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Formation Flying linear modelling[1]

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Abstract

An extension of the Hill-Clohessey-Wiltshire equations is presented. The equations refer to the relative motion of two satellites as seen by a Chief undergoing a circular Keplerian motion. Both the Earth oblateness and the air drag are included. The new set of equations is then used to study a simple 100m Leader-Follower formation in a near polar orbit at 600Km of altitude. As the system shows to be linear with time periodic coefficients, Floquet theory is used to determine the stability of the formation and the relative trajectory. Poincaré exponents are also determined and discussed.

Introduction

Dealing with formation-flying issues always arise the question whether it is more convenient to describe the trajectories with a full non-linear model or to use the linear model given by the Hill-Clohessey-Wiltshire (HCW) equations. Arguments in favour of both approaches can be easily given. The non-linear model has so much of the dynamic into it that the results given are clearly more accurate and precise, on the other hand, due to it complexity, it gives little insight into the physics of the problem and it makes quite difficult to generate control algorithms. The HCW equations, on the other hand, are very simple and can be fitted into the well established linear control theory but do not provide, in their classical formulation, any description of non Keplerian forces. Many articles have been written on how to overcame this problem extending the equations to more general cases. Schweighart and Sedwick [2] wrote the HCW equations with an averaged J_2 term and speeding up the Chief motion. Hughes and Mailhe [3] used the linear HCW equations to validate a more complex non-linear model. An important suggestion comes from Wiesel [5] who uses Floquet theory in conjunction with an extended set of HCW equations to model the dynamic of relative satellite motion. In the work of Vadali-Vaddi-Alfriend et al. [4] HCW equations are modified considering a perturbed mean motion (in-plane equations) and an analytical J_2 perturbation (out-of-plane). In the present paper we present an extension of HCW equations accounting for the drag force and the J_2 term. The equations found are linear with time-periodic coefficient and are used to study the stability of a Leader-Follower formation in a LEO near polar mission.

[1] Written in Cranfield as a chapter of MSc. MUSTANG report

Equations derivation

Let us introduce the Local Vertical Local Horizontal (LVLH) frame relative to a Chief satellite undergoing a Keplerian motion on a circular orbit and its associated vectrix $\mathcal{F}_b$ (for an explanation of vectrix notation see Hughes [1]). We then consider n satellites moving in a close orbit so that their position is "not too far away" from that of the Chief. The satellites are feeling both the J_2 perturbation and the drag. We can write the equation of the i-th satellite in the form:

$$\ddot{\vec{\mathbf{r}}}_{P_i} = \vec{\mathbf{g}}(\vec{\mathbf{r}}_{P_i}) + \vec{\mathbf{f}}_{\mathbf{D}}(\dot{\vec{\mathbf{r}}}_{P_i}) + \vec{\mathbf{J}}_2(\vec{\mathbf{r}}_{P_i}) \qquad i = 1..n \tag{1}$$

where $\vec{\mathbf{f}}_D$ is the drag acceleration, $\vec{\mathbf{J}}_2$ and $\vec{\mathbf{g}}$ represent the gravitational acceleration. If we linearize the terms in the second hand of the previous equations with respect to the reference state of the Chief (subscript Ω) and we subtract any two of the obtained equations we get:

$$\ddot{\vec{\delta\mathbf{r}}} = \nabla\vec{\mathbf{g}}|_{\vec{\mathbf{r}}_\Omega}\delta\vec{\mathbf{r}} + \nabla\vec{\mathbf{J}}_{\mathbf{2}}|_{\vec{\mathbf{r}}_\Omega}\delta\vec{\mathbf{r}} + \nabla\vec{\mathbf{f}}_{\mathbf{D}}|_{\dot{\vec{\mathbf{r}}}_\Omega}\delta\dot{\vec{\mathbf{r}}} \tag{2}$$

where $\delta\vec{\mathbf{r}}$ is the relative position vector of the two chosen satellites. In order to be able to evaluate the analytical expression of the gradients we must choose our state variables. A natural choice is to use the coordinates in $\mathcal{F}_b$ of $\delta\vec{\mathbf{r}}$ and their derivates (this is the choice that leads to write the HCW when the drag and the earth oblateness are neglected).

