REPORT NO. 160

May, 1963.

THE COLLEGE OF AERONAUTICS

CRANFIELD

Slip Flow Past Slender Pointed Bodies

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SUMMARY

Using Oseen's approximation to linearise the Navier-Stokes equations, a solution to the problem of axisymmetric slip flow past slender bodies of revolution at zero incidence has been obtained, in a manner similar to that used by Laurmann in his paper on the slip flow past a flat plate(6).

The drag coefficient has been evaluated for both incompressible and compressible flows, and this has been compared with normal boundary layer values. It is found that the skin friction drag is the dominant component of drag in the incompressible case and in the compressible case.

This work was conducted at the College of Aeronautics, Cranfield, in partial fulfilment of the requirements for the Diploma of the College.

CONTENTS

LIST OF SYMBOLS

a_∞	free stream speed of sound
$a_o(x)$, $a_1(x)$ etc.	functions of x
$A_o(p)$, $A_1(p)$ etc.	Fourier transforms of $a_o(x)$, $a_1(x)$, etc.
$b_o(x)$, $b_1(x)$ etc.	functions of x
B	$\sqrt{M^2 - 1}$
c_1, c_2	constants
C_D	$D/\frac{1}{2}\rho U^2 S$ - drag coefficient in slip flow
C_f	skin friction drag coefficient in continuum flow
C_{D_w}	wave drag coefficient
C_p	specific heat at constant pressure
D	$D_p + D_v = D_1 + D_2$ = total drag
k	$UL/2\nu$
K	thermal conductivity
$\bar{\ell}$	molecular mean free path
L	length of body
M	U/a_∞ - free stream Mach number
n	co-ordinate in direction normal to body surface
p	pressure
p	transform variable
q	component of velocity parallel to body surface
$\underset{\sim}{q}$	velocity vector
$\underset{\sim}{q}_1$, $\underset{\sim}{q}_2$, $\underset{\sim}{q}_3$	components of $\underset{\sim}{q}$
$\dot{q}_w$	heat transfer rate
r	radial co-ordinate
R_e	UL/ν - Reynolds number

List of Symbols (Continued)

Suffix ∞ denotes free stream value

Asterisk * denotes sum of free stream and perturbation values

No suffix etc. denotes perturbation values of u, v, w, p, ρ, μ, K and T

Fourier transforms of ϕ, x etc. are denoted by $\overline{\phi}$, $\overline{x}$ etc.

Dashes denote differentiation

List of Symbols (Continued)

$r_1(x)$	radius of body
R	radius of control cylinder
S	$2\pi \int_0^1 r_1(x)\, dx$ surface area of body
$S(x)$	$\pi \left[r_1(x) \right]^2$ body cross-sectional area
t	2δ
T	temperature
ΔT	temperature jump at surface
T_s	perturbation temperature of gas adjacent to surface
T_w	temperature of surface
U	free stream velocity
u, v, w	Cartesian velocity components
U_s	slip velocity
v_r	radial velocity component
x, y, z	Cartesian co-ordinates
α	$\tan^{-1} dr_1/dx$
β	$\sqrt{1 - M^2}$
γ	ratio of specific heats
Γ	.5772 ... Euler's constant
δ	maximum radius of body
λ	$\gamma M^2/2k\sigma$
μ	coefficient of viscosity
ν	μ/ρ kinematic viscosity
ρ	density
σ	$\mu C_p/K$ Prandtl number
ϕ, ϕ_1, ϕ_2	velocity potentials
χ	vorticity function
ψ	$e^{-kx} \chi$

1. Introduction

The full Navier-Stokes equations may be derived either from macroscopic arguments or from the kinetic theory of gases via the Boltzmann equation. Their use may therefore be extended into the field of semi-rarefied gas dynamics, in which slip effects occur at the boundaries of the fluid and even, to a certain extent, into free molecule flows where, however, they eventually cease to describe the physical conditions correctly and have to be replaced by other methods based on kinetic theory.

It is the object of this paper to investigate an axisymmetric solution to the Navier-Stokes equations in the slip flow regime, employing Oseen's approximation to simplify the equations, and because it is suited to dealing with the velocity discontinuity which appears at fluid boundaries.

The Oseen method linearises the equations by considering perturbations on a uniform stream, and allows the problem to be solved throughout the entire flow field, but the perturbations on the free stream must be everywhere small. Bodies must therefore produce only minor changes from free stream conditions, and the method is thereby restricted to slender configurations where the body cross-section is small compared with its length; in particular, the fluid velocity at boundaries, with such slender bodies, must be similar in magnitude to the free stream velocity, a condition which is not satisfied in boundary layer flows at normal densities but which, in slip flow, is permitted by the fluid velocity discontinuity at surfaces, which is characteristic of this regime.

Use of Oseen's approximation, therefore, demands that the slip velocity be of the same order of magnitude as that of the free stream. Since this slip velocity is a function of molecular mean free path and body shape which, as noted above, is itself restricted, the problem is confined to one in which the mean free path is limited to a small range of values and must, in fact, be large compared with the dimensions of the body, i.e. the Knudsen number is large. This is outside the range of Knudsen number normally associated with slip flow, but theoretical work in the kinetic theory of gases, by Wang Chang and Uhlenbeck[8], shows that the slip form of boundary condition is valid for all values of mean free path, although numerical coefficients may change; there is also some experimental evidence that this is so.

This work then, although ostensibly concerned with the slip flow regime ($.01 < $ Knudsen number < 1.0), is actually a description of fluids at a somewhat lower density, which is approaching that of free molecule flows, but in which it is assumed that the Navier-Stokes equations still hold and in which the slip boundary condition is valid.

2. The incompressible case

(a) The equations

The full Navier-Stokes equations for steady incompressible flow in three dimensions are

$$u^* \frac{\partial u^*}{\partial x} + v^* \frac{\partial u^*}{\partial y} + w^* \frac{\partial u^*}{\partial z} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x} + \nu \nabla^2 u^*$$

$$u^* \frac{\partial v^*}{\partial x} + v^* \frac{\partial v^*}{\partial y} + w^* \frac{\partial v^*}{\partial z} = -\frac{1}{\rho}\frac{\partial p^*}{\partial y} + \nu \nabla^2 v^*$$

$$u^* \frac{\partial w^*}{\partial x} + v^* \frac{\partial w^*}{\partial y} + w^* \frac{\partial w^*}{\partial z} = -\frac{1}{\rho}\frac{\partial p^*}{\partial z} + \nu \nabla^2 w^*$$

The equation of continuity is

$$\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z} = 0$$

If u^*, v^*, w^* and p^* are replaced by $U(1 + u)$, vU, wU and $p_\infty(1 + p)$, where u, v, w and p are small non-dimensional perturbations on a uniform stream U, these equations become, neglecting second order terms and using a non-dimensional Cartesian co-ordinate system

$$\frac{\partial u}{\partial x} = -\frac{p_\infty}{\rho U^2}\frac{\partial p}{\partial x} + \frac{1}{R_e}\nabla^2 u$$

$$\frac{\partial v}{\partial x} = -\frac{p_\infty}{\rho U^2}\frac{\partial p}{\partial y} + \frac{1}{R_e}\nabla^2 v$$

$$\frac{\partial w}{\partial x} = -\frac{p_\infty}{\rho U^2}\frac{\partial p}{\partial z} + \frac{1}{R_e}\nabla^2 w$$

where $R_e = \frac{UL}{\nu}$ and L is the length used to non-dimehsionalise the co-ordinate system.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Lamb[5] shows that, with cylindrical symmetry about the x-axis, the non-dimensional perturbation velocity vector $\underset{\sim}{q} = (u, v, w)$ may be split into two components, one irrotational being the gradient of a scalar potential function ϕ, and the other rotational being derived from a scalar vorticity function χ, such that

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \chi}{\partial x} - 2k\chi \qquad (2.1)$$

$$v = \frac{\partial \phi}{\partial y} + \frac{\partial \chi}{\partial y}$$

$$w = \frac{\partial \phi}{\partial z} + \frac{\partial \chi}{\partial z}$$

$$p = -\frac{\rho U^2}{p_\infty} \frac{\partial \phi}{\partial x} \qquad (2.2)$$

and also

$$v_r = \frac{\partial \phi}{\partial r} + \frac{\partial \chi}{\partial r} \qquad (2.3)$$

where v_r is the non-dimensional radial velocity component in axisymmetric flow.

ϕ and χ satisfy the equations

$$\nabla^2 \phi = 0 \qquad (2.4)$$

$$\left(\nabla^2 - 2k \frac{\partial}{\partial x} \right) \chi = 0 \qquad (2.5)$$

with $2\,k = \frac{UL}{\nu} = R_e$

The problem is reduced to finding solutions to equations (2.4) and (2.5), which are derived from second order partial differential equations; four independent boundary conditions will therefore be needed, and will be provided by conditions on the normal and tangential fluid velocity components at the body surface, and by conditions at infinity.

