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The Compressible Laminar Boundary Layer with Foreign Gas Injection

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SUMMARY

The equations of the steady compressible two-dimensional laminar boundary layer with foreign gas injection through a porous wall are solved, using an extended form of Lighthill's approximate method, for arbitrary main stream pressure gradient, wall temperature and injection velocity. The wall shear stress and heat transfer rate are obtained in the form of equations suitable for iteration.

It is shown that substantial reductions in skin friction and heat transfer rate can be obtained by the injection of a light gas instead of air.

CONTENTS

LIST OF SYMBOLS

a	speed of sound
A, A_1, A_2	constants
B, B_1, B_2	constants
c	concentration of foreign gas
c'	concentration gradient $\left(\frac{\partial c}{\partial y}\right)$ at the wall
C_p	specific heat at constant pressure
C	$\frac{\mu\rho\, p_o}{\mu_o \rho_o p}$
D_{12}	the binary diffusion coefficient
f_w	dimensionless injection parameter $= \dot{m}\,(x/\rho_a \mu_a u_a)^{\frac{1}{2}}$
$G(X, \psi)$	$Z - \int_0^X S(z, \psi)\, d\, U_1^2(z)$
h	specific enthalpy
h_s	stagnation enthalpy
$H(X, \psi)$	$G(X, \psi) + \int_0^X V_w(X) \frac{\partial G}{\partial \psi}\, dX$
k	thermal conductivity
Le	Lewis number $\rho\, \bar{C}_p D_{12}/k$
$\dot{m}$	injection mass flow rate per unit area
$m(x)$	$1 + \frac{\gamma - 1}{2}\, M^2(x)$
M	Mach number
p	pressure
$\dot{q}$	normal energy flux due to injection
$\dot{Q}_w(x)$	rate of heat transfer per unit area
$\dot{Q}_{wo}$	rate of heat transfer for zero injection
$s_w(x)$	$\dot{Q}_w(x)\left[x/\rho_a \mu_a u_a\right]^{\frac{1}{2}}$, the modified heat transfer rate
S	$1 - h/h_1$

Sc	Schmidt number $\mu/\rho D_{12}$
$t_w(x)$	non-dimensional wall shear stress, $\tau_w(x)\left[\dfrac{x}{\rho_a \mu_a u_a^3}\right]^{\frac{1}{2}}$
t_{wo}	non-dimensional wall shear stress for zero injection
T	temperature
u, v	velocity components in the compressible flow
U, V	velocity components in the transformed flow
v_w, V_w	normal velocity at the wall in the compressible and transformed flows. respectively
x, y	co-ordinates in the compressible flow
X, Y	co-ordinates in the transformed flow
Z	$U_1^2 - U^2$
γ	ratio of specific heats C_p/C_v
Δ	$\frac{1}{\sigma}(Le - 1)(h_e - h_i)$
μ	viscosity
ν	kinematic viscosity
ρ	density
σ	Prandtl number $\mu C_p/k$
ψ	stream function
τ_w	wall shear stress

Subscripts

o	stagnation value
1	value outside the boundary layer
w	value at the wall
a	reference condition
e	mainstream
i	injected gas

A bar over a quantity denotes its Laplace transform

1. Introduction

Recent studies* have suggested that injection of a gas into the boundary layer through a porous wall can be used to reduce the skin friction and the rate of heat transfer to the wall. The majority of the work on the laminar boundary layer with injection is theoretical and considers mainly the injection of air into air. The analyses are restricted severely by the assumption of particular stream-wise and injection velocity distributions in obtaining solutions of the equations. Since it is difficult to maintain a laminar boundary layer there is very little experimental evidence but such as exists (Ref. 2) lends support to the theoretical results.

Injection of a foreign gas into a two-dimensional laminar boundary layer has been considered by Smith(3), Eckert and Schneider(4) and Faulders(5). Each shows that injection of a light gas is much more effective than injection of air in reducing skin friction. Smith's solution does not give values of the wall shear stress explicitly but these can be found from the velocity profiles which are presented. Each solution is subject to some restrictive assumptions. Smith solves the boundary layer equations and the diffusion equation with the boundary conditions appropriate to the impermeable wall. The solution takes account of the foreign gas (the concentration of which is taken to be large at the wall) but paradoxically considers the injection velocity to be zero.

The solutions of Eckert and Schneider and of Faulders are restricted to the case of zero heat transfer and assume that the injection velocity varies inversely as $x^{\frac{1}{2}}$. A further assumption in Faulder's treatment is that the viscosity of the binary mixture is independent of concentration and varies linearly with temperature. The Schmidt number is taken to be unity.

The case of non-zero heat transfer is considered by Korobkin(16) in a study to determine which of the properties of the injected gas is of most importance in reducing skin friction and rate of heat transfer. Using the simple rigid sphere model for the molecular collision processes, the equations of motion are solved numerically for the case when the injection velocity varies inversely as $x^{\frac{1}{2}}$. In the results presented two of the three properties of the mixture, molecular weight, molecular diameter, and specific heat at constant pressure are given the value for air and the third is varied taking the value corresponding to the calculated concentration. This solution (to an approximate physical problem) shows that variations of C_p have a negligible effect on skin friction. The greatest reduction in skin friction is to be expected when the injected gas has low molecular weight and large molecular diameter. These properties coupled with high specific heat per unit mass should give the greatest reduction in the rate of heat transfer.

A more general formulation and solution of the problem of gas injection into a laminar boundary layer is possible using an approximate method originally developed by Lighthill(6) for the incompressible layer and extended to the compressible layer by Lilley(7). Both these solutions are for the impermeable wall. Stevenson(8) has used Lighthill's approach to solve approximately the equations of the incompressible laminar boundary layer with either suction or air injection through a porous wall. Arbitrary distributions of main stream velocity,

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* A comprehensive bibliography is contained in Ref. 1.

wall temperature and normal velocity at the wall are included in the solution which is extended in the same paper to the compressible case.

The present paper uses Lilley's simplified theory for a compressible laminar boundary layer as the starting point to consider foreign gas injection. Approximate solutions are obtained for the diffusion equation and the equations of the compressible laminar boundary layer with arbitrary external pressure gradient, wall temperature and injection velocity distributions. Expressions for the wall shear stress and heat transfer rate to the wall are obtained in the form of integral equations involving the concentration of the injected gas at the wall (which is obtained from a third integral equation). These integral equations are in a form suitable for numerical iteration.

2. The Boundary Layer Equations appropriate to Injection

It is assumed that both the injected and the mainstream gases are perfect and that chemical reactions are absent. Consequently we may consider the enthalpy h of the binary mixture to be related to the enthalpies of the two constituents by the equation

$$h = (1 - c)\, h_e + c h_i \tag{1}$$

where h_e is the enthalpy of the mainstream gas $\int_o^T C_{p_e}\, dT$

h_i is the enthalpy of the injected gas $\int_o^T C_{p_i}\, dT$

and c is the concentration of the injected gas expressed as a mass fraction

C_{p_e} and C_{p_i} are functions of T only.

If suffix $_1$ denotes local conditions outside the boundary layer, the equations governing the steady two-dimensional compressible boundary layer in the presence of a pressure gradient are

(i) continuity

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{2}$$

(ii) motion

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} - \rho_1 u_1 \frac{du_1}{dx} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{3}$$

$$\frac{\partial p}{\partial y} = 0 \tag{4}$$

(iii) energy

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} + u \rho_1 u_1 \frac{du_1}{dx} = \mu\left(\frac{\partial u}{\partial y}\right)^2 - \frac{\partial \dot{q}}{\partial y} \tag{5}$$

(iv) diffusion

$$\rho u \frac{\partial c}{\partial x} + \rho v \frac{\partial c}{\partial y} = \frac{\partial}{\partial y}\left(\rho D_{12} \frac{\partial c}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\mu}{Sc} \frac{\partial c}{\partial y}\right) \quad \ldots\ldots \tag{6}$$

where the Schmidt number Sc is defined as $\mu / \rho D_{12}$ and D_{12} is the binary diffusion coefficient for the mixture

$\dot{q}$ in equation (5) is the normal component of the energy flux component. In terms of the diffusion velocities of the two species, $\dot{q}$ may be written in the form

$$\dot{q} = \rho c v_i h_i + \rho(1 - c)v_e h_e - k\frac{\partial T}{\partial y} \tag{7}$$

where v_i and v_e are the diffusion velocities of the injected and main stream gases respectively.

In terms of the concentration and the concentration gradient the diffusion velocities may be written, if pressure and thermal diffusion effects are neglected,

$$c\, v_i = - D_{12}\frac{\partial c}{\partial y}$$

$$(1 - c)v_e = - D_{21}\frac{\partial}{\partial y}(1 - c) = D_{12}\frac{\partial c}{\partial y} \quad \text{since } D_{21} = D_{12}$$

Furthermore we may express $\frac{\partial T}{\partial y}$ in terms of enthalpy and concentration gradients. From (1)

$$\frac{\partial h}{\partial y} = (1 - c)C_{p_e}\frac{\partial T}{\partial y} + c\, C_{p_i}\frac{\partial T}{\partial y} - h_e\frac{\partial c}{\partial y} + h_i\frac{\partial c}{\partial y}$$

$$= \bar{C}_p\frac{\partial T}{\partial y} + (h_i - h_e)\frac{\partial c}{\partial y}\ ; \quad \bar{C}_p = c\, C_{p_i} + (1 - c)C_{p_e}$$

or

$$k\frac{\partial T}{\partial y} = \frac{\mu}{\sigma}\frac{\partial h}{\partial y} - (h_i - h_e)\frac{\mu}{\sigma}\frac{\partial c}{\partial y}$$

where the Prandtl number $\sigma = \frac{\mu\bar{C}_p}{k}$

Substituting these forms in (7) the normal energy flux can be written

$$\dot{q} = -\frac{\mu}{\sigma}\frac{\partial h}{\partial y} + \frac{\mu}{\sigma}(Le - 1)(h_e - h_i)\frac{\partial c}{\partial y}$$

or

$$\dot{q} = -\frac{\mu}{\sigma}\frac{\partial h}{\partial y} + \mu\Delta\frac{\partial c}{\partial y} \tag{8}$$

where $\Delta = \frac{1}{\sigma}(Le - 1)(h_e - h_i)$

and Le is the Lewis number $\rho\,\bar{C}_p\, D_{12}/k$

The boundary conditions are

(i) at the wall $y = 0$,

$$\begin{aligned} u &= 0 \\ v &= v_w(x) \\ c &= c_w(x) \\ T &= T_w(x) \end{aligned} \tag{9}$$

where the suffix w denotes the wall value.

(ii) at $y = \infty$

$$u = u_1(x)$$
$$c = 0 \qquad (10)$$
$$T = T_1(x)$$

$$\frac{\partial^n u}{\partial y^n} = \frac{\partial^n T}{\partial y^n} = \frac{\partial^n c}{\partial y^n} = 0 \qquad n \geqslant 1$$

If we define the stagnation enthalpy h_s by

$$h_s = h + u^2/2$$

it is possible to eliminate the pressure gradient in the energy equation by multiplying (3) by u and adding it to (5). The resulting equation for the stagnation enthalpy is

$$\rho u \frac{\partial h_s}{\partial x} + \rho v \frac{\partial h_s}{\partial y} = \frac{\partial}{\partial y}\left[\mu \frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)\right] - \frac{\partial \dot{q}}{\partial y}$$

or, on substituting for $\dot{q}$ from (8)

$$\rho u \frac{\partial h_s}{\partial x} + \rho v \frac{\partial h_s}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{\sigma}\frac{\partial h_s}{\partial y}\right) - \frac{\partial}{\partial y}\left\{\frac{\mu}{\sigma}\left[1-\sigma\right]\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)\right\} - \frac{\partial}{\partial y}\left(\mu\Delta\frac{\partial c}{\partial y}\right)$$

..... (11)

The external flow is assumed to be isentropic so that

$$\frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = h_{s1} = \frac{a_o^2}{\gamma - 1} \qquad (12)$$

where γ is the constant ratio of the specific heats in the external flow.

3. The Stewartson- Illingworth transformation

In the compressible flow the equation of continuity (2) can be satisfied by a stream function ψ defined by

$$\frac{\rho u}{\rho_o} = \frac{\partial \psi}{\partial y} \quad ; \quad \rho v - \rho_w v_w(x) = -\rho_o \frac{\partial \psi}{\partial x} \qquad (13)$$

where the suffix o denotes some constant reference condition and $\rho_w(x)$ is the density of the binary mixture at the wall.

