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A Method of Optimizing Aircraft Autostabilizer Systems

-by-

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SUMMARY

A novel procedure for the optimization of aircraft autostabilizer systems is presented. The procedure is straightforward, and its application does not result in demands for autostabilizer systems of prohibitive complexity. Many important non-linear effects may be included with only slight extra complication in the required calculations. The procedure is applicable, in the first place, to piloted aircraft, - the essence of the procedure being the assumption that the purpose of the autostabilizer is to reduce the effort demanded of the pilot in executing a given manoeuvre or attaining a given response. Although the presence of the pilot is explicitly taken into account in the calculations no form of pilot's transfer function need be specified.

It is shown how the procedure may be modified to form an approximate procedure for the optimization of autostabilizers for pilotless aircraft having linear autostabilizer characteristics and linear aircraft dynamics. The results of some calculations presented herein support a suggestion that this approximate optimization procedure may also be frequently applied with success to pilotless aircraft having certain non-linearities, either in the autostabilizer system or in the aircraft dynamics.

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LIST OF SYMBOLS

DERIVATIVES

The conventional British system of non-dimensionalized derivatives of Ref. 1 is employed with the following modifications.

(i) μ_1, is written simply as μ

(ii) The derivative $m_{\dot{w}}$ is μ times that of Ref.1.

(iii) The non-dimensionalized rate of pitch, $\frac{d\theta}{d\tau}$ is written as $\hat{q}$.

For Lateral Stability the supplementary notation of Mitchell (Ref.2.) is employed (with trifling alterations). In this system,

$$\ell_1 = -\frac{\ell_p}{i_{A'}}, \quad \ell_2 = \frac{\ell_r}{i_{A'}}, \quad n_1 = -\frac{n_p}{i_{C'}}, \quad n_2 = \frac{-n_r}{i_{C'}}, \quad \bar{y}_v = -y_v$$

$$\mathcal{N} = \frac{\mu_2 n_v}{i_{C'}}, \quad \mathcal{L} = -\frac{\mu_2 \ell_v}{i_{A'}}$$

$$\mathcal{C}_{\ell_\xi} = -\frac{\mu_2 \ell_\xi}{i_{A'}}, \quad \mathcal{C}_{n_\zeta} = -\frac{\mu_2 n_\zeta}{i_{C'}}, \quad \text{etc.,}$$

OTHER SYMBOLS

LOWER CASE

a	(i) see Equation 2.48; (ii) see Equation 3.23 — a negative constant
$a_0, a_1, a_2, a_3,$	see Equation 6.8.
a_1, a_3, a_5, a_7	Coefficients of the polynomial $P(x)$
e_0, e_1, e_2, e_3	see Equation 6.6.
h	$m_{\dot{w}}$ with autostabilization / basic $m_{\dot{w}}$
h_0, h_1, h_2, h_3	see Equation 6.9.
k (i)	$C_L/2$
k (ii)	x_S/x_M
k′	m_q with autostabilization / basic m_q

m_G	an impulsive pitching moment
m_1, m_3	see Equation 2.44.
n	see Equation 3.28.
p	see Equation 2.15. (a positive constant)
s	The Laplace Transform Variable
t	A measure of time (see Equation 3.34 and Chapter 4)
w_c	see Equation 3.27.
χ	The input signal to the autostabilizer.
χ_M	The maximum value of χ that need be considered.
χ_s	The value of χ at which saturation occurs.
y	A general control surface deflection due to the autostabilizer
y_A	The autostabilizer characteristic.
y_M	The limiting value of y
α (i) (ii)	Angle of incidence see Equation 6.14.
β (i) (ii)	(for Time Vectors) $= -\bar{v}$, where $\bar{v}$ = sideslip angle. see Equation 3.28
δ	the Cardinal Spectrum Interval
ϵ (i) (ii)	an arbitrarily short time $P(\chi) - y_A$
η_p	an elevator deflection produced by the pilot
$\Delta\,\eta(\tau)$	a step elevator deflection of magnitude $\delta\,\eta$
$\delta\,\eta$	see $\Delta\,\eta(\tau)$
η_s	an elevator deflection due to the autostabilizer
η_F	the amplitude of η_s for flicker autostabilization.

CAPITALS

A_1, A_3, A_5, A_7,	generalized forms of a_1, a_3, a_5, a_7
Ⓒ	a cardinal spectrum
D	the differentiation operator
E	defined from Equation 6.14.
H, H' =	n_p with autostabilization / basic n_p
I	an integral, as defined in the text
I_ϵ	" " " " " " "
I_1	an integral of the 1st kind (see Chapter 3. Section 4)
I_2	" " " " 2nd " (" " ")
I_∞	see Equation 3.30.
I_β	see Equation 3.31.
J (i)	$= K - 1$
(ii)	$= \sqrt{H - R^2}$ where $H = \frac{-\mu m_w}{i_B} + m_q z_w$
K, K'	= n_r with autostabilization / basic n_r
P	defined by Equation 3.28
P_1	$= e^{-0.434T} (\sin [T - 77.05^\circ])$ (from Equation 3.22)
P_2, P_4, P_6, P_8	Coefficients of A_1
P (x)	The polynomial approximation to the autostabilizer characteristic
Q	see Equation 3.20
Q_2, Q_4, Q_6, Q_8	Coefficients of A_3
Q (s)	a quadratic expression in s
R	$= \frac{1}{2} \left(\frac{-m_q}{i_B} - \frac{m_{\dot{w}}}{i_B} - z_w \right)$
R (τ)	a general response

$R_D(\tau)$	a general desired response
R_2, R_4, R_6, R_8	Coefficients of A_5
S_2, S_4, S_6, S_8	Coefficients of A_7
T (i)	a measure of time (frequently $11.51\ \tau$)
(ii)	a convenient (fixed) time
$\times$	denotes polymultiplication
$\Delta\ (s)$	a quartic expression in s
$\Delta\ \eta(\tau)$	a step elevator deflection of magnitude $\delta\ \eta$

SUFFICES

D	a quantity associated with the desired response. (e.g. $\acute{q}_D$, w_D,)
s	a control surface deflection due to the autostabilizer (e.g. η_s) (This should not be confused with the Laplace transform of an elevator deflection $\eta(\tau)$, which is written $\eta(s)$)
p	a control surface deflection due to the pilot (e.g. η_p)
p_D	(e.g. η_{p_D}) the control surface deflection demanded of the pilot to attain the desired response.

A few other transient symbols are defined as necessary in the text. The notation of Cardinal Spectrum Analysis is explained in Chapter 4.

Footnotes are not employed: superfixes in the text thus[1] refer to the section headed 'Notes on Chapters'.

Chapter 1. INTRODUCTION

It will be generally agreed that the subject of aircraft autostabilization has rapidly grown in importance in recent years. The reason for this is not hard to find; it is simply that recent great advances in aircraft performance have demanded aircraft configurations which often have inherently poor stability and response characteristics. Examples of this are legion; to quote only three, the adoption of sweepback as a means of increasing critical Mach number has led in some cases to an undesirably high value of ℓ_v at high C_Ls ; the high operational altitudes now common, result in poor damping of the lateral and longitudinal oscillations; and the inertia distribution of many modern high performance aircraft is such that inertial cross-coupling in roll is readily induced. The reader will doubtless be familiar with many further examples of this trend of reduced stability with increased performance.

The problem that confronts us is, then, how we may improve an aircraft's stability and response characteristics without sacrifice of performance. Some improvement can be achieved by careful design of the basic airframe. For example, the above-mentioned excessive ℓ_v due to sweep may be reduced by the adoption of anhedral, and a large fin may alleviate the undesired effects of inertial cross-coupling in roll. However, the gains that can be attained in this manner are limited by the restriction that the aircraft's performance must not be reduced, and in many cases autostabilization must be resorted to if satisfactory response and stability characteristics are to be attained.

If an autostabilizer system of unlimited weight, complication and expense were permitted the stability and response characteristics of a given aircraft could certainly be made quite satisfactory under all conditions. In practice, of course, all three of the above factors will be limited and it should be appreciated that any discussion which fails to take into account the possible effects of such limitations may be somewhat unrealistic. Thus, for example, a study of the effects of changing lateral derivatives may show that satisfactory lateral stability characteristics may be obtained with a value of n_v several times that of the basic (i.e. non-autostabilized) aircraft. But it may be that the power available for the autostabilizer system is inadequate to generate the control surface deflections required to attain this value of n_v at moderate and large angles of sideslip. Even if sufficient power is available, the aircraft designer may well decide to limit the maximum amplitudes of the control surface deflections due to the autostabilizer so that in the event of a run-away catastrophic divergence will not occur.

One important reason why comparatively little attention has been given to the more practical considerations of autostabilizer design such as the above-mentioned, is simply that the problem is non-linear; i.e. the mathematical formulation of such a problem reduces to a set of non-linear differential equations. Whereas linear dynamic systems of great complexity may be described by differential equations having fairly straightforward methods of solution, the complications involved in solving even a simple non-linear equation may be considerable. For aircraft motion having several degrees of freedom it is frequently found that no analytic solution of the resulting set of non-linear equations is known. Step-by-step numerical solution is usually a tedious process and recourse has usually to be made to analogue computation. The procedure then adopted is to solve the equations of motion for different values of the adjustable autostabilizer parameters within the preselected limits. The adjustable parameters of the autostabilizer system are then fixed at those values which have been shown to yield response characteristics acceptably close to the desired response characteristics.

The above procedure suffers the disadvantage of demanding analogue computer equipment - perhaps of considerable expense - and the procedure has a certain crudity in that the final (optimum) values of the adjustable autostabilizer parameters cannot be attained directly but are arrived at by trial and error. Nevertheless, for the design of autostabilizers for pilotless aircraft having non-linear equations of motion with several degrees of freedom this procedure is probably the best practical technique available. Analytic methods are only likely to prove superior in problems of very limited complexity.

For piloted aircraft even the recourse of analogue computation may fail. It is well known that the stability requirements for piloted and pilotless aircraft differ. For example, spiral instability may be quite acceptable in a piloted aircraft, whereas in a pilotless aircraft it would be catastrophic. In a piloted aircraft, longitudinal or lateral oscillations having a very short period may cause confusion and discomfort to the pilot whereas in a pilotless aircraft these characteristics may be quite unobjectionable - or even desirable, since such short periods are usually associated with rapid rates of response. Considerations such as these show that we cannot simply assume that the optimum setting of the adjustable parameters of an autostabilizer calculated on the assumption of pilotless flight will necessarily be suitable for piloted flight. The above-mentioned analogue computer equipment may (with further expense) be extended to form a flight simulator, but the representation of flight in such a device may be too limited to be satisfactory.[1] Actual flight tests using the autostabilizer equipment can, of course, only be undertaken when the aircraft is complete and it is obviously desirable to have the design of the autostabilizer equipment finalized (at least within limits to allow for possible inaccuracies in the data used for computation) well before the completion of the aircraft. Might we then attempt an analytical solution of the problem of optimizing the autostabilizer system of a piloted aircraft by representing the pilot mathematically by a suitable transfer function in the equations of motion ? For a purely linear system (i.e. linear aircraft and autostabilizer characteristics) this would be possible, but for a non-linear system the equations of motion would be even more complicated than in the case of a pilotless aircraft and the chances of an analytic solution being known even less. Apart from these considerations, however, at the present time no satisfactory transfer function to describe the pilot is available, though it has recently been shown that an expression for the transfer function of a pilot may be obtained for certain very restricted types of manoeuvre such as pure yawing.[2]

For more complicated manoeuvres it seems that the human pilot is actually able to vary his transfer function to suit the conditions of flight and the demands made on him. In view of this, the complications of such an analytical solution starting from the equations of motion, as suggested in this paragraph, become formidable.

From the above survey it might seem that an analytical solution of the problem of optimizing the autostabilizer system of a piloted airfract having non-linear characteristics (either in the aircraft dynamics, or in the autostabilizer system, or both) is, at present, hardly to be hoped for. In fact, this is not the case, and the purpose of this thesis is to present a straightforward technique developed by the present writer which yields exact solutions for the optimum values of the adjustable parameters of a specified autostabilizer system for linear cases and approximate solutions of good accuracy for many non-linear cases of importance. This technique is applicable to both piloted and pilotless aircraft having linear characteristics (including the autostabilizer system) and to piloted aircraft having certain non-linear characteristics (either in the aircraft dynamics, the autostabilizer, or both).

CHAPTER 2.

2.1. THE OPTIMIZATION PROCEDURE FOR PILOTED AIRCRAFT

The most general procedure consists of a number of steps as detailed below. In any particular example it will usually be possible to telescope two or more of the steps into one.

Step 1.

The response of the basic aircraft (the aircraft with no autostabilization) to a specified input is calculated. This response is called the 'basic response'.

Step 2.

The desired response to the input is specified, and compared with the basic response. In general the basic response will differ appreciably from the desired response and autostabilization will be required if the desired response is to be attained or closely approached.

Step 3.

The type of autostabilization to be used is selected (i.e. Δm_q, Δn_v, Δn_r, etc.) and any limitations or non-linearities specified.

Step 4.

It is assumed that the desired response is attained exactly, through the combined actions of the autostabilizer and the pilot.

Step 5.

The optimum adjustment of the variable parameters of the autostabilizer system is assumed to have been made when the effort demanded of the pilot is a minimum. We use the term 'effort' in a broad sense to include mental strain as well as physical exertion. The mathematical representation of effort by means of an 'effort function' is discussed later.

Step 6.

With this criterion equations for the optimum values of the adjustable parameters of the autostabilizer system are produced and solved.

Step 7.

The time history of the control surface deflections demanded of the pilot with the optimum autostabilization is calculated. If these appear difficult to attain it is necessary to proceed to Step 8.

Step 8.

The response to the specified input with the optimum autostabilization but with no pilot action (other than such as may be included in the specified input) is calculated. This response is then compared with the desired response. If it is acceptably close to the desired response the optimum autostabilization may be regarded as satisfactory; if not, we conclude that the type of autostabilization chosen is inherently incapable of producing a satisfactorily close approximation to the desired response even when adjusted to its optimum value, and some other type of autostabilization must therefore be selected.