Let us begin with the evaluation of the drag acceleration gradient. We consider the expression $\vec{\mathbf{f}}_D(\dot{\vec{\mathbf{r}}}) = -\frac{1}{2}\rho C_D \frac{A}{m}|\dot{\vec{\mathbf{r}}}|\dot{\vec{\mathbf{r}}}$. As the velocity vector $\dot{\vec{\mathbf{r}}}_{\mathbf{P}}$ depends on the whole state variable $\mathbf{q} = |x, y, z, \dot{x}, \dot{y}, \dot{z}|^T$ the drag force gradient will be the following 3×6 matrix:

$$\nabla\vec{\mathbf{f}}_D(\dot{\vec{\mathbf{r}}}_P) = -\frac{1}{2}\rho C_D A \begin{bmatrix} 0 & -n^2R & 0 & nR & 0 & 0 \\ 2n^2R & 0 & 0 & 0 & 2nR & 0 \\ 0 & 0 & 0 & 0 & 0 & nR \end{bmatrix} \tag{3}$$

The following expressions were used to compute the gradient:

$$\dot{\vec{\mathbf{r}}}_P = \mathcal{F}_b^T \begin{bmatrix} \dot{x} - ny \\ \dot{y} + nx + nR \\ \dot{z} \end{bmatrix}$$

and

$$|\dot{\vec{\mathbf{r}}}_{\mathbf{P}}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + n^2(y^2 + x^2) + R^2n^2 + 2Rn^2x + 2n(x\dot{y} - y\dot{x}) + 2Rn\dot{y}}$$

Eventually, the following expression is found:

$$\nabla\vec{\mathbf{f}}_{\mathbf{D}}|_{\dot{\vec{\mathbf{r}}}_\Omega}\delta\dot{\vec{\mathbf{r}}} = -\frac{1}{2}\rho C_D \frac{A}{m}\mathcal{F}_b^T \begin{bmatrix} -n^2Ry + nR\dot{x} \\ 2Rn^2x + 2Rn\dot{y} \\ nR\dot{z} \end{bmatrix} + O(2)$$

It is convenient to define the parameter $\chi := \frac{1}{4}\frac{\rho C_D A R}{m}$. Expressing equation (2) in terms of the frame $\mathcal{F}_b$ (neglecting the J_2 term) and premultiplying for the corresponding vectrix $\mathcal{F}_b$, we obtain the system:

$$\begin{array}{l} \ddot{x} - 2n\dot{y} + 2n\chi\dot{x} - 3n^2 x - 2n^2\chi y = 0 \\ \ddot{y} + 4n\chi\dot{y} + 2n\dot{x} + 4n^2\chi x = 0 \\ \ddot{z} + 2\chi n\dot{z} + n^2 z = 0 \end{array}$$

If we compare these equations to the the classical HCW we immediately observe the introduction of some viscous term in the z axis motion. Due to the extreme rarefaction of the air the damping ratio χ is always very poor resulting in an under damped system for the z motion. Putting the equation in a non dimensional form, we get:

$$\begin{cases} \ddot{\xi} - 2\dot{\eta} + 2\chi\dot{\xi} - 3\xi - 2\chi\eta = 0 \\ \ddot{\eta} + 4\chi\dot{\eta} + 2\dot{\xi} + 4\chi\xi = 0 \\ \ddot{\zeta} + 2\chi\dot{\zeta} + \zeta = 0 \end{cases}$$

We observe here that the dynamic matrix A_D of the previous system is:

$$A_D := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 2\chi & 0 & -2\chi & 2 & 0 \\ -4\chi & 0 & 0 & -2 & -4\chi & 0 \\ 0 & 0 & -1 & 0 & 0 & -2\chi \end{bmatrix} \tag{4}$$

whose determinant $8\chi^2$ is not zero although very small. There is not any zero eigenvalue nor any secular term, the system is therefore Lyapunov stable and the transient phase will therefore eventually vanish.