Following Stewartson[9] and Illingworth[10], the x, y co-ordinates of the compressible flow field are transformed to X, Y co-ordinates related to x, y by

$$X = \int_o^x \frac{a_1(x')\, p_1(x')}{a_o\, p_o}\, dx'$$
$$\qquad (14)$$
$$Y = \frac{a_1(x)}{a_o}\int_o^y \frac{\rho(x, y')}{\rho_o}\, dy'$$

The velocity components (U, V) in the X, Y plane are now related to those in the x, y plane. Thus

$$\frac{\rho u}{\rho_o} = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial\psi}{\partial Y}\frac{\partial Y}{\partial y} = \frac{a_1\rho}{a_o\rho_o}\frac{\partial\psi}{\partial Y}$$

and defining U as $\frac{\partial\psi}{\partial Y}$ we have

$$U = \frac{a_o u}{a_1(x)} \tag{15}$$

Also

$$-\frac{1}{\rho_o}(\rho v - \rho_w v_w) = \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial\psi}{\partial Y}\frac{\partial Y}{\partial x}$$

$$= \frac{a_1 p_1}{a_o p_o}\frac{\partial\psi}{\partial X} + \frac{a_o u}{a_1}\frac{\partial}{\partial x}\left(\frac{a_1}{a_o}\int_o^y \frac{\rho(x,y')}{\rho_o}\,dy'\right)$$

and therefore

$$-\frac{\partial\psi}{\partial X} = \frac{a_o p_o}{a_1 p_1 \rho_o}\left[\rho v - \rho_w v_w + \rho_o\frac{a_o u}{a_1}\frac{\partial}{\partial x}\left(\frac{a_1}{a_o}\int_o^y \frac{\rho}{\rho_o}dy'\right)\right]$$

If we define

$$V - V_w = -\frac{\partial\psi}{\partial X}$$

it follows that

$$V = \frac{a_o p_o}{a_1 p_1}\left[\frac{\rho v}{\rho_o} + \frac{a_o u}{a_1}\frac{\partial}{\partial x}\left(\frac{a_1}{a_o}\int_o^y \frac{\rho}{\rho_o}\,dy'\right)\right]$$

and

$$V_w = \frac{a_o p_o \rho_w}{a_1 p_1 \rho_o}\, v_w \tag{16}$$

Writing suffix o to denote stagnation conditions in the mainstream, equation (15) with $u = u_1$ substituted into (12) yields

$$a_1^2 = a_o^2 \Big/ \left(1 + \frac{\gamma - 1}{2}\cdot\frac{U_1^2}{a_o^2}\right) \tag{17}$$

Using the transformation equations (14 - 16) the equation of motion (3) becomes

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \frac{h_s}{h_{s1}}U_1\frac{\partial U_1}{\partial X} + \frac{p_o}{p_1}\nu_o\frac{\partial}{\partial Y}\left(\frac{\rho\mu}{\rho_o\mu_o}\frac{\partial U}{\partial Y}\right) \tag{18}$$

which can be simplified by putting

$$S = 1 - h_s/h_{s1} \tag{19}$$

and

$$C(X,Y) = \frac{p_o\rho\mu}{p_1\rho_o\mu_o} \tag{20}$$

giving

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = (1 - S)\, U_1 \frac{\partial U_1}{\partial X} + \nu_o \frac{\partial}{\partial Y}\left(C\frac{\partial U}{\partial Y}\right) \qquad (21)$$

Similarly transformed the diffusion equation (6) becomes

$$U\frac{\partial c}{\partial X} + V\frac{\partial c}{\partial Y} = \nu_o \frac{\partial}{\partial Y}\left(\frac{C}{Sc}\frac{\partial c}{\partial Y}\right) \qquad (22)$$

The transformed equation for the stagnation enthalpy is (from 11)

$$U\frac{\partial S}{\partial X} + V\frac{\partial S}{\partial Y} = \nu_o \frac{\partial}{\partial Y}\left[\frac{C}{\sigma}\frac{\partial}{\partial Y}S\right] + \nu_o \frac{\partial}{\partial Y}\left[\frac{C}{\sigma}(1-\sigma)\frac{\partial}{\partial Y}\left(\frac{U^2 a_1^2}{2\, a_o^2\, h_{s1}}\right)\right]$$
$$+ \nu_o \frac{\partial}{\partial Y}\left[\frac{C\Delta}{h_{s1}}\frac{\partial c}{\partial Y}\right] \qquad (23)$$

In equation 20 we can, by virtue of (4), replace p_1 by p. For the case of air injection C can be written

$$C = \frac{T_o \mu}{T \mu_o} = \left(\frac{T}{T_o}\right)^{\omega - 1}$$

if μ is taken to be proportional to T^{ω}. For foreign gas injection μ and ρ are concentration dependent as well as temperature dependent, and thus no simplification of C is possible.
The von Mises transformation

$$\left(\frac{\partial}{\partial X}\right)_Y = \left(\frac{\partial}{\partial X}\right)_\psi - (V - V_w)\left(\frac{\partial}{\partial \psi}\right)_X$$
$$\left(\frac{\partial}{\partial Y}\right)_X = U\frac{\partial}{\partial \psi} \qquad (24)$$

is now applied to transform from the "pseudo-incompressible" space co-ordinates (X, Y) to independent variables (X, ψ).
Putting

$$Z(X,\psi) = U_1^2(X) - U^2(X,\psi)$$

the equations of motion (21), diffusion (22) and stagnation enthalpy (23) become respectively

$$\frac{\partial Z}{\partial X} + V_w\frac{\partial Z}{\partial \psi} = S\frac{d\,U_1^2}{dX} + \nu_o U \frac{\partial}{\partial \psi}\left(C\frac{\partial Z}{\partial \psi}\right) \qquad (25)$$

$$\frac{\partial c}{\partial X} + V_w\frac{\partial c}{\partial \psi} = \nu_o \frac{\partial}{\partial \psi}\left(\frac{UC}{Sc}\frac{\partial c}{\partial \psi}\right) \qquad (26)$$

$$\frac{\partial S}{\partial X} + V_w\frac{\partial S}{\partial \psi} = \nu_o \frac{\partial}{\partial \psi}\left[\frac{UC}{\sigma}\frac{\partial S}{\partial \psi}\right] + \nu_o \frac{\partial}{\partial \psi}\left[\frac{UC}{\sigma}(1-\sigma)\frac{\partial}{\partial \psi}\left(\frac{U^2 a_1^2}{2a_o^2 h_{s1}}\right)\right]$$
$$+ \nu_o \frac{\partial}{\partial \psi}\left[\frac{UC\Delta}{h_{s1}}\frac{\partial c}{\partial \psi}\right] \qquad (27)$$

Equation (27) can be written alternatively in the form

$$\frac{\partial S}{\partial X} + V_w \frac{\partial S}{\partial \psi} - \nu_o \frac{\partial}{\partial \psi}\left(\frac{UC}{\sigma}\frac{\partial S}{\partial \psi}\right) = \frac{\nu_o(\gamma-1)}{a_1^2\left(1+\frac{\gamma-1}{2}\cdot\frac{U_1^2}{a_o^2}\right)}\frac{\partial}{\partial \psi}\left(UC\Delta\frac{\partial c}{\partial \psi}\right)$$

$$-\frac{\nu_o}{U_1^2}\left[\frac{\frac{\gamma-1}{2}\cdot\frac{U_1^2}{a_o^2}}{1+\frac{\gamma-1}{2}\frac{U_1^2}{a_o^2}}\right]\frac{\partial}{\partial \psi}\left(\frac{UC}{\sigma}\left[1-\sigma\right]\frac{\partial Z}{\partial \psi}\right) \tag{28}$$

In these equations the Prandtl number σ, the Schmidt number Sc, the Lewis number Le (in Δ) and the parameter C are concentration dependent. γ is the ratio of the specific heats of the mainstream gas and is a constant.

4. An approximate solution of the transformed equation of motion

The first term on the right hand side of the transformed equation of motion (25) can be written

$$S(X,\psi)\frac{d\,U_1^2(X)}{dX} = \frac{\partial}{\partial X}\int_o^X S(z,\psi)\,d\,U_1^2(z).$$

and thus (25) becomes

$$\frac{\partial}{\partial X}\left(Z - \int_o^X S(z,\psi)\,d\,U_1^2(z)\right) = \nu_o U\frac{\partial}{\partial \psi}\left(C\frac{\partial Z}{\partial \psi}\right) - V_w\frac{\partial Z}{\partial \psi} \tag{29}$$

Let us now consider the equation

$$\frac{\partial G}{\partial X}(X,\psi) = \nu_o U\frac{\partial}{\partial \psi}\left(C\frac{\partial G}{\partial \psi}\right) - V_w\frac{\partial G}{\partial \psi} \tag{30}$$

where

$$G(X,\psi) = Z - \int_o^X S(z,\psi)\,d\,U_1^2(z)$$

If we replace $S(z,\psi)$ by some suitably chosen average value $S^*(z)$ for small values of ψ and by zero for large values of ψ then equation (30) reduces approximately to (29) with

$$G(X,\psi)_{\psi\to o} = Z - \int_o^X S^*(z)\,d\,U_1^2(z)$$

and

$$G(X,\psi),\ \frac{\partial G}{\partial \psi},\ \frac{\partial^2 G}{\partial \psi^2} \to 0 \text{ as } \psi\to\infty \tag{31}$$

One further simplification can be made to equation (30). We may expand $\frac{\partial}{\partial \psi}\left(C\frac{\partial G}{\partial \psi}\right)$ so that (30) becomes

$$\frac{\partial}{\partial X}G(X,\psi) + \left[V_w(X) - \nu_o U\frac{\partial C}{\partial \psi}\right]\frac{\partial G}{\partial \psi}(X,\psi) = \nu_o UC\frac{\partial^2 G}{\partial \psi^2}(X,\psi) \tag{32}$$

Consider the term $\nu_o U \frac{\partial C}{\partial \psi}$. Reverting back to the original space co-ordinates (x, y)

$$\nu_o U \frac{\partial C}{\partial \psi} = \frac{\nu_o a_o \rho_o}{a_1 \rho} \frac{\partial}{\partial y} \left(\frac{p_o \rho \mu}{p_1 \rho_o \mu_o} \right)$$

$$= \frac{1}{\rho \rho_o} \cdot \frac{a_o p_o}{a_1 p_1} \frac{\partial}{\partial y} (\rho \mu)$$

Now ρ and μ are functions of temperature and concentration. Thus

$$\frac{\partial}{\partial y}(\rho \mu) = \frac{\partial}{\partial T}(\rho \mu) \frac{\partial T}{\partial y} + \frac{\partial}{\partial c}(\rho \mu) \frac{\partial c}{\partial y}$$

It is shown later (equation 75) that $\frac{\partial c}{\partial y}$ is small being directly proportional to the injection mass flow. From tables of properties of gas mixtures (Refs. 13, 14) it is seen that $\frac{\partial \mu}{\partial c}$ is very small for small concentrations of injected gas and it can be inferred that $\frac{\partial}{\partial c}(\rho \mu)$ is not large. $\frac{\partial}{\partial T}(\rho \mu)$ is small and $\frac{\partial T}{\partial y}$, which is related to the heat transfer rate, is known to be reduced by air injection. It is assumed (and proved by the later analysis) that a greater reduction is obtained by light gas injection. The condition under which it is possible to ignore $\nu_o U \frac{\partial C}{\partial \psi}$ can be assessed by considering the concentration profiles found by Eckert and Schneider[4] for hydrogen injected into air at zero heat transfer in incompressible flow. In terms of the similarity parameter $\eta = \frac{1}{2} y (U_1 / \nu_o x)$ we may write

$$V_w - \nu_o U \frac{\partial C}{\partial \psi} = U_1 \left[\frac{V_w}{U_1} - \frac{1}{2 R_x^{\frac{1}{2}}} \frac{\partial C}{\partial \eta} \right]$$

Plotting C against η for different wall concentrations of injected hydrogen (Fig. 1) it can be seen that $\frac{\partial C}{\partial \eta}$ is not greater than 0.2 . $\nu_o U \frac{\partial C}{\partial \psi}$ can be neglected in comparison with V_w when

$$\frac{R_x^{\frac{1}{2}} V_w}{U_1} >> 0.1$$

We may therefore approximate to $C(X, \psi)$ in (32) by its value at some value of ψ. In other words we will assume that C is a function of X only, its value having to be determined later. The equation of motion (32) becomes

$$\frac{\partial}{\partial X} G(X, \psi) = \nu_o U \, C(X) \frac{\partial^2 G}{\partial \psi^2}(X, \mu) - V_w(X) \frac{\partial G}{\partial \psi}(X, \psi) \qquad (33)$$

with the boundary conditions

(i) at the wall, $\psi = 0$, $\quad G(X, \psi) = U_1^2(X) - \int_o^X S^*(z) \, d\, U_1^2(z)$

(ii) at $\psi = \infty$ $\qquad G(X,\psi) = 0$

(iii) as $X \to 0$ $\qquad G(X,\psi) \to 0$

(iv) near the wall

$$G(X,\psi) = U_1^2(0+) - U^2(X,\psi) + \int_0^X \frac{h_s^*(z)}{h_{s1}(z)}\, d\,U_1^2(z)$$

where "the intermediate enthalpy" h_s^* is given by $S^* = 1 - \frac{h_s^*}{h_{s1}}$.

Provided complete velocity profiles are not required we may use the approximation to the velocity distribution near the wall used by Fage and Falkner[11] and by Lighthill[6] namely

$$U = \frac{\tau_w(X)Y}{\mu_o} = \sqrt{\frac{2\,\tau_w(X)}{\mu_o}}\;\psi^{\frac{1}{2}} \tag{34}$$

With this substitution the equation of motion (33) becomes

$$\frac{\partial G}{\partial X} = \sqrt{\frac{2\mu_o}{\rho_o^2}\cdot\tau_w(X)\,C^2(X)}\;\;\psi^{\frac{1}{2}}\,\frac{\partial^2 G}{\partial\psi^2} - V_w(X)\,\frac{\partial G}{\partial\psi} \tag{35}$$

with the boundary conditions

(i) as $\psi \to \infty$, $G \to 0$

(ii) as $X \to \infty$, $G \to 0$

(iii) as $\psi \to 0$

$$G = U_1^2(O) + \int_0^X \frac{h_s^*(z)}{h_{s1}}\, d\,U_1^2(z) - \frac{2\,\tau_w(X)}{\mu_o}\,\psi + O(\psi^{3/2}) \tag{36}$$

For small values of injection velocity, $\frac{\partial G}{\partial\psi}$ can be approximated by its value at the wall and we may regard it as a function of X only. Thus the second term on the right hand side of equation 35 is taken as a function of X only.