Discussion of the above procedure is delayed until the end of this chapter. Four examples now follow. In each we assume a pilot effort function of the form $\int_0^\infty \left[\eta_{p_D}(\tau) \right]^2 d\tau$, where $\eta_{p_D}(\tau)$ is the elevator deflection that must be applied by the pilot to attain the desired response, and τ is a measure of time. Thus we shall assume that the optimum autostabilization is that which minimizes this integral. (The procedure is by no means restricted to effort functions of this type and the use of other types of effort function is described later).

2.2. EXAMPLE 1. LONGITUDINAL SHORT-PERIOD RESPONSE TO AN IMPULSIVE PITCHING MOMENT

The standard non-dimensional equations of motion for short-period longitudinal response are

$$(D - z_w)\, w(\tau) - \hat{q}(\tau) = 0 \qquad (2.1)$$

$$-\left(\frac{m_{\dot{w}}}{\mu} D + m_w\right) w(\tau) + \left(\frac{i_B}{\mu} D - \frac{m_q}{\mu}\right) \hat{q}(\tau) = m(\tau) \qquad (2.2)$$

assuming $z(\tau)$ to be negligible

with $D = \frac{d}{d\tau}$

and $\hat{q}(\tau) = \frac{d\theta}{d\tau}$

We assume that an impulsive pitching moment (due to, for example, gun recoil) is applied such that $\int_0^\epsilon m(\tau)\, d\tau = 10^{-3}$ where ϵ may be made as small as we please.

Applying the Laplace transform to Equations 2.1 and 2.2 we obtain

$$(s - z_w)\, w(s) - \hat{q}(s) = 0 \qquad (2.3)$$

$$-\left(\frac{m_{\dot{w}}}{\mu} s + m_w\right) w(s) + \left(\frac{i_B}{\mu} s - \frac{m_q}{\mu}\right) \hat{q}(s) = 10^{-3} \qquad (2.4)$$

$$\text{whence } w(s) = \frac{\frac{\mu}{i_B}\, 10^{-3}}{s^2 + 2R_s + R^2 + J^2} \qquad (2.5)$$

where $s^2 + 2Rs + R^2 + J^2 = s^2 + s\left(-\frac{m_q}{i_B} - \frac{m_{\dot{w}}}{i_B} - z_w\right) - \mu\frac{m_w}{i_B} + m_q z_w$ (2.6)

For the aircraft of Appendix I, flying at M=0.9, 50,000' with the C.G. at 28% s.m.c., the following derivatives apply:-

$z_w = -2.35$, $m_w = -0.108$, $m_q = -0.2263$, $m_{\dot{w}} = -0.0895$,

$i_B = 0.298$, $\mu = 365.0$,

whence $R = 1.706$, $J = 11.51$,

Applying the inverse Laplace transform to Equation 2.5, we obtain

$$w(\tau) = \frac{\mu}{i_B} . 10^{-3} \; e^{-R\tau} \frac{\sin J\tau}{J} \qquad (2.7)$$

$$\therefore \quad w(\tau) = 1.225 \; e^{-1.706\tau} \frac{\sin 11.51\,\tau}{11.51} \qquad (2.8)$$

Equation 2.8 describes the basic response (i.e. the response to the selected input with no autostabilization) in w of the aircraft. Examination of the graph of Equation 2.8 (Fig. 1) shows that the response is markedly oscillatory and only moderately damped. Let us suppose that the desired response in w is described by

$$w_D(\tau) = 1.225 \; e^{-5.0\tau} \frac{\sin 11.51\,\tau}{11.51} \qquad (2.9)$$

We assume that the autostabilizer available is of such a type that an elevator deflection proportional to $\hat{q}$ may be produced. Thus (neglecting terms in z_η) the derivative m_q is at our disposal. In addition to the normal assumptions of linear theory implicit in Equations 2.1 and 2.2 we also assume that the motion is sufficiently small for saturation and limitation of control surface deflection effects in the autostabilizer system to be neglected.

$$\text{Let } k' = \frac{\text{total } m_q \text{ with autostabilization}}{m_q \text{ of the basic aircraft}}$$

The problem is to determine the optimum value of k'.

We have $w_D(\tau) = 0.1064\, e^{-5.0\tau} \sin 11.51\, \tau$ (2.9)

whence $D\, w_D(\tau) = e^{-5.0\tau}(-0.532 \sin 11.51\, \tau + 1.225 \cos 11.51\, \tau)$ (2.10)

From Equations 2.1, 2.9, and 2.10, we obtain

$\hat{q}_D(\tau) = e^{-5.0\tau}(-0.284 \sin 11.51\, \tau + 1.225 \cos 11.51\, \tau)$ (2.11)

whence $D\hat{q}_D(\tau) = e^{-5.0\tau}(-12.68 \sin 11.51\, \tau - 9.385 \cos 11.51\, \tau)$ (2.12)

Equation 2.2 may be written in the form

$$-\left(\frac{m_{\dot{w}}}{i_B} D + \mu \frac{m_w}{i_B}\right) w_D(\tau) + \left(D - \frac{m_q}{i_B} k'\right) \hat{q}_D(\tau) - \mu \frac{m_G}{i_B}(\tau) = \frac{\mu m_\eta}{i_B} . \eta_{P_D}(\tau) \qquad (2.13)$$

where m_G is the (non-dimensionalized) applied impulsive moment

and η_{P_D} is the elevator deflection that must be applied by the pilot to attain the desired response.

Substituting the numerical values of the derivatives into Equation 2.13, we obtain after some reduction,

$$\frac{\mu m_\eta}{i_B} . \eta_{P_D}(\tau) = e^{-5.0\tau}\Big[(+1.23 \sin 11.51\, \tau - 8.982 \cos 11.51\, \tau) + k'(-0.2135 \sin 11.51\, \tau + 0.930 \cos 11.51\, \tau)\Big] - \frac{\mu m_G}{i_B}(\tau) \qquad (2.14)$$

Now our criterion for the optimum value of k is that the integral $I = \int_0^\infty \left[\frac{\mu m_\eta}{i_B} . \eta_{P_D}(\tau)\right]^2 d\tau$, should be a minimum. As $m_G = 0$ for $\tau > \epsilon$, where ϵ may be as small as we please, we may eliminate $m_G(\tau)$ from the remainder of the calculation (with consequent simplification of the expressions to be dealt with) by the device of changing our criterion from that of k' being chosen to minimize I to the following criterion: k' is chosen so as to minimize the integral I_ϵ, where

$$I_\epsilon = \int_\epsilon^\infty \left[\frac{\mu m_\eta}{i_B} \cdot \eta_{P_D}(\tau)\right]^2 d\tau.$$

With negligible error, the following relations hold for ϵ arbitrarily small.

$$\int_{\epsilon}^{\infty} e^{-pt}\, dt = \frac{1}{p} \qquad (2.15)$$

$$\int_{\epsilon}^{\infty} e^{-pt} . \sin \omega t\, dt = \frac{\omega}{p^2 + \omega^2} \qquad (2.16)$$

$$\int_{\epsilon}^{\infty} e^{-pt} . \cos^2 \omega t\, dt = \frac{p^2 + 2\omega^2}{p\,(p^2 + 4\omega^2)} \qquad (2.17)$$

Squaring Equation 2.14 and integrating the result between limits of ϵ and ∞, making use of Equations 2.15, 16, 17, yields after some reduction an expression for I_ϵ of the form $I_\epsilon = 0.448\, k'^2 - 0.854\, k' +$ terms not involving k' (2.18)

For I_ϵ stationary $\frac{\partial I_\epsilon}{\partial k'} = 0$ when $k' = k'_{optimum} = 9.53$

We must now ascertain the nature of the control deflection demanded of the pilot with the optimum value of k' (Step 7).

$\eta_{p_D}(\tau)$ is obtained from Equation 2.14, noting that because of the impulsive nature of $m_G(\tau)$ the solution for $\eta_{p_D}(\tau)$ is, strictly, valid only for $\tau > \epsilon$.

Thus we obtain

$$\frac{\mu m_\eta}{i_B} . \eta_{p_D}(\tau) = e^{-5.\text{[illegible]}\tau} (-0.806 \sin 11.51\, \tau - 0.122 \cos 11.51\, \tau) \qquad (2.19)$$

Equation 2.19 is graphed in Fig.2. It will be seen that the required $\eta_{p_D}(\tau)$ can hardly be attained, if only because the initial $(\tau > \epsilon)$ amplitude is non-zero. However even if the pilot holds the stick quite fixed (Step 8), substitution of $k' = 9.53$ in Equation 2.15 et.seq. yields

$$\begin{matrix} w(\tau) \\ k' = 9.53 . \eta_p = 0, \end{matrix} = 1.225\, e^{-4.95\tau} \frac{\sin 11.17\tau}{11.17} \qquad (2.20)$$

which, as may be seen from the graph (Fig.2) is a close approximation to the desired response.

We conclude that a satisfactory approximation to the desired response is attained even if the pilot holds the stick quite fixed. The optimum autostabilization may, therefore, be regarded as satisfactory.

2.3 EXAMPLE 2. LONGITUDINAL SHORT-PERIOD RESPONSE TO A SHARP-EDGED GUST

(It is important to avoid the appearance of divergent integrals in the expression for the effort function. This, and the next example show how this may be accomplished even with specified inputs of infinite duration.)

It is assumed that at $\tau = 0$ the aircraft flies into a sharp-edged downgust of constant magnitude and infinite duration. The limited realism of this assumption should be appreciated - use of such a simplified representation leads to straightforward working, however, and there is no essential difficulty in extending the technique illustrated by this example to deal with more complicated forms of gusts.

If the initial change in incidence due to gust is $- W_o$, then application of the Laplace transform to Equations 2.1 and 2.2 yields, for the response in $W(\tau)$ of the basic aircraft.

$$(s - z_w)\, w(s) - \hat{q}(s) = -w_o \qquad (2.21)$$

$$-\left(\frac{m_{\dot{w}}}{\mu} s + m_w\right) w(s) + \left(\frac{i_B}{\mu} s - \frac{m_q}{\mu}\right) \hat{q}(s) = m(s) + \frac{m_{\dot{w}}}{\mu} . w_o \qquad (2.22)$$

$$\text{whence } \frac{w(s)}{w_o} = \frac{\left[-s + \dfrac{m_q + m_{\dot{w}}}{i_B} \right]}{s^2 + 2\,Rs + R^2 + J^2} \qquad (2.23)$$

with the same notation as the previous example for $m(s) = 0$ (i.e. no applied moment by the pilot or autostabilizer)

Using the same numerical data as Example 1. and applying the inverse Laplace transform to Equation 2.23 we obtain

$$\frac{w(\tau)}{w_o} = e^{-1.706\tau} \left(-\cos 11.51\,\tau + 0.645 \sin 11.51\tau\right) \qquad (2.24)$$

Equation 2.24 describes the response of the basic aircraft. The calculation now proceeds in a similar manner to Example 1. Note how in formulating the initial conditions for Equations 2.21 and 2.22 we have chosen the origin of w such that $w_{\tau \to \infty} = 0$.

In this way the appearance of divergent integrals in the expression for $\int_0^\infty \left[\eta_{p_D}(\tau)\right]^2 d\tau$ is avoided.

There is no further new point to be made by completing the example, so we pass on to Example 3. in which there is rather more difficulty in eliminating divergent integrals.

2.4. EXAMPLE 3. LONGITUDINAL SHORT-PERIOD RESPONSE TO A STEP DEFLECTION OF ELEVATOR

(This manoeuvre is of some importance as it may represent a stressing case. As we shall show, the steady-state response must be considered separately from the transient response due to the appearance of divergent integrals in the expression for $\int_0^\infty \left[\eta_{p_D}(\tau)\right]^2 d\tau$.

We write the equations of motion as

$$(D - z_w)\, w(\tau) - \hat{q}(\tau) = 0 \tag{2.1}$$

$$-\left(\frac{m_{\dot{w}}}{\mu} D + m_w\right) w(\tau) + \left(\frac{i_B}{\mu} D - \frac{m_q}{\mu}\right) \hat{q}(\tau) = m_\eta \cdot \eta_p(\tau) + m_\eta \cdot \Delta\eta(\tau) \tag{2.25}$$

where $\Delta\eta(\tau)$ represents the step deflection of the elevator,

$$\Delta\eta = 0 \text{ for } \tau < 0, \quad \Delta\eta = \delta\eta \quad \text{for } \tau > 0$$

and $\eta_p(\tau)$ is the additional elevator deflection due to the pilot.

Application of the Laplace transform to Equations 2.1, 2.25 yields for $\eta_p = 0$

$$(s - z_w)\, w(s) - q(s) = 0 \tag{2.3}$$

$$-\left(\frac{m_{\dot{w}}}{\mu} s + m_w\right) w(s) + \left(\frac{i_B}{\mu} s - \frac{m_q}{\mu}\right) \hat{q}(s) = \frac{m_\eta \cdot \delta_\eta}{s} \tag{2.26}$$

$$\therefore \quad w(s) = \frac{\frac{\mu}{i_B} \cdot m_\eta \cdot \delta\eta}{s\,(s^2 + 2Rs + R^2 + J^2)} \tag{2.27}$$

Applying the inverse Laplace transform to Equation 2.27 with the same notation as Example 1. we obtain for the basic response in $w(\tau)$,

$$w(\tau) = \frac{\mu\, m_\eta}{i_B} \cdot \delta_\eta \cdot \frac{1}{R^2 + J^2}\left[1 - e^{-R\tau}\left(\cos J\tau + \frac{R}{J}\sin J\tau\right)\right] \tag{2.28}$$

Using the numerical data of Example 1., we obtain,

$$w(\tau) = 9.13 \left[1 - e^{-1.706\tau} (\cos 11.51\ \tau + 0.1482 \sin 11.51\ \tau) \right] \quad (2.29)$$

$$= w_\infty - w_\infty \cdot e^{-1.706\tau} (\cos 11.51\ \tau + 0.1482 \sin 11.51\ \tau) \quad (2.30)$$

where w_∞ is the steady-state response in W

Now for the steady-state, Equations 2.1 and 2.25 become

$$- z_w \cdot w_\infty - \hat{q}_\infty = 0 \quad (2.31)$$

$$- m_{\dot{w}} w_\infty - \frac{m_q}{\mu} \cdot \hat{q}_\infty = m_\eta \cdot \delta_\eta \quad (2.32)$$

where $\hat{q}_\infty$ is the steady-state response in $\hat{q}$

Thus

$$w_\infty = \frac{\begin{vmatrix} 0 & -1 \\ m_\eta \cdot \delta\eta & -\frac{m_q}{\mu} \end{vmatrix}}{\frac{m_q z_w}{\mu} - m_w} = \frac{\mu\ m_\eta\ \delta\ \eta}{\mu\, m_w - m_q\ z_w} \quad (2.33)$$

(i.e. the steady state change of incidence is inversely proportional to the manoeuvre margin). Now re-writing Equation 2.25 in terms of the desired response,

$$m_\eta \cdot \eta_{p_D}(\tau) = -\left(\frac{m_{\dot{w}}}{\mu} D + m_w\right) w_D(\tau) + \left(\frac{i_B}{\mu} D - \frac{m_q}{\mu}\right) \hat{q}_D(\tau) - m_\eta \cdot \delta\ \eta \quad (2.34)$$

For $\int_0^\infty \left[\eta_{p_D}(\tau)\right]^2 d\tau$ to exist $\eta_{p_D}(\tau = \infty)$ must equal zero.