Inviscid flow equations, which are first order equations, are obtained when $k = \infty$ ($\nu = 0$); equation (2.5) is then reduced to $\chi = 0$, and equation (2.4) gives Adams and Sears solution[1] using the normal fluid velocity boundary condition only. In the following analysis, the solution will tend to the inviscid solution as $k \to \infty$, except inside layers of order $1/k$ in thickness adjacent to the body surface; i.e. as Reynolds number becomes large, the viscous effects are confined to a boundary layer. The case $k = 0$ is of no practical interest, but provides a check on the solutions for ϕ and χ, which should then be identical.

(b) General solutions

Equations (2.4) has been solved, using the exponential Fourier transform, by Adams and Sears[1] as follows

If $\bar{\phi}(p) = \int_{-\infty}^{\infty} \phi e^{ipx} \, dx$

then $\nabla^2 \phi = 0$

becomes $\frac{\partial^2 \bar{\phi}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\phi}}{\partial r} - p^2 \bar{\phi} = 0$

This is a modified Bessel equation of zero order, giving

$$\bar{\phi} = A_o(p)\, K_o(|p|\, r) \quad \text{since} \quad \bar{\phi} = 0 \text{ at } r = \infty$$

For small values of r

$$\bar{\phi} = - A_o(p) \left(\Gamma + \log \tfrac{1}{2} \, |p| \, r \right) + 0(r^2) \tag{2.6}$$

Applying the inverse transform

$$\phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\phi} \, e^{-ipx} \, dp$$ and using the Faltung theorem, then

$$\phi = a_o(x) \log \frac{r}{2} - \tfrac{1}{2} \int_{-\infty}^{x} \frac{\partial a_o}{\partial y} \log (x-y) \, dy + \tfrac{1}{2} \int_{x}^{\infty} \frac{\partial a_o}{\partial y} \log (y-x) dy \tag{2.7}$$

where $a_o(x) = - \frac{1}{2\pi} \int_{-\infty}^{\infty} A_o(p) \, e^{-ipx} \, dp$ is an arbitrary function, to be determined from the boundary conditions.

ϕ may conveniently be expressed

$$\phi = a_o(x) \log \frac{r}{2} + b_o(x) \tag{2.8}$$

where
$$b_o(x) = - \tfrac{1}{2} \int_{-\infty}^{x} \frac{\partial a_o}{\partial y} \log (x-y) \, dy + \tfrac{1}{2} \int_{x}^{\infty} \frac{\partial a_o}{\partial y} \log (y-x) \, dy \tag{2.9}$$

Equation (2.5) becomes

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial x^2} - k^2 \psi = 0$$ after making the substitution

$$x = e^{kx} \psi$$

Applying the Fourier transform

$$\frac{\partial^2 \bar{\psi}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\psi}}{\partial r} - (p^2 + k^2) \bar{\psi} = 0$$

giving

$$\bar{\psi} = A_1(p) \, K_o \left[\sqrt{p^2 + k^2} \; r \right]$$

For small values of kr

$$\bar{\psi} = - A_1(p) \left(\Gamma + \log \left[\tfrac{1}{2} \sqrt{p^2 + k^2} \; r \right] \right) + 0 \, (k^2 r^2) \tag{2.10}$$

As long as the previous restriction on r, required for equation (2.6), is observed, this restriction on kr does not limit the range of values of k that may be considered. k may always be as small as we please, but large k is equally acceptable, provided r is small enough.

Inviscid conditions are, however, excluded since, as $k \to \infty$ (viscosity $\to 0$), r must tend to zero i.e. only extremely thin bodies may be considered and, in the limit, the trivial flow of a uniform stream past a 'line' is obtained.

The inverse transform $\psi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\psi} \, e^{-ipx} \, dp$ is evaluated in Appendix C to give $$\psi = a_1(x) \log \frac{r}{2} + b_1(x)$$

where $$a_1(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} A_1(p)\, e^{-ipx}\, dp$$

and $$b_1(x) = -\tfrac{1}{2}\int_{-\infty}^{x} \frac{\partial a_1}{\partial y}\left\{ Ei\left[-k(x-y)\right] - \Gamma - \log k \right\} dy$$

$$+ \tfrac{1}{2}\int_{x}^{\infty} \frac{\partial a_1}{\partial y}\left\{ Ei\left[-k(y-x)\right] - \Gamma - \log k \right\} dy$$

$$= a_1(x)\left\{ \Gamma + \log k \right\} - \tfrac{1}{2}\int_{-\infty}^{x} \frac{\partial a_1}{\partial y}\, Ei\left[-k(x-y)\right] dy$$

$$+ \tfrac{1}{2}\int_{x}^{\infty} \frac{\partial a_1}{\partial y}\, Ei\left[-k(y-x)\right] dy \qquad (2.11)$$

where $Ei(\theta) = \int_{-\infty}^{\theta} \frac{e^t}{t}\, dt$ the exponential integral function.

Hence $$\chi = e^{kx}\,\psi$$

$$= a_1(x)\, e^{kx} \log \frac{r}{2} + b_1(x)\, e^{kx}$$

$$= a_2(x) \log \frac{r}{2} + b_2(x) \qquad (2.12)$$

where $$a_2(x) = a_1(x)\, e^{kx} \qquad (2.13)$$

$$b_2(x) = b_1(x)\, e^{kx} \qquad (2.14)$$

$b_1(x)$ and $b_2(x)$ may be expressed in the alternative forms (derived in Appendix A)

$$b_1(x) = -\frac{e^{-kx}}{2}\int_{0}^{x} \frac{\partial a_2}{\partial y} \log(x-y)\, dy + \frac{e^{kx}}{2}\int_{x}^{1} \frac{\partial}{\partial y}\left(a_2(y) e^{-2ky}\right) \log(y-x)\, dy \qquad \ldots\ldots (2.15)$$

$$b_2(x) = -\tfrac{1}{2}\int_{0}^{x} \frac{\partial a_2}{\partial y} \log(x-y) dy + \frac{e^{2kx}}{2}\int_{x}^{1} \frac{\partial}{\partial y}\left(a_2(y) e^{-2ky}\right) \log(y-x) dy \qquad (2.16)$$

From equations (2.1) and (2.3) we find

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \chi}{\partial x} - 2k\chi$$

$$= \left[a_0'(x) + a_2'(x) - 2k\, a_2(x) \right] \log \frac{r}{2} + \left[b_0'(x) + b_2'(x) - 2k\, b_2(x) \right] \qquad \ldots\ldots (2.17)$$

$$= a_3(x) \log \frac{r}{2} + b_3(x) \quad \text{say,} \qquad (2.18)$$

and $$v_r = \frac{\partial \phi}{\partial r} + \frac{\partial \chi}{\partial r}$$

$$= \frac{1}{r}\left[a_0(x) + a_2(x) \right] \qquad (2.19)$$

$$= \frac{a_4(x)}{r} \quad \text{say} \qquad (2.20)$$

(c) The boundary conditions

The boundary conditions will be applied to a slender body of revolution, pointed at both ends, of length L, and maximum thickness $2\delta L$, making the usual slender body assumptions concerning the body radius, slope of the body surface, etc. Let the radius of the body be $L\, r_1(x)$, with the axis of symmetry along the x-axis and upstream end at the origin.

$$S(x) = \pi \left[r_1(x) \right]^2 \qquad \text{and} \qquad 2\delta = t$$

The normal velocity condition gives

$$v_r = \frac{dr_1}{dx} (1 + u) \tag{2.21}$$

and the tangential velocity condition

$$U_s = (1 + u) \cos\alpha + v_r \sin\alpha \tag{2.22}$$

where U_s = non-dimensional slip velocity

and $\tan\alpha = \dfrac{dr_1}{dx}$

Since $\dfrac{dr_1}{dx} = 0(t)$ by the slender hypothesis, equation(2.21) shows that

$$v_r = 0(t) \tag{2.23}$$

We find therefore, that

$$a_o(x),\ a_1(x),\ a_2(x) = 0(t^2), \text{ from (2.19) and (2.13)}$$

$$\phi\,,x = 0(t^2 \log t), \text{ from (2.8) and (2.12)}$$

while $u = 0(t^2 \log t)$, from (2.17)

Equations (2.21) and (2.22) become

$$v_r = \frac{dr_1}{dx} + 0\,(t^3 \log t) \tag{2.24}$$

$$U_S = 1 + 0\,(t^2 \log t) \tag{2.25}$$

Hence $\dfrac{dr_1}{dx} = v_r = \dfrac{a_4(x)}{r_1(x)}$, from (2.20) and (2.24)

or
$$a_4(x) = \frac{1}{2\pi} \frac{d}{dx} (S(x)) = \frac{S'(x)}{2\pi} \tag{2.26}$$

From kinetic theory, the slip velocity $U_s = c_1\, \overline{\ell}\, \dfrac{\partial q}{\partial n}$

where c_1 = a constant, (approximately equal to unity)

$\overline{\ell}$ = the non-dimensional mean free path of the molecules = Knudsen number

q = component of the fluid velocity parallel to the body surface

n is in the direction normal to the body surface.