Putting $V_w(X) = 0$ in (35) gives the equation for the impermeable wall

$$\frac{\partial G}{\partial X} = \sqrt{\frac{2\mu_o}{\rho_o^2}\,\tau_w(X)\,C^2(X)}\;\psi^{\frac{1}{2}}\,\frac{\partial^2 G}{\partial\psi^2}$$

or, if $\qquad t = \int_0^X \sqrt{\frac{2\mu_o}{\rho_o^2}\,\tau_w(X)\,C^2(X)}\,dX,$

$$\frac{\partial}{\partial t}\,G(t,\psi) = \psi^{\frac{1}{2}}\,\frac{\partial^2 G}{\partial t^2}(t,\psi) \tag{37}$$

with the boundary conditions at the wall

$$G = F_1(X) = U_1^2(O) + \int_0^X \frac{h_s^*(z)}{h_{s1}} \, d\, U_1^2(z)$$

$$\frac{\partial G}{\partial \psi} = F_2(X) = -\frac{2\tau_w(X)}{\mu_o} \qquad (38)$$

Following Lighthill and using the Laplace transform method, in which $\overline{F}(p,\psi) = \int_0^\infty e^{-pt} F(t,\psi)dt$, the solution of this equation is

$$\overline{G} = (\tfrac{2}{3}p^{\frac{1}{2}})^{\frac{2}{3}} \; \psi^{\frac{1}{2}} \; \Gamma(\tfrac{1}{3}) \; I_{-\frac{2}{3}}(q)\, \overline{F}_1 + (\tfrac{2}{3}p^{\frac{1}{2}})^{-\frac{2}{3}} \psi^{\frac{1}{2}} \; \Gamma(\tfrac{5}{3})\, I_{\frac{2}{3}}(q)\, \overline{F}_2 \qquad (39)$$

where $I_{\frac{2}{3}}$ and $I_{-\frac{2}{3}}$ are modified Bessel Functions

and $\quad q = \frac{4}{3} p^{\frac{1}{2}} \psi^{\frac{3}{4}}$

The solution of the complete equation of motion (35) for injected flow can be obtained from (39) by the method of variation of parameters.

Let the solution of (35) be

$$\overline{G} = P_1(\psi)\,\overline{G}_1 + P_2(\psi)\,\overline{G}_2 \qquad (40)$$

where

$$\overline{G}_1 = \psi^{\frac{1}{2}} I_{-\frac{2}{3}}(q)$$

$$\overline{G}_2 = \psi^{\frac{1}{2}} I_{\frac{2}{3}}(q)$$

The equations for P_1 and P_2 are then

$$\frac{dP_1}{d\psi} = \frac{-\overline{G}_2 \overline{F}_3 \psi^{-\frac{1}{2}}}{\overline{G}_1' \overline{G}_2 - \overline{G}_1 \overline{G}_2'}$$

and

$$\frac{dP_2}{d\psi} = \frac{\overline{G}_1 \overline{F}_3 \psi^{-\frac{1}{2}}}{\overline{G}_1' \overline{G}_2 - \overline{G}_1 \overline{G}_2'}$$

where, from (35) and (34)

$$F_3(X) = V_w(X)\left(\frac{\partial G}{\partial \psi}\right)_{\psi=0} \left(\frac{\rho_o^2}{2\mu_o \; \tau_w(X)\, C^2(X)}\right)^{\frac{1}{2}} = -\frac{\rho_o V_w(X)}{\mu_o\, C(X)} \sqrt{\frac{2\,\tau_w(X)}{\mu_o}} \quad \ldots\ldots \quad (41)$$

and the prime ′ denotes partial differentiation with respect to ψ.

It can readily be shown that

$$\overline{G}_1' \, \overline{G}_2 - \overline{G}_1 \, \overline{G}_2' = -\frac{3}{2\pi} \sin \frac{2\pi}{3}$$

and thus

$$P_1 = -\frac{2\pi}{3\sin\frac{2\pi}{3}} \int_0^{\psi} \overline{F}_3\, I_{\frac{2}{3}}(q)\, d\psi$$

$$P_2 = +\frac{2\pi}{3\sin\frac{2\pi}{3}} \int_0^{\psi} \overline{F}_3\, I_{-\frac{2}{3}}(q)\, d\psi$$

giving the operational form of the solution of the equation of motion in the form

$$\begin{aligned}\overline{G}(p,\psi) = &-\frac{2\pi}{3\sin\frac{2\pi}{3}} \psi^{\frac{1}{2}} I_{-\frac{2}{3}}(q) \int_0^{\psi} \overline{F}_3\, I_{\frac{2}{3}}(q)\, d\psi \\ &+\frac{2\pi}{3\sin\frac{2\pi}{3}} \psi^{\frac{1}{2}} I_{\frac{2}{3}}(q) \int_0^{\psi} \overline{F}_3\, I_{-\frac{2}{3}}(q)\, d\psi \\ &+ A\psi^{\frac{1}{2}} I_{-\frac{2}{3}}(q) + B\psi^{\frac{1}{2}} I_{\frac{2}{3}}(q)\end{aligned} \tag{41}$$

where A and B must be determined from the boundary conditions.

In the limit as $\psi \to 0$, and comparing with (38), equation (41) gives

$$\overline{F}_1 = A(\tfrac{2}{3}\, p^{\frac{1}{2}})^{-\frac{2}{3}} \Big/ \Gamma(\tfrac{1}{3})$$

Differentiating (41) and taking the limit as $\psi \to 0$

$$\overline{F}_2 = B\,(\tfrac{2}{3}\, p^{\frac{1}{2}})^{\frac{2}{3}} \Big/ \Gamma(\tfrac{5}{3})$$

Hence (41) becomes

$$\begin{aligned}\overline{G}(p,\psi) = &-\frac{2\pi\,\overline{F}_3}{3\sin\frac{2\pi}{3}} \psi^{\frac{1}{2}} \left[I_{-\frac{2}{3}}(q) \int_0^{\psi} I_{\frac{2}{3}}(q)\, d\psi - I_{\frac{2}{3}}(q) \int_0^{\psi} I_{-\frac{2}{3}}(q)\, d\psi \right] \\ &+ (\tfrac{2}{3}\, p^{\frac{1}{2}})^{\frac{2}{3}} \Gamma(\tfrac{1}{3})\, \overline{F}_1\, \psi^{\frac{1}{2}} I_{-\frac{2}{3}}(q) + (\tfrac{2}{3}\, p^{\frac{1}{2}})^{-\frac{2}{3}}\, \Gamma(\tfrac{5}{3})\, \overline{F}_2\, \psi^{\frac{1}{2}} I_{\frac{2}{3}}(q)\end{aligned} \tag{42}$$

Since $\overline{G} \to 0$ as $\psi \to \infty$, the coefficients of $I_{\frac{2}{3}}(q)$ & $I_{-\frac{2}{3}}(q)$ must be equal in magnitude and opposite in sign yielding

$$-\frac{2\pi\overline{F}_3}{3\sin\frac{2\pi}{3}} \int_0^{\infty} \left(I_{\frac{2}{3}}(q) - I_{-\frac{2}{3}}(q) \right) \frac{d\psi}{dq}\, dq + (\tfrac{2}{3}\, p^{\frac{1}{2}})^{\frac{2}{3}}\, \Gamma(\tfrac{1}{3})\, \overline{F}_1 + (\tfrac{2}{3}\, p^{\frac{1}{2}})^{-\frac{1}{3}}\, \Gamma(\tfrac{2}{3})\, \overline{F}_2 = 0 \quad \ldots\ldots \tag{43}$$

Now
$$\begin{aligned}\int_0^{\infty} \left[I_{\frac{2}{3}}(q) - I_{-\frac{2}{3}}(q) \right] \frac{d\psi}{dq}\, dq &= -\frac{2}{\pi} \sin\frac{2\pi}{3}\, (\tfrac{3}{4})^{\frac{1}{3}}\, p^{-\frac{2}{3}} \int_0^{\infty} q^{\frac{1}{3}} K_{\frac{2}{3}}(q)\, dq \\ &= -\Gamma(\tfrac{1}{3}) \sin\frac{2\pi}{3}\, p^{-\frac{2}{3}} .\, \frac{2^{\frac{1}{3}}}{\pi} (\tfrac{3}{4})^{\frac{1}{3}}\end{aligned} \tag{44}$$

$K_{\frac{2}{3}}(q)$ is a modified Bessel function of the third kind.

Using (44), (43) becomes

$$\overline{F}_1 + \frac{\overline{F}_3}{p} = -\left(\tfrac{2}{3}\right)^{-4/3} p^{-\frac{2}{3}} \frac{\Gamma(\frac{5}{3})}{\Gamma(\frac{1}{3})} \overline{F}_2 \tag{45}$$

Taking the inverse transforms of (45) we have

$$U_1^2(O+) + \int_0^X \frac{h_s^*(z)}{h_{s1}} d\, U_1^2(z) - \frac{2}{\mu_o} \int_0^X V_w(z)\, \tau_w(z) dz$$

$$= \frac{2.3^{\frac{1}{3}}}{\Gamma(\frac{1}{3})(\rho_o \mu_o)^{\frac{2}{3}}} \int_0^X C(X_1)\, \tau_w^{3/2}(X_1) \left[\int_{X_1}^X \tau_w^{\frac{1}{2}}(z)\, C(z) dz\right]^{-\frac{1}{3}} dX \quad \ldots\ldots \tag{46}$$

Equation (46) is an integral equation for the wall shear stress in terms of the external flow conditions, and the intermediate enthalpy distribution.

5. An alternative solution for the equation of motion

If we put $H(X,\psi) = G(X,\psi) + \int_0^X V_w(X) \frac{\partial G}{\partial \psi} dX$

in equation (35), the equation of motion becomes

$$\frac{\partial H}{\partial X} = \sqrt{\frac{2\mu_o}{\rho_o^2} \tau_w(X)\, C^2(X)}\, \psi^{\frac{1}{2}} \frac{\partial^2 H}{\partial \psi^2} \tag{47}$$

with boundary conditions

(i) as $\psi \to \infty$ $\quad H \to 0$

(ii) as $X \to 0$ $\quad H \to 0$

(iii) as $\psi \to 0$

$$H = U_1^2(O+) + \int_0^X \frac{h_s^*(z)}{h_{s1}} d\, U_1^2(z) - 2\int_0^X \frac{V_w(z)\tau_w(z)}{\mu_o} dz - \frac{2\tau_w}{\mu_o}\psi + 0(\psi^{3/2})$$

$$= H_1(X) + H_2(X)\psi\,.$$

In defining $H(X,\psi)$ it is assumed that $\frac{\partial G}{\partial \psi}$ is given its wall value and is thus a function of X only.

Using the operational techniques of the previous section, (47) becomes

$$p\,\overline{H} = \psi^{\frac{1}{2}} \frac{\partial^2 \overline{H}}{\partial \psi^2}$$

which has the solution

$$\overline{H} = \left(\tfrac{2}{3}\, p^{\frac{1}{2}}\right)^{\frac{2}{3}} \psi^{\frac{1}{2}}\, \Gamma(\tfrac{1}{3})\, I_{-\frac{2}{3}}(q)\, \overline{H}_1 + \left(\tfrac{2}{3}\, p^{\frac{1}{2}}\right)^{-\frac{2}{3}} \psi^{\frac{1}{2}}\, \Gamma(\tfrac{5}{3})\, I_{\frac{2}{3}}(q)\, \overline{H}_2$$

and since $\overline{H} \to 0$ as $\psi \to \infty$ the coefficients of the Bessel functions must be equal in magnitude and opposite in sign.
Thus

$$\overline{H}_1 = -\left(\tfrac{2}{3}p^{\frac{1}{2}}\right)^{-4/3} \frac{\Gamma(\frac{5}{3})}{\Gamma(\frac{1}{3})} \overline{H}_2 \tag{48}$$

Taking the inverse transforms we obtain

$$U_1^2(O+) + \int_0^X \frac{h_s^*(z)}{h_{s1}} \, d\,U_1^2(z) - 2\int_0^X \frac{V_w(z)\,\tau_w(z)}{\mu_o} dz$$
$$= \frac{2.3^{\frac{1}{3}}}{\Gamma(\frac{1}{3})(\rho_o\mu_o)^{\frac{2}{3}}} \int_0^X C(X_1)\,\tau_w^{3/2}(X_1)\left[\int_{X_1}^X \tau_w^{\frac{1}{2}}(z)\,C(z)dz\right]^{-\frac{1}{3}} dX_1$$

which is identical with equation (46).

6. The wall shear stress

We now transform equation (46) for the wall shear stress into its compressible form by using relations stemming from the Stewartson-Illingworth transformation (14)

$$\frac{dX}{dx} = \left[m_1(x)\right]^{-\frac{3\gamma-1}{2(\gamma-1)}} \qquad \text{where } m_1(x) = 1 + \frac{\gamma-1}{2} M_1^2(x)$$
$$U_1(X) = a_o M_1(x) \quad ; \quad V_w(X) = \frac{\rho_w v_w}{\rho_o} m_1^{\frac{3\gamma-1}{2(\gamma-1)}} \tag{49}$$
$$\tau_w(X) = \frac{\tau_w(x)}{C_w(x)} m_1^{\frac{2\gamma-1}{\gamma-1}} \quad \text{where } C_w(x) = \frac{p_o\rho_w\mu_w}{p_1\rho_o\mu_o}$$

Consistent with the previous approximations we put $C_w = C$.

