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Robust Controller Design Using H_∞ Loop-Shaping and the Method of Inequalities

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Abstract

A new approach to robust controller design is proposed. By using plant weighting functions as the design parameters, the approach combines the method of inequalities with robust stabilization of normalized coprime factor descriptions of the weighted plant to design explicitly for closed-loop performance and stability robustness. A procedure for the design of robust two degree-of-freedom controllers is presented, and is illustrated on a high-purity distillation column example.

1 Introduction

Specifications for the performance of feedback control systems are often expressed in terms of inequalities which need to be satisfied. This fact resulted in the development of the method of inequalities [20], a design method where the design objectives are expressed explicitly as a set of algebraic inequalities representing desired bounds on a set of performance indices. For a successful design, the inequalities must be satisfied. A separate development has been the use of H_∞-optimization in a variety of approaches to design robust control systems. One such approach is the loop-shaping design procedure (LSDP) [11, 12]. This approach involves the robust stabilization to additive perturbation in the sense of H_∞-norm of normalized coprime factors of a weighted plant. The weighted plant singular values are shaped by adjusting the weighting functions to give a desired open-loop shape which gives good closed-loop performance with stability robustness.

Certain aspects of the LSDP make it suitable to combine this approach with the method of inequalities to design directly for both closed-loop performance and stability robustness. This paper describes this new approach, and applies the proposed method to the design of a control system for a high purity distillation column, a plant which has received considerable attention in the literature of late [6, 10, 15, 16, 19].

2 Normalized Coprime Factorization

The plant model $G = \tilde{M}^{-1}\tilde{N}$, is a normalized left coprime factorization (NLCF) of G if $\tilde{M}, \tilde{N} \in RH_\infty$; there exists $V, U \in RH_\infty$ such that $\tilde{M}V + \tilde{N}U = I$; and $\tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I$ where for a real rational function of s, X^* denotes $X'(-s)$.

Using the notation

$$G(s) = D + C(sI - A)^{-1}B \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \tag{1}$$

then as shown in [11]

$$[\,\tilde{N} \quad \tilde{M}\,] \overset{s}{=} \left[\begin{array}{c|cc} A + HC & B + HD & H \\ \hline R^{-1/2}C & R^{-1/2}D & R^{-1/2} \end{array}\right], \tag{2}$$

is a normalized coprime factorization of G where $H = -(BD' + ZC')R^{-1}$, $R = I + DD'$, and the matrix $Z \geq 0$ is the unique stabilizing solution to the algebraic Riccati equation (ARE)

$$(A - BS^{-1}D'C)Z + Z(A - BS^{-1}D'C)' - ZC'R^{-1}CZ + BS^{-1}B' = 0, \tag{3}$$

where $S = I + D'D$.

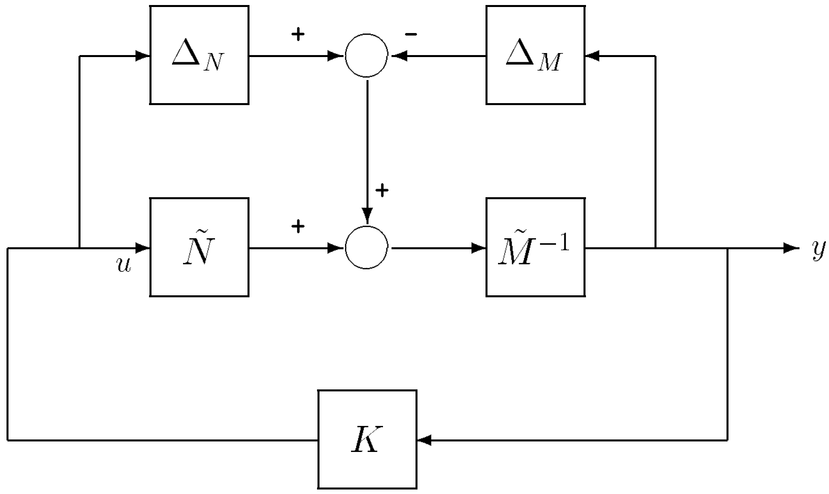

Figure 1: Robust stabilization with respect to coprime factor uncertainty

A perturbed model G_P is defined as

$$G_P = (\tilde{M} + \Delta_M)^{-1}(\tilde{N} + \Delta_N) \tag{4}$$

where $\Delta_M, \Delta_N \in RH_\infty$. To maximize the class of perturbed models defined by (4) such that the configuration of Figure 1 is stable, we need to find the controller K which stabilizes the nominal closed-loop system and which minimizes γ where

$$\gamma = \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \tilde{M}^{-1} \right\|_\infty . \tag{5}$$

