquad (22)$$

at time 0.25 units.

The only wave is a left depression wave. The free surface touches the bed, tangentially, at the point $x = (u_L + 2a_L)t$.

There are now at least two ways of utilising the exact solution of the dry-bed Riemann problem to solve, numerically, the general initial-boundary value problem for (3). Front tracking is one of them. Here we shall adopt the front capturing alternative.

4. EXTENSION OF THE WAF NUMERICAL METHOD

The weighted average flux method, or WAF, was presented by this author in Toro (1989). This is a convservative, shock capturing high resolution method based on local solutions to Riemann problems.

Consider a control volume labelled i of dimensions Δx by Δt in the x - t plane as shown in Fig. 5. Evaluation of the integral form (5) of the conservation laws (3) gives

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x}\left(F_{i-1/2} - F_{i+1/2} \right) \qquad (23)$$

This is an explicit conservative scheme whose numerical flux $F_{i+1/2}$ remains to be defined. The WAF method has two possibilities for defining a numerical flux. Here we adopt

$$F_{i+1/2} = F(V_{i+1/2}) \tag{24}$$

where

$$V_{i+1/2} = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \tilde{V} \, (x/\frac{\Delta t}{2}, V_i , V_{i+1}) \, dx \tag{25}$$

$\tilde{V}$ is the solution to the Riemann problem with data V_i (left), V_{i+1} (right) at time $t = \Delta t/2$. There is freedom in choosing the variable V. The choice $V = (u, a)^T$ is convenient here, for the exact integration in (25) can be easily performed. Details of the WAF method as applied to the strictly hyperbolic equations can be found in Toro (1990). Here we derive the intercell flux (24) - (25) for the dry-bed case. We first evaluate the integral averages (25) for u and a for case 1 of the previous section. Fig. 6 shows the integration path for (25) which can be written as

$$V_{i+1/2} = W_1 V_i + \frac{1}{\Delta x} \int_{x_L}^{x_R} \tilde{V} dx + W_3 V_{i+1} \tag{26}$$

where

$$W_1 = \frac{1}{2} (1 + \nu_1) , \quad W_3 = \frac{1}{2} (1 - \nu_2) \tag{27}$$

$$\nu_1 = \frac{\Delta t}{\Delta x}(u_i - a_i) \quad , \quad \nu_2 = \frac{\Delta t}{\Delta x}(u_i + 2a_i) \tag{28}$$

$$x_L = (u_i - a_i)\frac{\Delta t}{2} \quad , \quad x_R = (u_i + 2a_i)\frac{\Delta t}{2} \tag{29}$$

W_1 and W_2 are the normalised geometric extents of the intervals [A, B] and [C, D] of Fig. 6 respectively; ν_1 and ν_2 are the Courant numbers associated with the bounding wave speeds present in the solution of the Riemann problem; x_L and x_R are distances transversed in time $\Delta t/2$ by the speed $S_L = u_i - a_i$ and $S_R = u_i + 2a_i$.

The second term in (26) remains to be found. The integrals for each component $u_{i+1/2}$ and $a_{i+1/2}$ of $V_{i+1/2}$ can be evaluated exactly using (15) and (16). Setting

$$I_u = \frac{1}{\Delta x}\int_{x_L}^{x_R} \tilde{u}\, dx = \frac{1}{\Delta x}\int_{x_L}^{x_R} \frac{1}{3}\left(u_i + 2a_i + \frac{4x}{\Delta t}\right) dx$$

and after some algebra we obtain

$$I_u = \frac{3a_i\ \Delta t}{2\Delta x}(u_i + a_i)$$

It is convenient to express this as

$$I_u = \frac{3a_i\ \Delta t}{2\Delta x}(u_i + a_i) = W_2\ u_2$$

where $W_2 = \frac{x_R - x_L}{\Delta x}$ is the normalised extent of the path of integration BC in Fig. 6 across the depression fan. This is accomplished by

$$\left.\begin{aligned} I_u &= W_2\, u_2 \\ W_2 &= \frac{3a_i\, \Delta t}{2\Delta x} \quad , \quad u_2 = u_i + a_i \end{aligned}\right\} \qquad (27)$$