Equation (46) becomes

$$a_o^2\left[M_1^2(O) + \int_0^x \frac{h_s^*(z)}{h_{s1}} \, d\,M_1^2(z)\right] = \frac{2}{\mu_o\rho_o}\int_0^x \frac{\rho_w v_w \tau_w(z)}{C(z)} m_1^{\frac{2\gamma-1}{\gamma-1}} dz$$
$$+ \frac{2.3^{\frac{1}{3}}}{\Gamma(\frac{1}{3})(\mu_o\rho_o)^{\frac{2}{3}}} \int_0^x \frac{\tau_w^{3/2}(x_1)}{C^{\frac{1}{2}}(x_1)} m_1^{\frac{3\gamma-2}{2(\gamma-1)}} \left\{\int_{x_1}^x \frac{\tau_w^{\frac{1}{2}}(z)\,C^{\frac{1}{2}}(z)}{m_1^{\gamma/2(\gamma-1)}} dz\right\}^{-\frac{1}{3}} dx_1 \quad \ldots\ldots \tag{50}$$

We define a wall shear stress parameter $t_w(x)$ by

$$t_w(x) = \tau_w(x)\left(x/\rho_a\mu_a u_a^3\right)^{\frac{1}{2}} \tag{51}$$

and an injection parameter $f_w(x)$ by

$$f_w(x) = \dot{m}\left(x/\rho_a\mu_a u_a\right)^{\frac{1}{2}}; \quad \dot{m} = \rho_w v_w \tag{52}$$

where the suffix a refers to an arbitrary reference condition in the external stream and $\dot{m}(x)$ is the mass flow of injected gas per unit area.

Furthermore

$$\frac{\rho_a \mu_a a_a^2}{\rho_o \mu_o a_o^2} = \frac{\mu_a T_o}{\mu_o T_a} \Big/ m_a^{\frac{2\gamma-1}{\gamma-1}} \tag{53}$$

and

$$\left[\frac{\rho_a \mu_a a_a^3}{\rho_o \mu_o a_o^3}\right]^{\frac{2}{3}} = \left(\frac{\mu_a T_o}{\mu_o T_a}\right)^{\frac{2}{3}} \Big/ m_a^{\frac{5\gamma-3}{3(\gamma-1)}} \tag{54}$$

Substituting in (50) we obtain

$$\frac{M_1^2(o)}{M_a^2} + \int_o^x \frac{h_s^*(z)}{h_{s1}}\, d\left(\frac{M_1^2(z)}{M_a^2}\right) = 2\int_o^x \frac{f_w(z)\, t_w(z)}{C_a\, z} \left(\frac{m_1}{m_a}\right)^{\frac{2\gamma-1}{\gamma-1}} dz$$

$$+ \frac{2.3^{\frac{1}{3}}}{\Gamma(\frac{1}{3})} \int_o^x \frac{t_w^{3/2}(x_1)}{x_1^{\frac{3}{4}} C_a^{\frac{1}{2}}(x_1)} \left(\frac{m_1}{m_a}\right)^{\frac{3\gamma-2}{2(\gamma-1)}} \left\{ \int_{x_1}^x \frac{C_a^{\frac{1}{2}}(z) t_w^{\frac{1}{2}}(z)}{z^{\frac{1}{4}}} \left(\frac{m_a}{m_1}\right)^{\gamma/2(\gamma-1)} dz \right\}^{-\frac{1}{3}} dx_1 \quad \ldots\ldots \tag{55}$$

where

$$C_a = \frac{p_a \rho \mu}{p_1 \rho_a \mu_a} \quad ; \quad m_a(x) = 1 + \frac{\gamma - 1}{2} M_a^2(x)$$

If we put C_a equal to its wall value for air injection, i.e. $\frac{\mu_w T_a}{\mu_a T_w}$, equation (55) is identical with Stevenson's equation B.6 (Ref. 8). For the impermeable wall $f_w = 0$ in which case (55) becomes the same as Lilley's equation 30 (Ref. 7).

Equation (55) can be simplified by approximating to the value of the inner integral in the second term on the right hand side by writing

$$\int_{x_1}^x F(z)dz = (x - x_1)\, F(x)$$

The equation for $t_w(z)$ becomes

$$\frac{M_1^2(o)}{M_a^2} + \int_o^x \frac{h_s^*(z)}{h_{s1}}\, d\left(\frac{M_1^2(z)}{M_a^2}\right) = 2\int_o^x \frac{f_w(z)\, t_w(z)}{z\, C_a(z)} \left(\frac{m_1(z)}{m_a}\right)^{\frac{2\gamma-1}{\gamma-1}} dz$$

$$+ \frac{2.3^{\frac{1}{3}}}{\Gamma(\frac{1}{3})} \int_o^x \frac{t_w^{4/3}(x_1)}{x_1^{\frac{2}{3}}(x - x_1)^{\frac{1}{3}}} \cdot \frac{\left(m_1(z)/m_a\right)^{\frac{5\gamma-3}{3(\gamma-1)}}}{C_a^{\frac{2}{3}}(z)}\, dz \quad \ldots\ldots \tag{56}$$

An alternative form of the wall shear stress equation can be obtained by writing (45) as

$$\overline{F}_2 = -\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{3})}\left(\tfrac{2}{3}p^{\frac{1}{2}}\right)^{4/3}\overline{F}_1 - \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{4}{3})}\left(\tfrac{2}{3}\right)^{4/3}p^{-\frac{1}{3}}\overline{F}_3$$

or equation (48) as

$$\overline{H}_2 = -\left(\tfrac{2}{3}p^{\frac{1}{2}}\right)^{4/3}\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{3})}\overline{H}_1$$

Taking the inverse transforms of either equation, we obtain

$$\tau_w(X) = \frac{(\rho_o\mu_o)^{\frac{2}{3}}}{3^{\frac{1}{3}}\Gamma(\frac{2}{3})}\left\{\tfrac{1}{2}\int_o^X\left(\int_{X_1}^X C(z)\,\tau_w^{\frac{1}{2}}(z)\,dz\right)^{-\frac{2}{3}} d\left[U_1^2(X_1) + \int_o^{X_1}\frac{h_s^*(z)}{h_{s1}}\,d\,U_1^2(z)\right]\right.$$
$$\left. - \frac{1}{\mu_o}\int_o^X \tau_w(X_1)\,V_w(X_1)\left(\int_{X_1}^X C(z)\,\tau_w^{\frac{1}{2}}(z)\,dz\right)^{-\frac{2}{3}} dX_1\right\} \quad \ldots\ldots \quad (57)$$

Reverting to the compressible flow co-ordinates (x, y) using the relations (49) and introducing the shear stress and injection parameters defined in (51) and (52), equation 57 becomes

$$t_w(x) = \frac{x^{\frac{1}{2}}\,C_a(x)}{3^{\frac{1}{3}}\,\Gamma(\frac{2}{3})}\left(\frac{m_a}{m_1}\right)^{\frac{2\gamma-1}{\gamma-1}}\left\{\tfrac{1}{2}\int_o^x\left(\int_{x_1}^x \frac{C_a^{\frac{1}{2}}(z)\,t_w^{\frac{1}{2}}(z)}{z^{\frac{1}{4}}}\left(\frac{m_a}{m_1}\right)^{\gamma/2(\gamma-1)} dz\right)^{-\frac{2}{3}}\right.$$
$$d\left[\frac{M_1^2(x_1)}{M_a^2} + \int_o^{x_1}\frac{h_s^*(z)}{h_{s1}}\,d\left(\frac{M_1^2(z)}{M_a^2}\right)\right] - \int_o^x \frac{t_w(x_1)}{x_1\,C_a(x_1)}\left(\frac{m_1}{m_a}\right)^{\frac{2\gamma-1}{\gamma-1}} f_w(x_1)$$
$$\left.\left(\int_{x_1}^x \frac{C_a^{\frac{1}{2}}(z)\,t_w^{\frac{1}{2}}(z)}{z^{\frac{1}{4}}}\left(\frac{m_a}{m_1}\right)^{\gamma/2(\gamma-1)} dz\right)^{-\frac{2}{3}} dx_1\right\} \quad \ldots\ldots \quad (58)$$

or, again approximating to the inner integrals, we have an expression for the wall shear stress which lends itself to an iterative evaluation.

$$t_w(x) = \frac{x^{\frac{1}{2}}C_a(x)}{3^{\frac{1}{3}}\;\Gamma(\frac{2}{3})}\left[\frac{m_a(x)}{m_1(x)}\right]^{\frac{2\gamma-1}{\gamma-1}}\left\{\tfrac{1}{2}\int_o^x C_a^{-\frac{1}{3}}(x_1)\,t_w^{-\frac{1}{3}}(x_1)\,\frac{x_1^{1/6}}{(x-x_1)^{\frac{2}{3}}}\left(\frac{m_1(x_1)}{m_a}\right)^{\gamma/3(\gamma-1)}\right.$$
$$\left. d\left[\frac{M_1^2(x_1)}{M_a^2} + \int_o^{x_1}\frac{h_s^*(z)}{h_1}\,d\left(\frac{M_1^2(z)}{M_a^2}\right)\right] - \int_o^x \frac{t_w^{\frac{2}{3}}(x_1)}{C_a^{4/3}(x_1)}\cdot\frac{f_w(x_1)}{x_1^{5/6}(x-x_1)^{\frac{2}{3}}}\left(\frac{m_1}{m_a}\right)^{\frac{7\gamma-3}{3(\gamma-1)}} dx_1\right\} \quad \ldots\ldots \quad (59)$$

If conditions in the free stream are known together with the knowledge of the injection mass flow and intermediate enthalpy, equation (59) can be solved only when C_a is known. This requires a solution to the diffusion equation, for with the concentration of foreign gas determined, it is then possible to calculate the values of density and viscosity at the wall and from these to obtain C_a.

7. An approximate solution of the diffusion equation

Again taking a value of C and Schmidt number independent of ψ, the diffusion equation (26) becomes

$$\frac{\partial c}{\partial X} + V_w(X)\frac{\partial c}{\partial \psi} = \nu_o \frac{C(X)}{Sc} \frac{\partial}{\partial \psi}\left(U\frac{\partial c}{\partial \psi}\right)$$

The velocity near the wall has previously (eqn. 34) been taken as

$$U = \sqrt{\frac{2\tau_w(X)}{\mu_o}} \quad \psi^{\frac{1}{2}}$$

leading to the diffusion equation in the form

$$\frac{\partial c}{\partial X} - \frac{C(X)}{\rho_o Sc}\sqrt{2\tau_w(X)\mu_o} \frac{\partial}{\partial \psi}\left(\psi^{\frac{1}{2}}\frac{\partial c}{\partial \psi}\right) = -V_w(X)\frac{\partial c}{\partial \psi} \qquad (60)$$

for which the boundary conditions are

(i) as $\psi \to \infty$, $c \to 0$

(ii) as $X \to 0$, $c \to 0$

(iii) as $\psi \to 0$, $c = c_w(X) + c'(X)\left(\frac{2\mu_o}{\tau_w(X)}\right)^{\frac{1}{2}} \psi^{\frac{1}{2}} + \ldots$

where $c'(X) = \left.\frac{\partial c}{\partial Y}\right|_{Y=0}$ i.e. at the wall.

If $t = \int_o^X \frac{C(z)}{Sc\rho_o}\sqrt{2\tau_w(z)\mu_o}\, dz$, (60) becomes

$$\frac{\partial}{\partial t} c(t,\psi) - \frac{\partial}{\partial \psi}\left(\psi^{\frac{1}{2}}\frac{\partial c}{\partial \psi}\right) = \psi^{-\frac{1}{2}} F_4(t) \qquad (61)$$

$$\text{where } F_4(X) = -\frac{V_w(X)\, c'(X)\, \rho_o .\, Sc}{2\tau_w(X)\, C(X)} \qquad (62)$$

In the notation of the Laplace transform

$$p\bar{c} - \frac{\partial}{\partial \psi}\left(\psi^{\frac{1}{2}}\frac{\partial \bar{c}}{\partial \psi}\right) = \psi^{-\frac{1}{2}} \bar{F}_4(p) \qquad (63)$$

The solution of

$$p\bar{c} = \frac{\partial}{\partial \psi}\left(\psi^{\frac{1}{2}}\frac{\partial \bar{c}}{\partial \psi}\right)$$

is given by Lighthill as

$$\bar{c} = a\,\psi^{\frac{1}{4}} I_{-\frac{1}{3}}(q) + b\,\psi^{\frac{1}{4}} I_{\frac{1}{3}}(q) \; ; \; q = \frac{4}{3} p^{\frac{1}{2}} \psi^{\frac{3}{4}}$$

where a and b are constants to be found from the boundary conditions. Using Lighthill's solution for the homogeneous equation we can solve (63) by the method of variation of parameters.

Let the solution of (63) be

$$\overline{c} = P_3(\psi)\,\overline{c}_1 + P_4(\psi)\,\overline{c}_2 \tag{64}$$

$$\text{where } \overline{c}_1 = \psi^{\frac{1}{4}} I_{\frac{1}{3}}(q)$$

$$\overline{c}_2 = \psi^{\frac{1}{4}} I_{-\frac{1}{3}}(q)$$

P_3 and P_4 are derived from the equations

$$\frac{d\,P_3}{d\,\psi} = - \frac{\overline{c}_2\,\psi^{-1}\,\overline{F}_4(p)}{\overline{c}_1'\,\overline{c}_2 - \overline{c}_1\,\overline{c}_2'} \tag{65}$$

$$\frac{d\,P_4}{d\psi} = \frac{\overline{c}_1\,\psi^{-1}\,\overline{F}_4(p)}{\overline{c}_1'\,\overline{c}_2 - \overline{c}_1\,\overline{c}_2'} .$$

where the prime $'$ denotes partial differentiation with respect to ψ .