Thus w_∞ and δ_η must satisfy Equation 2.33

i.e. $w_{D\infty} = w_\infty$, $\hat{q}_{D\ \infty} = \hat{q}_\infty$

Hence, the steady-state response must be adjusted to the desired value by autostabilization or other means before attempting to improve the transient response.

Let us suppose this has been done, so that w_∞ in Equation 2.30 is equal to the desired steady-state incidence change $w_{D\infty}$. Thus (strictly) the manoeuvre margin has been fixed and any further autostabilization that may be introduced to improve the transient response must leave the value of the manoeuvre margin unchanged. Hence the derivatives contained in the expression for the manoeuvre margin are no longer at our disposal[1] and any autostabilization that we may wish to introduce must either consist of the variation of $m_{\dot{w}}$ or the effective introduction of derivatives such as $m_{\dot{q}}$, $m_{\ddot{q}}$. etc. In practice, however, it is unlikely that the steady-state response will be specified exactly, as we have assumed here, - it is more likely that the static margin only will be specified, leaving derivatives other than m_w at our disposal for autostabilization purposes. In these circumstances the desired response should be specified in such a manner that its steady state is zero (e.g. $D\,w_D(\tau)$ should be specified, rather than $w_D(\tau)$ as in this example.

Equation 2.29 is graphed in Fig.3. The basic response in $w(\tau)$ is seen to be markedly oscillatory with a large initial overshoot. We assume a desired response of the form

$$\frac{w_D(\tau)}{m_{\eta}.\delta_{\eta}} = 9.13\left[1 - e^{-5.0\tau}(\cos 11.51\tau + 0.1504 \sin 11.51\,\tau)\right] \qquad (2.34)$$

whence
$$\frac{Dw_D(\tau)}{m_{\eta}.\delta_{\eta}} = e^{-5.0\tau}(112.0 \sin 11.51\tau + 30.8 \cos 11.51\,\tau) \qquad (2.35)$$

From Equations 2.1, 2.34, 2.35,

$$\frac{\hat{q}_D(\tau)}{m_{\eta}.\delta_{\eta}} = 21.45 + e^{-5.0\tau}(109.84 \sin 11.51\tau + 9.35 \cos 11.51\tau) \qquad (2.36)$$

whence

$$\frac{D\hat{q}_D}{m\eta.\delta_\eta} = e^{-5.0\,\tau}(1,218.2 \cos 11.51\,\tau - 656.5 \sin 11.51\,\tau) \qquad (2.37)$$

We assume that the autostabilizer available is of such a type that an elevator deflection proportional to Dw may be produced. Then putting $h = \frac{m_{\dot{w}} \text{ with autostabilizer}}{m_{\dot{w}} \text{ basic}}$ we may solve Equation 2.25 for $\eta_{P_D}(\tau)$

Thus,

$$\frac{\mu}{i_B} \cdot \frac{1}{\delta_\eta} \cdot \eta_{p_D}(\tau) = e^{-5.0\tau}\left[(15.81 + 9.26\text{ h}) \cos 11.51\tau + (-760.9 + 33.7\text{ h}) \sin 11.51\,\tau\right]$$

+ a negligible constant term due to rounding-off errors (2.38)

Putting $T = 11.51\tau$ we obtain after some reduction

$$\left[\frac{\mu}{i_B} \cdot \frac{1}{\delta_\eta} \cdot \eta_{p_D}(T)\right]^2 = (2.93.6\text{ h} + 85.75\text{ h}^2)e^{-.868T}\cos^2 T$$
$$+ (-51,300\text{ h} + 1,136.0\text{ h}^2)\,e^{-.868T}\sin^2 T$$
$$+ (-6,473\text{ h} + 312.0\text{ h}^2)e^{-.868T}\sin 2T$$

+ terms not involving h (2.39)

Integrating Equation 2.39 between 0 and ∞, making use of the integral formulae of Equations 2.15, 2.16, 2.17 with $\varepsilon \to 0$ we obtain eventually

$$\int_0^\infty \left[\frac{\mu}{i_B} \cdot \frac{1}{\delta_\eta} \cdot \eta_{p_D}(T)\right]^2 = 710\text{ h}^2 - 27,525\text{ h}$$

+ terms not involving h (2.40)

For this integral to be stationary

$$h = h_{\text{optimum}} = \frac{27,525}{710 \times 2} = 19.6 \qquad (2.41)$$

Substituting this value of h in Equation 2.38, the elevator deflection demanded of the pilot with the optimum autostabilization, $\eta_{p_D}(\tau)$, is given by

$$\frac{\mu}{i_B}\,\frac{1}{\delta_\eta}\,\eta_{p_D}(\tau) = e^{-5.0\tau}\left[198.71 \cos 11.51\tau - 100.9 \sin 11.51\tau\right] \qquad (2.42)$$

Owing to the non-zero initial amplitude this deflection cannot be attained. However, proceeding to Step 8, we find that the stick-fixed response with the optimum autostabilization is described by

$$\frac{w(\tau)}{m_\eta\,\delta_\eta} = 9.13\left[1 - e^{-4.50\tau}\left(\cos 10.8\tau + 0.417 \sin 10.8\tau\right)\right] \qquad (2.43)$$

Equation 2.43 is graphed in Fig.4. It will be seen that it approximates well to the desired response. The optimum autostabilization may, therefore, be regarded as satisfactory.

2.5. EXAMPLE 4. SHORT-PERIOD LONGITUDINAL RESPONSE TO AN IMPULSIVE PITCHING MOMENT OF AN AIRCRAFT HAVING A NON-LINEAR VARIATION OF PITCHING MOMENT WITH INCIDENCE

(This example illustrates how the optimization procedure may be extended to deal with non-linear derivatives and serves as an introduction to the technique used in the succeeding chapters for the optimization of autostabilizer systems having non-linear characteristics.)

Let the static variation of pitching moment coefficient with incidence be of the form $C_m = A\alpha + B\alpha^3$, where A and B are constants. Then, for a conventional aircraft having the wing positioned near the C.G., we may allow for this non-linearity in the equations of motion simply by replacing the term $m_w . w(\tau)$ in Equation 2.2 by a cubic expression in $w(\tau)$. With this exception, we use the numerical data of Example 1.

The equations of motion become

$$(D - z_w)\, w(\tau) - \hat{q}(\tau) = 0 \qquad (2.1.)$$

$$-\frac{m_{\dot{w}}}{\mu} Dw(\tau) + m_1 w(\tau) + m_3 . w^3(\tau) + \left(\frac{i_B}{\mu} D - \frac{m_q}{\mu}\right) \hat{q}(\tau) = m(\tau) \qquad (2.44)$$

For the same specified input as Example 1, and with the same desired response in w

$$w_D(\tau) = 0.1064\, e^{-5.0\tau} \sin 11.51\tau$$

Choosing $\frac{\mu}{i_B} m_1 = 132.3$ and $\frac{\mu}{i_B} m_3 = -13,230.0$

the static variation of C_m with α is of the form shown in Fig.5. (This form is chosen in order to represent the characteristics of 'pitch-up'). The elevator deflection demanded of the pilot if the desired response is to be attained exactly is given by

$$\frac{\mu m_\eta}{i_B} \cdot \eta_{p_D}(\tau) = \frac{-m_{\dot{w}}}{i_B} . Dw(\tau) + m_1 . w_D(\tau) + m_3 . \left[w_D(\tau) \right]^3 + D\hat{q}_D(\tau) - k' \frac{m_q}{i_B} . \hat{q}_D(\tau) - \frac{\mu m_G}{i_B}(\tau) \qquad (2.45)$$

Substituting the numerical data of Example 1, we obtain

$$\frac{\mu m}{i_B}\eta . \ {}^{\eta}p_D(\tau) = e^{-0.434T}\left[(1.23 \sin T - 8.982 \cos T) + k'(-0.2135 \sin T + 0.930 \cos T) \right] - 15.98\, e^{-1.302T} \sin^3 T - \frac{\mu}{i_B} . m_G(\tau) \qquad (2.46)$$

where $T = 11.51\,\tau$

As in Example 1, our criterion for the optimum k' is that

$$\int_{\epsilon}^{\infty} \left[\frac{\mu m}{i_B}\eta . {}^{\eta}p_D(T) \right]^2 dT$$ is to be a minimum, with ϵ as small as we please.

Squaring Equation 2.46 we obtain after some reduction

$$\left[\frac{\mu m}{i_B}\eta . \ {}^{\eta}p_D(T)\right]^2 = (e^{-0.434T}\sin T)^2 \ (-0.8205\, k'^2 + 16.176\, k') + e^{-0.868T}(0.866\, k'^2 - 16.7\, k') + e^{-0.868T}\sin 2T\,(3.06\, k' - 0.1984\, k'^2) + 2.96\, k'\, e^{-1.736T}\sin^3 T \cos T + \text{terms not involving } k' \qquad (2.47)$$

With negligible error, for very small ϵ,

$$\int_{\epsilon}^{\infty} e^{-pt}\, dt = \frac{1}{p}, \qquad \int_{\epsilon}^{\infty} e^{-pt} \sin \omega t\, dt = \frac{\omega}{p^2 + \omega^2} \qquad (2.15,16)$$

$\int_{\epsilon\to 0}^{\infty} (e^{-pt}.\sin t)^{2n}\, dt$ may be evaluated by means of a table of Laplace transforms or may be read from the graphs (Figs. 18 & 19). The remaining integral is evaluated by integration by parts, thus:

$$\int_{\epsilon}^{\infty} e^{4at}.\sin^3 t.\cos t.\, dt = \left[\frac{e^{4at}}{4}.\sin^4 t \right]_{\epsilon}^{\infty} - \int_{\epsilon}^{\infty} 4a.\, e^{4at}.\, \frac{\sin^4 t}{4}\, dt \qquad (2.48)$$

$$= -a \int_{\epsilon}^{\infty} (e^{at} \sin t)^4\, dt \qquad (2.49)$$

which last integral is evaluated as described above.

Integrating Equation 2.47, making use of the above integrals, we obtain after reduction

$$\int_{\epsilon}^{\infty} \left[\frac{\mu m_\eta}{i_B} . \eta_{p_D}(T) \right] = 0.5255\, k'^2 - 10.785\, k' + \text{terms not involving } k' \qquad (2.50)$$

whence $k'_{optimum} = \frac{10.785}{2 \times 0.525} = 10.25$

Substituting this value of k' in Equation 2.43 we find that the elevator deflection demanded of the pilot if the desired response is to be attained exactly is given by,

$$\frac{\mu m_\eta}{i_B} . \eta_{p_D}(T) = e^{-0.434T}(0.558 \cos T - 0.955 \sin T) - 15.98\, e^{-1.302T} \sin^3 T \qquad (2.51)$$

This is graphed in Fig.5. The required $\eta_{p_D}(T)$ can hardly be attained, mainly because the initial $(\tau > \epsilon)$ amplitude is non-zero. It is necessary, therefore, to proceed to step 8 of the optimization procedure.

Step 8 presents more difficulty than hitherto due to the non-linearity of the equations of motion. The procedure adopted is as follows.

Equations 2.1, 2.44 are combined to give

$$\left[D^2 + \left(-\frac{m_{\dot w}}{i_B} - \frac{m_q}{i_B} k' - z_w\right) D + \frac{\mu m}{i_B} + \frac{m_q}{i_B} k' z_w \right] w(\tau) + \frac{\mu m}{i_B^3} w^3(\tau) = \frac{\mu m_G}{i_B}(\tau) \qquad (2.52)$$

where $m_G(\tau)$ is the (non-dimensionalized) applied impulsive moment

Since $\int_0^{\epsilon} m_G(\tau)\, d\tau = 10^{-3}$, $\int_0^{\epsilon} \frac{\mu m_G(\tau)}{i_B}\, d\tau = 1.225 \qquad (2.53)$

Substituting the appropriate numerical data in Equation 2.52 we obtain

$$\left[D^2 + 10.44\, D + 150.6 \right] w(\tau) - 13{,}230\, w^3(\tau) = 1.225 \perp \qquad (2.54)$$

where $\perp$ denotes a unit impulse

Equation 2.54 is solved by Tustin's regression equation technique (see Ref. 8) - an extension of Cardinal Spectrum Analysis - using a step interval of $\frac{1}{10}$ of an airsecond and replacing the unit impulse by a triangular pulse of equal strength (i.e. 'area') and of base $\frac{1}{10}$ airsec. The solution is graphed in Fig.6. Owing to the limitations of finite-difference techniques such as that employed, the initial value is in error as (to a lesser extent) is the second value. Allowance has been made for this in drawing the curve describing the solution of Equation 2.54. These errors could be reduced (but never eliminated) by taking a smaller step-interval.

Comparison of the desired response (Fig.1) with that graphed in Fig.6, shows good agreement between the two. We conclude therefore that the optimum autostabilization is satisfactory.

2.6. Discussion of the Optimization Procedure

Step 1.

The purpose of this step is merely to confirm the necessity for autostabilization. For non-linear cases the calculation required for this step may be considerable and if it is reasonably certain that the basic response is unsatisfactory this step may be omitted. Thus in Example 4 since the non-linearity is mild for small w it is reasonable to suppose that the basic response of the aircraft will be somewhat similar to the basic response of Example 1., and Step 1 may safely be omitted.

Step 2.