We now evaluate the J_2 gradient in the same coordinates. The $\vec{\mathbf{J}}_2$ vector is given by:

$$\vec{\mathbf{J}}_2 = -\frac{3}{2}\frac{J_2\mu R_E^2}{r_P^4}\mathcal{F}_f^T \begin{bmatrix} 1 - 3\sin^2 i \sin^2\psi \\ 2\sin^2 i \sin\psi\cos\psi \\ 2\sin i \cos i \sin\psi \end{bmatrix}$$

We have introduced the symbols R_E for the earth equatorial radius and used the well known coordinate system r, ψ, i and its natural basis $\mathcal{F}_f$. The origin of $\mathcal{F}_b$ will be denoted, in this coordinates, by the triple $R, \hat{\psi}, \hat{i}$. As what we need is the expression for $\vec{\mathbf{J}}_2$ in the $\mathcal{F}_b$ reference frame, we have:

$$\mathcal{F}_b^T J_{2_b} = -\mathcal{F}_b^T \frac{3}{2}\frac{J_2\mu R_E^2}{r^4} C_{bf} \begin{bmatrix} 1 - 3\sin^2 i \sin^2\psi \\ 2\sin^2 i \sin\psi\cos\psi \\ 2\sin i \cos i \sin\psi \end{bmatrix} \tag{5}$$

where J_{2_b} represents a vector containing the components of $\vec{\mathbf{J}}_2$ in the $\mathcal{F}_b$ reference frame. The matrix C_{bf} is defined by the equation $\mathcal{F}_b = C_{bf}\mathcal{F}_f$ and can be supposed to be the sum of two infinitesimal rotations around axis z and the line of the nodes. The mathematical form of this matrix is:

$$C_{bf} = \begin{bmatrix} 1 & -(\psi-\hat{\psi}) & -(i-\hat{i})\sin\psi \\ (\psi-\hat{\psi}) & 1 & -\cos\psi(i-\hat{i}) \\ (i-\hat{i})\sin\psi & +\cos\psi(i-\hat{i}) & 1 \end{bmatrix} \tag{6}$$

Substituting expression (6) into (5) we finally have an expression for J_{2_b} in terms of our state variables (note that the coordinates r, i, ψ depend on x, y, z). It is therefore possible to evaluate the desired expression for the gradient evaluated in $R, \hat{\psi}, \hat{i}$:

$$\nabla\vec{\mathbf{J}}_{\mathbf{2}}|_{\vec{\mathbf{r}}_Q} = \\ = \frac{4k}{R}\begin{bmatrix} 1-3s^2\hat{i}s^2\hat{\psi} & s^2\hat{i}s2\hat{\psi} & s\hat{\psi}s2\hat{i} \\ s^2\hat{i}s2\hat{\psi} & -\frac{1}{4}+s^2\hat{i}\left(\frac{7}{4}s^2\hat{\psi}-\frac{1}{2}\right) & -\frac{c\hat{\psi}s2\hat{i}}{4} \\ s\hat{\psi}s2\hat{i} & -\frac{c\hat{\psi}s2\hat{i}}{4} & -\frac{3}{4}+s^2\hat{i}\left(\frac{1}{2}+\frac{5}{4}s^2\hat{\psi}\right) \end{bmatrix} \tag{7}$$

with $k = \frac{3}{2}\frac{J_2\mu R_E^2}{R^4}$. Schweighart and Sedwick in [2] evaluated this gradient: their result is slightly different from the one we obtained here, we will assume that our expression is correct. We note that as the reference orbit is circular $\hat{\psi} = \tau$ (initial conditions are supposed to be given at the ascending node).

Numerical results

We have expressed all the term in eq.(2) with respect to the reference frame $\mathcal{F}_b$, we therefore can get the following ODE:

$$\dot{\mathbf{q}} = (A_D + 4KA_J)\,\mathbf{q} \tag{8}$$

where the matrix A_D is defined by eq.(4) and A_J is:

$$A_J = \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1-3s^2\hat{i}s^2\tau & s^2\hat{i}s2\tau & s\tau s2\hat{i} & 0 & 0 & 0 \\ s^2\hat{i}s2\tau & -\frac{1}{4}+s^2\hat{i}\left(\frac{7}{4}s^2\tau-\frac{1}{2}\right) & -\frac{c\tau s2\hat{i}}{4} & 0 & 0 & 0 \\ s\tau s2\hat{i} & -\frac{c\tau s2\hat{i}}{4} & -\frac{3}{4}+s^2\hat{i}\left(\frac{1}{2}+\frac{5}{4}s^2\tau\right) & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