Now $\qquad \overline{c}_1'\,\overline{c}_2 - \overline{c}_1\,\overline{c}_2' = -\psi^{-\frac{1}{2}} \cdot \frac{3}{2\pi}\sin\pi/3$

Thus, from (65),

$$P_3 = - \frac{2\pi}{3\sin\frac{\pi}{3}} \int_0^{\psi} \psi^{-\frac{1}{4}}\;\overline{F}_4(p)\, I_{\frac{1}{3}}(q)\, d\psi$$

$$P_4 = \frac{2\pi}{3\sin\frac{\pi}{3}} \int_0^{\psi} \psi^{-\frac{1}{4}}\;\overline{F}_4(p)\, I_{-\frac{1}{3}}(q)\, d\psi \tag{66}$$

and the solution of (63) is

$$\overline{c}(p,\psi) = \psi^{\frac{1}{4}} I_{-\frac{1}{3}}(q) \int_0^{\psi} \frac{2\pi\overline{F}_4(p)}{3\sin\frac{\pi}{3}}\,\psi^{-\frac{1}{4}}\, I_{\frac{1}{3}}(q)\,d\psi - \psi^{\frac{1}{4}} I_{\frac{1}{3}}(q) \int_0^{\psi} \frac{2\pi\overline{F}_4(p)}{3\sin\frac{\pi}{3}}\,\psi^{-\frac{1}{4}}\, I_{-\frac{1}{3}}(q)\,d\psi$$

$$+ A_1\,\psi^{\frac{1}{4}} I_{-\frac{1}{3}}(q) + B_1\,\psi^{\frac{1}{4}} I_{\frac{1}{3}}(q) \tag{67}$$

where A_1 and B_1 have to be determined from boundary conditions.

The boundary condition as $\psi \to 0$ can be written in the transform notation as

$$\overline{c}(p,\psi) = \overline{F}_5(p) + 2\psi^{\frac{1}{2}}\;\overline{F}_6(p) \tag{68}$$

$$\text{where } F_5(X) = c_W(X)$$

$$F_6(X) = c'(X)\sqrt{\frac{\mu_0}{2\,\tau_W(X)}}$$

From (67) and (68) in the limit as $\psi \to 0$

$$\begin{aligned} \frac{A_1}{\Gamma(\frac{2}{3})\, p^{1/6}\, (\frac{2}{3})^{\frac{1}{3}}} &= \bar{c}(p,0) = \bar{F}_5(p) \\ \frac{B_1\, p^{1/6}\, (\frac{3}{2})^{\frac{2}{3}}}{\Gamma(\frac{1}{3})} &= \psi^{\frac{1}{2}} \frac{\partial}{\partial \psi} \bar{c}(p,\psi) = \bar{F}_6(p) \end{aligned} \tag{69}$$

Furthermore as $\psi \to \infty$, $c \to 0$ and hence the coefficients of $I_{-\frac{1}{3}}(q)$ and $I_{\frac{1}{3}}(q)$ must be equal in magnitude and opposite in sign, i.e.

$$A_1 + B_1 + \frac{2\pi \bar{F}_4(p)}{3 \sin \frac{\pi}{3}}\, p^{-\frac{1}{2}} \int_o^\infty \left(I_{\frac{1}{3}}(q) - I_{-\frac{1}{3}}(q) \right) dq = 0$$

Now
$$\int_o^\infty \left(I_{\frac{1}{3}}(q) - I_{-\frac{1}{3}}(q) \right) dq = -\frac{\Gamma(\frac{2}{3})\, \Gamma(\frac{1}{3})}{\pi} \sin \pi/3$$

and therefore

$$A_1 + B_1 = \tfrac{2}{3}\, \bar{F}_4(p)\, p^{-\frac{1}{2}}\, \Gamma(\tfrac{2}{3})\, \Gamma(\tfrac{1}{3})$$

or using (69)

$$\bar{F}_5(p) = (\tfrac{2}{3})^{\frac{2}{3}}\, \Gamma(\tfrac{1}{3})\, p^{-\frac{2}{3}}\, \bar{F}_4(p) - (\tfrac{2}{3})^{\frac{1}{3}} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})}\, p^{-\frac{1}{3}}\, \bar{F}_6(p) \tag{70}$$

Taking the inverse transforms of (70) gives an equation for the wall concentration of foreign gas

$$\begin{aligned} c_w(X) = &- \frac{(\mu_o \rho_o)^{\frac{1}{3}}}{3^{\frac{2}{3}}} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \int_o^X \frac{V_w(X_1)\, c'(X_1)}{\tau_w^{\frac{1}{2}}(X_1)} \left(\int_{X_1}^X \frac{C(z)}{Sc} \cdot \tau_w^{\frac{1}{2}}(z)\, dz \right)^{-\frac{1}{3}} dX_1 \\ &- \frac{1}{\Gamma(\frac{2}{3})} \cdot \left(\frac{\mu_o^2}{3\rho_o} \right)^{\frac{1}{3}} \int_o^X \frac{c'(X_1)\, C(X_1)}{Sc} \left(\int_{X_1}^X \frac{C(z)}{Sc}\, \tau_w^{\frac{1}{2}}(z)\, dz \right)^{-\frac{2}{3}} dX_1 \end{aligned} \tag{71}$$

Now
$$c'(X) = \frac{\partial c}{\partial Y}_{Y=0} = \frac{\partial c}{\partial y} \frac{dy}{dY} = c'(x) \frac{\rho_o\, p_1}{\rho_w\, p_o}\, m_1^{\frac{3\gamma - 1}{2(\gamma-1)}}$$

and, using the transformations (49) from (X, Y) to (x, y) co-ordinates, equation (71) becomes

$$\begin{aligned} c_w(x) = &- \frac{(\mu_o \rho_o)^{\frac{1}{3}}}{3^{\frac{2}{3}}} \cdot \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \int_o^x \frac{\dot{m}\, c'(x_1)}{\rho_o} \cdot \frac{\rho_o p_1}{\rho_w p_o}\, m_1^{\gamma/2(\gamma-1)} \frac{C^{\frac{1}{2}}(x_1)}{\tau_w^{\frac{1}{2}}(x_1)} \\ &\left(\int_{x_1}^x \frac{C^{\frac{1}{2}}(z)\, \tau_w^{\frac{1}{2}}(z)\, dz}{Sc\, m_1^{\gamma/2(\gamma-1)}} \right)^{-\frac{1}{3}} dx_1 - \frac{1}{\Gamma(\frac{2}{3})} \left(\frac{\mu_o^2}{3\rho_o} \right)^{\frac{1}{3}} \int_o^x \frac{C(x)}{Sc} \frac{\rho_o p_1}{\rho_w p_o}\, c'(x_1) \\ &\left(\int_{x_1}^x \frac{C^{\frac{1}{2}}(z)\, \tau_w^{\frac{1}{2}}(z)}{Sc.\ m_1^{\gamma/2(\gamma-1)}} \right)^{-\frac{2}{3}} dx_1 \qquad \ldots\ldots \end{aligned} \tag{72}$$

It is now necessary to determine another equation for the concentration gradient $c'(x)$. The diffusion equation (6) can be written

$$\frac{\partial}{\partial x}(\rho u c) + \frac{\partial}{\partial y}(\rho v c) = \frac{\partial}{\partial y}\left(\frac{\mu}{Sc}\frac{\partial c}{\partial y}\right)$$

which at the wall becomes

$$\rho_w v_w c_w - \left(\frac{\mu}{Sc}\frac{\partial c}{\partial y}\right)_{y=0} = \rho_i v_i \tag{73}$$

for the injected species, and

$$\rho_w v_w (1 - c_w) - \left(\frac{\mu}{Sc}\frac{\partial}{\partial y}(1 - c)\right)_{y=0} = 0 \tag{74}$$

for the stream gas.

Adding (73) and (74) shows that

$$\dot{m} = \rho_w v_w = \rho_i v_i$$

i.e. the mass flow normal to the wall at the wall in the boundary layer is equal to the mass flow through the wall.

Subtracting gives

$$c'(x) = \left(\frac{\partial c}{\partial y}\right)_{y=0} = \frac{Sc_w}{\mu_w}\,\dot{m}\,(c_w(x) - 1) \tag{75}$$

Eliminating $c'(x)$ in (72) and introducing the wall shear stress and injection parameters t_w and f_w defined by (51) and (52) the equation for the wall concentration is

$$c_w(x) = \frac{1}{3^{\frac{2}{3}}} \cdot \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \int_o^x \frac{f_w^2(x_1)}{x_1^{\frac{3}{4}}} \frac{\left[1 - c_w(x_1)\right]}{t_w^{\frac{1}{2}}(x_1)} \cdot \frac{Sc}{C_a^{\frac{1}{2}}(x_1)} \left(\frac{m_1}{m_a}\right)^{\gamma/2(\gamma-1)}$$
$$\left(\int_{x_1}^x \frac{C_a^{\frac{1}{2}}(z)}{Sc} \cdot \frac{t_w^{\frac{1}{2}}(z)}{z^{\frac{1}{4}}} \left(\frac{m_a}{m_1}\right)^{\gamma/2(\gamma-1)} dz\right)^{-\frac{1}{3}} dx_1 + \frac{1}{3^{\frac{1}{3}}\Gamma(\frac{2}{3})} \int_o^x \frac{f_w(x_1)}{x_1^{\frac{1}{2}}}$$
$$\left[1 - c_w(x_1)\right]\left(\int_{x_1}^x \frac{C_a^{\frac{1}{2}}(z)}{Sc} \cdot \frac{t_w^{\frac{1}{2}}(z)}{z^{\frac{1}{4}}} \cdot \left(\frac{m_a}{m_1}\right)^{\gamma/2(\gamma-1)} dz\right)^{-\frac{2}{3}} dx_1 \qquad \ldots\ldots (76)$$

and, approximating to the interior integrals,

$$c_w(x) = \frac{1}{3^{\frac{2}{3}}} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \int_o^x \frac{f_w^2(x_1)\left[1 - c_w(x_1)\right]}{x_1^{\frac{2}{3}}(x - x_1)^{\frac{1}{3}} t_w^{\frac{2}{3}}(x_1)} \cdot \frac{Sc^{4/3}}{C_a^{\frac{2}{3}}(x_1)} \cdot \left(\frac{m_1(x)}{m_a}\right)^{\frac{2\gamma}{3(\gamma-1)}} dx_1$$
$$+ \frac{1}{3^{\frac{1}{3}}\Gamma(\frac{2}{3})} \int_o^x \frac{f_w(x_1)\left[1 - c_w(x_1)\right]}{x_1^{\frac{1}{3}}(x - x_1)^{\frac{2}{3}} t_w^{\frac{1}{3}}(x_1)} \cdot \frac{Sc^{\frac{2}{3}}}{C_a^{\frac{1}{3}}(x_1)} \left(\frac{m_1(x)}{m_a}\right)^{\frac{\gamma}{3(\gamma-1)}} dx_1 \qquad \ldots\ldots (77)$$

which with equation (59) makes possible the iterative process to determine the wall shear stress.

8. An approximate solution of the stagnation enthalpy equation

Considering the function C, the Lewis number and Prandtl number to be dependent on X only, being obtained from the intermediate enthalpy, the equation for the stagnation enthalpy (28) is

$$\frac{\partial S}{\partial X} - \frac{\nu_o C}{\sigma} \frac{\partial}{\partial \psi}\left(U \frac{\partial S}{\partial \psi}\right) + V_w \frac{\partial S}{\partial \psi} = \frac{\nu_o(\gamma-1)C}{a_1^2\left(1+\frac{\gamma-1}{2}\frac{U_1^2}{a_o^2}\right)} \frac{\partial}{\partial \psi}\left(U\Delta\frac{\partial c}{\partial \psi}\right)$$
$$- \nu_o \frac{C(1-\sigma)}{\sigma U_1^2}\left[\frac{(\gamma-1)\,U_1^2/2a_o^2}{1+\frac{\gamma-1}{2}\frac{U_1^2}{a_o^2}}\right]\frac{\partial}{\partial \psi}\left(U\frac{\partial Z}{\partial \psi}\right) \qquad \ldots\ldots \quad (78)$$

with the boundary conditions

(i) at $\psi = \infty$, $\quad S(X,\infty) = 0$

(ii) as $X \to 0$, $\quad S \to 0$

(iii) as $\psi \to 0$, $\quad S = 1 - \dfrac{h_s^*(X)}{h_{s_1}} + \dfrac{\sigma}{h_{s_1}}\dfrac{\mu_o}{\mu_w}\dot{Q}_w(X)\sqrt{\dfrac{2\psi}{\mu_o \tau_w(X)}} + \ldots.$ $\qquad \ldots\ldots \quad (79)$

where the rate of heat transfer from the wall to the boundary layer is, in the X, Y co-ordinates,

$$\dot{Q}_w(X) = -\left(k_w \frac{\partial T}{\partial Y}\right)_{Y=0}$$

The right hand side of equation 78 can be written

$$\frac{\nu_o C}{h_{s_1}}\left[\frac{\partial}{\partial \psi}\left(U\Delta\frac{\partial c}{\partial \psi}\right) - \frac{a_1^2}{2a_o^2}\left(\frac{1-\sigma}{\sigma}\right)\frac{\partial}{\partial \psi}\left(U\frac{\partial Z}{\partial \psi}\right)\right] \qquad (80)$$

since $\qquad h_{s_1} = \dfrac{a_1^2}{\gamma-1}\left(1+\dfrac{\gamma-1}{2a_o^2}U_1^2\right)$

Consider $U\Delta\dfrac{\partial c}{\partial \psi}$ near the wall. U has been assumed (eqn. 34) to be $\left(\dfrac{2\tau_w(X)}{\mu_o}\psi\right)^{\frac{1}{2}}$ and Δ has been defined in equation (8) as

$$\Delta = \frac{Le-1}{\sigma} \cdot (h_e - h_i)$$

which can be rewritten in the form

$$\Delta = \frac{1}{c}\left(\frac{Le - 1}{\sigma}\right)\left(h_{s_1} S - \frac{a_1^2(X)}{2\,a_o^2} Z\right)$$

where, near the wall, the concentration of injected gas is given by equation (60)

$$c = c_w(X) + c'(X)\left(\frac{2\,\mu_o}{\tau_w(X)}\,\psi\right)^{\frac{1}{2}} + 0(\psi^{3/2})$$

and S is given in equation (79)

$$U\Delta\,\frac{\partial c}{\partial \psi} = \frac{Le - 1}{\sigma}\cdot\left[h_{s_1} - h_s^* - \frac{a_1^2(X)}{2\,a_o^2}\,U_1^2 + \frac{\sigma\dot{Q}_w(X)\mu_o}{\mu_w}\sqrt{\frac{2}{\mu_o\tau_w}}\,\psi^{\frac{1}{2}} + \frac{a_1^2}{a_o^2}\cdot\frac{\tau_w}{\mu_o}\,\psi\right]$$

$$\cdot\left(2\,\frac{\tau_w(X)}{\mu_o}\,\psi\right)^{\frac{1}{2}}\frac{\partial}{\partial\psi}(\log c)$$