This is the problem of robust stabilization of normalized coprime factor plant descriptions as introduced in [5]. From the small gain theorem, the closed-loop system will remain stable if

$$\| [\, \Delta_N \quad \Delta_M \,] \|_\infty < \gamma^{-1}. \tag{6}$$

The minimum value of γ for all stabilizing controllers K is

$$\gamma_0 = \inf_{K \text{ stabilizing}} \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \tilde{M}^{-1} \right\|_\infty . \tag{7}$$

It is shown in [5] that

$$\gamma_0 = \left(1 + \lambda_{\max}(ZX)\right)^{1/2} . \tag{8}$$

where $\lambda_{\max}(\cdot)$ represents the maximum eigenvalue, and $X \geq 0$ is the unique stabilizing solution of the ARE

$$(A - BS^{-1}D'C)'X + X(A - BS^{-1}D'C) - XBS^{-1}B'X + C'R^{-1}C = 0. \tag{9}$$

A controller which achieves γ_0 is given in [11] by

$$K \stackrel{s}{=} \left[\begin{array}{c|c} A + BF + \gamma_0^2(Q')^{-1}ZC'(C + DF) & \gamma_0^2(Q')^{-1}ZC' \\ \hline B'X & -D' \end{array} \right], \tag{10}$$

where $F = -S^{-1}(D'C + B'X)$, and $Q = (1 - \gamma_0^2)I + XZ$.

From the above, the optimum controller is synthesized by the solution of two ARE's, unlike most H_∞ problems, which require an iterative search on γ to find the optimum.

In practice, to design control systems using normalized coprime factorizations, the plant needs to be weighted to meet closed-loop performance requirements. A design procedure has been developed [11, 12], known as the loop-shaping design procedure (LSDP), to choose the weights by studying the open-loop singular values of the plant, and augmenting the plant with weights so that the weighted plant has an open-loop shape which will give good closed-loop performance.

The nominal plant G is augmented with pre- and post-compensators W_1 and W_2 respectively, so that the augmented plant G_s is $G_s = W_2GW_1$. Using the procedure outlined earlier, an optimum feedback controller K_s is synthesized which robustly stabilizes the NLCF of G_s given by $(\tilde{N}_s, \tilde{M}_s)$ where $G_s = \tilde{M}_s^{-1}\tilde{N}_s$. The final feedback controller K is then constructed by simply combining K_s with the weights to give

$$K = W_1K_sW_2. \tag{11}$$

Note that from [11],

$$\gamma_0 = \inf_{K \text{ stabilizing}} \left\| \begin{bmatrix} W_1^{-1}K \\ W_2 \end{bmatrix} (I - GK)^{-1} \begin{bmatrix} W_2^{-1} & GW_1 \end{bmatrix} \right\|_\infty. \tag{12}$$

Essentially, with the LSDP, the weights W_1 and W_2 are the design parameters which are chosen both to give the augmented plant a 'good' open-loop shape and to ensure that γ_0 is not too large. γ_0 is a design indicator of the success of the loop-shaping as well as a measure of the robustness of the stability property.

3 The Method of Inequalities

Performance specifications for control systems are frequently given in terms of algebraic or functional inequalities, rather than in the minimization of some objective function.

For example, the system may be required to have a rise-time of less than 1 second, a settling time of less than 5 seconds and an overshoot of less than 10%. In such cases, it is obviously more logical and convenient if the design problem is expressed explicitly in terms of such inequalities.

The method of inequalities (MOI) [20] is a computer-aided multi-objective design approach, where desired performance is represented by such a set of algebraic inequalities, and where the aim of the design is to simultaneously satisfy these inequalities. The design problem is expressed as

$$\phi_i(p) \leq \varepsilon_i \quad \text{for} \quad i = 1 \ldots n, \tag{13}$$

where ε_i are real numbers, $p \in \mathcal{P}$ is a real vector $(p_1, p_2, \ldots, p_q)$ chosen from a given set $\mathcal{P}$ and ϕ_i are real functions of p. The functions ϕ_i are performance indices, the components of p represent the design parameters and ε_i are chosen by the designer and represent the largest tolerable values of ϕ_i. The aim is the satisfaction of the set of inequalities in order that an acceptable design p is reached.

For control system design, the functions $\phi_i(p)$ may be functionals of the system step response, for example the rise-time, overshoot or the integral absolute error, or functionals of the frequency response, such as the bandwidth. They can also represent measures of the system stability, such as the maximum real part of the closed-loop poles. Additional inequalities which arise from the physical constraints of the system can also be included, to restrict for example, the maximum control signal. In practice, the constraints on the design parameters p which define the set $\mathcal{P}$ are also included in the inequality set, e.g. to constrain the possible values of some of the design parameters, or to limit the search to stable controllers only.