For the celerity a we have

$$I_a = \frac{1}{\Delta x} \int_{x_L}^{x_R} \tilde{a}\, dx = \frac{1}{\Delta x} \int_{x_L}^{x_R} \frac{1}{3} (u_i + 2a_i - \frac{2x}{\Delta t})\, dx$$

which, after performing the integration gives

$$I_a = W_2\, a_2$$

$$W_2 = \frac{3a_i\, \Delta t}{2\Delta x} \quad , \quad a_2 = \frac{1}{2} a_i$$

Hence, the integral average (25) or (26) for case 1 (dry bed is on right hand side) is now complete. The averaged vector $V_{i+1/2} = (u_{i+1/2}\ ,\ a_{i+1/2})^T$ is given by

$$\begin{bmatrix} u_{i+1/2} \\ a_{i+1/2} \end{bmatrix} = W_1 \begin{bmatrix} u_i \\ a_i \end{bmatrix} + W_2 \begin{bmatrix} u_2 \\ a_2 \end{bmatrix} + W_3 \begin{bmatrix} u_{i+1} \\ a_{i+1} \end{bmatrix} \qquad (28)$$

where

$$\left.\begin{aligned}
&S_L = u_i - a_i \quad , \quad S_R = u_i + 2a_i \\
&\nu_1 = \frac{\Delta t}{\Delta x} S_L \quad , \quad \nu_2 = \frac{\Delta t}{\Delta x} S_R \\
&W_1 = \frac{1}{2} (1 + \nu_1) \quad , \quad W_3 = \frac{1}{2} (1 - \nu_2) \\
&W_2 = \frac{3}{2} a_i \frac{\Delta t}{\Delta x} = 1 - (W_1 + W_3) \\
&u_2 = u_i + a_i \quad , \quad a_2 = \frac{1}{2} a_i
\end{aligned}\right\} \qquad (29)$$

For case 2, in which the dry bed is on the left hand side, the corresponding result is as in (28) with

$$\left.\begin{aligned}
&S_L = u_{i+1} - 2a_{i+1} \quad , \quad S_R = u_{i+1} + a_{i+1} \\
&\nu_1 = \frac{\Delta t}{\Delta x} S_L \quad , \quad \nu_2 = \frac{\Delta t}{\Delta x} S_R \\
&W_1 = \frac{1}{2} (1 + \nu_1) \quad , \quad W_3 = \frac{1}{2} (1 - \nu_2) \\
&W_2 = \frac{3}{2} a_{i+1} \frac{\Delta t}{\Delta x} = 1 - (W_1 + W_3) \\
&u_2 = u_{i+1} - a_{i+1} \quad , \quad a_2 = \frac{1}{2} a_{i+1}
\end{aligned}\right\} \qquad (30)$$

The numerical fluxes $F_{i+1/2} = F(V_{i+1/2})$ in (24) can now be computed according to the definitions (4) for the physical fluxes and noting definition (7) for the celerity a. These fluxes make the scheme (23) second order accurate. Consequently, spurious oscillations are expected near high gradients. The oscillation-free version of the method (see Toro, 1990) modifies the weights W_k in (28) by modifying the wave speeds, or equivalently, the Courant numbers ν_1, ν_2. For the particular case in which the local Riemann problem (i, i+1) is that of a dry bed, cases (29) or (30), the modification of the weights should be performed in the usual way for W_1 and W_3 and then set $W_2 = 1 - (W_1 + W_3)$.

We now apply the numerical method just described to solve problem (22). Fig. 7 shows a comparison between the exact solution (shown by the full line) and the numerical solution (shown in symbols). The agreement is excellent. A very small entropy glitch is observed in the numerical solution. Locally sonic flow occurs in this problem at the position $x = 1/2$, where the initial discontinuity was positioned at time zero. Since the method is entropy satisfying (Toro, 1990) the problem would tend to disappear with fine grids. In practice, care is required in defining a dry bed. This is related to the zero of the particular computer in use.