Expanding log c as a power series,

$$U\Delta\frac{\partial c}{\partial\psi} = \frac{Le - 1}{\sigma}\cdot\left(\frac{2\,\tau_w}{\mu_o}\right)^{\frac{1}{2}}\left[\frac{1}{2}\,\frac{c'}{c_w}\left(\frac{2\mu_o}{\tau_w}\right)^{\frac{1}{2}}\left(h_{s_1} - h_s^* - \frac{1}{2}\,\frac{a_1^2}{a_o^2}\,U_1^2\right)\right.$$

$$\left. + \psi^{\frac{1}{2}}\left\{\frac{1}{2}\,\frac{c'}{c_w}\left(\frac{2\mu_o}{\tau_w}\right)^{\frac{1}{2}}\frac{\sigma\dot{Q}_w\mu_o}{\mu_w}\sqrt{\frac{2}{\mu_o\tau_w}} + \left(\frac{c'}{c_w}\right)^2\frac{2\mu_o}{\tau_w}\left(h_{s_1} - h_s^* - \frac{a_1^2}{2a_o^2}\,U_1^2\right)\right\} + 0(\psi)\right]$$

and thus

$$\frac{\partial}{\partial\psi}\left(U\Delta\frac{\partial c}{\partial\psi}\right) = \frac{Le - 1}{\sigma}\left(\frac{\mu_o}{2\tau_w}\right)^{\frac{1}{2}}\left[\frac{c'\sigma\dot{Q}_w}{c_w\mu_w} + 2\left(\frac{c'}{c_w}\right)^2\left(h_{s_1} - h_s^* - \frac{a_1^2}{2a_o^2}\,U_1^2\right)\right]\psi^{-\frac{1}{2}} + 0(1) \qquad \ldots\ldots \quad (81)$$

Also

$$\frac{\partial}{\partial\psi}\left(U\,\frac{\partial Z}{\partial\psi}\right) = -\left(\frac{2}{\psi}\right)^{\frac{1}{2}}\left(\frac{\tau_w}{\mu_o}\right)^{3/2} \qquad (82)$$

Substituting from (81) and (82) into (80), the right hand side of (78) becomes, . near the wall,

$$\frac{\nu_o C}{h_{s_1}}\left[\frac{Le - 1}{\sigma}\cdot\left(\frac{\mu_o}{2\tau_w}\right)^{\frac{1}{2}}\left\{\frac{c'\sigma\dot{Q}_w}{c_w\,\mu_w} + 2\left(\frac{c'}{c_w}\right)^2\left(h_{s_1} - h_s^* - \frac{a_1^2}{2a_o^2}\,U_1^2\right)\right\}\right.$$

$$\left. + \frac{a_1^2}{2^{\frac{1}{2}}a_o^2}\left(\frac{1 - \sigma}{\sigma}\right)\left(\frac{\tau_w}{\mu_o}\right)^{3/2}\right]\psi^{-\frac{1}{2}} + 0(1) \qquad \ldots\ldots \quad (83)$$

and the stagnation enthalpy equation can be written, using (79)

$$\frac{\partial S}{\partial X} - \frac{\nu_o C}{\sigma}\frac{\partial}{\partial\psi}\left(U\frac{\partial S}{\partial\psi}\right) = \frac{\dot{Q}_w(X)\sigma}{h_{s1}\mu_w}\left(\frac{\mu_o}{2\tau_w(X)}\right)^{\frac{1}{2}}\left[\nu_o C(X)\frac{c'(X)}{c_w(X)}\left(\frac{Le-1}{\sigma}\right) - V_w(X)\right]\psi^{-\frac{1}{2}}$$

$$+ \frac{2\nu_o C(X)}{h_{s1}}\left(\frac{Le-1}{\sigma}\right)\left(\frac{\mu_o}{2\tau_w}\right)^{\frac{1}{2}}\left(\frac{c'(X)}{c_w(X)}\right)^2\left(h_{s1} - h_s^* - \frac{a_1^2}{2a_o^2}U_1^2\right)\psi^{-\frac{1}{2}}$$

$$+ \frac{\nu_o C(X)}{2^{\frac{1}{2}}h_{s1}} \cdot \frac{a_1^2}{a_o^2} \cdot \frac{1-\sigma}{\sigma}\left(\frac{\tau_w}{\mu_o}\right)^{3/2}\psi^{-\frac{1}{2}} \qquad \ldots\ldots (84)$$

The last term in (84) is put zero by Lilley for the impermeable wall on the assumption that the recovery enthalpy is independent of the wall temperature distribution.

Taking the approximate form for U given by equation (34) and putting

$$t = \int_o^X \frac{C(z)}{\rho_o\sigma}\left(2\mu_o\tau_w(z)\right)^{\frac{1}{2}} dz$$

and

$$V_{w1} = \tfrac{1}{2}\left\{\frac{V_w(X)}{\nu_o C(X)} - \left(\frac{Le-1}{\sigma}\right)\frac{c'(X)}{c_w(X)}\right\}\frac{\sigma^2\mu_o}{h_{s1}\mu_w}\cdot\frac{\dot{Q}_w(X)}{\tau_w(X)}$$

$$-\frac{\sigma}{h_{s1}}\left(\frac{Le-1}{\sigma}\right)\cdot\frac{\mu_o}{\tau_w}\left(\frac{c'(X)}{c_w(X)}\right)^2\left(h_{s1} - h_s^* - \frac{a_1^2(X)}{2a_o^2}U_1^2\right) \qquad (85)$$

$$-\frac{1-\sigma}{2h_{s1}}\cdot\frac{a_1^2(X)}{a_o^2}\cdot\frac{\tau_w(X)}{\mu_o}$$

the approximate stagnation enthalpy equation (84) is

$$\frac{\partial S}{\partial t}(t,\psi) - \frac{\partial}{\partial\psi}\left(\psi^{\frac{1}{2}}\frac{\partial S}{\partial\psi}\right) = -V_{w1}(t)\,\psi^{-\frac{1}{2}} \qquad (86)$$

which, in the notation of the Laplace transform, becomes

$$p\,\bar{S} - \frac{\partial}{\partial\psi}\left(\psi^{\frac{1}{2}}\frac{\partial\bar{S}}{\partial\psi}\right) = \bar{F}_7(p)\,\psi^{-\frac{1}{2}} \qquad (87)$$

where $F_7(t) = -V_{w1}(t)$

This equation is similar to the transformed diffusion equation (63) and its solution is, similarly,

$$\bar{S}(p,\psi) = \psi^{\frac{1}{4}} I_{-\frac{1}{3}}\frac{2\pi}{3\sin\frac{\pi}{3}}\int_o^\psi \bar{F}_7(p)\,\psi^{-\frac{1}{4}} I_{\frac{1}{3}}(q)\,d\psi - \psi^{\frac{1}{4}} I_{\frac{1}{3}}(q)\frac{2\pi}{3\sin\frac{\pi}{3}}\int_o^\psi \bar{F}_7(p)\,\psi^{-\frac{1}{4}} I_{-\frac{1}{3}}(q)\,dq$$

$$+ A_2\,\psi^{\frac{1}{4}} I_{-\frac{1}{3}}(q) + B_2\psi^{\frac{1}{4}} I_{\frac{1}{3}}(q) \qquad \ldots\ldots (88)$$

where A_2 and B_2 have to be determined from the boundary conditions.

Now, as $\psi \to 0$,

$$S \to 1 - \frac{h_s^*(X)}{h_{s1}(X)} = F_8(X)$$

and thus, from (88)

$$\frac{A_2}{\Gamma(\frac{2}{3})\, p^{1/6} (\frac{2}{3})^{\frac{1}{3}}} = \overline{S}(p,\ 0) = \overline{F}_8(p) \tag{89}$$

Also, as $\psi \to 0$,

$$\frac{\partial S}{\partial \psi} \to \frac{\sigma}{h_{s1}} \left(\frac{\mu_o}{2\,\tau_w(X)} \right)^{\frac{1}{2}} \frac{\dot{Q}_w(X)}{\mu_w}\ \psi^{-\frac{1}{2}} = F_9(X)\, \psi^{-\frac{1}{2}}$$

and thus, from (88)

$$\frac{B_2\, p^{1/6} (\frac{3}{2})^{\frac{2}{3}}}{\Gamma(\frac{1}{3})} = \left[\psi^{\frac{1}{2}}\ \frac{\partial}{\partial \psi}\, \overline{S}(p,\ \psi) \right]_{\psi=0} = \overline{F}_9(p) \tag{90}$$

Furthermore as $\psi \to \infty$, $S \to 0$ and hence $\overline{S}(p, \infty) \to 0$. This implies that the coefficients of $I_{-\frac{1}{3}}(q)$ and $I_{\frac{1}{3}}(q)$ in (88) must be equal in magnitude but opposite in sign.

i.e. $\qquad A_2 + B_2 = \frac{2}{3}\, p^{-\frac{1}{2}}\, \Gamma(\frac{1}{3})\Gamma(\frac{2}{3})\, \overline{F}_7(p)$

or using (89) and (90)

$$\overline{F}_9(p) = (\tfrac{2}{3})^{\frac{1}{3}}\, \Gamma(\tfrac{2}{3}) p^{-\frac{1}{3}}\, \overline{F}_7(p) - (\tfrac{2}{3})^{-\frac{1}{3}}\, p^{\frac{1}{3}}\, \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\, \overline{F}_8(p) \tag{91}$$

Taking the inverse transforms of (91) we obtain an integral equation for the rate of heat transfer in the X, Y co-ordinates

$$\begin{aligned}\frac{\mu_o}{\mu_w}\, \dot{Q}_w(X) = &- \frac{(3\mu_o \rho_o)^{\frac{1}{3}}}{\sigma\,\Gamma(\frac{1}{3})} \cdot \tau_w^{\frac{1}{2}}(X)\, h_{s1}(X) \int_o^X \left(\int_{X_1}^X \frac{C(z)}{\sigma}\, \tau_w^{\frac{1}{2}}(z) dz \right)^{-\frac{1}{3}} d\left[1 - \frac{h_s^*(X_1)}{h_{s1}(X_1)} \right] \\ &- 2\left(\frac{\mu_o^2}{3\rho_o} \right)^{\frac{1}{3}} \frac{h_{s1}(X)}{\sigma}\ \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\ \tau_w^{\frac{1}{2}}(X) \int_o^X \frac{C(X_1)}{\sigma}\, V_{w1}(X_1)\, \tau_w^{\frac{1}{2}}(X_1) \\ &\qquad \left(\int_{X_1}^X \frac{C(z)}{\sigma}\, \tau_w^{\frac{1}{2}}(z)\, dz \right)^{-\frac{2}{3}} dX_1 \qquad \ldots.(92)\end{aligned}$$

Now
$$\dot{Q}_w(X) = \left(-k_w \frac{\partial T}{\partial Y}\right)_{Y=0}$$
$$= \left(-k_w \frac{\partial T}{\partial y}\right)_{y=0} \left(\frac{\partial y}{\partial Y}\right)_{y=0}$$
$$= \dot{Q}_w(x) \frac{\rho_o p_1}{\rho_w p_o} \cdot \frac{a_o p_o}{a_1 p_1}$$

and hence
$$\frac{\mu_o}{\mu_w} \dot{Q}_w(X) = \frac{\dot{Q}_w(x)}{C(x)} m_1^{\frac{3\gamma-1}{2(\gamma-1)}} \tag{93}$$

Therefore, substituting for $V_{w_1}(X)$ in (92) from (85) and reverting to the compressible flow co-ordinates (x, y) using equations (49) and (93), the equation for the heat transfer rate from the wall to the flow is

$$\frac{\dot{Q}_w(x)}{\tau_w^{\frac{1}{2}}(x)} = \frac{(3\mu_o\rho_o)^{\frac{1}{3}} C^{\frac{1}{2}}(x) h_{s1}}{\sigma\,\Gamma(\frac{1}{3})\, m_1^{\gamma/2(\gamma-1)}} \int_o^x \left(\int_{x_1}^{x} \frac{C^{\frac{1}{2}}(z)\,\tau_w^{\frac{1}{2}}(z)\,dz}{\sigma\, m_1^{\gamma/2(\gamma-1)}} \right)^{-\frac{1}{3}} d\left[\frac{h_s^*(x_1)}{h_{s1}} - 1\right]$$
$$+ \frac{2\,C^{\frac{1}{2}}(x)\,h_{s1}}{(3\mu_o\rho_o)^{\frac{1}{3}}\sigma m_1^{\gamma/2(\gamma-1)}} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \int_o^x \left(\int_{x_1}^{x} \frac{C^{\frac{1}{2}}(z)\tau_w^{\frac{1}{2}}(z)\,dz}{\sigma\, m_1^{\gamma/2(\gamma-1)}} \right)^{-\frac{2}{3}} \frac{m_1^{\gamma/2(\gamma-1)}}{C^{\frac{1}{2}}(x_1)\, h_{s1}} \cdot$$
$$\cdot \left[\frac{1-\sigma}{2\sigma} \cdot \tau_w^{3/2}(x_1) + \left(\frac{\mu_w c'(x_1)}{c_w(x_1)}\right)^2 \left(\frac{Le-1}{\sigma}\right) \frac{h_{s1} - h_s^* - u_1^2/2}{\tau_w^{\frac{1}{2}}(x_1)} \right.$$
$$\left. + \frac{\sigma\dot{Q}_w(x_1)}{2\,\tau_w^{\frac{1}{2}}(x_1)} \left\{ \frac{Le-1}{\sigma} \cdot \frac{\mu_w c'(x_1)}{c_w(x_1)} - \rho_w v_w \right\} \right] dx \qquad \ldots\ldots \tag{94}$$