There should be little difficulty in specifying the form of the desired response - though there may be considerable difficulty in assessing the merit of any particular form of response chosen. For example, the author chose the form of the desired response in Example 1.

$$w_D(\tau) = 0.1064\ e^{-5.0\tau} \sin 11.51\ \tau$$

simply because it is smooth and highly damped and therefore likely to be pleasing to the pilot. Many other similar forms of response would have been equally acceptable and it is not easy to formulate a numerical criterion for the relative merits of the possible response forms.

It is true that (military) aircraft specifications frequently demand that certain response and stability criteria should be met -

for example, a minimum value of the logarithmic decrement of the longitudinal and lateral oscillations is commonly specified. Criteria of this kind are chosen on the basis of pilots experience and preferences, (see for example Ref. 5) but although these criteria define the boundaries between acceptable and unacceptable response characteristics they provide little guidance on the relative merits of various acceptable responses. Optimum forms of response are commonly specified for servomechanisms (see Ref. 10) usually forms which minimize a certain function of output error (E) such as

$$\int_0^\infty E^2 \, d\tau$$

- but owing to the large number of freedoms possessed by an aircraft and the wide range of flight conditions under which it may operate it hardly seems practicable to extend this concept of optimum response to aircraft flight. Certainly any attempt to do so would be beyond the scope of this present report.

Whilst this difficulty of assessing the merit of a given response should not be overlooked, we believe that it is of a philosophical rather than of practical importance. For any given aircraft one will always be able to suggest a suitable form for the desired response, even though one may be unable to define the optimum response.

Step 3.

It is obviously desirable that the type of autostabilization chosen should be capable of attaining the desired response without making excessive demands on the pilot. Otherwise, effort will have been wasted in fruitless calculation. For linear systems the time vector method of presentation provides an excellent means of predicting the probable effects of various types of autostabilization. The vector polygons for the short period longitudinal oscillation of the aircraft of Examples 1 to 3 are given in Fig.31. From inspection of these polygons one can deduce the type of autostabilization most likely to achieve the high damping associated with the desired response. Although for the short period longitudinal oscillation one could deduce as much from the coefficients of the auxiliary equation

$$\lambda^2 - \left(\frac{m_q}{i_B} + \frac{m_{\dot{w}}}{i_B} + z_w\right)\lambda - \frac{\mu m_w}{i_B} + m_q z_w = 0$$

for the more complicated lateral oscillation this is hardly possible and the use of time vector presentation is very desirable if an intelligent approach is to be made to the problem of choosing the type of autostabilization most likely to achieve the desired response.

Step 4.

For general (i.e. non-optimum) autostabilization this assumption must be regarded as a mathematical artifice rather than an assertion of what is physically feasible. A check on the validity of this assumption with the optimum adjustment of the autostabilization is provided by Step 7.

Step 5.

A good discussion of the effects of pilot effort on aircraft response is contained in Ref. 4 which see. This supports our view that the purpose of the autostabilizer is to relieve the strain on the pilot so that more of his attention may be devoted to tasks such as navigation, weapon aiming, etc. and so that he may have greater reserves available for emergencies. Unless this view is accepted it is hardly possible to optimize the autostabilizer system of a piloted aircraft as such, and one is reduced to improving the response characteristics of the (same) aircraft in the (supposed) absence of a pilot. As shown in the Introduction this procedure may be somewhat unrealistic. (An optimization procedure for pilotless aircraft is developed later in this report.)

The choice of effort function must be made on empirical grounds, as there is insufficient data at present available on the psychological and physical strain experienced by a pilot in attempting a given task. In Example 1 to 6 an 'integrated displacement-squared' effort function of the form $\int_0^\infty \left[\eta_{P_D}(\tau)\right]^2 d\tau$ was employed. Although the pilot actions demanded to attain the desired response were in each example physically unattainable, with the optimum autostabilization the amplitude of η_{p_D} in each example was so small that the stick-fixed response was very close to the desired response. This state of affairs is more likely to be achieved by the use of an effort function which is a function of control deflection rather than by the use of a more refined effort function dependent on time derivatives of the pilot's control deflections. For this reason the writer prefers to use simple 'displacement' effort functions rather than more refined types. Use of a more refined type of effort function would result in a more physically feasible η_{p_D} but the results of the pilot failing to achieve this would, in general, be less satisfactory.

However, many other types of effort functions may be employed. Duddy (in Ref. 4) suggests that the best control system (autostabilizer system in the present context) would be that which demands the simplest transfer function of the pilot. This criterion is much less easy to use as a basis for optimizing a given type of autostabilizer system since there is great difficulty in formulating a quantitative criterion for the complexity of transfer functions.

Step 6.

The minimization of the effort functions of Examples 1 to 6 has been performed by the standard procedure of the differential calculus. This will generally be possible(as we later demonstrate) even if the desired response is specified in a non-analytic manner, provided the effort function is of an analytic form, e.g. $\int_0^\infty \left[\eta_{p_D}(\tau)\right]^2 d\tau$ rather than, say $\int_0^\infty \left|\eta_{p_D}(\tau)\right| d\tau$. With a non-analytic effort function recourse may have to be made to iterative methods of minimization.

Step 7.

Although the main purpose of this step is to examine the feasibility of the demanded pilot action, it also checks that the stationary value of the effort function obtained in Step 6 is a minimum and not a maximum.

In Examples 1 to 6 inspection of $\eta_{p_D}(\tau)$ reveals the amplitude to be so small that one might well surmise that the effect of the pilot taking no action whatsoever would be to cause only a slight divergence from the desired response. However it is always desirable to prove this by proceeding to Step 8, particularly so for non-linear systems, where a pilot input of small amplitude may produce an unexpectedly large change in the aircraft response.

Step 8.

It is worthy of remark that, for non-linear systems, Step 8 (together with Step 1) will probably be the most tedious part of the calculation.

More general comments on the procedure as a whole and comparisons with published work are given towards the end of this report.

CHAPTER 3.

3.1. OPTIMIZATION OF SOME NON-LINEAR AUTOSTABILIZER SYSTEMS

INTRODUCTION

In general, the amplitude of the control surface deflection generated by the autostabilizer system will be limited. The limits may be chosen deliveratley so as to avoid catastrophic divergence in the event of an autostabilizer run-away, or may arise through limitations of available jack effort, or through installation difficulties. Provided the control surface deflection required to attain the desired response does not exceed the limiting value the methods of the previous chapter may be applied, and the limits and the non-linearities arising therefrom need not be taken into account.

In this chapter we show how the general optimization procedure for piloted aircraft may be used to obtain the optimum values of the adjustable parameters of such a 'limited' autostabilizer system for the more general case when these non-linearities cannot be excluded from the analysis. The procedure is applicable both to 'limited' or 'saturable' autostabilizer systems of the type described above, and to 'flicker' or 'flip-flop' autostabilizer systems in which the magnitude of the control surface deflection is constant, its sign, at any instant, being the same as that of some preselected response parameter. In the latter case, we assume that the system parameter to be optimized is the magnitude of the control surface deflection.

We adopt the term 'Autostabilizer Characteristic' to denote the graph of the autostabilizer output (i.e. control surface deflection) against the input signal to the autostabilizer. It is first necessary to derive a family of continuous characteristic curves which approximate to the discontinuous characteristic of the actual autostabilizer. The seventh-power polynomial approximation presented in the following section has been found to give solutions of good accuracy without introducing too great complication into the calculation required for the optimization procedure.

3.2 To determine the Coefficients of the Polynomial Approximation to the Autostabilizer Characteristic.

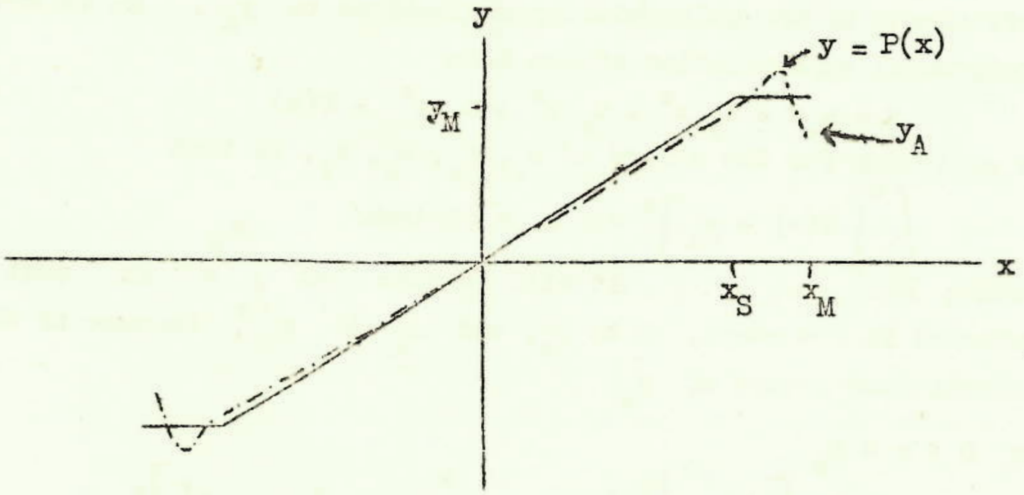


FIG. 3.1.

In Fig. 3.1.

y = the control surface deflection due to the autostabilizer

x = the input signal to the autostabilizer (Thus for, say, m_q autostabilization, y would be elevator deflection, and x, rate of pitch)

y_A is the actual autostabilizer characteristic,

y_M is the limiting value of y,

x_M is the maximum value of the input signal that need be considered in any particular example, (Thus for m_q autostabilization, x_M would be the maximum value of $\hat{q}_D(\tau)$)

x_s is the 'saturation' x ,

If the maximum amplitude of control surface deflection is limited to prevent catastrophic divergence ensuing from autostabilizer failure and run-away, then y_M is fixed and the only parameter of y_A at our disposal is x_s . For 'flicker' or 'flip-flop' autostabilization we assume that the parameter at our disposal is y_M.

In this section we seek to obtain expressions for the coefficients of the polynomial approximations to y_A . We select a polynomial approximation of the form

$$y = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 = P(x) \tag{3.1}$$

Our criterion for the choice of a_1, a_3, a_5, a_7, is that

$$\int_0^\infty \left[P(x) - y_A \right]^2 dx = \text{a minimum} \tag{3.2}$$

Putting $P(x) - y_A = \epsilon$, it will be seen that $\int_0^{x_M} \epsilon^2 \, dx$ must be evaluated in two steps, 0 to x_s, and x_s to x_M, because of the discontinuous nature of y_A.

For $0 \leq x \leq x_s$

$$\epsilon^2 = \left[(a_1 - \frac{y_M}{x_s}) x + a_3 x^3 + a_5 x^5 + a_7 x^7 \right]^2 \tag{3.3}$$

whence

$$\int_0^\infty \epsilon^2 \, dx = \frac{1}{3} (a_1 - \frac{y_M}{x_s})^2 x_s^3 + \frac{2a_3}{5}(a_1 - \frac{y_M}{x_s}) x_s^5 + \left| \frac{2a_5}{7} (a_1 - \frac{y_M}{x_s}) + \frac{a_3^2}{7} \right| x_s^7$$

$$+ \frac{2}{9} \left[a_7 (a_1 - \frac{y_M}{x_s}) + a_3 a_5 \right| x_s^9 + \left| \frac{2}{11} a_3 a_7 + \frac{a_5^2}{11} \right] x_s^{11}$$

$$+ \frac{2a_5 a_7}{13} x_s^{13} + \frac{a_7^2}{15} x_s^{15} \tag{3.4}$$

For $x_s < x < x_M$

$$\epsilon^2 = \left[P(x) \right]^2 + y_M^2 - 2y_M P(x) \qquad (3.5)$$

whence

$$\int_{x_s}^{x_M} \epsilon^2 \, dx = y_M^2 x_M - y_M a_1 x_M^2 - \frac{1}{2} y_M a_3 x_M^4 - \frac{1}{3} y_M a_5 x_M^6 - \frac{1}{4} y_M a_7 x_M^8$$
$$+ \frac{a_1^2}{3} x_M^3 + \frac{2a_1 a_3}{5} x_M^5 + \left[\frac{2a_1 a_5 + a_3^2}{7} \right] x_M^7 + \frac{2}{9} \left[a_1 a_7 + a_3 a_5 \right] x_M^9$$
$$+ \frac{2a_3 a_7 + a_5^2}{11} x_M^{11} + \frac{2a_5 a_7}{13} x_M^{13} + \frac{a_7^2}{15} x_M^{15}$$
$$- y_M^2 x_s + y_M a_1 x_s^2 + \frac{1}{2} y_M a_3 x_s^4 + \frac{1}{3} y_M a_5 x_s^6 + \frac{1}{4} y_M a_7 x_s^8$$
$$- \frac{a_1^2 x_s^3}{3} - \frac{2a_1 a_3}{5} x_s^5 - \left[\frac{2a_1 a_5 + a_3^2}{7} \right] x_s^7 - \frac{2}{9} \left[a_1 a_7 + a_3 a_5 \right] x_s^9$$
$$- \left[\frac{2a_3 a_7 + a_5^2}{11} \right] x_s^{11} - \frac{2a_5 a_7 x_s^{13}}{13} - \frac{a_7^2}{15} x_s^{15} \qquad (3.6)$$

Summing Equations 3.4 and 3.6 we obtain

$$I = \int_0^{x_M} \epsilon^2 \, dx = - \frac{y_m}{x_s} \left[\frac{2}{3} a_1 x_s^3 + \frac{2}{5} a_3 x_s^5 + \frac{2}{7} a_5 x_s^7 + \frac{2}{9} a_7 x_s^9 \right] + y_m^2 (x_M - x_s)$$
$$- y_M \left[a_1 x_M^2 + \frac{1}{2} a_3 x_M^4 + \frac{1}{3} a_5 x_M^6 + \frac{1}{4} a_7 x_M^8 \right]$$
$$+ y_M \left[a_1 x_s^2 + \frac{1}{2} a_3 x_s^4 + \frac{1}{3} a_5 x_s^6 + \frac{1}{4} a_7 x_s^8 \right]$$
$$+ \frac{a_1^2}{3} x_M^3 + \frac{2a_1 a_3}{5} x_M^5 + \left[\frac{2a_1 a_5 + a_3^2}{7} \right] x_M^7 + \frac{2}{9} \left[a_1 a_7 + a_3 a_5 \right] x_M^9$$
$$+ \left[\frac{2a_3 a_7 + a_5^2}{11} \right] x_M^{11} + \frac{2a_5 a_7}{13} x_M^{13} + \frac{a_7^2 x_M^{15}}{15} \qquad (3.7)$$