Extension of the numerical method to more realistic models, including for instance variable beds, bottom roughness, or two space dimensions, is possible following the ideas set out in Ref. 4 (Toro, 1990).

CONCLUSIONS

The weighted average flux (WAF) numerical method has been extended to deal with problem of propagation of water flows into a dry bed. This is accomplished by solving exactly the special Riemann problem in which one of the data states is a dry bed. This local solution is incorporated into the front capturing approach of the WAF method. Numerical fluxes are derived by taking exact integral averages of the particle velocity u and the celerity a. Application of the method to a one dimensional test problem with exact solution gives very satisfactory results.

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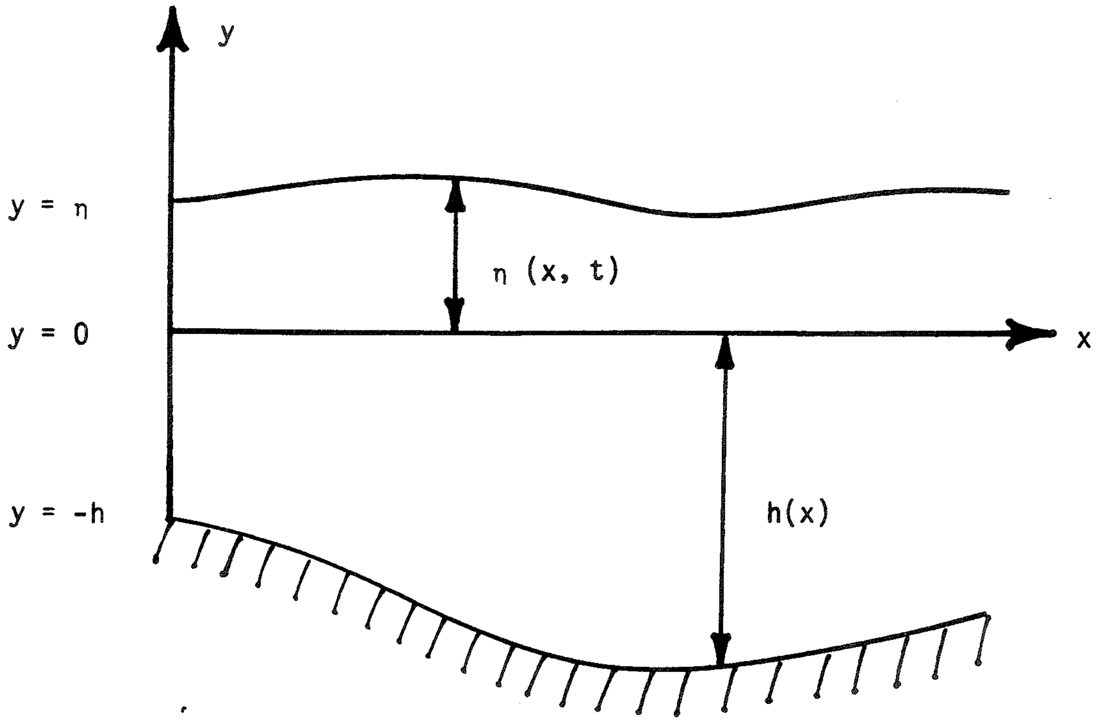

Fig. 1 Flow configuration. Undisturbed depth is $y = 0$, bed profile is $y = -h$ and free surface elevation is $y = \eta$, the total depth is $\eta + h$.