It has been shown in equation (75) that

$$\mu_w \frac{c'(x)}{c_w(x)} = \dot{m}\, Sc\left[1 - \frac{1}{c_w(x)}\right]$$

Hence

$$\left(\mu_w \frac{c'(x)}{c_w(x)}\right)^2 \left(\frac{Le-1}{\sigma}\right) \frac{h_{s1} - h_s^* - u_1^2/2}{\tau_w^{\frac{1}{2}}(x_1)} + \frac{\sigma\dot{Q}_w(x_1)}{2\tau_w^{\frac{1}{2}}(x_1)} \left\{\frac{Le-1}{\sigma} \frac{\mu_w c'(x_1)}{c_w(x_1)} - \rho_w v_w\right\}$$
$$= \tau_w^{-\frac{1}{2}}(x_1) \left\{ \left(\frac{\dot{m}\,Sc}{c_w}\right)^2 \cdot \frac{Le-1}{\sigma} (1-c_w)^2 (h_1 - h_s^*) + \frac{\dot{m}\,Sc}{c_w}(1 - Le - c_w) \frac{\dot{Q}_w(x_1)}{2} \right\}$$

and equation (94) becomes

$$\frac{\dot{Q}_w(x)}{\tau_w^{\frac{1}{2}}(x)} = \frac{(3\mu_o\rho_o)^{\frac{1}{3}}\, C^{\frac{1}{2}}(x)}{\sigma\,\Gamma(\frac{1}{3})\, m_1^{\gamma/2(\gamma-1)}} \int_o^x \left(\int_{x_1}^x \frac{C^{\frac{1}{2}}(z)\, \tau_w^{\frac{1}{2}}(z)}{\sigma\, m_1^{\gamma/2(\gamma-1)}}\, dz \right)^{-\frac{1}{3}} d\left[h_s^*(x_1) - h_{s1} \right]$$

$$+ \frac{2\, C^{\frac{1}{2}}(x)}{(3\mu_o\rho_o)^{\frac{1}{3}}\sigma\, m_1^{\gamma/2(\gamma-1)}} \; \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \int_o^x \left(\int_{x_1}^x \frac{C^{\frac{1}{2}}(z)\tau_w^{\frac{1}{2}}(z)}{\sigma\, m_1^{\gamma/2(\gamma-1)}}\, dz \right)^{-\frac{2}{3}} \frac{m_1^{\gamma/2(\gamma-1)}}{C^{\frac{1}{2}}(x_1)} \, .$$

$$. \left\{ \left(\frac{\dot{m}\, Sc}{c_w}\right)^2 \frac{Le - 1}{\sigma \tau_w^{\frac{1}{2}}(x_1)} \, . \, (1 - c_w)^2 (h_1 - h_s^*) + \frac{\dot{m}\, Sc}{c_w}(1 - Le - c_w)\, \frac{\dot{Q}_w(x_1)}{2\,\tau_w^{\frac{1}{2}}(x_1)} \right.$$

$$\left. + \frac{1 - \sigma}{2\sigma}\, \tau_w^{3/2}(x_1) \right\} dx_1 \qquad \ldots\ldots \qquad (95)$$

We now put $\dot{Q}_w(x)$ in a modified form $s_w(x)$ defined by

$$s_w(x) = \dot{Q}_w(x) \left[x/\rho_a \mu_a u_a \right]^{\frac{1}{2}} \qquad (96)$$

and introduce the non-dimensional wall shear stress and injection parameters t_w and f_w defined in equations (51) and (52). In this manner (95) becomes,

$$\frac{s_w(x)}{t_w^{\frac{1}{2}}(x)} = \frac{3^{\frac{1}{3}}\, x^{\frac{1}{4}}}{\sigma\,\Gamma(\frac{1}{3})}\, C_a^{\frac{1}{2}}(x) \left(\frac{m_a}{m_1}\right)^{\gamma/2(\gamma-1)} \int_o^x \left(\int_{x_1}^x \frac{C_a^{\frac{1}{2}}(z)\, t_w^{\frac{1}{2}}(z)}{\sigma\, z^{\frac{1}{4}}} \left(\frac{m_a}{m_1}\right)^{\gamma/2(\gamma-1)} dz \right)^{-\frac{1}{3}}$$

$$d\left[h_s^*(x_1) - h_{s1} \right] + \frac{2\, C_a^{\frac{1}{2}}(x)}{3^{\frac{1}{3}}\sigma} \; \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\, x^{\frac{1}{4}} \left(\frac{m_a}{m_1}\right)^{\gamma/2(\gamma-1)} \int_o^x \left(\int_{x_1}^x \frac{C_a^{\frac{1}{2}}(z) t_w^{\frac{1}{2}}(z)}{\sigma\, z^{\frac{1}{4}}} \left(\frac{m_1}{m_a}\right)^{\gamma/2(\gamma-1)} dz \right)^{-\frac{2}{3}}$$

$$\left(\frac{m_1}{m_a}\right)^{\gamma/2(\gamma-1)} \, . \; \frac{x_1^{-\frac{3}{4}}}{C_a^{\frac{1}{2}}(x_1)} \, . \left\{ \left(\frac{f_w Sc}{c_w}\right)^2 \frac{Le-1}{\sigma t_w^{\frac{1}{2}}(x_1)} \, . \, (1-c_w)^2 (h_1 - h_s^*) + \frac{f_w\, Sc}{c_w}(1 - Le - c_w) \frac{s_w(x_1)}{2 t_w^{\frac{1}{2}}(x_1)} \right.$$

$$\left. + \frac{1-\sigma}{2\sigma} \, . \, t_w^{3/2}(x_1)\, u_a^2 \right\} dx_1 \qquad \ldots\ldots \qquad (97)$$

Approximating to the inner integrals

$$\frac{s_w(x)}{t_w^{\frac{1}{2}}(x)} = \frac{3^{\frac{1}{3}}\, x^{\frac{1}{4}}}{\sigma\,\Gamma(\frac{1}{3})}\, C_a^{\frac{1}{2}}(x) \left(\frac{m_a}{m_1(x)}\right)^{\gamma/2(\gamma-1)} \int_o^x \frac{x_1^{1/12}\, \sigma^{\frac{1}{3}}}{(x - x_1)^{\frac{1}{3}}} \, . \, \frac{(m_1(x_1)/m_a)^{\gamma/6(\gamma-1)}}{C_a^{1/6}(x_1) t_w^{1/6}(x_1)} \; d\left[h_s^*(x_1) - h_{s1} \right]$$

$$+ \frac{2 x^{\frac{1}{4}}}{3^{\frac{1}{3}}\sigma}\; C_a^{\frac{1}{2}}(x) \left(\frac{m_a}{m_1}\right)^{\gamma/2(\gamma-1)} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \int_o^x \frac{\sigma^{\frac{2}{3}}\, C_a^{-5/6}(x_1)\, t_w^{-\frac{1}{3}}(x_1)}{(x - x_1)^{\frac{2}{3}}\, x_1^{7/12}} \left(\frac{m_1}{m_a}\right)^{\frac{5\gamma}{6(\gamma-1)}} \, .$$

$$. \left\{ \left(\frac{f_w Sc}{c_w}\right)^2 \frac{Le - 1}{\sigma t_w^{\frac{1}{2}}(x_1)} \, . \, (1 - c_w)^2 (h_1 - h_s^*) + \frac{f_w\, Sc}{c_w}(1 - Le - c_w) \frac{s_w(x_1)}{2 t_w^{\frac{1}{2}}(x_1)} \; + \right.$$

$$\left. + \frac{1 - \sigma}{2\sigma}\, t_w^{3/2}(x_1)\, u_a^2 \right\} dx_1 \qquad \ldots\ldots \qquad (98)$$

Thus the heat transfer rate can be obtained by an iterative process from the given external flow conditions once the wall shear stress and wall concentration of injected gas are known.

9. Numerical solutions for the wall shear stress and heat transfer rate

The wall shear stress and the heat transfer rate must now be found from equations (58) and (98) with (77) using an iterative process. Stevenson[8] was able to obtain, for air injection, relations between f_w, t_w and Nusselt number in precise form when it is assumed that the free stream speed, the wall shear stress and the wall temperature vary as some power of x. This is possible since the viscosity in the boundary layer can be related to the temperature only. In the analysis presented in this paper such a treatment is not possible since the density and viscosity of the boundary layer are dependent also upon the concentration of the injected gas.

To assess the accuracy of the method the value of t_w has been calculated for hydrogen injected into an incompressible layer with zero heat transfer and zero pressure gradient. The injection velocity is assumed to be proportional to $x^{-\frac{1}{2}}$ and C_a is given its value at the wall since, in the absence of complete concentration profiles, it is not possible to obtain its value elsewhere. The relation of t_w/t_{wo} to f_w is compared in Fig. 2 with the result due to Eckert and Schneider[4]. It is seen that the difference between these results is approximately the same as that between the exact results for air injection found by Donoughe and Livingood[12] and the approximate results obtained by Stevenson[8]. The agreement between the two solutions for hydrogen injection can be improved if the value of C_a is increased by some 40% above its wall value. Values of t_w calculated on this basis are given in curve 3 of Fig. 2. Even closer agreement would be possible if the percentage increase of C_a was changed with increase of the injection parameter. Using the curves $\rho\mu/\rho_o\mu_o$ obtained from the concentration profiles of Ref. 4 it is seen that the required 40% increase in C_a is obtained when η = 0.8 approximately.

For helium injected into the laminar boundary layer on a cooled wall at M = 6, the results of the present paper are compared in Fig. 3 with those of Korobkin[16] obtained by considering the variation of the molecular weight of the mixture. Since Korobkin's results are approximate it is not possible to assess, in this case, the error of the method at M = 6 or to estimate the alteration necessary to the value of C_a.

The process of iteration is started by substituting in the concentration equation (77) the value of t_w for air injection corresponding to the chosen value of f_w and the external flow conditions. Such substitution gives an integral of the form

$$\int_0^1 x^{m-1} (1 - x)^{n-1} \, dx$$

which is the Beta function, the value of which is immediately obtainable from tables of the gamma function. The resulting value of the wall concentration is then used to determine the first approximation for wall values of density and viscosity by methods given by Hirschfelder et al[13]. These density and viscosity values are substituted in equation (59) to give a second approximation to t_w and in (98) to obtain a second approximation to the heat transfer rate. The higher order approximations are obtained similarly. It was found that five iterations gave an accuracy of convergence of better than five per cent. In Fig. 2 the values of t_w for helium injection are also given. In this calculation the values of viscosity, Prandtl number and thermal conductivity were obtained from tables recently calculated by Eckert, Ibele and Irvine[14].

To illustrate the order of magnitude of the reduction in skin friction and heat transfer rate to be obtained at supersonic speeds with foreign gas injection, the ratio t_w/t_{wo} has been calculated for $M_1 = 4$ with zero heat transfer (Fig. 4) and for the cooled wall, $T_w = T_1$ (Fig. 5). For the cooled wall the ratio of heat transfer rates $\dot{Q}_w/\dot{Q}_{wo}$ is shown in Fig. 6. In each case the pressure gradient is zero. The corresponding exact results for air injection due to Lew and Fanucci[15] and Stevenson's approximate results are shown for comparison.

10. Conclusions

The equations for foreign gas injection into a compressible steady laminar boundary layer have been solved approximately for arbitrary pressure gradient and arbitrary distributions of wall temperature and injection velocity.

It is shown that substantial reductions in skin friction and heat transfer rate can be obtained by injection of a light gas instead of air.

11. Acknowledgements

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12. References

1. Craven, A.H. — Boundary layers with suction and injection. College of Aeronautics Report 136, 1960.

2. Libby, P.A., Kaufman, L., Harrington, R.P. — An experimental investigation of the isothermal laminar boundary layer on a porous flat plate. Jour. Aero. Sci., vol. 19, 1952, pp 127-134.

3. Smith, J.W. — Effect of diffusion fields on the laminar boundary layer. Jour. Aero. Sci. vol. 21, 1954, pp 154-162 and pp 640-641.

4. Eckert, E.R.G., Schneider, P.J. — Effect of diffusion in an isothermal boundary layer. Jour. Aero. Sci. vol. 23, 1956, pp 384-387.

5. Faulders, C.R. — A note on laminar boundary layer skin friction under the influence of foreign gas injection. Jour. Aero/Space Sci. vol. 28, 1961, pp 166, 167.

6. Lighthill, M.J. — Contributions to the theory of heat transfer through a laminar boundary layer. Proc. Royal Society, Series A, vol. 202, 1950, pp 359-377.

7. Lilley, G.M. — A simplified theory of skin friction and heat transfer for a compressible laminar boundary layer. College of Aeronautics Note No. 93, 1959.

8. Stevenson, T.N. — The laminar boundary layer with injection through a permeable wall. College of Aeronautics Report 145, 1961.

9. Stewartson, K. — Correlated incompressible and compressible boundary layers. Proc. Royal Society (A), vol. 200, 1949, pp 84 - 100.

10. Illingworth, C.R. — Steady flow in the laminar boundary layer. Proc. Royal Society (A), vol. 199, 1949, pp 533-558.

11. Fage, A., Falkner, V.M. — On the relation between heat transfer and surface friction for laminar flow. R & M. 1408, 1931.