Each inequality $\phi_i(p) \leq \varepsilon_i$ of the set of inequalities (13) defines a set $\mathcal{S}_i$ of points in the q-dimensional space $\mathbb{R}^q$ and the co-ordinates of this space are $p_1, p_2, \ldots, p_q$, so

$$\mathcal{S}_i = \{p : \phi_i(p) \leq \varepsilon_i\}. \tag{14}$$

The boundary of this set is defined by $\phi_i(p) = \varepsilon_i$. A point $p \in \mathbb{R}^q$ is a solution to the set of inequalities (13) if and only if it lies inside every set $\mathcal{S}_i$, $i = 1, 2, \ldots, n$ and hence inside the set $\mathcal{S}$ which denotes the intersection of all the sets $\mathcal{S}_i$,

$$\mathcal{S} = \bigcap_{i=1}^{n} \mathcal{S}_i. \tag{15}$$

$\mathcal{S}$ is called the admissible set and any point p in $\mathcal{S}$ is called an admissible point denoted p_s.

The objective is thus to find a point p such that $p \in \mathcal{S}$. Such a point satisfies the set of inequalities (13) and is said to be a solution. In general, a point p_s is not unique unless the subset $\mathcal{S}$ is a point in the space $\mathbb{R}^q$. In some cases, there is no solution to the problem, i.e. $\mathcal{S}$ is an empty set. It is then necessary to relax the boundaries of some of the inequalities, i.e. increase some of the numbers ε_i, until an admissible point p_s exists.

The actual solution to the set of inequalities (13) may be obtained by means of numerical search algorithms, such as the moving boundaries process (MBP), details of the MBP may be found in [17] and [20]. The procedure for obtaining a solution is interactive, in that it requires supervision and intervention from the designer. The designer needs to choose the configuration of the design, which determines the dimension of the design parameter vector p, and initial values for the design parameters. The progress of the search algorithm should be monitored, and, if a solution is not found, the designer may either change the starting point, relax some of the desired bounds ε or change the design configuration. Alternatively, if a solution is found easily, to improve the quality of the design, the bounds could be tightened or additional design objectives could be included in (13). The design process is thus a two way process, with the MOI providing information to the designer about conflicting design requirements, and the designer making decisions about the 'trade-offs' between design requirements based on this information as well as on the designer's knowledge, experience and intuition about the particular problem. The designer can be supported in this role by various graphical displays [14] which provide information about the progress of the search algorithm and about the conflicting design requirements.

In some previous applications of the MOI, the design parameter p has parameterized a controller with a particular structure. For example, $p = (p_1, p_2)$ could parameterize a PI controller $p_1 + p_2/s$. This has meant that the designer has had to choose the structure of the control scheme and the order of the controllers. In general, the lower the dimension of the design vector p, the easier it is for the numerical search algorithm to find a solution, if one exists. Although this does give the designer some flexibility and leads to simple controllers, and is of particular value when the structure of the controller is constrained in some way, it does mean that better solutions may exist with more complicated and higher order controllers. A further limitation of using the MOI in this way is that a stability point must be located as a pre-requisite to searching the parameter space to improve the index set ϕ, this issue is addressed in more detail in [20].

4 Robust Design Using a Coprime Factor Plant Description with the Method of Inequalities

Two aspects of design using robust stabilization of normalized coprime factor descriptions of the weighted plant make it amenable to combine this approach with the MOI. Firstly, unlike most H_∞-optimization problems, the H_∞-optimal controller for the weighted plant can be synthesized from the solution of just two ARE's and does not require time-consuming γ-iteration. Secondly, in the LSDP described in Section 2, the weighting functions are chosen by considering the open-loop response of the weighted plant, so effectively the weights W_1 and W_2 are the design parameters. This means that the design problem can be formulated as in the method of inequalities, with the weighting parameters used as the design parameters p to satisfy some set of *closed-loop* performance inequalities.

Such an approach to the MOI overcomes the limitations to the MOI described at the end of Section 3. The designer does not have to choose the order or structure of the controller, but instead chooses the structure and order of the weighting functions. With low-order weighting functions, high order controllers can be synthesized which often lead to significantly better performance or robustness than if simple low order controllers were used. Additionally, the problem of finding a stability point does not exist because stability is guaranteed through the solution to the robust stabilization problem, provided that the weighting functions do not cause undesirable pole/zero cancellations.