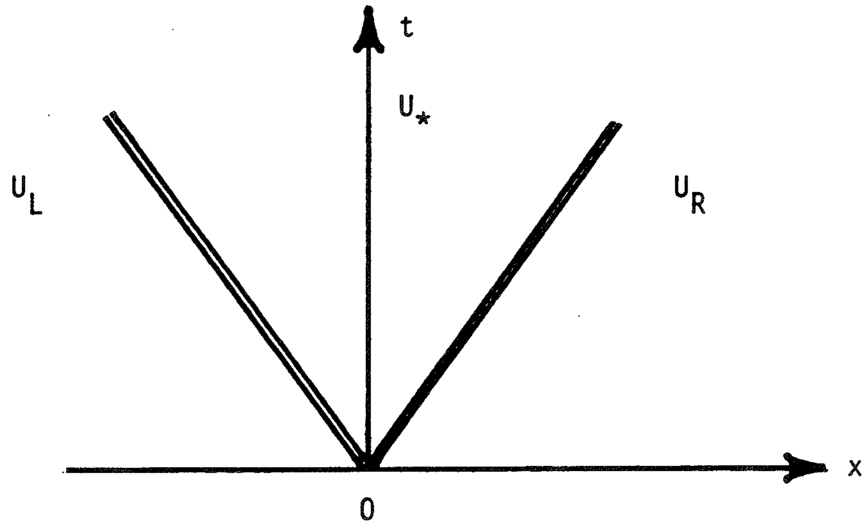

Fig. 2 Structure of solution of Riemann problem with data U_L, U_R for the strictly hyperbolic case $\eta > -h$.

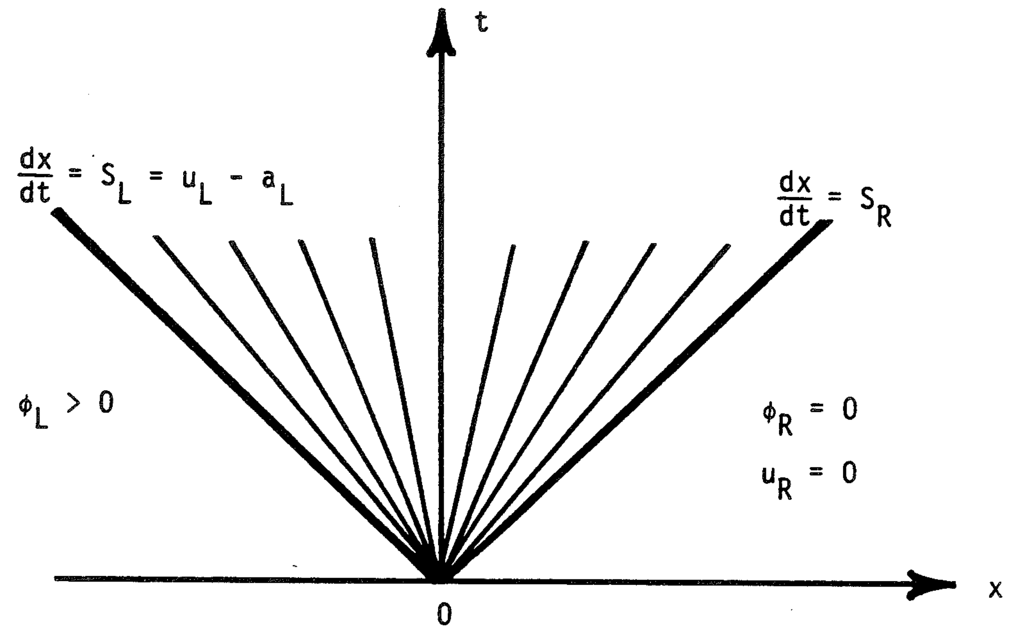

Fig. 3 Structure of solution of dry-bed Riemann problem.
Dry-bed is on right-hand side.