12. Donoughe, P. L., Livingood, J. N. B. — Exact solutions of laminar boundary layer equations with constant property values for porous wall with variable temperature. NACA T.N. 3151, 1954.

13. Hirschfelder, J. O., Curtiss, C. F., Bird, R. B. — Molecular theory of gases and liquids. John Wiley and Sons Inc. 1954.

14. Eckert, E. R. G., Ibele, W. E., Irvine, T. F. — Prandtl number, thermal conductity and viscosity of air-helium mixtures. NASA T.N. D-533, 1960.

15. Lew, H. G., Fanucci, J. B. — On the laminar compressible boundary layer over a flat plate with suction or injection. Jnl. Aero. Sci., vol. 22, 1955, pp 589-597.

16. Korobkin, I. — The effects of the molecular properties of an injected gas on compressible laminar boundary layer skin friction and heat transfer. U.S. Naval Ordnance Laboratory Report 7410, 1961.

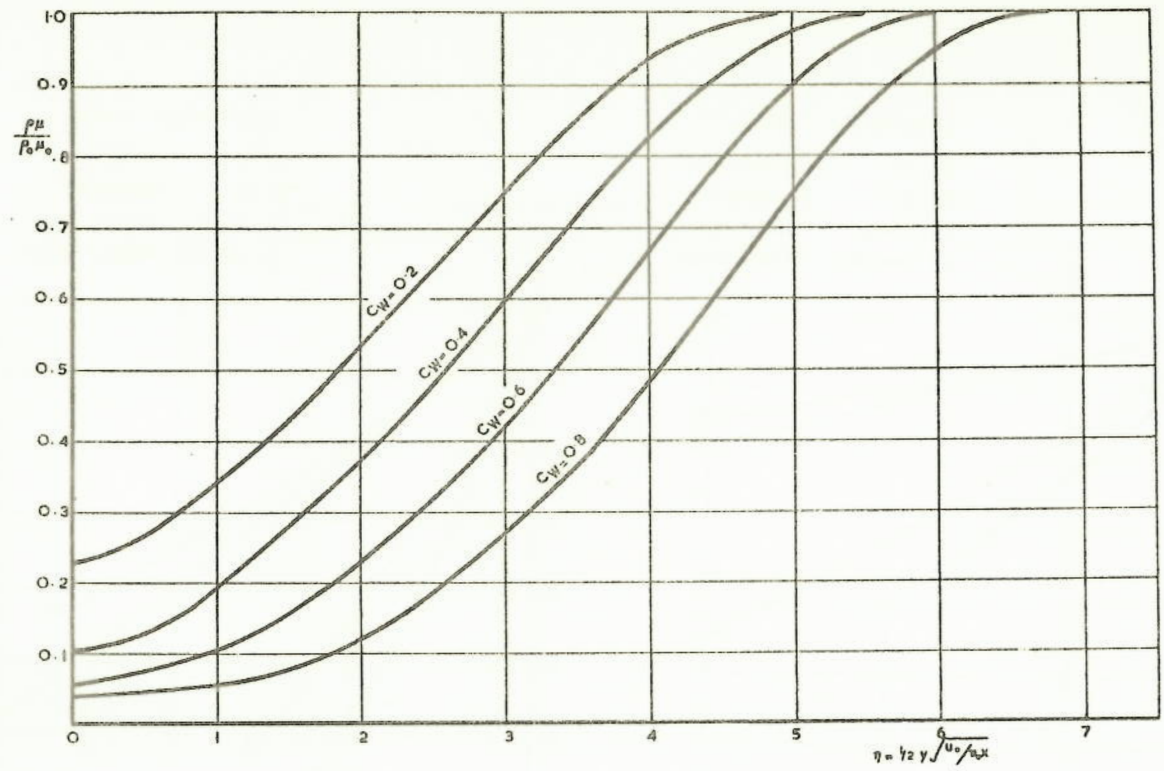

FIG. 1. VARIATION OF $\frac{\rho\mu}{\rho_o\mu_o}$ IN AN INCOMPRESSIBLE BOUNDARY LAYER WITH HYDROGEN INJECTION (from Eckert & Schneider[4])

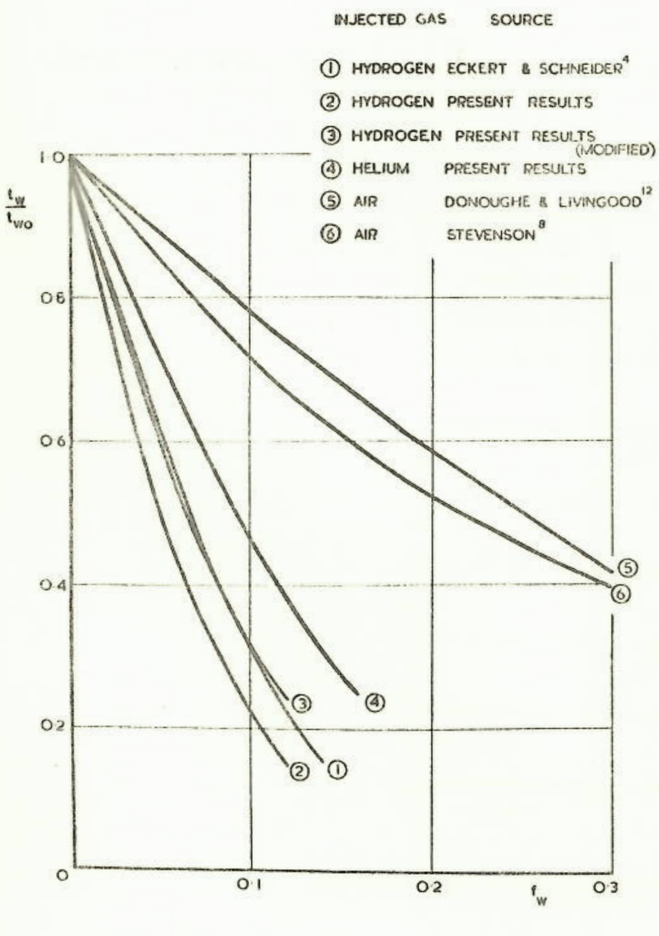

FIG. 2. EFFECT OF FOREIGN GAS INJECTION ON SKIN FRICTION (Incompressible flow, zero heat transfer, v_w proportional to $x^{-\frac{1}{2}}$)

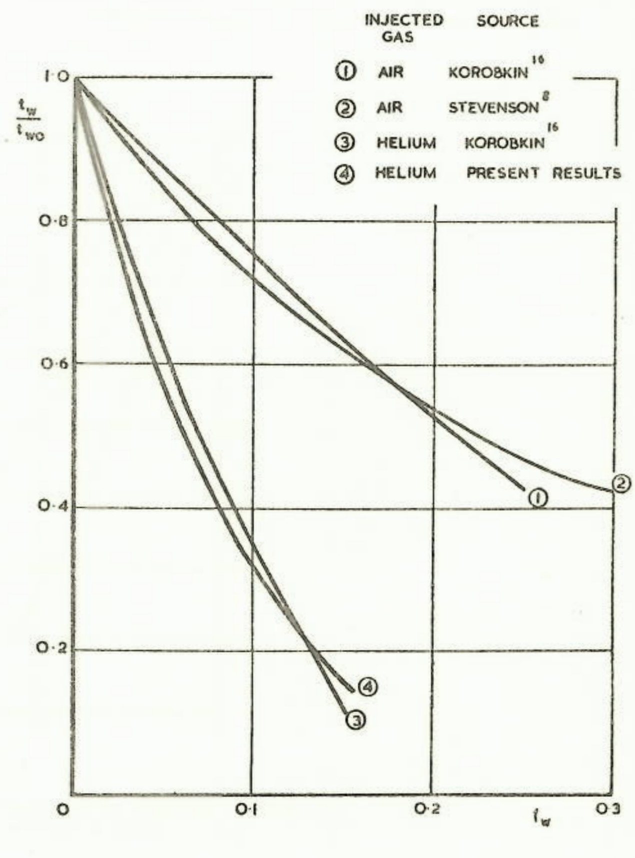

FIG. 3. EFFECT OF FOREIGN GAS INJECTION ON SKIN FRICTION ($M = 6$, $T_w/T_1 = 0.5$, v_w proportional to $x^{-\frac{1}{2}}$)

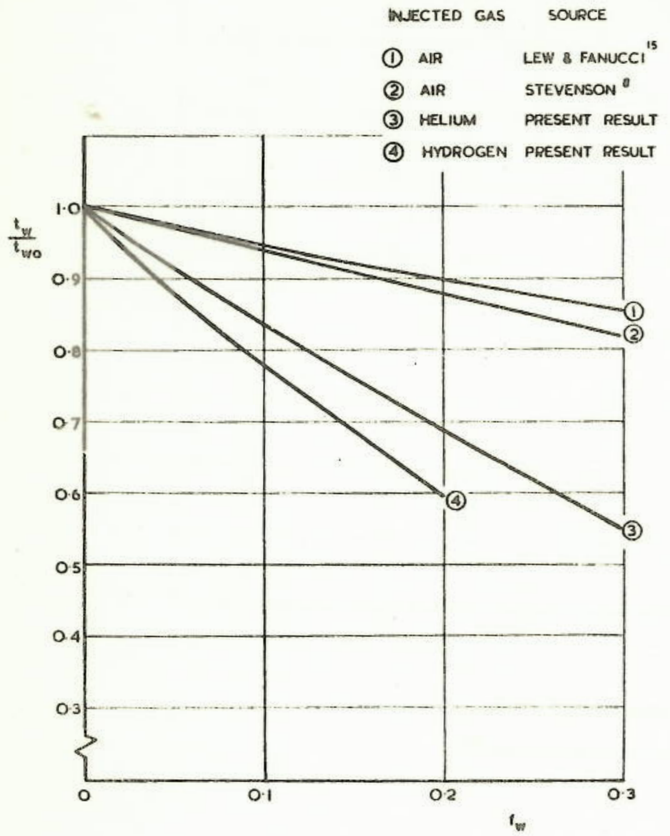

FIG. 4. EFFECT OF FOREIGN GAS INJECTION ON SKIN FRICTION
(Uniform injection velocity, $M_1 = 4$, zero heat transfer)

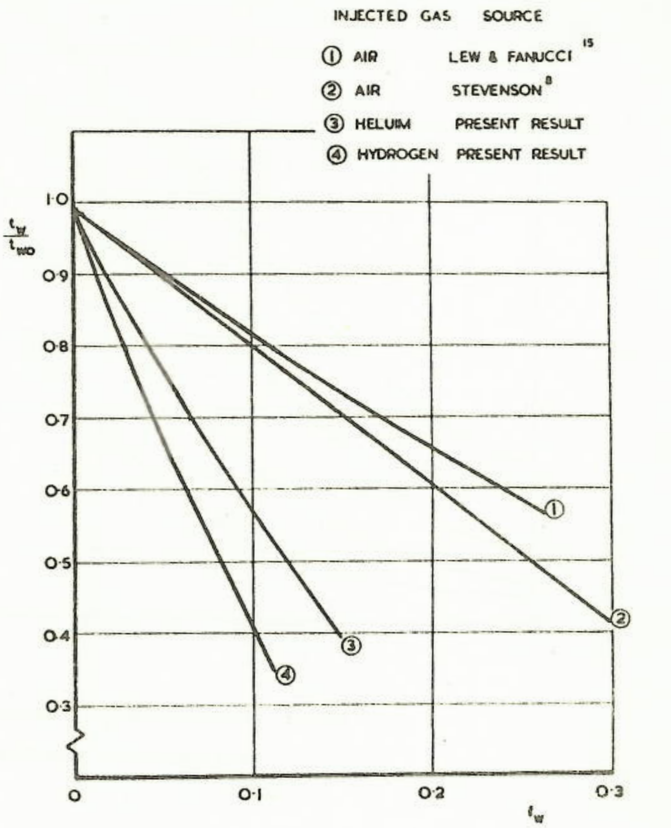

FIG. 5. EFFECT OF FOREIGN GAS INJECTION ON SKIN FRICTION
(Uniform injection velocity $M_1 = 4$, $T_w = T_1$)

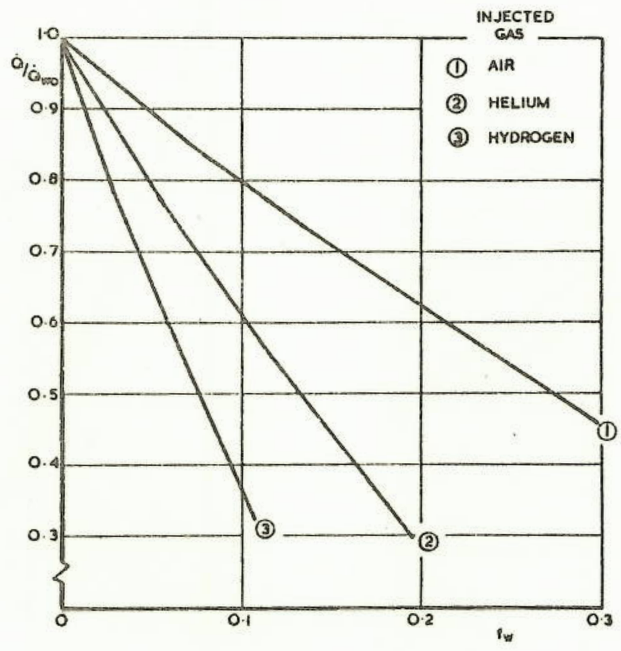

FIG. 6. EFFECT OF FOREIGN GAS INJECTION ON HEAT TRANSFER RATE
(Uniform injection velocity, $M_1 = 4$, $T_w = T_1$)