We introduced a non dimensional parameter $K = \frac{k}{n^2R} = \frac{3}{2}J_2\left(\frac{R_E}{R}\right)^2$. Although linear the equation has a time dependent dynamic matrix. The time appears only through periodic functions and the dynamic matrix is therefore periodic too. Floquet theory could be used to study this problem, as suggested by Wiesel [5] but we will here first attempt a numeric simulation. The main problem with this approach is the accuracy and the stability of the numeric integrator, the extreme difference in the magnitude of

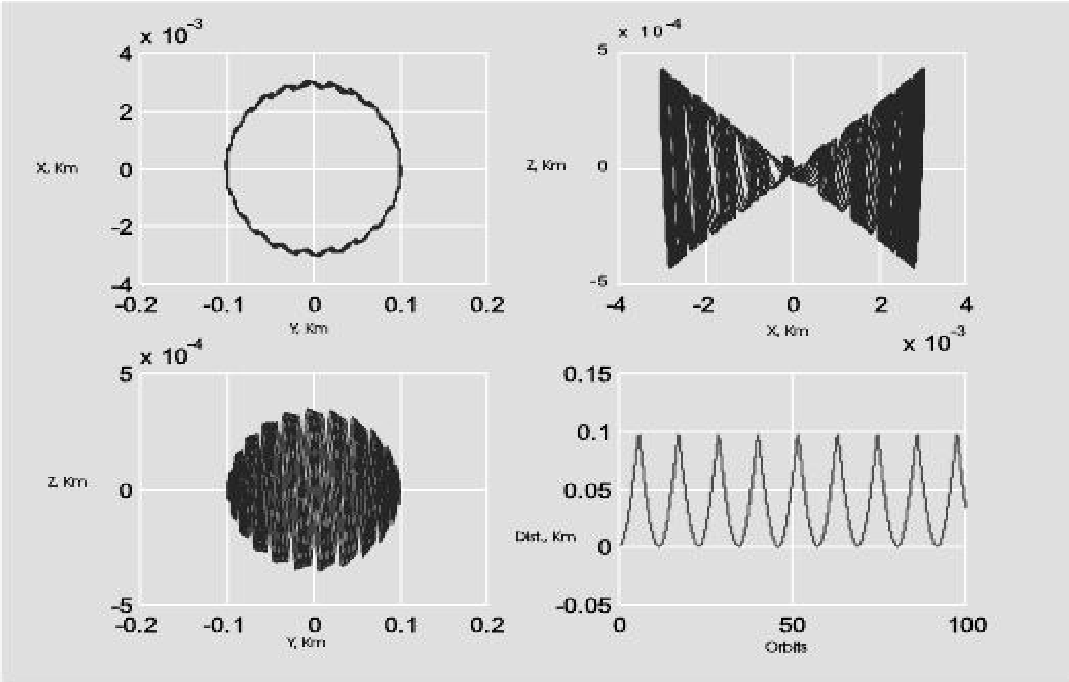

Figure 1: 2-D plot of the relative position of the two satellites. The distance difference between the actual position and the desired 100m Formation Flying is also shown.

the various terms appearing in the equation make it non trivial to pursue the numerical simulation. We considered a simple 100m Leader-Follower formation in a circular reference orbit inclined of 82^o and having a semi-major axis of 6978Km. The results are shown in figure 1. We observe that leaving the satellite uncontrolled results in an oscillating relative distance with a fundamental frequency of roughly 10 orbits. The satellites, initially separated by 100m would begin to get closer and closer (up to 6m) to then invert the formation to a "Follower-Leader" one. The simulation was run for roughly 150 orbits, after this time length numerical instability begins to corrupt the solution.

Poincaré exponents in LEO near polar case.