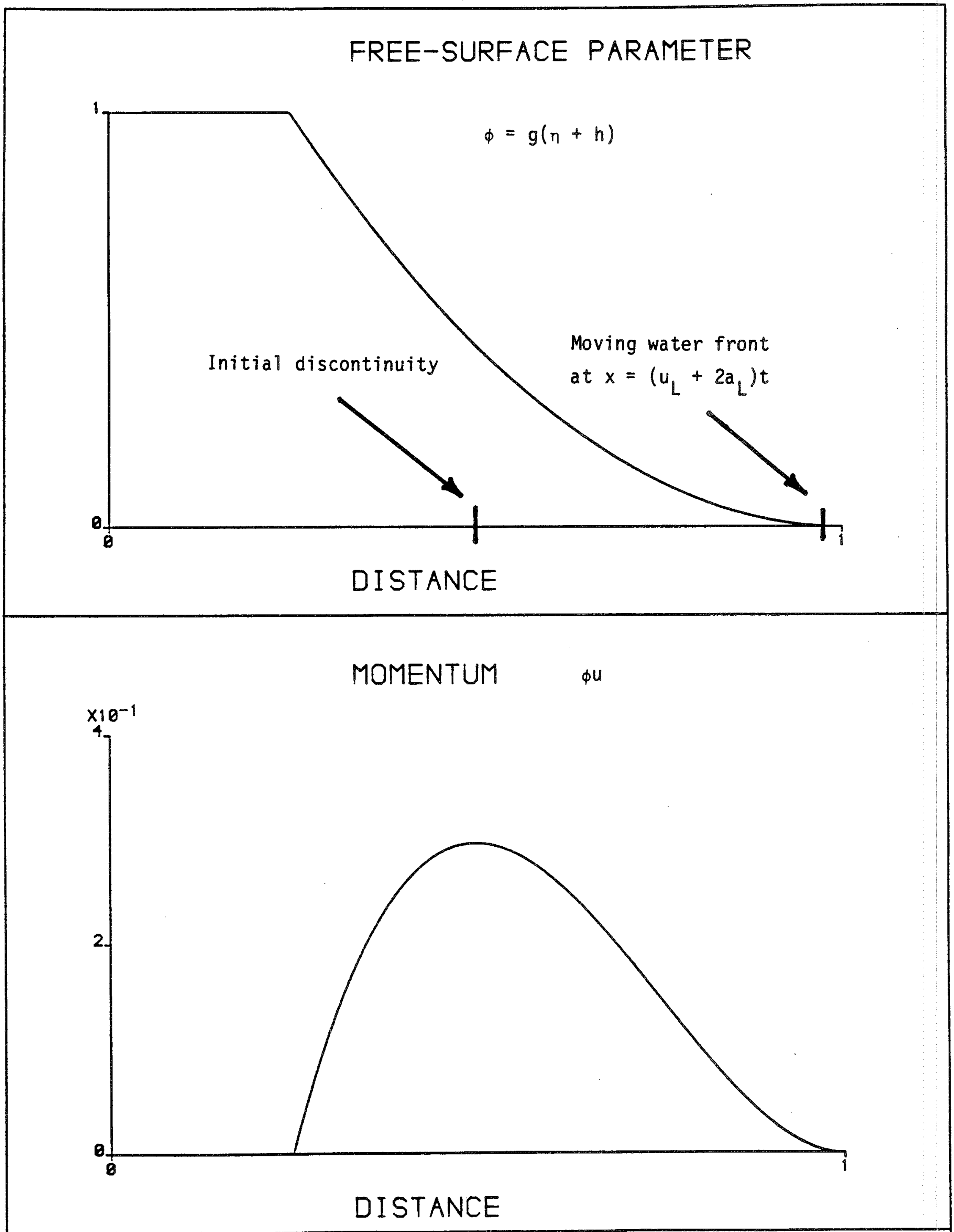

Fig.4 Exact solution for ϕ and ϕu at time 0.25. Dry bed is on right-hand side.