Improved performance for tracking systems may be obtained with a 2 degree-of-freedom (2 DOF) scheme. Tis is done by including a pre-compensator K_p on the reference input, as shown in Figure 2. The pre-compensator is parameterized with a sub-set of the design parameters; the controller K_s is the solution to the weighted normalized coprime factor approach already described, with the weighting functions W_1 and W_2 parameterized with the remaining design parameters. An analytical method of designing a 2 DOF controller with the LSDP is described in [6], this method can also be combined with the MOI [13, 18].

The design problem is now stated as follows:

Problem

For the system of Figure 2, find a (W, K_p) such that

$$\gamma_0(W) \leq \varepsilon_\gamma, \tag{16}$$

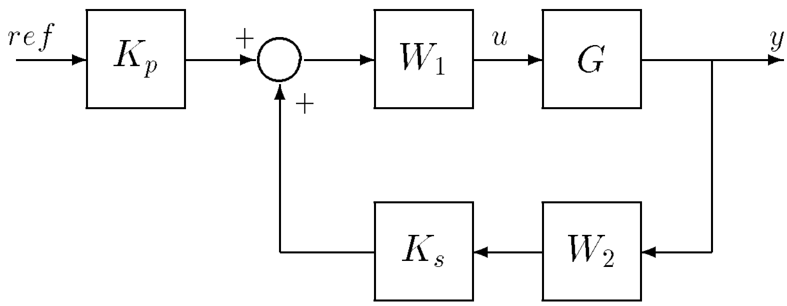

Figure 2: The 2 DOF scheme

and

$$\phi_i(W, K_p) \le \varepsilon_i \quad \text{for} \quad i = 1 \ldots n, \tag{17}$$

where

$$\gamma_0(W) = \inf_{K \text{ stabilizing}} \left\| \begin{bmatrix} W_1^{-1} K \\ W_2 \end{bmatrix} (I - GK)^{-1} \begin{bmatrix} W_2^{-1} & GW_1 \end{bmatrix} \right\|_\infty , \tag{18}$$

and $\phi_i(W, K_p)$ are functionals of the 2 DOF closed-loop system, ε_γ, ε_i are real numbers representing desired bounds on γ_0 and ϕ_i respectively, $W = (W_1, W_2)$ is a pair of fixed order weighting functions with real parameters $w = (w_1, w_2, \ldots, w_q)$ and K_p is a pre-compensator with a fixed structure and order and with real parameters $p = (p_1, p_2, \ldots, p_r)$.

Design Procedure

A design procedure to solve the above problem is:

i) Define the plant G, and define the functionals ϕ_i.

ii) Define the values of ε_γ and ε_i

iii) Define the form and order of the weighting functions W_1 and W_2. Bounds should be placed on the values of w_i to ensure that W_1 and W_2 are stable and minimum phase to prevent undesirable pole/zero cancellations. The order of the weighting functions, and hence the value of q, should initially be small.

iv) Define the form and order of the pre-compensator K_p. Bounds may be placed on the values of p_i if desired. The order of the pre-compensator transfer function, and hence the value of r, should initially be small.

v) Define initial values of w_i based on the open-loop frequency response of the plant. Define initial values of p_i.

vi) Implement the MBP in conjunction with (8) and (10) to find a W and K_p which satisfy inequalities (16) and (17). If a solution is found, the design is satisfactory. If no solution is found, either increase the order of the weighting functions or pre-compensator, relax one or more of the desired bounds ε, or try again with different initial values of w and p.

vii) With satisfactory weighting functions W_1 and W_2, a satisfactory feedback controller is obtained from (11).

5 Example : The Distillation Column

The proposed design method is used to design a control system for the high purity distillation column described in [9]. The column is considered in just one of its configurations, the LV configuration, for which the following model is relevant

$$G_D(s, k_1, k_2, \tau_1, \tau_2) = \frac{1}{75s+1}\begin{bmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{bmatrix}\begin{bmatrix} k_1 e^{-\tau_1 s} & 0 \\ 0 & k_2 e^{-\tau_2 s} \end{bmatrix} \tag{19}$$

where $0.8 \leq k_1, k_2 \leq 1.2$ and $0 \leq \tau_1, \tau_2 \leq 1$, and all time units are in minutes. The time delay and actuator gain values used in the nominal model G are $k_1 = k_2 = 1.0$ and $\tau_1 = \tau_2 = 0.5$. The time delay element is approximated by a first-order Padé approximation.