Now
$$\frac{\partial q}{\partial n} = \frac{\partial u}{\partial r} \cos^2 \alpha + \left(\frac{\partial v_r}{\partial r} - \frac{\partial u}{\partial x}\right) \cos \alpha \sin \alpha - \frac{\partial v_r}{\partial x} \sin^2 \alpha$$
$$= \frac{\partial u}{\partial r} + \frac{\partial v_r}{\partial r} \cdot \frac{dr_1}{dx} + 0\,(t^3 \log t)$$

therefore
$$U_s = c_1 \overline{\ell} \left\{ \frac{\partial u}{\partial r} + \frac{\partial v_r}{\partial r} \cdot \frac{dr_1}{dx} \right\} = 1, \text{ from (2.25)}$$
$$= c_1 \overline{\ell} \left\{ \frac{a_0'(x) + a_2'(x) - 2k\, a_2(x)}{r_1(x)} - \frac{a_4(x)}{r_1^2(x)} r_1'(x) \right\},$$

from (2.17) and (2.20).

Since the bracket contains terms of order t, it follows that $\overline{\ell}$ must be of order t^{-1} i.e. the Knudsen number must be large. This restriction is a consequence of the use of Oseen's approximation, which requires the fluid velocity at all points to be of the same order of magnitude as the free stream velocity.

The large Knudsen number does not prevent the use of slip boundary conditions.

The second boundary condition therefore gives

$$\frac{a_0'(x) + a_2'(x) - 2k\, a_2(x)}{r_1(x)} - \frac{a_4(x)}{r_1^2(x)} r_1'(x) = \frac{1}{c_1 \overline{\ell}} \tag{2.27}$$

and it follows that

$$a_2(x) = \frac{1}{2k} \left\{ \frac{S''(x)}{2\pi} - \left[r_1'(x) \right]^2 - \frac{r_1(x)}{c_1 \overline{\ell}} \right\} + 0(t^4 \log t)$$
$$= \frac{1}{2k} \left\{ r_1(x)\, r_1''(x) - \frac{r_1(x)}{c_1 \overline{\ell}} \right\} \tag{2.28}$$

$$a_0(x) = \frac{S'(x)}{2\pi} - \frac{1}{2k} \left\{ r_1(x)\, r_1''(x) - \frac{r_1(x)}{c_1 \overline{\ell}} \right\} + 0(t^4 \log t) \tag{2.29}$$

$a_0(o)$, $a_2(o)$, $a_0(1)$ and $a_2(1)$ are all zero, provided $r_1(x)\ r_1''(x) = 0$ at $x = 0$ and $x = 1$; this condition is satisfied if the body is pointed at both ends.

Then, from (2.18) and (2.20), we find

$$u = \left\{ \frac{r_1(x)}{c_1 \overline{\ell}} + \left[r_1'(x) \right]^2 \right\} \log \frac{r}{2} + b_3(x) \tag{2.30}$$

and $$v_r = \frac{S'(x)}{2\pi r} \tag{2.31}$$

For a general pointed body, $b_3(x)$ will contain logarithmic singularities at $x = 0$ and at $x = 1$, and the local velocity and pressure at the nose and tail will be physically untenable and contrary to the small perturbation hypothesis. The effect is, however, confined to a restricted region and its net influence on drag force is negligible.

The singularities may be removed by the imposition of the additional restriction that the nose and tail should be cusped, i.e. no streamwise discontinuities in curvature, but since this paper is ultimately concerned with drag force, the more general analysis of a pointed nose has been followed.

(d) Drag

The drag may be evaluated by considering the pressure, momentum flux and viscous stresses at the surfaces of a cylinder of length L and radius R which just encloses the body. Let S_1 and S_3 denote the upstream and downstream plane faces of this cylinder, and S_2 the curved surface.

The drag due to the viscous stresses at the cylinder surfaces is

$$\frac{D_v}{\frac{1}{2}\rho U^2} = -\frac{4\mu}{\rho UL}\int_{S_1}\left(\frac{\partial u}{\partial x}\right)_{x=0} dS_1 + \frac{2\mu}{\rho UL}\int_{S_2}\left(\frac{\partial u}{\partial r}+\frac{\partial v_r}{\partial x}\right)_{r=R} dS_2 + \frac{4\mu}{\rho UL}\int_{S_3}\left(\frac{\partial u}{\partial x}\right)_{x=1} dS_3$$

$$= \frac{2\pi}{k}\int_0^1\left\{\frac{r_1(x)}{c_1\,\overline{\ell}} + \left[r_1'(x)\right]^2 + \frac{S''(x)}{2\pi}\right\}dx + 0\,(t^4\log t)$$

$$= \frac{2\pi}{k}\int_0^1\left\{\frac{r_1(x)}{c_1\,\overline{\ell}} + \left[r_1'(x)\right]^2\right\}dx \tag{2.32}$$

$$= 0(t^2)$$

The drag due to pressure and momentum change through the cylinder is

$$\frac{D_p}{\frac{1}{2}\rho U^2} = \int_{S_1}\left[\frac{p^*}{\frac{1}{2}\rho U^2} + 2(1+u)^2\right]_{x=0} dS_1 - 2\int_{S_2}\left[v_r(1+u)\right]_{r=R} dS_2 - \int_{S_3}\left[\frac{p^*}{\frac{1}{2}\rho U^2} + 2(1+u)^2\right]_{x=1} dS_3$$

Using the continuity equation, and equation (2.2) this becomes

$$\frac{D_p}{\frac{1}{2}\rho U^2} = \int_{S_1}\left[-2\frac{\partial\phi}{\partial x} + 2u\right]_{x=0} dS - 2\int_{S_2}\left[u\,v_r\right]_{r=R} dS_2 - \int_{S_3}\left[-2\frac{\partial\phi}{\partial x} + 2u\right]_{x=1} dS_3 + 0(t^6\log^2 t)$$

$$= 0(t^4\log t)$$

∴ Total Drag D is given by

$$\frac{D}{\frac{1}{2}\rho U^2} = \frac{D_p + D_v}{\frac{1}{2}\rho U^2} = \frac{2\pi}{k} \int_0^1 \left\{ \frac{r_1(x)}{c_1 \bar{\ell}} + \left[r_1'(x)\right]^2 \right\} dx + 0(t^4 \log t)$$

This total drag is due to the normal and tangential viscous stresses at the body surface.

The first term, $D_1 = \frac{2\pi}{k\, c_1 \bar{\ell}} \int_0^1 r_1(x)\, dx$, may be compared with Laurmann's result for a flat plate (Ref. 6).

It gives a drag coefficient, based on body surface area $S = 2\pi \int_0^1 r_1(x)\, dx$,

$$C_{D_1} = \frac{D_1}{\frac{1}{2}\rho U^2 S} = \frac{1}{k\, c_1\, \bar{\ell}}$$

which is exactly the drag coefficient for one side of a flat plate. The situation is similar to that of laminar boundary layer theory where skin friction drag coefficient for one side of a flat plate $\frac{1.328}{\sqrt{R_e}}$ is the same as that for a slender body of revolution (Ref. 3).

C_{D_1} will remain finite as Reynolds number increases to large values because $\bar{\ell}$ is proportional to $\frac{1}{k}$.

The second term, $D_2 = \frac{2\pi}{k} \int_0^1 \left[r_1'(x)\right]^2 dx$, is a viscous term which becomes small at high Reynolds numbers. It is, therefore, a low Reynolds number term, due to the three dimensional body shape.

3. The compressible case

(a) The equations

The steady three-dimensional Navier-Stokes equations and the continuity equation for compressible flow are

$$\rho^* \left(u^* \frac{\partial u^*}{\partial x} + v^* \frac{\partial u^*}{\partial y} + w^* \frac{\partial u^*}{\partial z} \right) = - \frac{\partial p^*}{\partial x} + \mu^* \nabla^2 u^* + \frac{1}{3}\mu^* \frac{\partial}{\partial x}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z} \right)$$
$$- \frac{2}{3} \frac{\partial \mu^*}{\partial x}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z} \right) + 2\frac{\partial \mu^*}{\partial x} \frac{\partial u^*}{\partial x}$$
$$+ \frac{\partial \mu^*}{\partial y}\left(\frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x} \right) + \frac{\partial \mu^*}{\partial z}\left(\frac{\partial u^*}{\partial z} + \frac{\partial w^*}{\partial x} \right)$$

and two similar equations.

$$\rho^* \left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z} \right) = -\left(u^* \frac{\partial \rho^*}{\partial x} + v^* \frac{\partial \rho^*}{\partial y} + w^* \frac{\partial \rho^*}{\partial z} \right)$$