A necessary condition for Equation 3.2. to be satisfied is that

$$\frac{\partial I}{\partial a_1} = \frac{\partial I}{\partial a_3} = \frac{\partial I}{\partial a_5} = \frac{\partial I}{\partial a_7} = 0 \qquad (3.8)$$

Differentiating Equation 3.7 we obtain

$$\frac{\partial I}{\partial a_1} = - y_M x_M^2 + y_M x_s^2 + \frac{2a_1}{3} x_M^3 + \frac{2a_3}{5} x_M^5 + \frac{2a_5}{7} x_M^7 + \frac{2a_7}{9} x_M^9 - \frac{2}{3} y_M x_s^2 \qquad (3.9)$$

$$\frac{\partial I}{\partial a_3} = - \frac{1}{2} y_M x_M^4 + \frac{1}{2} y_M x_s^4 + \frac{2a_1}{5} x_M^5 + \frac{2a_3}{7} x_M^7 + \frac{2a_5}{9} x_M^9 + \frac{2a_7}{11} x_M^{11} - \frac{2}{5} y_M x_s^4 \qquad (3.10)$$

$$\frac{\partial I}{\partial a_5} = - \frac{1}{3} y_M x_M^6 + \frac{1}{3} y_M x_s^6 + \frac{2a_1}{7} x_M^7 + \frac{2a_3}{9} x_M^9 + \frac{2a_5}{11} x_M^{11} + \frac{2a_7}{13} x_M^{13} - \frac{2}{7} y_M x_s^6 \qquad (3.11)$$

$$\frac{\partial I}{\partial a_7} = - \frac{1}{4} y_M x_M^8 + \frac{1}{4} y_M x_s^8 + \frac{2a_1}{9} x_M^9 + \frac{2a_3}{11} x_M^{11} + \frac{2a_5}{13} x_M^{13} + \frac{2a_7}{15} x_M^{15} - \frac{2}{9} y_M x_s^8 \qquad (3.12)$$

For stationary I, the following matrix equation results.

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \frac{1}{9} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \frac{1}{11} \\ \frac{1}{7} & \frac{1}{9} & \frac{1}{11} & \frac{1}{13} \\ \frac{1}{9} & \frac{1}{11} & \frac{1}{13} & \frac{1}{15} \end{bmatrix} \begin{bmatrix} a_1 x_M \\ a_3 x_M^3 \\ a_5 x_M^5 \\ a_7 x_M^7 \end{bmatrix} = y_M \begin{bmatrix} \frac{1}{2} - \frac{k^2}{6} \\ \frac{1}{4} - \frac{k^4}{20} \\ \frac{1}{6} - \frac{k^6}{42} \\ \frac{1}{8} - \frac{k^8}{72} \end{bmatrix} \qquad (3.13)$$

$$\text{where } k = \frac{x_s}{x_M} \qquad (3.14)$$

This equation may be generalized for all x_M and y_M by putting $\frac{a_1 x_M}{y_M} = A_1$, $\frac{a_3 x_M^3}{y_M} = A_3$, etc. The solutions for A_1, A_3, etc. may be written in the form

$$\begin{aligned} A_1 &= P_0 + P_2 k^2 + P_4 k^4 + P_6 k^6 + P_8 k^8 \\ A_3 &= Q_0 + Q_2 k^2 + \dots\dots\dots + Q_8 k^8 \\ A_5 &= R_0 + \dots\dots\dots\dots\dots + R_8 k^8 \\ A_7 &= S_0 + \dots\dots\dots\dots\dots + S_8 k^8 \end{aligned} \qquad (3.15)$$

The coefficients P_0 , P_2 , ...S_6 , S_8 are given in Table I both as fractions and as numbers correct to 4 decimal places. It will be observed that the determinants required for the solution of Equation 3.13 by Cramer's rule are ill-conditioned -hence the elements of these determinants are left as fractions throughout the solution of Equation 3.13, only the final solutions for the coefficients P_0, P_2, ... S_6, S_8 being converted to decimal form.

Graphs of $y = A_1 x + A_3 x^3 + A_5 x^5 + A_7 x^7$ are plotted in Figs. 7-16 for k = 0, .1, .2, .3, .4, .5, .6, .7, .8, .9 It will be seen that except for the lowest ks the polynomial approximation to the exact autostabilizer characteristic is very close and we believe that the accuracy of this approximation is sufficient for most practical calculations. We later give an example for the $k = 0$ case (flicker autostabilization), for which the approximation is least accurate, the results of which support this view.

3.3 Example 5. Longitudinal Short-Period Response to an Impulsive Pitching Moment with Flicker m_q Autostabilization

(This example illustrates how a flicker autostabilizer system may be optimized using the same type of effort function used in the previous examples.)

We shall employ the numerical data of Example 1 and the same magnitude of the applied impulsive pitching moment as in Example 1. The non-dimensionalized equations of motion may be written as

$$(D - z_w)\, w(\tau) - \hat{q}\,(\tau) = 0 \qquad (2.1)$$

$$-\left(\frac{m_{\dot{w}}}{\mu} D + m_w\right) w\,(\tau) + \left(\frac{i_B}{\mu} D - \frac{m_q}{\mu}\right) \hat{q}\,(\tau) = m_G(\tau) + m_\eta .\ \eta_s(\tau) + m_\eta .\ \eta_p(\tau) \qquad (3.16)$$

where $m_G(\tau)$ is the impulsive moment

η_s is the elevator deflection due to the autostabilizer

η_p is the elevator deflection due to the pilot

From Example 1 the basic response in $w(\tau)$ is known to be

$$w(\tau) = 0.1064\, e^{-1.706\tau} \sin 11.51\, \tau \tag{2.8}$$

We assume a similar desired response to Example 1, i.e.

$$w_D(\tau) = 0.1064\, e^{-5.0\tau} \sin 11.51\tau \tag{2.9}$$

The associated responses in $Dw_D(\tau)$, $D\hat{q}_D(\tau)$ & $\hat{q}_D(\tau)$ are as given in Equations 2.10, 2.11, 2.12. In particular we have $\left[\hat{q}_D(\tau) = e^{-5.0\tau} (-0.284 \sin 11.51\tau + 1.225 \cos 11.51\, \tau).\right]$

By taking the maximum value of the desired response in $\hat{q}$, $\hat{q}_{D\ max}$, as 1.4 (the exact value is 1.225 but this is not critical)[1] the coefficients of the polynomial approximation to the autostabilizer characteristic are given by

$$\eta_s = \eta_F \left[5.3883\ \frac{\hat{q}_D}{1.4} - 19.7388 \left(\frac{\hat{q}_D}{1.4}\right)^3 + 30.7925\left(\frac{\hat{q}_D}{1.4}\right)^5 - 15.7104 \left(\frac{\hat{q}_D}{1.4}\right)^7\right] \tag{3.17}$$

where η_F = the amplitude of the elevator deflection generated by the autostabilizer

η_F is the system parameter at our disposal for optimization purposes. For $\tau > \epsilon$ (ϵ is the duration of the impulsive moment) substitution of the numerical data in Equation 3.16 yields, for $m_\eta = -0.205$, and for $T = 11.51\tau$

$$\begin{aligned}\frac{\mu\, m_\eta}{i_B} \cdot \eta_{p_D}(T) &= 0.301\, e^{-0.434T} (-0.532 \sin T + 1.225 \cos T) \\ &+ 132.3 \times 0.1064\, e^{-0.434T} \sin T \\ &+ e^{-0.434T} (-12.68 \sin T - 9.385 \cos T) \\ &+ 0.760\, e^{-0.434T} (-0.284 \sin T + 1.225 \cos T) \\ &+ 251.0\eta_F \left[5.3833\left(\frac{\hat{q}_D}{1.4}\right) - 19.7388\left(\frac{\hat{q}_D}{1.4}\right)^3 \right. \\ &\left. + 30.7925\left(\frac{\hat{q}_D}{1.4}\right)^5 - 15.7104 \left(\frac{\hat{q}_D}{1.4}\right)^7\right]\end{aligned} \tag{3.18}$$

$$\begin{aligned}&= e^{-0.434T}(1.0165 \sin T - 8.052 \cos T) \\ &+ 251.0\, \eta_F \left[5.3833\, Q - 19.7388\, Q^3 + 30.7925\, Q^5 - 15.7104\, Q^7\right]\end{aligned} \tag{3.19}$$

where $$Q = e^{-0.434T}(0.875 \cos T - 0.2039 \sin T) \qquad (3.20)$$

whence,

$$-\eta_{P_D}(T) = -0.003387\, e^{-0.434T} \sin T + 0.0366625\, P_1 - \eta_F \left[5.3833 \times (0.898\, P_1) - 19.7388 \times (0.898\, P_1)^3 + 30.7925 \times (0.898\, P_1)^5 - 15.7104 \times (0.898\, P)^7\right] \qquad (3.21)$$

with $$P_1 = \frac{-Q}{0.898} = e^{-0.434T} \sin(T - 77.05^\circ) \qquad (3.22)$$

Squaring Equation 3.21 we obtain

$$\left[\eta_{P_D}(T)\right]^2 = \eta_F^2 \,(23.3695\, P_1^2 - 138.2968\, P_1^4 + 378.6333\, P_1^6 - 586.6059\, P_1^8 + 536.0218\, P_1^{10} - 266.7702\, P_1^{12} + 54.9029\, P_1^{14})$$
$$+ \eta_F\,(-0.158493\, P_1^2 + 0.470428\, P_1^4 - 0.592057\, P_1^6 + 0.2436978\, P_1^8 + 0.16375\, P_1\, e^{-0.434T} \sin T - 0.04843\, P_1^3\, e^{-0.434T} \sin T + 0.060978\, P_1^5\, e^{-.434T} \sin T - 0.0250995\, P_1^7\, e^{-0.434T} \sin T) \qquad (3.23)$$
$$+ \text{ terms not involving } \eta_F$$

To obtain the effort function $\int_0^\infty \left[\eta_{P_D}(T)\right]^2 dT$ it is necessary to evaluate integrals of the form $\int_0^\infty P_1^{2n}\, dT$, and of the form $\int_0^\infty P_1^{2n-1} \cdot e^{-0.434T} \sin T \, dT$. In order to avoid too great a digression at this point, description of the evaluation of these integrals is held over to the next section of this chapter. Suffice to state that general formulae and graphs are given therein for the evaluation of integrals of the forms

$$\int_0^\infty P^{2n}\, dT \quad \text{and} \quad \int_0^\infty P^{2n-1}\, e^{aT} \sin T\, dT \quad \text{where}$$

$P = e^{aT} \sin(T + \beta)$, a and β being constants.

Using the results of the next section with $P = P_1$ we obtain,

$$\int_0^\infty P_1^2\, dT = 1.097, \quad \int_0^\infty P_1^4\, dT = 0.908, \quad \int_0^\infty P_1^6\, dT = 0.852,$$
$$\int_0^\infty P_1^8\, dT = 0.807, \quad \int_0^\infty P_1^{10}\, dT = 0.774, \quad \int_0^\infty P_1^{12}\, dT = 0.737$$
$$\int_0^\infty P_1^{14}\, dT = 0.706,$$

$$\int_0^\infty P_1 e^{-0.434T} \sin T \, d\, T = 0.2461, \quad \int_0^\infty P_1^{\,3} .e^{-0.434T} \sin T \, d\, T = 0.366,$$

$$\int_0^\infty P_1^{\,5} .e^{-0.434T} \sin T \, d\, T = 0.4105, \quad \int_0^\infty P_1^{\,7} .e^{-0.434T} \sin T \, d\, T = 0.422$$

Integrating Equation 3.23, making use of the above integrals, we obtain after some reduction,

$$\int_0^\infty \left[\eta_{p_D}(\tau)\right]^2 \, d\, T = 2.3522 \; \eta_F^2 - 0.54959 \; \eta_F \qquad (3.24)$$

$$+ \text{ terms not involving } \eta_F$$

whence

$$\eta_{F \text{ optimum}} = \frac{0.54959 \times 57.296}{2 \times 2.3522} = +0.67^\circ \qquad (3.25)$$

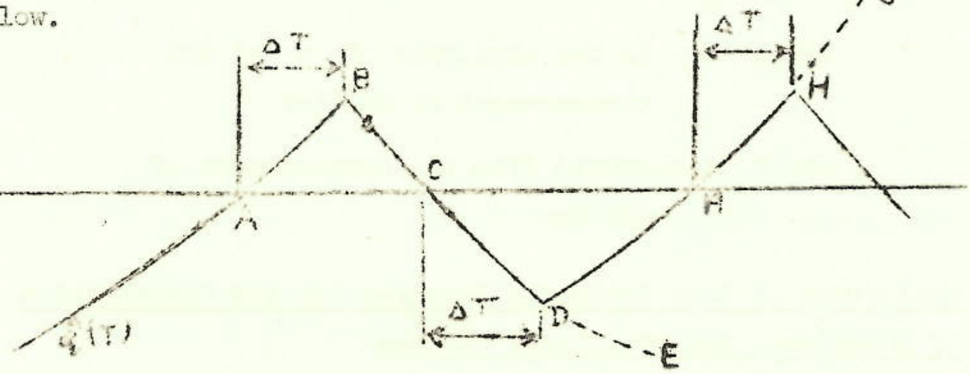
(The positive sign indicates that the sign of η_s at any instant is the same as that of $\hat{q}$.)

Since the total elevator deflection, $\eta_{p_D} + \eta_s$, must be of a smooth nature in order to obtain the desired response exactly, and since $\eta_s(\tau)$ is discontinuous, it follows that $\eta_{p_D}(\tau)$ must also be discontinuous. The pilot will certainly be unable to provide such a discontinuous $\eta_{p_D}(\tau)$, so we may proceed at once to Step 8 of the optimization procedure, without actually calculating the time history of η_{p_D} , with the optimum adjustment of η_F.