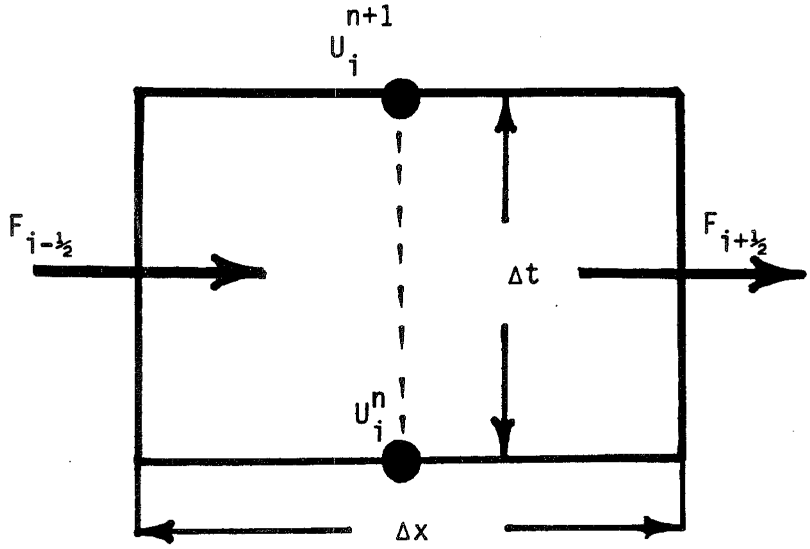

Fig. 5 Control volume i in the x-t plane of dimensions Δx by Δt. Conservation laws are satisfied in i.

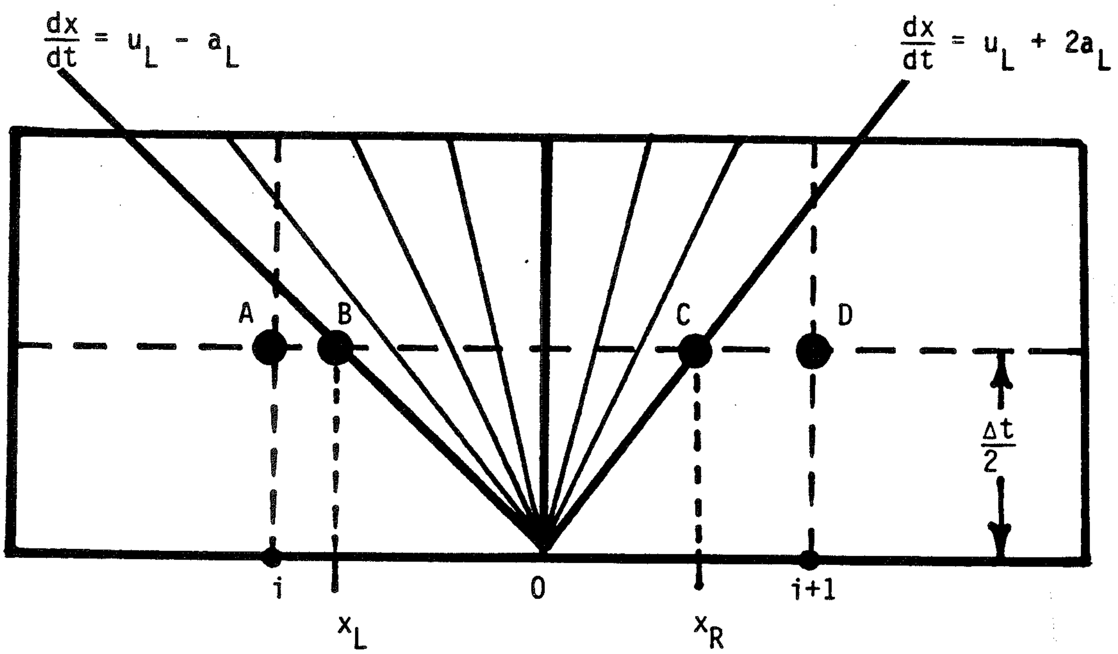

Fig. 6 Integration path for the calculation of the averages $u_{i+\frac{1}{2}}$ and $a_{i+\frac{1}{2}}$