These, and the equations of energy and state, may be linearised by considering small perturbations from conditions in a uniform stream defined by

$$
\begin{aligned}
u^* &= U\,(1 + u)\\
v^* &= Uv\\
w^* &= Uw\\
p^* &= p_\infty(1 + p)\\
\rho^* &= \rho_\infty(1 + \rho)\\
\mu^* &= \mu_\infty(1 + \mu)
\end{aligned}
$$

giving $$\frac{\partial u}{\partial x} = -\frac{1}{\gamma M^2}\frac{\partial p}{\partial x} + \frac{1}{R_e}\nabla^2 u + \frac{1}{3R_e}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial x\,\partial z}\right)$$

where $M = \dfrac{U}{a_\infty}$, the free stream Mach number and the co-ordinate system is non-dimensionalised with respect to L.

$$\frac{\partial v}{\partial x} = -\frac{1}{\gamma M^2}\frac{\partial p}{\partial y} + \frac{1}{R_e}\nabla^2 v + \frac{1}{3R_e}\left(\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y\partial z}\right) \qquad (3.1)$$

$$\frac{\partial w}{\partial x} = -\frac{1}{\gamma M^2}\frac{\partial p}{\partial z} + \frac{1}{R_e}\nabla^2 w + \frac{1}{3R_e}\left(\frac{\partial^2 u}{\partial x\partial w} + \frac{\partial^2 v}{\partial y\partial w} + \frac{\partial^2 w}{\partial z^2}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{\partial \rho}{\partial x} \qquad (3.2)$$

These equations may be combined into

$$\frac{\partial^2 \rho}{\partial x^2} - \frac{1}{\gamma M^2}\nabla^2 p = \frac{4}{3R_e}\nabla^2\left(\frac{\partial \rho}{\partial x}\right) \qquad (3.3)$$

The non-dimensionalised linearised energy equation becomes

$$\frac{\partial T}{\partial x} + (\gamma - 1)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \frac{\gamma}{\sigma R_e}\nabla^2 T \qquad (3.4)$$

where T is the non-dimensional perturbation temperature and

$$\sigma = \text{Prandtl number} = \frac{\mu_\infty C_p}{K_\infty}$$

The equation of state

$$\frac{p^*}{p_\infty} = \frac{\rho^*}{\rho_\infty}\,\frac{T^*}{T_\infty}$$

becomes $$p = \rho + T \qquad (3.5)$$

From (3.2) and (3.4)

$$\frac{\partial T}{\partial x} - (\gamma - 1)\frac{\partial \rho}{\partial x} = \frac{\gamma}{\sigma R_e} \nabla^2 T \qquad (3.6)$$

Eliminating T from (3.5) and (3.6)

$$\frac{\partial p}{\partial x} - \gamma \frac{\partial \rho}{\partial x} = \frac{\gamma}{\sigma R_e} \left[\nabla^2 p - \nabla^2 \rho\right] \qquad (3.7)$$

which, with (3.3) gives

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{M^2}\nabla^2 p = \frac{\gamma}{\sigma R_e}\nabla^2\left(\frac{\partial p}{\partial x}\right) + \frac{\gamma}{R_e}\left(\frac{4}{3} - \frac{1}{\sigma}\right)\nabla^2\left(\frac{\partial \rho}{\partial x}\right) \qquad (3.8)$$

If it is assumed that $\sigma = \frac{3}{4}$ (it is .72 for air), the last term of the above equation vanishes and there remains the pressure equation

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{M^2}\nabla^2 p = \frac{\gamma}{\sigma R_e}\nabla^2\left(\frac{\partial p}{\partial x}\right) \qquad (3.9)$$

Elimination of p and ρ from (3.5), (3.6) and (3.9) gives

$$\left(\frac{\partial}{\partial x} - \frac{1}{\sigma R_e}\nabla^2\right)\left(\frac{\partial^2 T}{\partial x^2} - \frac{1}{M^2}\nabla^2 T - \frac{\gamma}{\sigma R_e}\nabla^2\frac{\partial T}{\partial x}\right) = 0$$

The solution of this, the temperature equation, will be a linear combination of the solutions to

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{M^2}\nabla^2 T - \frac{\gamma}{\sigma R_e}\nabla^2\frac{\partial T}{\partial x} = 0 \qquad (3.10)$$

and

$$\frac{\partial T}{\partial x} - \frac{1}{\sigma R_e}\nabla^2 T = 0 \qquad (3.11)$$

As in the incompressible case, with cylindrical symmetry the perturbation velocity vector may be split into two components, derived from a potential function and a vorticity function. Now, however, the potential function ϕ is divided into two parts, ϕ_1 and ϕ_2, each associated with one of the two temperature equations (3.10) and (3.11).

$$\text{Thus} \quad \underset{\sim}{q} = \underset{\sim}{q}_1 + \underset{\sim}{q}_2 + \underset{\sim}{q}_3 = (u, v_r)$$
$$= \nabla\phi_1 + \nabla\phi_2 + \underset{\sim}{f}(x)$$

This splitting of the linearised flow equations was first carried out by Lagerstrom, Cole and Trilling[4]. They neglected the effect of thermal conductivity and so did not divide the potential function into two parts, but they obtained separate solutions in the form of waves for ϕ and x, which they termed 'longitudinal' and 'transverse' waves respectively. Trilling[7] showed how thermal conductivity could be included, to give a method of solving a linearised flow field problem in terms of the three functions ϕ_1, ϕ_2 and χ defined above; Laurmann[6] follows this method in his paper on slip flow over a flat plate, calling the three solutions of ϕ_1, ϕ_2 and χ, pressure, temperature and viscous waves respectively.

Details of this splitting process are given in Appendix B and result in the following equations for ϕ_1, ϕ_2 and χ, and the associated non-dimensional perturbation velocity vectors, pressures, temperatures and densities in axi-symmetric flow.

Pressure wave

$$\nabla^2\phi_1 - M^2\frac{\partial^2\phi_1}{\partial x^2} + \frac{\gamma M^2}{2k\sigma}\nabla^2\frac{\partial\phi_1}{\partial x} = 0 \qquad (3.12)$$

$$\underset{\sim}{q}_1 = \left(\frac{\partial\phi_1}{\partial x}, \frac{\partial\phi_1}{\partial r}\right)$$

$$p_1 = \gamma M^2\left[\frac{2}{3k}\nabla^2 - \frac{\partial}{\partial x}\right]\phi_1$$

$$T_1 = -(\gamma - 1)M^2\frac{\partial\phi_1}{\partial x}$$

$$\rho_1 = M^2\left[\frac{2\gamma}{3k}\nabla^2 - \frac{\partial}{\partial x}\right]\phi_1$$

with $\sigma = \frac{3}{4}$

Temperature wave

$$\nabla^2\phi_2 - \frac{3k}{2}\frac{\partial\phi_2}{\partial x} = 0 \qquad (3.13)$$

$$\underset{\sim}{q}_2 = \left(\frac{\partial\phi_2}{\partial x}, \frac{\partial\phi_2}{\partial r}\right)$$

$$p_2 = 0$$

$$T_2 = \frac{3k}{2}\phi_2$$

$$\rho_2 = -\frac{3k}{2}\phi_2$$

Viscous wave

$$\nabla^2 \chi - 2k \frac{\partial \chi}{\partial x} = 0 \qquad (3.14)$$

$$\underset{\sim}{q}_3 = \left(\frac{\partial \chi}{\partial x} - 2k\chi, \frac{\partial \chi}{\partial r} \right)$$

$$p_3 = 0$$

$$T_3 = 0$$

$$\rho_3 = 0$$

The viscous wave is therefore independent of compressibility.

(b) General solutions

(i) The pressure wave equation (3.12) is

$$\left(\nabla^2 - M^2 \frac{\partial^2}{\partial x^2} + \frac{\gamma M^2}{2k\sigma} \nabla^2 \frac{\partial}{\partial x} \right) \phi_1 = 0$$

Let $\dfrac{\gamma M^2}{2k\sigma} = \lambda$

Then
$$\lambda \frac{\partial}{\partial x} \left(\frac{\partial^2 \phi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_1}{\partial r} + \frac{\partial^2 \phi_1}{\partial x^2} \right) + (1 - M^2) \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_1}{\partial r} = 0 \qquad \ldots\ldots (3.15)$$

Clarke[(2)] solves a similar equation in his paper on relaxation effects on slender bodies; he uses Fourier transforms to get equation (3.15) in the form

$$\frac{\partial^2 \bar{\phi}_1}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\phi}_1}{\partial r} - p^2 \frac{(1 - M^2 - i p\lambda)}{1 - i p\lambda} \bar{\phi}_1 = 0$$

Again, this is a Bessel equation of zero order giving

$$\bar{\phi}_1 = A_5(p) K_0 \left[p \sqrt{\frac{1 - M^2 - i p\lambda}{1 - i p\lambda}} \; r \right]$$

For small values of r,

$$\bar{\phi}_1 = - A_5(p) \left[\Gamma + \log \left\{ \frac{1}{2} p\, r \sqrt{\frac{1 - M^2 - i p\lambda}{1 - i p\lambda}} \right\} \right] + 0(r^2)$$

and there are no restrictions on M, k or λ in making this approximation. To obtain the inverse transform, it is necessary to consider two cases depending on whether $M < 1$ or $M > 1$.