The design specifications are to design a controller which guarantees for all $0.8 \leq k_1, k_2 \leq 1.2$ and $0 \leq \tau_1, \tau_2 \leq 1$:

i) Closed-loop stability.

ii) The output response to a step demand $h(t)\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ satisfies $y_1(t) \leq 1.1$ for all t, $y_1(t) \geq 0.9$ for all $t > 30$ and $y_2(t) \leq 0.5$ for all t.

iii) The output response to a step demand $h(t)\left[\begin{smallmatrix}0.4\\0.6\end{smallmatrix}\right]$ satisfies $y_1(t) \leq 0.5$ for all t, $y_1(t) \geq 0.35$ for all $t > 30$, $y_2(t) \leq 0.7$ for all t and $y_2(t) \geq 0.55$ for all $t > 30$.

iv) The output response to a step demand $h(t)\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ satisfies $y_1(t) \leq 0.5$ for all t, $y_2(t) \leq 1.1$ for all t and $y_2(t) \geq 0.9$ for all $t > 30$.

v) Zero steady state error.

vi) The frequency response of the closed-loop transfer function between demand input and plant input is gain limited to 50 dB and the unity gain cross over frequency of its largest singular value should be less than 150 rad/min.

The first design attempt was to use the MOI to satisfy the performance design specifications for the nominal plant G using the configuration of Figure 2. The design criteria were, from (16) and (17),

$$\gamma_0(W) \;\leq\; \varepsilon_\gamma, \tag{20}$$

$$\phi_i(G, W, K_p) \;\leq\; \varepsilon_i, \quad \text{for} \quad i = 1, 2, \ldots, 12, \tag{21}$$

where the prescribed bound for γ_0 is not fixed, but for stability robustness, it should not be too large [11], and is here taken as

$$\varepsilon_\gamma = 5.0. \tag{22}$$

The performance functionals $\phi_i(G_D, W, K_p)$ are defined in the Appendix, and the respective prescribed bounds are decided from the design specifications and are shown in Table 2 (note that ε_{15} is in dB and ε_{16} is in rad/sec).

An integrator term was included in W_1, which ensures that the final controller has integral action and the steady state specifications are satisfied. To ensure the steady state error specifications are met, the pre-compensator is set to be the gain matrix $K_p = K_s(0)W_2(0)$ where

$$K_s(0)W_2(0) = \lim_{s\to 0} K_s(s)W_2(s). \tag{23}$$

With weighting functions $W_1 = w_1(s + w_2)/s(s + w_3)I_2$, and $W_2 = I_2$; the design procedure described in Section 4 was implemented in Matlab on a Sun SPARCstation, and a design that successfully satisfied inequalities (20) and (21) obtained easily. The performance was then tested with various values of τ_1, τ_2, k_1 and k_2, and the design was found to be not very robust.

To obtain robust performance, the next attempt was to satisfy the performance design specifications for several plant models each at an extreme of the parameter range. The design criteria were hence amended to

$$\gamma_0(W) \;\leq\; \varepsilon_\gamma, \tag{24}$$

$$\phi_i(G_j, W, K_p) \;\leq\; \varepsilon_i, \quad \text{for} \quad j = 1, 2, 3, 4, \;\; i = 1, 2, \ldots, 12, \tag{25}$$

where plants G_j, $j = 1, 2, 3, 4$ have actuator time delays and gains shown in Table 1. These extreme plant models were chosen because they were judged to be the most difficult to simultaneously obtain good performance.

With weighting functions as above, a satisfactory design was not achieved, so the order of the weighting functions W_1 and W_2 was increased to give

$$W_1 = \frac{s^2 + w_1 s + w_2}{s(s^2 + w_3 s + w_4)} \begin{bmatrix} w_5 & 0 \\ 0 & w_6 \end{bmatrix}, \tag{26}$$

	τ_1	τ_2	k_1	k_2
G_1	0	0	0.8	0.8
G_2	1	1	0.8	1.2
G_3	1	1	1.2	0.8
G_4	1	1	1.2	1.2

Table 1: Extreme plants G_j, $j = 1, 2, 3, 4$

and

$$W_2 = w_7 \frac{s + w_8}{s + w_9} I_2. \tag{27}$$

To ensure that the weightings are stable and minimum phase, the following inequalities were included in the inequality set (25):