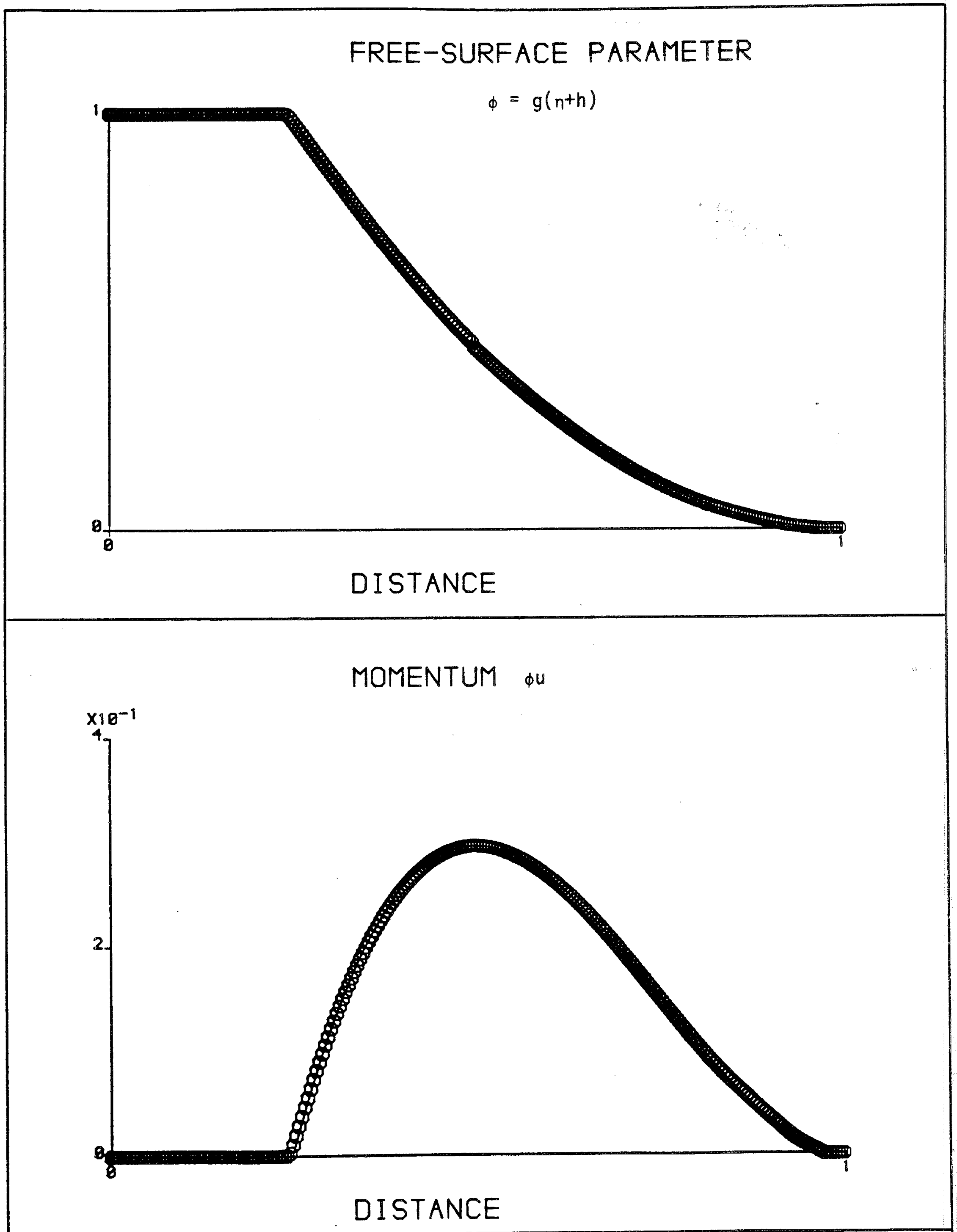

Fig.7 Numerical (symbol) and exact solutions (line) at time t = 0.25. Dry bed is on right-hand side.