Case 1 $M < 1$

Now, $\bar{\phi}_1$ must be expressed as

$$\bar{\phi}_1 = -A_5(p)\left[\Gamma + \log\left\{\frac{1}{2}\ |p|\ r\ \sqrt{\frac{p + i\beta^2/\lambda}{p + i/\lambda}}\right\}\right]$$

where $\beta^2 = 1 - M^2$

and, from Clarke's paper, the inverse transform is

$$\phi_1 = a_5(x)\log\frac{r}{2} - \frac{1}{2}\frac{\partial}{\partial x}\int_{-\infty}^{x} a_5(y)\log(x-y)\,dy + \frac{1}{2}\frac{\partial}{\partial x}\int_{x}^{\infty} a_5(y)\log(y-x)\,dy$$

$$-\frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\beta^2}{\lambda}\right)\int_{-\infty}^{x} a_5(y)\,e^{-\beta^2(x-y)/\lambda}\log(x-y)\,dy$$

$$+\frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{\lambda}\right)\int_{-\infty}^{x} a_5(y)\,e^{-(x-y)/\lambda}\log(x-y)\,dy$$

If $a_5(x) = 0$ for $x < 0$ at $x > 1$

$$\phi_1 = a_5(x)\log\frac{r}{2} - \frac{1}{2}\int_{0}^{x}\frac{\partial a_5}{\partial y}\log(x-y)dy + \frac{1}{2}\int_{x}^{1}\frac{\partial a_5}{\partial y}\log(y-x)\,dy$$

$$-\frac{\beta^2}{2\lambda}\int_{0}^{x} a_5(y)\,e^{-\beta^2(x-y)/\lambda}\log(x-y)dy + \frac{1}{2\lambda}\int_{0}^{x} a_5(y)\,e^{-(x-y)/\lambda}\log(x-y)dy$$

$$-\frac{1}{2}\int_{0}^{x}\frac{\partial a_5}{\partial y}\,e^{-\beta^2(x-y)/\lambda}\log(x-y)dy + \frac{1}{2}\int_{0}^{x}\frac{\partial a_5}{\partial y}\,e^{-(x-y)/\lambda}\log(x-y)dy$$

$$= a_5(x)\log\frac{r}{2} - \frac{1}{2}\int_{0}^{x}\frac{\partial a_5}{\partial y}\log(x-y)dy + \frac{1}{2}\int_{x}^{1}\frac{\partial a_5}{\partial y}\log(y-x)\,dy$$

$$-\frac{1}{2}\int_{0}^{x}\frac{a_5(y)}{(x-y)}\left\{e^{-\beta^2(x-y)/\lambda} - e^{-(x-y)/\lambda}\right\}dy$$

$$= a_5(x)\log\frac{r}{2} + b_5(x) \tag{3.16}$$

where
$$b_5(x) = -\frac{1}{2}\int_{0}^{x}\frac{\partial a_5}{\partial y}\log(x-y)dy + \frac{1}{2}\int_{x}^{1}\frac{\partial a_5}{\partial y}\log(y-x)\,dy$$

$$-\frac{1}{2}\int_{0}^{x}\frac{a_5(y)}{(x-y)}\left\{e^{-\beta^2(x-y)/\lambda} - e^{-(x-y)/\lambda}\right\}dy \tag{3.17}$$

Case 2 $\quad M > 1$

Now, $$\bar{\phi}_1 = - A_5(p)\left[\Gamma + \tfrac{1}{2}\log\left\{ pr\sqrt{\frac{p - iB^2/\lambda}{p + i/\lambda}}\right\}\right]$$

where $\quad B^2 = M^2 - 1$

Again, from Clarke, the inverse transform is

$$\phi_1 = a_5(x)\log\frac{r}{2} - \frac{\partial}{\partial x}\int_{-\infty}^{x} a_5(y)\log(x-y)dy + \tfrac{1}{2}\left(\frac{\partial}{\partial x} - \frac{B^2}{\lambda}\right)\int_{x}^{\infty} a_5(y)e^{-B^2(y-x)\lambda}\log(y-x)dy$$
$$+ \tfrac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{\lambda}\right)\int_{-\infty}^{x} a_5(y)\, e^{-(x-y)/\lambda}\log(x-y)\,dy$$

$$= a_5(x)\log\frac{r}{2} - \int_{o}^{x}\frac{\partial a_5}{\partial y}\log(x-y)dy - \tfrac{1}{2}\int_{x}^{1}\frac{a_5(y)}{y-x}e^{-B^2(y-x)/\lambda}dy + \tfrac{1}{2}\int_{o}^{x}\frac{a_5(y)}{x-y}e^{-(x-y)/\lambda}dy$$

$$= a_5(x)\log\frac{r}{2} + b_5(x) \qquad \ldots\ldots \quad (3.18)$$

where $$b_5(x) = -\int_{o}^{x}\frac{\partial a_5}{\partial y}\log(x-y)dy - \tfrac{1}{2}\int_{x}^{1}\frac{a_5(y)}{y-x}e^{-B^2(y-x)/\lambda}dy + \tfrac{1}{2}\int_{o}^{x}\frac{a_5(y)}{x-y}e^{-(x-y)/\lambda}dy$$

$$\ldots\ldots \quad (3.19)$$

(ii) The temperature wave equation (3.13) is

$$\left(\nabla^2 - \frac{3k}{2}\frac{\partial}{\partial x}\right)\phi_2 = 0$$

This equation was solved in section 2(b) giving

$$\phi_2 = a_6(x)\log\frac{r}{2} + b_6(x) \qquad \ldots\ldots \quad (3.20)$$

where $$b_6(x) = -\tfrac{1}{2}\int_{o}^{x}\frac{\partial a_6}{\partial y}\log(x-y)\,dy + \frac{e^{3kx/2}}{2}\int_{x}^{1}\frac{\partial}{\partial y}\left[e^{-3ky/2}a_6(y)\right]\log(y-x)dy$$
$$\ldots\ldots \quad (3.21)$$

(iii) The viscous wave equation (3.14) has the solution

$$x = a_7(x)\log\frac{r}{2} + b_7(x) \qquad \ldots\ldots \quad (3.22)$$

where $$b_7(x) = -\tfrac{1}{2}\int_{o}^{x}\frac{\partial a_7}{\partial y}\log(x-y)dy + \frac{e^{2kx}}{2}\int_{x}^{1}\frac{\partial}{\partial y}\left[e^{-2ky}a_7(y)\right]\log(y-x)dy$$
$$\ldots\ldots \quad (3.23)$$

The two slender body solutions above restrict k to finite values; this restriction was discussed in section 2(b) and the same conclusions hold here.

$$u = \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial x} + \frac{\partial x}{\partial x} - 2kx$$

$$= \left[a_5'(x) + a_6'(x) + a_7'(x) - 2k\, a_7(x) \right] \log \frac{r}{2} + \left[b_5'(x) + b_6'(x) + b_7'(x) - 2k\, b_7(x) \right] \qquad \ldots\ldots \quad (3.24)$$

$$= a_8(x) \log \frac{r}{2} + b_8(x) \quad \text{say} \qquad (3.25)$$

$$v_r = \frac{\partial \phi_1}{\partial r} + \frac{\partial \phi_2}{\partial r} + \frac{\partial x}{\partial r}$$

$$= \left[a_5(x) + a_6(x) + a_7(x) \right] \frac{1}{r} \qquad (3.26)$$

$$= \frac{a_9(x)}{r} \quad \text{say} \qquad (3.27)$$

and

$$T = -(\gamma - 1) M^2 \frac{\partial \phi_1}{\partial x} + \frac{3k}{2} \phi_2$$

$$= -(\gamma - 1) M^2 \left[a_5'(x) \log \frac{r}{2} + b_5'(x) \right] + \frac{3k}{2} \left[a_6(x) \log \frac{r}{2} + b_6(x) \right] \qquad (3.28)$$

$$= a_{10}(x) \log \frac{r}{2} + b_{10}(x) \quad \text{say} \qquad (3.29)$$

(c) The boundary conditions

The same slender body of revolution as that described in section 2(c) will be considered here. Three boundary conditions are now available, viz. normal and tangential velocities, and temperature jump at the surface.

The normal condition gives

$$v_r = \frac{dr_1}{dx} (1 + u)$$

and the tangential condition gives

$$U_s = (1 + u) \cos \alpha + v_r \sin \alpha$$

The temperature discontinuity ΔT is given by

$$\Delta T = (1 + T_s) - T_w$$

where T_s is the non-dimensional perturbation temperature of the fluid adjacent to the wall and T_w is the non-dimensional wall temperature.

By considering orders of magnitude, these equations can be simplified to

$$v_r = \frac{dr_1}{dx} \qquad (3.30)$$

$$U_s = 1 \qquad (3.31)$$

and

$$\Delta T = 1 - T_w \qquad (3.32)$$

(3.27) and (3.30) show that

$$a_9(x) = \frac{S'(x)}{2\pi} \tag{3.33}$$

As before, $$U_s = c_1 \bar{\ell} \left(\frac{\partial u}{\partial r} + \frac{\partial v_r}{\partial r} \cdot \frac{dr_1}{dx} \right)$$

ignoring the term in temperature gradient along the surface.