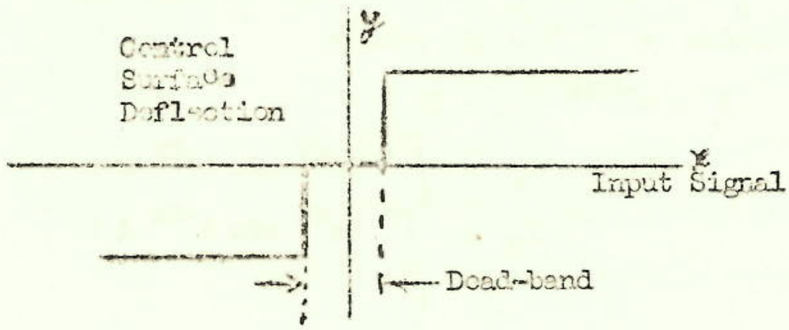
The stick-fixed response for $\eta_F = 0.67^\circ$ and with the exact autostabilizer characteristic (not the polynomial approximation) has been calculated by piecewise application of standard linear response theory, the 'pieces' being the intervals between successive zeroes of $\hat{q}(\tau)$. The resulting time histories of $w(\tau)$, and $\hat{q}(\tau)$ are plotted in Fig.17. It will be seen that the response in $w(\tau)$ approximates well to the desired response and the optimum autostabilization may therefore be regarded as satisfactory.

It will be observed that the solution for $\hat{q}$ 'ends' at the second zero of $\hat{q}\,(\tau)$. This is a familiar phenomenon in discontinuous automatic control systems may be explained as follows.

Due to the presence of (unavoidable) time lags in the autostabilizer circuit 'switching' of the elevator does not occur until a short time ΔT <u>after</u> $\hat{q}$ (T) changes sign. Hence the graph of $\hat{q}$ (T) against T at the second zero of $\hat{q}$ is of the form drawn below.



Switching occurs at the point B, when the $\hat{q}$ (T) graph commences to follow the path BCDE. But switching in the opposite sense occurs at point D, whereupon $\hat{q}$ (T) commences to follow the path DFHG, which it does as far as the next switching point H. The conditions at H are similar to those at B and so the cyclic variation of $\hat{q}$ is repeated ad infinitum. This phenomenon is known as 'chattering' and, for this example, is of theoretical rather than practical interest since an exact flicker characteristic is not practically attainable, and the presence of unavoidable imperfections such as dead-bands (see below) in the autostabilizer characteristic generally obviates chattering.



The time history of $w(\tau)$ with chattering in $\hat{q}$ (T), and η(T), may be found from Equation 2.1, assuming that the chattering amplitude of $\hat{q}$ is negligible in comparison with $w(\tau)$, and Dw (τ). With this assumption Equation 2.1 becomes

$$(D - z_w)\ w\ (\tau) = 0 \tag{3.26}$$

This has the solution

$$w\ (\tau) = w_c \cdot e^{+z_w \cdot \tau} \tag{3.27}$$

where w_c is the amplitude of w at the commencement of chatter

and τ is measured from the commencement of chatter

3.4. Evaluation of Some Integrals Required for the Optimization of Non-linear Autostabilizer Systems

As demonstrated in Example 5 we require to evaluate integrals of two kinds, (i) $\int_0^\infty P^{2n}\ d\,T$, and (ii) $\int_0^\infty P^{2n-1}\ e^{aT} \sin T\ d\,T$, where $P = e^{aT} \sin (T + \beta)$

It will be found that integrals of these kinds are frequently required when optimizing non-linear systems in which the non-linearity is expressible as a finite power series in some response parameter and it is convenient to evaluate these integrals once and for all for a range of a, β, & n rather than separately for each example.

Evaluation of Integrals of the First Kind

Denoting these by I_1, we have

$$I_1 = \int_0^\infty P^{2n}\ d\,T = \int_0^\infty e^{2naT} \cdot \sin^{2n} (T + \beta)\ d\,T \tag{3.28}$$

The substitution $t = T + \beta$ yields

$$I_1 = e^{-2na\beta} (I_\infty - I_\beta) \tag{3.29}$$

where

$$I_\infty = \int_0^\infty (e^{at} \sin t)^{2n}\ d\,t \tag{3.30}$$

$$I_\beta = \int_0^\beta (e^{at} \sin t)^{2n}\ d\,t \tag{3.31}$$

I_∞ may be obtained from a table of Laplace transforms (e.g. Ref.18) as

$$I_\infty = \int_0^\infty e^{2nat} \sin^{2n} t \, d\, t = \frac{(2n)!}{(-2na)(\overline{2na}^2 + 2^2)(\overline{2na}^2 + 4^2)\ldots.(\overline{2na}^2 + \overline{2n}^{\,2})} \tag{3.32}$$

I_∞ is plotted for $n = 1,2,3,4,5,6,7$ and $-0.5 < a < -0.05$ in Figs. 18 & 19.

I_β has been evaluated by graphical integration for the same n and a as I_∞, and for $0 < \beta < 2\pi$. Carpets of the variation of I_β with these parameters are given in Fig.s 20 - 26

Evaluation of Integrals of the Second Kind

Denoting these by I_2, we have

$$I_2 = \int_0^\infty P^{2n-1} e^{at} \sin (T + \beta)\, dT = \int_0^\infty e^{2nat}. \sin^{2n-1} (T + \beta). \sin T.\, d\, T \tag{3.33}$$

The substitution $t = T + \beta$ yields

$$I_2 = e^{-2na\beta} \int_\beta^\infty e^{2nat} \sin^{2n-1} t.\ \sin (t-\beta)\, dt \tag{3.34}$$

$$= e^{-2na\beta} \int_0^\infty e^{2nat} \left[\sin^{2n} t \cos\beta - \sin \beta.\ \sin^{2n-1} t.\ \cos t \right] dt \tag{3.35}$$

$$= e^{-2na\beta} \cos \beta \,(I_\infty - I_\beta)$$
$$- e^{-2na\beta} \sin \beta \int_\beta^\infty e^{2nat}.\ \sin^{2n-1} t.\ \cos t.dt \tag{3.36}$$

The last integral is evaluated by integration by parts which yields with $a < o$ (i.e. a stable desired response)

$$\int_\beta^\infty e^{2nat} \sin^{2n-1} t \cos t\, dt = \frac{-1}{2n} \sin^{2n} \beta.\ e^{2na\beta} - a \int_\beta^\infty e^{2nat} \sin^{2m} t\, dt \tag{3.37}$$

Thus substituting from Equation 3.37 in Equation 3.36 we obtain

$$I_2 = e^{-2na\beta} \cos \beta\, (I_\infty - I_\beta) + \frac{1}{2n} \sin^{2n+1} \beta + a \sin \beta.\ e^{-2na\beta} (I_\infty - I_\beta) \tag{3.38}$$

$$\therefore I_2 = \frac{1}{2n} \sin^{2n+1}\beta + e^{-2na\beta} (\cos \beta + a \sin\beta) (I_\infty - I_\beta) \tag{3.39}$$

This equation enables us to express the integrals of the second kind in terms of integrals of the first kind, as below

$$I_2 = \frac{1}{2n} \sin^{2n+1}\beta + (\cos \beta + a \sin \beta)\, I_1 \tag{3.40}$$

It will sometimes be found that for large a, n, and for β close to 2π, $I_\infty - I_\beta$ is given as a small difference between two approximately equal numbers, and in such circumstances difficulty will occur in evaluating I_1 and I_2 accurately. Rather than attempting great accuracy in the evaluation of I_∞ and I_β, the best procedure is then to replace β by a negative angle $-2\pi + \beta = \bar{\beta}$ and to evaluate $I_{\bar{\beta}} = \int_{-2n+\beta}^{0} (e^{at} \sin t)^{2n} \, d\,t$ graphically or numerically.[2]

3.5. Example 6. Longitudinal Short-period Response to an Impulsive Moment with Limited m_q Autostabilization.

(This example illustrates how a 'limited' or saturable autostabilizer system may be optimized by the use of the seventh-power polynomial approximation developed earlier in this chapter. An effort function similar to that of the previous examples is employed.)

With the numerical data, and desired response, of Example 1, and assuming a limiting elevator deflection of $\pm 1.05^\circ$, the calculation proceeds similarly to Example 5, except that Equation 3.17 is replaced by

$$\eta_s = \frac{1.05}{57.296}\left[P_0 \left(\frac{\hat{q}_D}{1.4}\right) + P_2\, k^2 \left(\frac{\hat{q}_D}{1.4}\right) + \ldots\ldots + S_8\, k^8 \left(\frac{\hat{q}_D}{1.4}\right)^7 \right] \tag{3.42}$$

where $k = \dfrac{\hat{q}\ \mathrm{SAT.}}{1.4}$, $\hat{q}$ SAT. being the 'saturation' value of $\hat{q}$

and the coefficients $P_0 \ldots . S_8$ are as listed in Table I

The problem is to find the optimum k. Putting $k^2 = h$ the equation corresponding to Equation 3.21 of Example 5 is

$$\begin{aligned} -\eta_{p_D}(T) \times \frac{57.296}{-1.05} = {} & \frac{57.296}{-1.05}\,(-0.003387\, e^{-0.434T} \sin T + 0.040827\, Z) \\ & + \left[P_0 Z + Q_0 Z^3 + R_0 Z^5 + S_0 Z^7 \right] \\ & + h \left[P_2 Z + Q_2 Z^3 + R_2 Z^5 + S_2 Z^7 \right] \\ & + h^2 \left[P_4 Z + Q_4 Z^3 + R_4 Z^5 + S_4 Z^7 \right] \\ & + h^3 \left[P_6 Z + Q_6 Z^3 + R_6 Z^5 + S_6 Z^7 \right] \\ & + h^4 \left[P_8 Z + Q_8 Z^3 + R_8 Z^5 + S_8 Z^7 \right] \end{aligned} \tag{3.42}$$

where $Z = 0.898\ P_1$, P_1 being defined by Equation 3.22.
To evaluate the effort function, $\int_0^\infty \left[\eta_{p_D} (T) \right]^2 d\,T$
we write Equation 3.42 as

$$-\eta_{p_D}(T) \times \frac{57.296}{-1.05} = a + b$$

where $a = \frac{57.296}{-1.05} \; (-0.003387\ e^{-0.434T} \sin T + 0.040827\ Z)$ (3.43)

and b = the remainder of the r.h.s. of Equation 3.42

Then

$$\int_0^\infty \left[\eta_{p_D}(T) \times \frac{57.296}{1.05} \right]^2 dT = \int_0^\infty a^2\, d\,T + \int_0^\infty 2\,a\,b\,d\,T + \int_0^\infty b^2\, d\,T \qquad (3.44)$$

In Equation 3.44 $\int_0^\infty a^2\, d\,T$ need not be evaluated since it does not involve h , and therefore will not appear in the equation for the optimum h, $\frac{\partial}{\partial h} \int_0^\infty \left[\eta_{p_D}(T) \right]^2 d\,T = 0.$

The evaluation of $\int_0^\infty 2\,a\,b\,dT$ is straightforward since the integrals required are of the forms $\int_0^\infty P_1^{\,2n}\, dT$, $\int_0^\infty P_1^{2n-1}\, e^{-0.434T} \sin T\, dT$, and these may be read from the list on Page 29 or more generally, for other problems, evaluated by means of the carpets of Figs. 20 to 26 . Using these integrals $\int_0^\infty b^2\, d\,T$ is easily calculated once b^2 is known. However, b is comprised of no less than twenty terms and it would be very tedious to have to evaluate b^2 anew for each problem. b^2 has therefore been evaluated once and for all, for a general Z, the result being tabulated in Table II.

Making use of this table, we eventually obtain

$$\int_0^\infty \left[\frac{57.296}{1.05}\ \eta_{p_D}(T) \right]^2 dT = -50.50289h + 328.96168h^2 + 2{,}512.19225\ h^3 + 2{,}293.68167\ h^4 - 2{,}828.8175\ h^5 + 2{,}098.11575\ h^6 - 861.2662\ h^7 + 150.35609\ h^8 \qquad (3.45)$$

(Note that it is desirable to leave rounding-off until late in the calculation, as far as possible. This is the reason for the appearance of such a large number of significant figures in the coefficients of the above equation.)

Differentiating Equation 3.45 with respect to h, and equating the result to zero, the only solution between 0 and +1 is h = 0.0484

$\therefore$ <u>k = 0.22</u> $\qquad (\therefore \ \hat{q}_{SAT.} = 1.4 \times 0.22 = 0.308)$

The total elevator deflection demanded to attain the desired response, $\eta_{p_D} + \eta_s$ is of a smooth nature. Since η_s is discontinuous it follows that η_{p_D} must also be discontinuous. It is presumed that the pilot will be unable to provide such a discontinuous $\eta_{p_D}(\tau)$, (although the discontinuity is less severe than in Example 5) and we therefore proceed at once to Step 8 of the optimization procedure, omitting Step 7.

The stick-fixed response for $\hat{q}_{SAT} = 0.308$ is calculated by piecewise application of standard linear response theory, the time histories of $\hat{q}(T)$ and $w(T)$ being graphed in Figs. 27 - 28. The autostabilizer is initially ($\tau \leq 0$) unsaturated, but owing to the impulsive nature of the applied moment the saturation $\hat{q}$ is attained in a very short time ($\tau < \epsilon$). Thus for purposes of calculation, only two 'pieces' are necessary, $\epsilon \leq \tau \leq \frac{0.886}{11.51}$ airsec, and $\tau \geq \frac{0.886}{11.51}$ airsec, since $\hat{q}_{SAT}$ is not attained for $\tau > \frac{0.886}{11.51}$ airsec.

It will be seen that the stick-fixed response in w closely approaches the desired response, and the optimum autostabilization may, therefore, be regarded as satisfactory.

<u>CHAPTER 4</u>

4.1. <u>A BRIEF EXPOSITION OF CARDINAL SPECTRUM ANALYSIS</u>

The purpose of this chapter is twofold. Firstly, it is intended to provide the reader having no previous knowledge of cardinal spectrum analysis with sufficient background to follow the application of this technique in the following chapters, and secondly it serves to introduce the nomenclature and symbols used therein. The treatment is a highly condensed version of that of Ref. 7 with such changes in nomenclature and symbolism as have been found to be desirable. For a more rigorous and extensive discussion of Cardinal Spectrum Analysis Refs. 6 and 7 should be consulted.

Basic Theory of Cardinal Spectrum Analysis

1. Definition of a Cardinal Spectrum

The cardinal spectrum of a function of time is simply a series of numbers corresponding to the heights of successive ordinates of the function measured at equal time intervals.