$$\mathrm{Re}\left\{-w_1 + \sqrt{w_1^2 - 4w_2}\right\} < 0, \tag{28}$$

$$\mathrm{Re}\left\{-w_3 + \sqrt{w_3^2 - 4w_4}\right\} < 0, \tag{29}$$

$$-w_8 < 0, \tag{30}$$

$$-w_9 < 0. \tag{31}$$

To attempt to satisfy all the performance specifications, a 2nd degree-of-freedom is introduced by setting K_p to be a four state dynamic pre-compensator. K_p has a steady state gain of $K_s(0)W_2(0)$ to ensure that the steady state error specifications are met. After implementing the MBP, a design that successfully satisfied inequalities (24) and (25) was easily obtained. However, inspection of the closed-loop step responses showed a very large amount of undershoot, so four additional inequalities to restrict the minimum undershoot to -0.1 were included. The undershoot functionals, $\phi_i(G_D, W, K_p)$, $i = 13, \ldots, 16$, are defined in the Appendix, and the prescribed bounds for the undershoot functionals are $\varepsilon_i = 0.1$ for $i = 13, \ldots, 16$. The design criteria were hence amended to

$$\gamma_0(W) \leq \varepsilon_\gamma, \tag{32}$$

$$\phi_i(G_j, W, K_p) \leq \varepsilon_i, \quad \text{for} \quad j = 1, 2, 3, 4, \;\; i = 1, 2, \ldots, 16. \tag{33}$$

After the MBP had iterated for about two hours, the performance shown in Table 2 was obtained with the weights

$$W_1 = \frac{(s + 0.311 \pm 0.733j)}{s(s + 0.922 \pm 0.714j)} \begin{bmatrix} 114.2 & 0 \\ 0 & 103.9 \end{bmatrix}, \tag{34}$$

i	ε_i	$\phi_i(G_1)$	$\phi_i(G_2)$	$\phi_i(G_3)$	$\phi_i(G_4)$
1	1.1	1.036	1.036	1.017	0.998
2	−0.9	−0.841	−0.925	−0.853	−0.930
3	0.5	0.372	0.279	0.465	0.343
4	0.5	0.433	0.427	0.425	0.422
5	−0.35	−0.397	−0.398	−0.399	−0.400
6	0.7	0.602	0.601	0.603	0.600
7	−0.55	−0.591	−0.591	−0.597	−0.596
8	0.5	0.394	0.445	0.324	0.364
9	1.1	1.023	1.007	1.032	0.998
10	−0.9	−0.900	−0.899	−0.958	−0.956
11	50.0	50.53	50.60	50.43	49.59
12	150.0	147.3	147.3	147.3	147.3
13	0.1	0.00	0.056	0.032	0.017
14	0.1	0.029	0.085	0.040	0.020
15	0.1	0.029	0.028	0.075	0.016
16	0.1	0.00	0.027	0.081	0.021

Table 2: Performance requirements and final performance

and

$$W_2(s) = 2.737\frac{(s+0.532)}{(s+1.617)}I_2, \tag{35}$$

and pre-compensator

$$K_p \stackrel{s}{=} \left[\begin{array}{cccc|cc} -0.4169 & 0 & 0 & 0 & -0.2374 & -0.05372 \\ 0 & -4.153 & 0 & 0 & -2.189 & -4.586 \\ 0 & 0 & -0.8368 & 0 & 0.0293 & -2.038 \\ 0 & 0 & 0 & -2.359 & 0.0269 & 0.0515 \\ \hline 0.2222 & 3.609 & -1.199 & -1.073 & 3.083 & 1.119 \\ -1.792 & -0.838 & 0.3027 & 4.9136 & -1.529 & 0.4734 \end{array}\right] K_s(0)W_2(0). \tag{36}$$

The resulting optimal compensator K_s had 13 states.

All the step response criteria were satisfied except for $\phi_2(G_1)$ and $\phi_2(G_3)$. The 50 dB gain limit was marginally exceeded by $\phi_{11}(G_1)$, $\phi_{11}(G_2)$ and $\phi_{11}(G_3)$. The step responses of the 16 possible extreme plants, using 5th order Padé approximants for the time delays, are shown in Figure 3 along with the maximum singular values of $(I-KG)^{-1}W_1K_p$, the demand to plant input transfer function. The prescribed bounds on the responses are also shown in the plots. Over all the extreme plants, the overshoot, rise-time and cross-coupling in the simulations are no worse than for the four extreme plants used for the design, however, this is not the case for the undershoot. To reduce the undershoot, more extreme plants could have been included in the MOI, but this would be at the expense of additional computational effort. The results compare favorably with other designs for the same problem [6, 19]. It was found that the prescribed gain and bandwidth bounds, ε_{11} and ε_{12}, were the most significant factors in restricting the performance, if these bounds were sufficiently increased, all the performance specifications could be met

6 Concluding Remarks

The use of numerical methods to design the weights in an H_∞-optimization problem appears to be new. The proposed method combines the flexibility of numerical optimization-type techniques with analytical optimization in an effective and practical manner, as demonstrated by the design of a controller for a high purity distillation column. The MOI is interactive, thus providing flexibility to the designer in formulating a realistic problem and in determining design trade-offs. Unlike the LSDP, closed-loop performance is *explicitly* considered in the formulation of the design problem, and can include both time and frequency domain performance indices. However, it was found in practice that the initial choice of weighting function parameters is very important in the subsequent progress of the MBP, and the LSDP approach was seen as useful in choosing the initial parameters and the weighting function structures.