From (3.25) and (3.31),

$$a_8(x) = \frac{r_1(x)}{c_1 \bar{\ell}} + \left[r_1'(x) \right]^2 \tag{3.34}$$

The temperature jump $\Delta T = c_2 \bar{\ell} \left(\frac{\partial T}{\partial r}\right)_{r=r_1} + 0(t^2 \log t)$

From (3.29) and (3.32),

$$a_{10}(x) = \frac{(1 - T_w)\, r_1(x)}{c_2 \bar{\ell}} \tag{3.35}$$

Finally, from equations (3.24) to (3.29), (3.33), (3.34) and (3.35), u, v_r and T may be deduced and $a_5(x)$, $a_6(x)$ and $a_7(x)$ may be expressed

$$u = \left(\frac{r_1(x)}{c_1 \bar{\ell}} + \left[r_1'(x) \right]^2 \right) \log \frac{r}{2} + b_8(x) \tag{3.36}$$

$$v_r = \frac{S'(x)}{2\pi r} \tag{3.37}$$

$$T = \frac{(1 - T_w)\, r_1(x)}{c_2 \bar{\ell}} \log \frac{r}{2} + b_{10}(x) \tag{3.38}$$

$$a_7(x) = \frac{1}{2k} \left\{ \frac{S''(x)}{2\pi} - \frac{r_1(x)}{c_1 \bar{\ell}} - \left[r_1'(x) \right]^2 \right\}$$

$$= \frac{1}{2k} \left\{ r_1(x)\, r_1''(x) - \frac{r_1(x)}{c_1 \bar{\ell}} \right\}$$

$$a_5(x) = \frac{e^{-3kx/2(\gamma-1)M^2}}{(\gamma - 1)M^2} \int_0^x e^{3kx/2(\gamma-1)M^2}$$

$$\left[\frac{3k}{2} \left(\frac{S'(x)}{2\pi} - \frac{r_1(x)\, r_1''(x)}{2k} + \frac{r_1(x)}{2k\, c_1 \bar{\ell}} \right) - \frac{(1 - T_w)\, r_1(x)}{c_2 \bar{\ell}} \right] dx$$

$$a_6(x) = \left\{ \frac{S'(x)}{2\pi} - \frac{r_1(x)\, r_1''(x)}{2k} + \frac{r_1(x)}{2k\, c_1 \bar{\ell}} \right\} - a_5(x)$$

(d) Drag

As before, the viscous and pressure drags, associated with a cylinder of length L and radius R enclosing the body, will be considered separately.

The drag due to the viscous stresses at the cylinder surfaces is

$$\frac{D_v}{\frac{1}{2}\rho U^2} = \frac{2}{3k}\int_{S_1}\left(\frac{\partial u}{\partial x}+\frac{\partial v_r}{\partial r}+\frac{v_r}{r}\right)_{x=0} dS_1 - \frac{2}{k}\int_{S_1}\left(\frac{\partial u}{\partial x}\right)_{x=0} dS_1 + \frac{1}{k}\int_{S_2}\left(\frac{\partial u}{\partial r}+\frac{\partial v_r}{\partial x}\right)_{r=R} dS_2$$

$$- \frac{2}{3k}\int_{S_3}\left(\frac{\partial u}{\partial x}+\frac{\partial v_r}{\partial r}+\frac{v_r}{r}\right)_{x=1} dS_3 + \frac{2}{k}\int_{S_3}\left(\frac{\partial u}{\partial x}\right)_{x=1} dS_3$$

$$= \frac{2\pi}{k}\int_0^1\left\{\frac{r_1(x)}{c_1\bar{\ell}} + \left[r_1'(x)\right]^2 + \frac{S''(x)}{2\pi}\right\} dx + 0(t^4\log t)$$

$$= \frac{2\pi}{k}\int_0^1\left\{\frac{r_1(x)}{c_1\bar{\ell}} + \left[r_1'(x)\right]^2\right\} dx$$

$$= 0(t^2)$$

which is identical with the incompressible case.

The drag due to the momentum flux and the pressure at the cylinder surfaces is $0(t^4\log t)$, as in the incompressible case, giving for the total drag

$$\frac{D}{\frac{1}{2}\rho U^2} = \frac{2\pi}{k}\int_0^1\left\{\frac{r_1(x)}{c_1\bar{\ell}} + \left[r_1'(x)\right]^2\right\} dx$$

(e) Heat transfer rate

By Fourier's law, the rate of heat transfer to the body surface is

$$\dot{q}_w = -\frac{K^* T_\infty}{L}\left(\frac{\partial T}{\partial n}\right)_{wall}$$

$$= -\frac{K_\infty T_\infty}{L}\left\{\left(\frac{\partial T}{\partial r}\right)_{wall} + 0(t^2\log t)\right\}$$

Thus the Stanton number $= \dfrac{\dot{q}_w}{\rho_\infty U(h_w - h_\infty)}$ with $h = C_p T$

$$= \frac{K_\infty}{c_2\bar{\ell} L\rho_\infty U C_p}$$

$$= \frac{1}{c_2\bar{\ell}\sigma R_e}$$

which is a constant for all regions of the body surface and decreases with increase in mean free path $\bar{\ell}$.

(f) Drag - an example

Finally, a few calculations will be made in order to compare the drag of a slender body under continuum flow conditions with drag under slip flow conditions.

A specific example of a body of length L with meridian shape given by

$$r_1(x) = 4\delta\,(x - x^2)$$, a parabola, will be considered.

δ is small.

For boundary layer flow, Goldstein[(3)] gives

$$C_f = \frac{1.328}{R_e^{\frac{1}{2}}}$$

for the laminar skin friction drag coefficient, based on surface area S, of a slender body of revolution; form drag may be neglected.

At a typical continuum Reynolds number of 10^6,

$$C_f = .0013$$

At supersonic speeds, the wave drag is

$$\frac{D_w}{\frac{1}{2}\rho U^2} = -\frac{1}{2\pi}\int_o^1\int_o^1 S''(x)\,S''(y)\log|x - y|\,dx\,dy$$

giving

$$C_{D_w} = \frac{D_w}{\frac{1}{2}\rho U^2 S}$$

$$= 32\,\delta^3 \qquad \text{for our example.}$$

Thus at normal densities, the supersonic pressure drag coefficient is appreciably larger than the skin friction coefficient for values of δ of order .1. This is the reverse of the situation in slip flow where, as we have seen, pressure drag is small compared with viscous drag.

For slip flow

$$\frac{D}{\frac{1}{2}\rho U^2} = \frac{2\pi}{k}\int_o^1\left[\frac{r_1(x)}{c_1\bar{\ell}} + \left[r_1'(x)\right]^2\right]dx$$

which, for the parabolic body, gives

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 S} = \frac{1}{kc_1\bar{\ell}} + \frac{8\delta}{k} = \frac{2\delta}{R_e}\left[\frac{1}{c_1\bar{\ell}\delta} + 8\right]$$

It has been shown in section 2(c) that $\bar{\ell} = 0(t^{-1})$ i.e. the mean free path is large compared with the body length and so $\bar{\ell}$, for the purposes of this example, will be taken to be $\frac{1}{\delta}$; also $c_1 \simeq 1$

$$\therefore \quad C_D \simeq \frac{18\delta}{R_e} \tag{3.39}$$

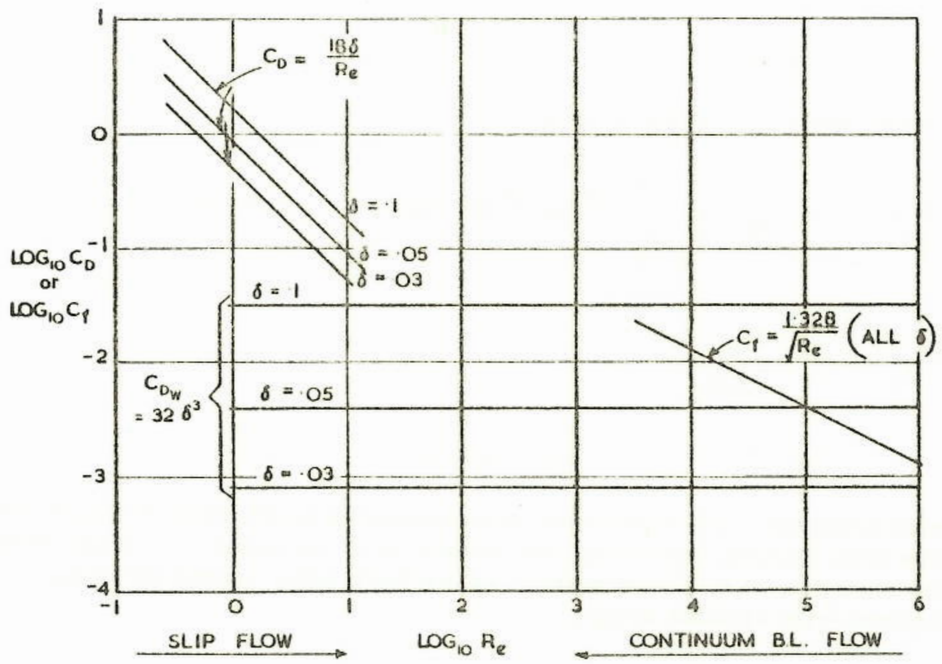
For a typical slip flow Reynolds number of 1,

$$C_D = 18\,\delta$$

Using the relation $M = \bar{\ell}\, R_e \sqrt{\frac{2}{\gamma\pi}}$, which is derived from kinetic theory, equation (3.39) may be expressed in the form

$$C_D\, M \simeq 12$$

These results are plotted in the following graph of $\log_{10} C_D$ against $\log_{10} R_e$.