Thus, denoting the cardinal spectrum of F(t) by Ⓒ F(t), we have,

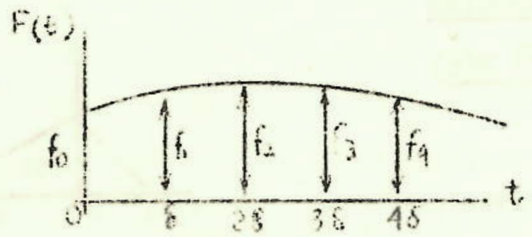


FIG.4.1.

$$Ⓒ\ F(t) = (f_0, f_1, f_2, \ldots\ldots.)$$

2. Triangular Pulse Interpolation

We may approximate to the area under the curve F(t) by summing a series of triangles of base 2δ as shown in Fig.4.2.

It is advantageous to define not only the area under the curve but the curve itself by summing a series of

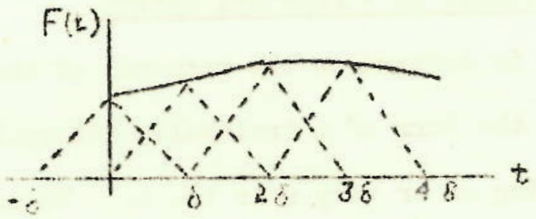


FIG.4.2.

triangles in this way, because it enables us to describe a curve numerically and uniquely. Using cardinal spectra alone, this is not possible. For example, given Ⓒ F(t) = (3,5,6,2,4, ...) one could draw F(t) as any curve passing through these points. Defining F(t) as the sum of a series of triangular pulses is equivalent to joining up the successive ordinates of the spectrum by straight lines.

Expressed mathematically this is the equation

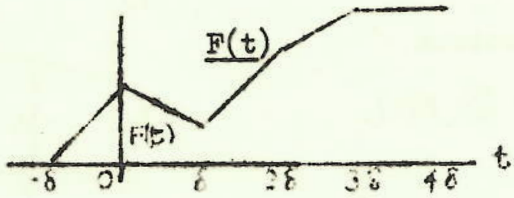
$$F(t) = \sum_{k=o}^{k=n} f_k . \Delta_k(t)$$

where $f_k . \Delta_k(t)$ denotes the triangular pulse having its peak at $t = k\,\delta$. The 'value' of the interpolation pulse $f_k . \Delta_k$ is defined as $f_k . \delta$. This is an approximation to the area under F(t) from $t = (k - \frac{1}{2})\delta$ to $t = (k + \frac{1}{2})\delta$

2 (a) Examples

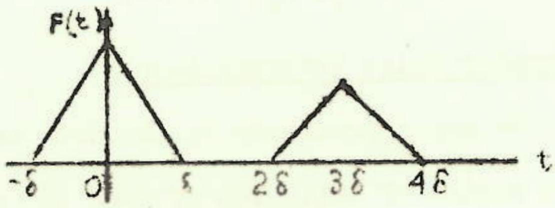
(i) Ⓒ F(t)

(2,1,3,4,4,.....)



(ii)

(3,0,0,2,0,0,......)



3. Pulse Admittance of a Physical System

This is defined as the response of the system to an impulse having the form of a traingular interpolation pulse of value δ occurring at or very near t = o. We use the symbol M(t) to denote the pulse admittance of a given system and, where necessary, we shall represent it by its cardinal spectrum. We use the notation Ⓒ $M(t) = (m_0, m_1, m_2, m_3, ...)$. If the instant of application of the input pulse occurs at $t = n\delta$, the response of the system is the 'displaced pulse admittance' denoted by $M(t - n\delta)$

In this case

$$Ⓒ\ M(t - n\delta) = (0, 0, 0, ...0, m_0, m_1, m_2, ...)$$

m_0 is here the n^{th} term in the cardinal spectrum.

4. To obtain the Response of a Linear System to a General Input. (Polymultiplication)

We shall now describe how the response $G(t)$ of a linear system, having a known pulse admittance $M(t)$, to an input $F(t)$ can be determined.

$$Ⓒ\ F(t) = (f_0, f_1, f_2, ...)$$

$$F(t) \doteq \sum_{k=0}^{k=n} f_k . \Delta k(t)$$

Δ_0 produces the response spectrum Ⓒ $M(t) = (m_0, m_1, m_2 ...)$. Hence the impulse $f_0 m_0$ produces the response spectrum $(f_0 m_0, f_0 m_1, f_0 m_2, ...)$ (assuming the system is linear).

Similarly, $f_1 m_1$ produces the response spectrum $(0, f_1 m_0, f_1 m_1, f_1 m_2, ...)$ and, in general, the impulse $f_i \Delta_i$ produces the response spectrum

$(0,0,0,......,0, f_i m_0, f_i m_1, ...)$ beginning at $t = k\delta$.

The superposition of these partial response spectra at $t = k\delta$ is the sum

$$gk = \sum_{i=0}^{k} f_i m_{k-i}$$

Thus, the calculation of Ⓒ $G(t) = (g_0, g_1, g_2, ...)$ can be tabulated up to $t = k\delta$ as follows:-

Ⓒ $M(t)$	m_0	m_1	$m_2 \ldots\ldots m_k \ldots\ldots$
Ⓒ $F(t)$	f_0	f_1	$f_2 \ldots\ldots f_k \ldots\ldots$
f_0 Ⓒ $M(t)$	f_0m_0	f_0m_1	$f_0m_2 \ldots\ldots f_0m_k \ldots\ldots$
f_1 Ⓒ $M(t)$	.	f_1m_0	$f_1m_1 \ldots\ldots f_1m_{k-1} \ldots$
f_2 Ⓒ $M(t)$	.	.	$f_2m_0 \ldots\ldots f_2m_{k-2} \ldots$
			$\ldots\ldots$
f_k Ⓒ $M(t)$	.	.	. $\ldots\ldots f_km_0 \ldots$
Ⓒ $G(t)$ =	f_0m_0	$f_0m_1 + f_1m_o$	$\sum_{i=o}^{k} f_im_{k-i}$
	"	"	"
	g_0	g_1	g_k

This tabulated process is rather analogous to the multiplication of two polynomials since

$$(f_0 + f_1x + f_2x^2 + \ldots\ldots f_kx^k)\ (m_0 + m_1x + m_2x^2 + \ldots\ldots m_kx^k)$$
$$= f_om_o + (f_1m_o + f_om_1)x + \ldots\ldots \qquad \sum_{i=o}^{k} f_im_{k-i}\ x^k$$

For this reason we call the process by which Ⓒ $M(t)$ is combined with Ⓒ $F(t)$ to yield Ⓒ$G(t)$, polymultiplication, and we describe the process symbolically thus :-

$$Ⓒ\ G(t) = Ⓒ\ F(t)\ X\ Ⓒ\ M(t).$$

5. To Obtain the Pulse Admittance of a System from its Response to a known Input. (Polydivision)

i.e. Knowing $G(t)$ and $F(t)$ we require to find $M(t)$.

We have Ⓒ $G(t)$ = Ⓒ $F(t)$ X Ⓒ $M(t)$

Let us define Polydivision from the equation

$$\textcircled{C}\ M(t) \;=\; \vdash\!\frac{\textcircled{C}\ G(t)}{\textcircled{C}\ F(t)}\!\dashv$$

($\vdash\!\!\longrightarrow\!\dashv$ denotes polydivision, as usual ——— denotes ordinary division). Hence polydivision means the process tabulated below :-

$$\begin{array}{lll}
 & \quad m_0 \qquad m_1 \qquad\quad m_2 & = \textcircled{C}\ M(t) \\
C\ \ F(t) = f_0, f_1, f_2 & \big)\,\overline{f_0m_0,\ f_0m_1+f_1m_0,\ f_0m_2+f_1m_1+f_2m_0} & = \textcircled{C}\ G(t) \\
 & \qquad\qquad \underline{f_1m_0 \qquad\qquad\quad f_2m_0} & \\
 & \qquad\qquad f_0m_1 \quad f_0m_2+f_1m_1 & \\
 & \qquad\qquad \underline{f_0m_1 \qquad\quad f_1m_1} & \\
 & \qquad\qquad\qquad\quad f_0m_2 \ \ldots\ldots\ldots & \\
 & \qquad\qquad\qquad\quad \underline{f_0m_2 \qquad\qquad} &
\end{array}$$

Of course, in any realistic problem $\textcircled{C}\ G(t)$ will not be given in the obvious form $(f_0m_0,\ f_0m_1+f_1m_0 \ldots\ldots)$ but in the form $(g_0,\ g_1,\ g_2 \ldots\ldots)$ and the process of polydivision must then be tabulated as follows :-

$$\begin{array}{lll}
 & \dfrac{g_0}{f_0}, \qquad \dfrac{g_1f_0 - g_0f_1}{f_0^{\,2}} \qquad \ldots\ldots & = \textcircled{C}\ M(t) \\
\textcircled{C}\ F(t) = f_0,\ f_1,\ f_2\ldots & & \\
 & \big)\,\overline{g_0 \qquad\qquad g_1 \qquad\qquad g_2 \qquad \ldots\ldots} & \\
 & g_0 \qquad\qquad \dfrac{g_0}{f_0}.\ f_1 \quad \dfrac{g_0}{f_0}.\ f_2 \quad \ldots\ldots & \\
 & \overline{\qquad (g_1 - \dfrac{g_0}{f_0}.\ f_1)\ (g_2 - \dfrac{g_0}{f_0}\ .\ f_2)\ \ldots\ldots} &
\end{array}$$

.......ETC. ETC.

The polydivision process is continued until the required number of terms in C $M(t)$ has been obtained.

6. Addition and Subtraction of Cardinal Spectra.

Simply add (or subtract) the corresponding terms

Thus, if Ⓒ $F(t) = (f_0, f_1, f_2 \ldots\ldots\ldots)$

and Ⓒ $H(t) = (h_0, h_1, h_2 \ldots\ldots\ldots)$

then Ⓒ$(F(t) \pm H(t)) = (f_0 \pm h_0, f_1 \pm h_1, f_2 \pm h_2 \ldots\ldots)$

Note that the commutative, associative and distributive laws hold for the polymultiplication, polydivision, addition and subtraction of cardinal spectra.

e.g. Ⓒ $M(t)$ X Ⓒ$(F(t) + H(t))$ = Ⓒ $M(t)$ X Ⓒ $F(t)$ + Ⓒ $M(t)$ X Ⓒ $H(t)$

7. Integration and Differentiation of Cardinal Spectra.

It may be shown that the cardinal spectrum operation

$$(h_0, h_1, h_2, \ldots\ldots) = \frac{\delta}{2} \cdot \frac{(1,+1)}{(1,-1)} \mathbb{X} (f_0, f_1, f_2, \ldots\ldots)$$

is an approximation to the integration operation

$$H(t) = \int_0^t F(t).dt$$

The cardinal spectrum operation

$$(f_0, f_1, f_2, \ldots) = \frac{2}{\delta} \cdot \frac{(1,-1)}{(1,+1)} \mathbb{X} (h_0, h_1, h_2, \ldots\ldots)$$

may similarly be shown to be an approximation to the differentiation operation

$$F(t) = \frac{d\,H(t)}{dt}$$

Repeated integration and differentiation may be performed by the use of such expressions as

$$\frac{d^2}{dt^2} \equiv D^2 \equiv \frac{2^2}{\delta^2} \frac{(1,-1)^2}{(1,+1)^2} = \frac{4}{\delta^2} \cdot \frac{(1,\ -2,\ +1)}{(1,\ +2,\ +1)}$$

or with greater accuracy by the following expressions

$$D^2 \equiv \frac{6}{\delta^2} \cdot \frac{(1,\ -1)^2}{(1,\ +4,\ +1)}$$

$$D^3 \quad \frac{24}{\delta^2} \cdot \frac{(1,\ -1)^3}{(1,\ +11,\ +11,\ +1)}$$

The appropriate reciprocal may be used for repeated integration.

CHAPTER 5. 5.1. THE APPLICATION OF CARDINAL SPECTRUM ANALYSIS TO THE OPTIMIZATION PROCEDURE

If the desired response is specified as an exponential function of time the autostabilizer system is most conveniently optimized by the procedure illustrated in the previous examples, in which the effort function was in the form of an infinite integral. Because of the rapid attenuation of the integrand (due to the high damping of the desired response) the unrealism of such an effort function was not objectionable. However, should it be desired to employ an effort function having the form of a finite integral it will generally be found to be more convenient to perform the optimization by means of Cardinal Spectrum Analysis. Should the desired response not be given in a convenient analytical form, Cardinal Spectrum Analysis must be employed in the optimization procedure, and in such cases the effort function must have the form of a finite integral.

The following example illustrates how a lateral autostabilizer system may be optimized. Effort functions for lateral response may be more complicated than those appropriate to longitudinal response since the aircraft may be controlled by independent deflections of ailerons and rudder, and a lateral, rather than a longitudinal, autostabilizer is selected so that this consideration may be examined. The Cardinal Spectrum technique used in this example is however, equally applicable to longitudinal response.

5.2. EXAMPLE 7. LATERAL RESPONSE TO A SHARP-EDGED SIDEGUST

With the portmanteau notation of Ref. 2 the non-dimensionalized equations of motion can be written as,

$$\begin{bmatrix} D + \bar{y}_v & -k & 1 \\ \mathcal{L} & D^2 + l_1 D & \epsilon_A D^2 + l_2 \\ -\mathcal{N} & \epsilon_C D^2 + n_1 D & D + n_2 \end{bmatrix} \begin{bmatrix} \bar{v}(\tau) \\ \phi(\tau) \\ \hat{r}(\tau) \end{bmatrix} = \begin{bmatrix} C_y(\tau) \\ C_l(\tau) \\ C_n(\tau) \end{bmatrix} \qquad (5.1)$$

To obtain the basic response to a sharp-edged sidegust we assume that at $\tau = 0$, $\phi = \hat{r} = D\phi = 0$, $\bar{v} = \bar{v}_0$ with $\bar{v} = 0$ for $\tau < 0$

Applying the Laplace transform to Equation 5.1 we obtain

$$\begin{bmatrix} s + \bar{y}_v & -k & 1 \\ \mathcal{L} & s^2 + l_1 s & \epsilon_A s + l_2 \\ -\mathcal{N} & \epsilon_C s^2 + n_1 s & s + n_2 \end{bmatrix} \begin{bmatrix} \bar{v}(s) \\ \phi(s) \\ \hat{r}(s) \end{bmatrix} = \begin{bmatrix} \bar{v}_0 \\ 0 \\ 0 \end{bmatrix} \tag{5.2}$$

where $$v(s) = v_0 \frac{\left[(s^2 + l_1 s)(s + n_2) - (\epsilon_A s + l_2)(\epsilon_C s^2 + n_1 s)\right]}{\Delta(s)} \tag{5.3}$$

where $\Delta(s)$ is the determinant of the s -matrix.