The approach suggested here has some advantages over the usual implementation of the MOI. In the usual implementation, a search is conducted in a set of fixed order controllers to try and find a feasible point. In the approach here, the search is restricted to controllers which are already robustly stable, thus the problem of finding a stability region does not exist.

Other multi-objective numerical methods exist which may be used to solve similar formulations of the problem. These include the goal-attainment method [2, 4], the vector performance optimization method [7, 8] and a class of numerical convex optimization techniques [1]. Other methods are summarized in [14].

The use of numerical methods to design the weighting functions is particularly suited to the NLCF approach because no γ-iteration is required. The MOI can be combined with H_∞-optimization methods which require γ-iteration [13, 18], but the process is considerably slower. The use of sub-optimal controllers would speed up the process, however the functionals are infinitely discontinuous, so traditional search techniques cannot be used. Genetic algorithms can be used to solve multiobjective control problems [3], and, because genetic algorithms do not require the objective functions to be continuous, investigations are being conducted into using genetic algorithms to design the weighting functions for sub-optimal H_∞ problems.

The implementation of the MOI suggested in Section 4 requires the choice of a nominal plant. Another possibility is to include some of the nominal plant parameters as design parameters, the search algorithm would then attempt to choose the 'best' nominal plant out of a set of possible nominal plants.

In the example chosen here, the performance was evaluated for a selection of extreme plant models chosen by the designer. The problem of efficiently determining the worst-case performance over the range of plants still exists, although more general measures of performance robustness could be included in the performance set. This would be of particular importance when the range of plant perturbations is less well-known.

Appendix - Closed Loop Performance Functionals

A set of closed-loop performance functionals $\{\phi_i(G_D, W, K_p),\ i = 1, 2, \ldots, 16\}$, are defined based on the design specifications given in Section 5.

Functionals ϕ_1 to ϕ_{10} are measures of the step response specifications. Functionals ϕ_1, ϕ_4, ϕ_6 and ϕ_9 are measures of the overshoot; ϕ_2, ϕ_5, ϕ_7 and ϕ_{10} are measures of the rise-time, and ϕ_3 and ϕ_8 are measures of the cross-coupling. Denoting the output response of the closed loop system with a plant G_D at a time t to a reference step demand $h(t)\left[\begin{smallmatrix} h_1 \\ h_2 \end{smallmatrix}\right]$ by $y_i([h_1\ h_2]', t)$, $i = 1, 2$, the step response functionals are

$$\begin{aligned}
\phi_1 &= \max_t y_1([1\ 0]', t), && (37)\\
\phi_2 &= -\min_{t>30} y_1([1\ 0]', t), && (38)\\
\phi_3 &= \max_t y_2([1\ 0]', t), && (39)\\
\phi_4 &= \max_t y_1([0.4\ 0.6]', t), && (40)\\
\phi_5 &= -\min_{t>30} y_1([0.4\ 0.6]', t), && (41)\\
\phi_6 &= \max_t y_2([0.4\ 0.6]', t), && (42)
\end{aligned}$$

$$\begin{aligned}
\phi_7 &= -\min_{t>30} y_2([0.4\ 0.6]', t), &(43)\\
\phi_8 &= \max_t y_1([0\ 1]', t), &(44)\\
\phi_9 &= \max_t y_2([0\ 1]', t), &(45)\\
\phi_{10} &= -\min_{t>30} y_2([0\ 1]', t). &(46)
\end{aligned}$$

The steady state specifications are satisfied automatically by the use of integral action.