THIS GRAPH SHOWS CLEARLY THE DIFFERENCE BETWEEN THE ORDERS OF MAGNITUDE OF THE DRAG COEFFICIENTS.

4. References

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APPENDIX A

Alternative forms for the functions $b_1(x)$ and $b_2(x)$

Use is made of the boundary condition that

$$a_1(x) = a_2(x) = 0 \qquad \text{for} \quad x \leqslant 0, \quad x \geqslant 1.$$

From equation(2.11)

$$b_1(x) = a_1(x)\left\{\Gamma + \log k\right\} - \tfrac{1}{2}\int_0^x \frac{\partial a_1}{\partial y}\,\mathrm{Ei}\left[-k(x-y)\right]dy + \tfrac{1}{2}\int_x^1 \frac{\partial a_1}{\partial y}\,\mathrm{Ei}\left[-k(y-x)\right]dy$$

$$= a_1(x)\left\{\Gamma + \log k\right\} - \tfrac{1}{2}\underset{\epsilon\to 0}{\mathrm{Lt}}\left\{\left[a_1(y)\,\mathrm{Ei}\left[-k(x-y)\right]\right]_0^{x-\epsilon} + \int_0^{x-\epsilon}\frac{a_1(y)\,e^{-k(x-y)}}{x-y}\,dy\right\}$$

$$+ \tfrac{1}{2}\underset{\delta\to 0}{\mathrm{Lt}}\left\{\left[a_1(y)\,\mathrm{Ei}\left[-k(y-x)\right]\right]_{x+\delta}^{1} - \int_{x+\delta}^{1}\frac{a_1(y)\,e^{-k(y-x)}}{y-x}\,dy\right\}$$

$$= a_1(x)\left\{\Gamma + \log k\right\} - \tfrac{1}{2}\underset{\epsilon\to 0}{\mathrm{Lt}}\left\{a_1(x-\epsilon)(\Gamma + \log k + \log\epsilon) + \int_0^{x-\epsilon}\frac{a_1(y)\,e^{-k(x-y)}}{x-y}\,dy\right\}$$

$$+ \tfrac{1}{2}\underset{\delta\to 0}{\mathrm{Lt}}\left\{-a_1(x+\delta)(\Gamma + \log k + \log\delta) - \int_{x+\delta}^{1}\frac{a_1(y)\,e^{-k(y-x)}}{y-x}\,dy\right\}$$

$$= -\tfrac{1}{2}\underset{\epsilon\to 0}{\mathrm{Lt}}\left\{a_1(x-\epsilon)\log\epsilon + \left[-\log(x-y)\,a_1(y)\,e^{-k(x-y)}\right]_0^{x-\epsilon} + e^{-kx}\int_0^{x-\epsilon}\log(x-y)\,\frac{d}{dy}\left[a_1(y)\,e^{ky}\right]dy\right\}$$

$$+ \tfrac{1}{2}\underset{\delta\to 0}{\mathrm{Lt}}\left\{-a_1(x+\delta)\log\delta - \left[\log(y-x)\,a_1(y)\,e^{-k(y-x)}\right]_{x+\delta}^{1} + e^{kx}\int_{x+\delta}^{1}\log(y-x)\,\frac{d}{dy}\left[a_1(y)\,e^{-ky}\right]dy\right\}$$

$$= -\frac{e^{-kx}}{2}\int_0^x \log(x-y)\,\frac{d\,a_2(y)}{dy}\,dy + \frac{e^{kx}}{2}\int_x^1 \log(y-x)\,\frac{d}{dy}\left[a_2(y)\,e^{-2ky}\right]dy$$

and

$$b_2(x) = e^{kx}\,b_1(x) = -\tfrac{1}{2}\int_0^x \frac{da_2}{dy}\log(x-y)\,dy + \frac{e^{2kx}}{2}\int_x^1 \frac{d}{dy}\left[a_2(y)e^{-2ky}\right]\log(y-x)\,dy.$$

APPENDIX B

Fluid velocity, pressure, temperature and density in linearised viscous flow in terms of the functions ϕ_1, ϕ_2 and χ

$$\underset{\sim}{q} = \underset{\sim}{q}_1 + \underset{\sim}{q}_2 + \underset{\sim}{q}_3,$$

$$= \nabla\phi_1 + \nabla\phi_2 + \underset{\sim}{f}(x)$$

$\underset{\sim}{f}(x)$ may be expressed in the form $\nabla\wedge\underset{\sim}{\omega}$ where $\underset{\sim}{\omega}$ is a vector function related to χ.

$\therefore \quad \nabla \underset{\sim}{q}_3 = 0$, and so from the continuity equation, $\dfrac{\partial \rho_3}{\partial x} = 0$

It may be assumed that $\rho_3 = 0$, and p_3 and T_3 may also be taken to be zero.

Equations (3.1) then give for $\underset{\sim}{q}_3$

$$\frac{\partial u_3}{\partial x} = \frac{1}{R_e} \nabla^2 u_3$$

$$\frac{\partial v_3}{\partial x} = \frac{1}{R_e} \nabla^2 v_3$$

$$\frac{\partial w_3}{\partial x} = \frac{1}{R_e} \nabla^2 w_3$$

which are the same equations as those obtained for the rotational component of $\underset{\sim}{q}$ in incompressible flow (Ref. 5).

χ therefore satisfies

$$\left(\nabla^2 - 2k \frac{\partial}{\partial x} \right) \chi = 0$$

where

$$2k = \frac{UL}{\nu_\infty} = R_e$$

$$u_3 = \frac{\partial \chi}{\partial x} - 2k\chi$$

$$v_{r_3} = \frac{\partial \chi}{\partial r}$$

From equations (3.10) and (3.11)

$$\frac{\partial^2 T_1}{\partial x^2} - \frac{1}{M^2} \nabla^2 T_1 = \frac{\gamma}{\sigma R_e} \nabla^2 \frac{\partial T_1}{\partial x} \tag{1}$$

$$\frac{\partial T_2}{\partial x} = \frac{1}{\sigma R_e} \nabla^2 T_2 \tag{2}$$

and from (3.6)

$$\frac{\partial T_2}{\partial x} - (\gamma - 1) \frac{\partial \rho_2}{\partial x} = \frac{\gamma}{\sigma R_e} \nabla^2 T_2 \tag{3}$$

Combining (2) and (3)

$$\frac{\partial T_2}{\partial x} + \frac{\partial \rho_2}{\partial x} = 0$$

$$\therefore \quad T_2 + \rho_2 = \text{constant} \tag{4}$$

$$\therefore \quad p_2 = \text{constant} = 0 \text{ say.}$$

After removal of terms dependent on x from the x-component of equations (3.1), we have

$$\frac{\partial}{\partial x}(u_1 + u_2) + \frac{1}{\gamma M^2} \frac{\partial p_1}{\partial x} - \frac{1}{R_e} \nabla^2 (u_1 + u_2) = \frac{1}{3R_e} \frac{\partial}{\partial x} \left\{ \frac{\partial}{\partial x}(u_1 + u_2) + \frac{\partial}{\partial y}(v_1 + v_2) + \frac{\partial}{\partial z}(w_1 + w_2) \right\} \tag{5}$$

and since $\underset{\sim}{q}_1 = \nabla \phi_1$ and $\underset{\sim}{q}_2 = \nabla \phi_2$

$$\left(\nabla^2 - \frac{3k}{2} \frac{\partial}{\partial x}\right) \phi_2 = 0 \tag{6}$$

From the continuity equation (3.2)

$$\frac{\partial \rho_2}{\partial x} + \nabla^2 \phi_2 = 0$$

$$\frac{\partial \rho_2}{\partial x} + \frac{3k}{2} \frac{\partial \phi_2}{\partial x} = 0 \qquad \text{from (6)}$$

$$\rho_2 = -\frac{3k}{2} \phi_2$$

and from (4) $\qquad T_2 = \frac{3k}{2} \phi_2$

The ϕ_1 dependent terms of equation (5) are

$$\frac{\partial^2 \phi_1}{\partial x^2} = -\frac{1}{\gamma M^2} \frac{\partial p_1}{\partial x} + \frac{4}{3R_e} \nabla^2 \frac{\partial \phi_1}{\partial x}$$

$$p_1 = -\gamma M^2 \left\{ \frac{\partial \phi_1}{\partial x} - \frac{4}{3R_e} \nabla^2 \phi_1 \right\} \qquad (7)$$