In this section we will approach the solution of eq.(8) using Floquet theory. We evaluate Poincaré exponents ω_i and discuss the stability of the system, that is of the Leader-Follower formation-flying. We get:

$$\omega_{1,2} = \pm 0.04498i$$
$$\omega_{3,4} = \pm 0.001356$$
$$\omega_{5,6} = \pm 0.0005619i$$

The exponent ω_3 is placed in the right half of the complex plane determining the Lyapunov instability of the system. The characteristic time of the instable exponent $\frac{1}{\omega_3}$ corresponds to roughly 120 orbits, that is why instability is not visible in the trajectory plotted in figure 1.

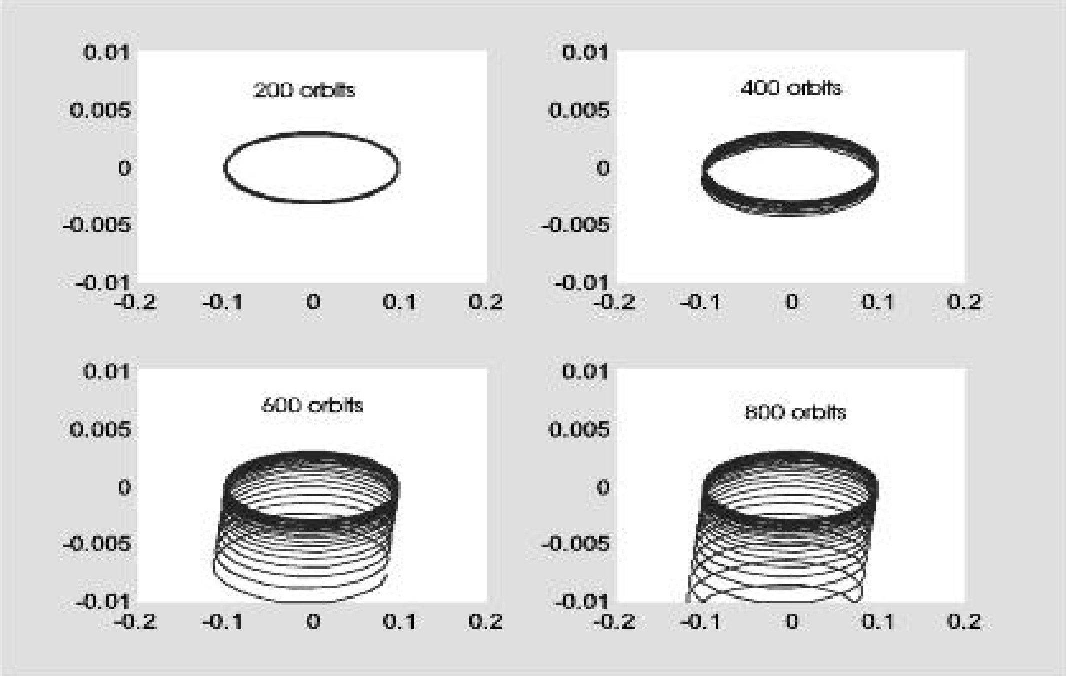

Figure 2: Instability rise in the relative position of the uncontrolled leader-follower formation, *y*,*x* plot. The plot was done by using Floquet analysis. It would not be possible to get the same result by direct numerical integration as the algorithm would be unstable after 150 orbits.

If we prolong the integration time to $200 - 300$ orbits instability begins to be visible as shown in figure 2. The periodic terms that will form the solution comes from the periodic matrix F of the Floquet transition matrix decomposition and from the complex Poincaré exponents. The periods of these terms is easily found and correspond to $1, \frac{1}{2}$ orbits those associated with the F matrix and $22, 1785$ orbits those associated with Poincaré exponents. The term with a 22-periodicity is the one responsible for the distance fluctuation plotted in figure 1. The linear system is therefore Lyapunov instable, such a result extends to the non linear case. The instability is visible after roughly 200 orbits as shown in figure 2.

Conclusions

An extension of Hill-Clohessey-Wiltshire equations is presented. Both the Earth oblateness and the air drag are taken into account. The resulting linear system has time-periodic coefficients and its solution is presented both via numerical integration and Floquet analysis. The studied case, a Leader-Follower formation at 100m distance, in a circular near polar orbit, is found to be unstable, the instability manifesting very slowly after 200 orbits. The equations found are particularly appealing for the design of a control strategy. An introduction to the control of time-periodic systems is given by Calico-Wiesel in [6].

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