From the gain requirement in the design specifications, ϕ_{11} is the H_∞-norm (in dB) of the closed-loop transfer function between the reference and the plant input,

$$\phi_{11} = \sup_\omega \bar{\sigma}\left((I - K(j\omega)G_D(j\omega))^{-1}W_1(j\omega)K_p(j\omega)\right). \tag{47}$$

From the bandwidth requirement in the design specifications, ϕ_{12} is defined (in rad/min) as

$$\phi_{12} = \max\{\omega\} \ \text{ such that } \bar{\sigma}\left((I - K(j\omega)G_D(j\omega))^{-1}W_1(j\omega)K_p(j\omega)\right) \geq 1. \tag{48}$$

Four additional performance functionals are defined to restrict the undershoot

$$\begin{aligned}
\phi_{13} &= -\min_t y_1([1\ 0]', t), &(49)\\
\phi_{14} &= -\min_t y_2([1\ 0]', t), &(50)\\
\phi_{15} &= -\min_t y_1([0\ 1]', t), &(51)\\
\phi_{16} &= -\min_t y_2([0\ 1]', t). &(52)
\end{aligned}$$

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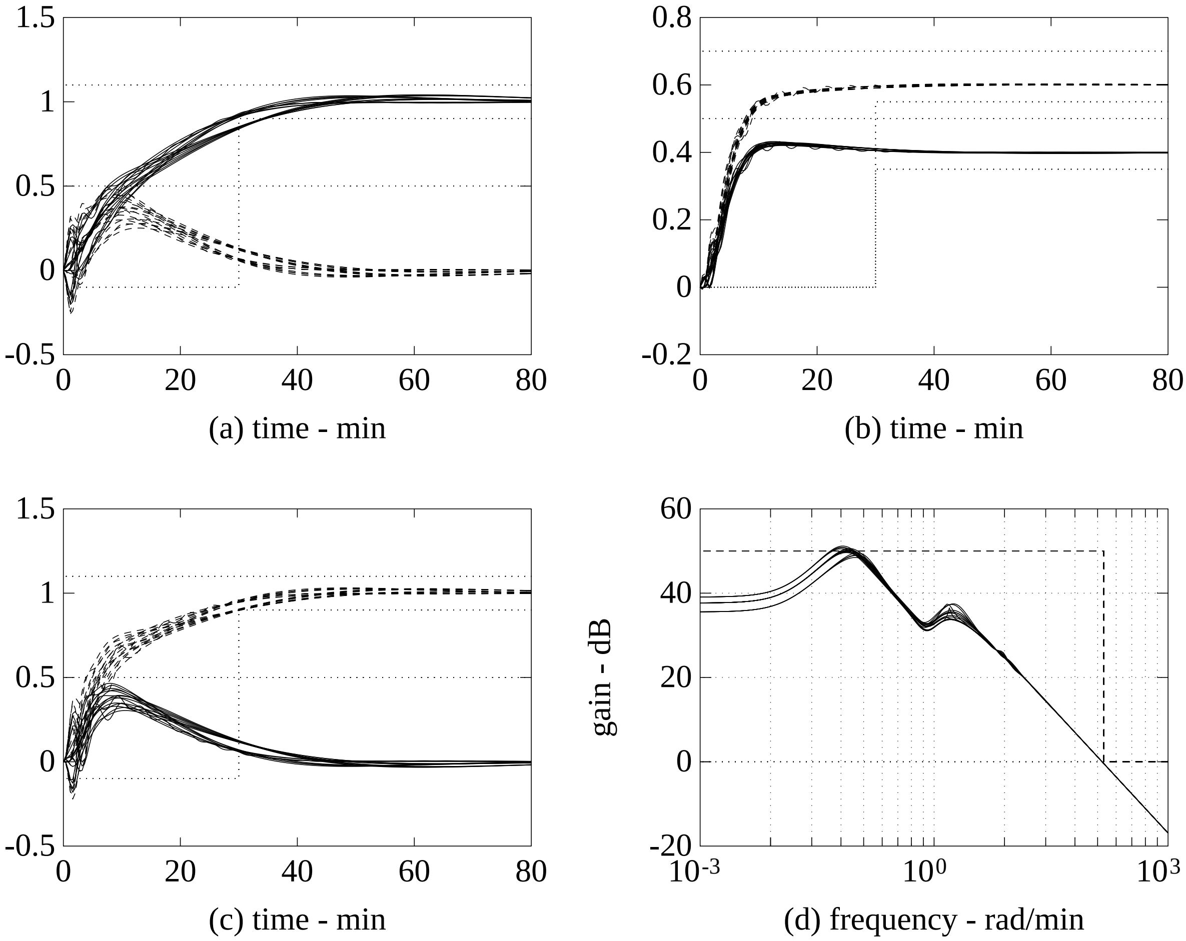

Figure 3: Responses of y_1 (——) and y_2 (- - -) of all extreme plants to (a) input $h(t)\begin{bmatrix}1\\0\end{bmatrix}$, (b) input $h(t)\begin{bmatrix}0.4\\0.6\end{bmatrix}$, (c) input $h(t)\begin{bmatrix}0\\1\end{bmatrix}$ and (d) maximum singular values of all extreme plants for $(I-KG)^{-1}W_1K_p$.