But p_1 satisfies (3. 9)

$$\frac{\partial^2 p_1}{\partial x^2} - \frac{1}{M^2} \nabla^2 p_1 = \frac{\gamma}{\sigma R_e} \nabla^2 \frac{\partial p_1}{\partial x}$$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2} - \frac{1}{M^2} \nabla^2 - \frac{\gamma}{\sigma R_e} \nabla^2 \frac{\partial}{\partial x} \right) \left(\nabla^2 - \frac{3R_e}{4} \frac{\partial}{\partial x} \right) \phi_1 = 0$$

ϕ_2 satisfies the second bracket.

$$\therefore \quad \left(\nabla^2 - M^2 \frac{\partial^2}{\partial x^2} + \frac{\gamma M^2}{\sigma R_e} \nabla^2 \frac{\partial}{\partial x} \right) \phi_1 = 0$$

Equation (7) gives p_1 as

$$p_1 = \gamma M^2 \left[\frac{2}{3k} \nabla^2 \phi_1 - \frac{\partial \phi_1}{\partial x} \right]$$

From (3.6)

$$\frac{\partial T_1}{\partial x} = (\gamma - 1) \frac{\partial \rho_1}{\partial x} + \frac{\gamma}{\sigma R_e} \nabla^2 T_1 \qquad (8)$$

and

$$\frac{\partial p_1}{\partial x} + \frac{\partial T_1}{\partial x} = \frac{\partial \rho_1}{\partial x} \quad \text{from (3.5)} \qquad (9)$$

Elimination of p_1 and ρ_1 from (7), (8) and (9) gives

$$T_1 = -(\gamma - 1) M^2 \frac{\partial \phi_1}{\partial x} \qquad (10)$$

Finally, from (7) (9) and (10) ,

$$\rho_1 = M^2 \left[\frac{2\gamma}{3k} \nabla^2 \phi_1 - \frac{\partial \phi_1}{\partial x} \right]$$

APPENDIX C

The inverse transform of $\bar{\psi}$

Equation (2.10) gives

$$\bar{\psi} = - A_1(p)\left(\Gamma + \log \tfrac{1}{2} \sqrt{p^2 + k^2}\, r\right)$$

The inverse transform is

$$\psi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\psi}\, e^{-ipx}\, dp$$

$$= - \frac{1}{2\pi} \int_{-\infty}^{\infty} A_1(p)\left(\Gamma + \log \tfrac{1}{2} r + \tfrac{1}{2} \log (p^2 + k^2)\right) e^{-ipx}\, dp$$

$$= a_1(x) \log \tfrac{1}{2} r - \frac{1}{2\pi} \int_{-\infty}^{\infty} A_1(p)\left(\Gamma + \tfrac{1}{2} \log (p^2 + k^2)\right) e^{-ipx}\, dp$$

where $a\,(x) = - \frac{1}{2\pi} \int_{-\infty}^{\infty} A_1(p)\, e^{-ipx}\, dp$

Now

$$- \frac{1}{2\pi} \int_{-\infty}^{\infty} A_1(p)\left(\Gamma + \tfrac{1}{2} \log (p^2 + k^2)\right) e^{-ipx}\, dp$$

$$= - \frac{1}{2\pi} \int_{-\infty}^{\infty} - ip\, A_1(p) \left[\frac{\Gamma + \frac{1}{2} \log (p^2 + k^2)}{- i\, p}\right] e^{-ipx}\, dp$$

$$= \int_{-\infty}^{\infty} \frac{\partial\, a_1(y)}{\partial\, y}\, f\,(x-y)\, dy \qquad \text{by the Faltung theorem}$$

where

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{\Gamma + \frac{1}{2} \log (p^2 + k^2)}{- i\, p}\right] e^{-ipx}\, dp$$

$$= \frac{1}{2\pi}\left\{ \int_{-\infty}^{0} \left[\frac{\Gamma + \frac{1}{2} \log (p^2 + k^2)}{- i\, p}\right] e^{-ipx} dp + \int_{0}^{\infty} \left[\frac{\Gamma + \frac{1}{2} \log(p^2 + k^2)}{- i\, p}\right] e^{-ipx} dp \right\}$$

Change the dummy variable p to -u in the first integral and to +u in the second.

$$f(x) = - \frac{1}{2\pi i}\left\{ \int_{\infty}^{0} \left[\frac{\Gamma + \frac{1}{2} \log(u^2 + k^2)}{u}\right] e^{iux} du + \int_{0}^{\infty} \left[\frac{\Gamma + \frac{1}{2} \log(u^2 + k^2)}{u}\right] e^{-iux} du \right\}$$

$$= - \frac{1}{2\pi i}\left\{ \int_{0}^{\infty} \left[\frac{\Gamma + \frac{1}{2} \log(u^2 + k^2)}{u}\right]\left(e^{-iux} - e^{iux}\right) du \right\}$$

$$= \frac{2i}{2\pi i} \int_{0}^{\infty} \left[\frac{\Gamma + \frac{1}{2} \log(u^2 + k^2)}{u}\right] \sin u\, x\, du$$

$$= \frac{1}{\pi} \int_0^\infty \left[\frac{\Gamma + \frac{1}{2}\log(u^2+k^2)}{u}\right] \sin u|x|\, dx \quad \operatorname{sgn} x$$

$$= \frac{1}{\pi} \operatorname{sgn} x \left\{ (\Gamma + \log k) \int_0^\infty \frac{\sin u|x|}{u} + \tfrac{1}{2} \int_0^\infty \log\left(1 + \frac{u^2}{k^2}\right)\frac{\sin u|x|}{u} du \right\}$$

$$= \frac{1}{\pi} \operatorname{sgn} x \left\{ (\Gamma + \log k)\ \frac{\pi}{2} + \tfrac{1}{2} - \pi \ \mathrm{Ei}\left[-k|x|\right]\right\}$$

where $\quad \mathrm{Ei}(\theta) = \int_{-\infty}^{\theta} \frac{e^t}{t}\, dt$

$$= \tfrac{1}{2} \operatorname{sgn} x \left\{ \Gamma + \log k - \mathrm{Ei}\left[-k|x|\right] \right\}$$

$$= -\tfrac{1}{2} \operatorname{sgn} x \left\{ \mathrm{Ei}\left[-k|x|\right] - \Gamma - \log k \right\}$$

$$\therefore\ \psi = a_1(x) \log \frac{r}{2} - \tfrac{1}{2} \operatorname{sgn}(x-y) \int_{-\infty}^{\infty} \frac{\partial a_1}{\partial y} \left\{ \mathrm{Ei}\left[-k|x-y|\right] - \Gamma - \log k \right\} dy$$

$$= a_1(x) \log \frac{r}{2} - \tfrac{1}{2} \int_{-\infty}^{x} \frac{\partial a_1}{\partial y} \left(\mathrm{Ei}\left[-k(x-y)\right] - \Gamma - \log k \right) dy$$
$$+ \frac{1}{2} \int_x^\infty \frac{\partial a_1}{\partial y} \left(\mathrm{Ei}\left[-k(y-x)\right] - \Gamma - \log k \right) dy$$

Thus $\quad \psi = a_1(x) \log \frac{r}{2} + b_1(x)$

where
$$b_1(x) = -\tfrac{1}{2} \int_{-\infty}^{x} \frac{\partial a_1}{\partial y} \left(\mathrm{Ei}\left[-k(x-y)\right] - \Gamma - \log k \right) dy$$
$$+ \frac{1}{2} \int_x^\infty \frac{\partial a_1}{\partial y} \left(\mathrm{Ei}\left[-k(y-x)\right] - \Gamma - \log k \right) dy$$

$$= \tfrac{1}{2}(\Gamma + \log k) \left[\int_{-\infty}^{x} \frac{\partial a_1}{\partial y} dy - \int_x^\infty \frac{\partial a_1}{\partial y} dy \right] - \tfrac{1}{2} \int_{-\infty}^{x} \frac{\partial a_1}{\partial y} \mathrm{Ei}\left[-k(x-y)\right] dy$$
$$+ \tfrac{1}{2} \int_x^\infty \frac{\partial a_1}{\partial y} \mathrm{Ei}\left[-k(y-x)\right] dy$$

$$= a_1(x)(\Gamma + \log k) - \tfrac{1}{2} \int_{-\infty}^{x} \frac{\partial a_1}{\partial y} \mathrm{Ei}\left[-k(x-y)\right] dy + \tfrac{1}{2} \int_x^\infty \frac{\partial a_1}{\partial y} \mathrm{Ei}\left[-k(y-x)\right] dy$$

giving equation (2.11).