The time history of $\bar{v}(\tau)$ may be obtained by means of a partial fraction expansion or, more easily, by use of Interpretation Formulae such as those listed in Ref. 9. The basic response in $\bar{v}$ is graphed in Fig. 29. It will be seen that the response is lightly damped with a large initial overshoot. The desired response in $\bar{v}$ is graphed in Fig.29. The desired response may be obtained by means of rudder and aileron deflections ζ_D and ξ_D, where

$$\text{Ⓒ}(\bar{v}_D - \bar{v}_{basic}) = \text{Ⓒ Pulse Admittance of } \bar{v} \text{ to } \zeta \times \text{Ⓒ}\zeta_D + \text{Ⓒ Pulse Admittance of } \bar{v} \text{ to } \xi \times \text{Ⓒ}\xi_D \tag{5.4}$$

The Pulse Admittances are those appropriate to the basic aircraft and are conveniently evaluated by use of the Laplace transform. Note that if the spectrum interval δ is small the form of the impulsive admittance closely approximates to that of the pulse admittance, which may be deduced therefrom by multiplication by a factor of δ .

It is now necessary to select the effort function. Three possible effort functions are,

$$\text{(i)} \int_0^T \xi_{p_D}^2 d\tau \quad , \text{(ii)} \int_0^T \zeta_{p_D}^2 \, d\tau \quad , \text{(iii)} \int_0^T \left(\xi_{p_D} + q\zeta_{p_D}\right)^2 d\tau$$

where T is a convenient time

and q is a constant

The most realistic of these is (iii), but there is difficulty in assigning a value to q which, in effect, describes the relative preference of the pilot for aileron and rudder movements. For well chosen autostabilization we may expect to be able to reduce the magnitudes of the control deflections demanded of the pilot to very small values, so in these circumstances a less realistic effort function may be tolerated. In this example we shall, therefore, employ (ii). ξ is then assumed to be zero throughout the motion, when from Equation 5.4 we obtain,

$$\text{(C)} \quad C_{n_\zeta} \cdot \zeta_D(\tau) = (0, -23{\cdot}162, -11{\cdot}36, 19{\cdot}70, 14{\cdot}128, 5{\cdot}496, -3{\cdot}033, -5{\cdot}355, -1{\cdot}737, 2{\cdot}3384, 2{\cdot}794, 1{\cdot}009, -0{\cdot}782, \ldots) \tag{5.5}$$

with a δ of 0·1 airsec.,

From the time vector polygons of Fig. 32 we see that the increased damping of the oscillatory mode that characterises the desired response is likely to be achieved by an autostabilizer system which provides a rudder deflection of such phasing that the derivatives n_r and n_p are effectively changed as follows.

(a) n_r is multiplied by K, $K > 1$

or (b) n_p is multiplied by $-H$, $H > 1$

or (c) n_r and n_p are multiplied by K′ & H′ with $n_r - n_p \left| \frac{p}{r} \right| < K' n_r - H' n_p \left| \frac{p}{r} \right|$

Solving for the optimum K in (a), we have,

$$\text{Ⓒ } r_D(\tau) = \text{Ⓒ Basic response in } \hat{r} + \text{Ⓒ Admittance of } \hat{r} \text{ to } \zeta \; X \; \text{Ⓒ} \zeta_D \quad (5.6)$$

whence

$$\text{Ⓒ } n_2 \, \hat{r}_D(\tau) = (0,\ 5{\cdot}95,\ 3{\cdot}445,\ -1{\cdot}18,\ -3{\cdot}32,\ -1{\cdot}184,\ 0{\cdot}409,\ 1{\cdot}208,\ -0{\cdot}506,\ -0{\cdot}452,\ -0{\cdot}727,\ -0{\cdot}3688,\ 0{\cdot}0536,\ldots) \quad (5.7)$$

Since

$$\frac{\mu_2 \, n_\zeta}{i_c'} \cdot \zeta_s = -(K-1)\,\frac{n_r}{i_c'} \cdot \hat{r}_D \quad (5.8)$$

where ζ_s is the rudder deflection due to the autostabilizer.

$$\text{Ⓒ } \frac{\mu_2 \, n_\zeta}{i_c'} \cdot \zeta_{p_D} = \text{Ⓒ } \frac{\mu_2 \, n_\zeta}{i_c'} \cdot \zeta_D + \text{Ⓒ } (K-1)\,\frac{n_r}{i_c'} \cdot \hat{r}_D \quad (5.9)$$

The effort function $\int_0^T \left(\frac{\mu_2 \, n_\zeta}{i_c'} \cdot \zeta_{p_D} \right)^2 d\tau$ is evaluated by squaring each element of the right-hand side of Equation 5.9., (approximate) integration being performed by summing the elements of the resulting Cardinal Spectrum,

For a T of 1·2 airsecs,

$$\int_0^T \left(\frac{\mu_2 \, n_\zeta}{i_c} \cdot \zeta_{p_D} \right)^2 d\tau \simeq (0)^2,\ +(-23{\cdot}162 + 5{\cdot}95\,\overline{K-1})^2,\ +(-11{\cdot}36 + 3{\cdot}445\,\overline{K-1})^2 + \ldots\ldots\ldots\ldots +(-0{\cdot}782 - 0{\cdot}0536\,\overline{K-1})^2 \quad (5.10)$$

$$= -534{\cdot}51(K-1) + 63{\cdot}932(K-1)^2 + \text{a constant term} \quad (5.11)$$

Therefore, for $\frac{\partial}{\partial(K-1)} \int_0^{1.2} \frac{\mu_2 n_\zeta}{i_c'} . \zeta_{p_D}^2 \; d\tau = 0, \quad K-1 = \frac{534 \cdot 51}{2x63 \cdot 932} = 4.18,$

Thus the optimum value of K is 5·18.

As the effort function is admittedly unrealistic Step 7 may be omitted. The response in $\bar{v}$ to the specified input with no pilot action and with the optimum adjustment of the autostabilizer is graphed in Fig. 30. It closely approximates to the desired response and the optimum autostabilization may, therefore, be regarded as satisfactory. A similar procedure may be employed to obtain the optimum value of H for n_p autostabilization.

For combined n_p and n_r autostabilization (which can be produced by canting the axis of the autostabilizer rate gyro) the rudder deflection demanded of the pilot is given by:

$$\text{Ⓒ} \frac{\mu_2 n_\zeta}{i_c'} . \zeta_{p_D} = \text{Ⓒ} \frac{\mu_2 n_\zeta}{i_c'} . \zeta_D + \text{Ⓒ} (K'-1) \frac{n_r}{i_c'} . \hat{r}_D + \text{Ⓒ} (H'-1) \frac{n_p}{i_c'} . D\phi_D \tag{5.12}$$

The optimum H′ and K′ are obtained by solving the simultaneous equations for H′ and K′ which result from equating to zero the appropriate partial derivatives of the effort function.

It is more difficult to formulate a realistic effort function for lateral response than for short-period longitudinal response, since both aileron and rudder control is available to the pilot. The effort function selected should, therefore, be of the 'integrated displacement-squared' type so that the effect of the pilot failing to provide the demanded control surface deflections will be to induce only a slight divergence from the desired response.

5.3. OPTIMIZATION OF NON-LINEAR SYSTEMS BY MEANS OF CARDINAL SPECTRUM ANALYSIS

Cardinal Spectrum Analysis may be employed to optimize non-linear autostabilizer systems by a similar procedure to that of Examples 4, 5 and 6. Thus, for example, in Example 5 Equation 3.23 would be replaced by:-

$$\sum Ⓒ \left[\eta_{P_D}(T)\right]^2 = \eta_F^2 \left(23{\cdot}3695 \sum Ⓒ P_1^{\,2} - 138{\cdot}2968 \sum Ⓒ P_1^{\,4} + \ldots \ldots + 54{\cdot}9029 \sum Ⓒ P_1^{\,14}\right) + \eta_F \left(-0{\cdot}158493 \sum Ⓒ P_1^{\,2} + \ldots \ldots - 0{\cdot}0250995 \sum Ⓒ P_1^{\,7} e^{-0{\cdot}434T} \sin T\right) + \text{terms not involving } \eta_F \tag{5.13}$$

where Σ Ⓒ denotes the sum of successive ordinates of the appropriate cardinal spectrum.

The procedure for minimization of the effort function is thenceforward similar to that illustrated in Example 7.

CHAP. 6 6.1. THE OPTIMIZATION OF AUTOSTABILIZER SYSTEMS FOR PILOTLESS AIRCRAFT

Let the desired response to a specified input of a pilotless aircraft be $R_D(\tau)$. We may regard the optimum values of the adjustable parameters of the autostabilizer system as having been attained when the actual response $R(\tau)$ most closely approaches $R_D(\tau)$. Hence a suitable criterion for optimization would be,

$$I = \int_0^T \left| R_D(\tau) - R(\tau) \right| d\tau = \text{a minimum} \tag{6.1}$$

or, alternatively

$$I = \int_0^T \left[R_D(\tau) - R(\tau) \right]^2 \; d\tau = \text{a minimum} \qquad (6.2)$$

where T is any convenient time

It will generally be found that $R(\tau)$ is a function of the adjustable parameters of the autostabilizer k_1, k_2, etc., of such a nature that differentiation of Equation 6.2 yields complicated expressions for $\frac{\partial I}{\partial k_1}$, $\frac{\partial I}{\partial k_2}$, etc., the zeros of which are difficult to obtain. A simpler method of optimization, employing an approximate form of the criterion of Equation 6.1, has therefore been developed and is presented below. The method is, strictly, only valid for completely linear systems (i.e. linear aircraft and autostabilizer dynamics) but, as we shall explain, it appears that it may often be applied to non-linear systems with success.

Consider (for example) the longitudinal motion of an aircraft fitted with m_q autostabilization. For a specified input $\Delta M(\tau)$ we have in Cardinal Spectrum Analysis notation,

$$\text{Ⓒ}\,\hat{q} = \text{Ⓒ Admittance of } \hat{q} \text{ to } \eta \times \text{Ⓒ}\,\eta + \text{Ⓒ Admittance of } \hat{q} \text{ to } \Delta M \times \text{Ⓒ}\,\Delta M(\tau) \qquad (6.3)$$

$$\text{Ⓒ}\,\hat{q}_D = \text{Ⓒ Admittance of } \hat{q} \text{ to } \eta \times \text{Ⓒ}\,\eta_D + \text{Ⓒ Admittance of } \hat{q} \text{ to } \Delta M \times \text{Ⓒ}\,\Delta M(\tau) \qquad (6.4)$$

Whence, with $\eta = k\hat{q}$

$$\text{Ⓒ}\,(\hat{q}_D - \hat{q}) = \text{Ⓒ Admittance of } \hat{q} \text{ to } \eta \times \text{Ⓒ}\,(\eta_D - k\hat{q}) \qquad (6.5)$$

Adopting the criterion of Equation 6.1, the optimum k is that which minimizes $\int_0^T \left| \hat{q}_D - \hat{q} \right| d\tau$

The Cardinal Spectrum of $\hat{q}_D - \hat{q}$ may be written as:

$$Ⓒ\ (\hat{q}_D - \hat{q}) \equiv (e_0, e_1, e_2, ...) \tag{6.6}$$

Whence

$$\int_0^{T=2n\delta} | \hat{q}_D - \hat{q} | d\tau \doteq |e_0| + |e_1| + |e_2| + |e_3| + ...+ |e_{2n}| \tag{6.7}$$

In Equation 6.5, we put

$$Ⓒ\ \text{Admittance of } \hat{q} \text{ to } \eta = (a_0, a_1, a_2, ...) \tag{6.8}$$

and

$$Ⓒ\ (\eta_D - k\hat{q}) = (h_0, h_1, h_2, ...) \tag{6.9}$$

Then

$$Ⓒ\ (\hat{q}_D - \hat{q}) = (a_0, a_1, a_2, ...) \times (h_0, h_1, h_2, ...) \tag{6.10}$$

and

$$\begin{aligned} e_0 &= a_0 h_0 \\ e_1 &= a_0 h_1 + a_1 h_0 \\ &\vdots \\ e_n &= a_0 h_n + a_1 h_{n-1} + ... + a_n h_0 \\ &\vdots \\ e_{2n} &= a_n h_n \end{aligned} \tag{6.11}$$

$$\begin{aligned} \therefore\ |e_0| + |e_1| + ... + |e_{2n}| &= a_0 (|h_0| + |h_1| + |h_2| + ... + |h_n|) \\ &+ |a_1| (|h_0| + |h_1| + |h_2| + ... + |h_n|) \\ &\vdots \\ &+ |a_n| (|h_0| + |h_1| + |h_2| + ... + |h_n|) \\ &= A (|h_0| + |h_1| + |h_2| + ... + |h_n|) \end{aligned}$$

where $A = |a_0| + |a_1| + ... + |a_n|$

Hence $\int_0^{T=2n\delta} |\hat{q}_D - \hat{q}| \, d\tau$ is minimized when k is chosen so as to minimize $|h_0| + |h_1| + |h_2| + ... + |h_n| \doteq \int_0^{n\delta} | \eta_D - k\hat{q} | \, d\tau$

For well-chosen autostabilization with the adjustable parameter k close to its optimum value, $\eta_D - k\,\hat{q} \simeq \eta_D - k\,\hat{q}_D$

Therefore, by choosing k such that $\int_0^{n\delta} |\eta - k\,\hat{q}_D|\,d\tau$ is minimized,

an approximate value for the optimum k (as defined by Equation 6.1) is obtained. It is interesting to note that for a piloted aircraft we should have $\eta_{p_D} = \eta_D - k\,\hat{q}_D$ and k would then be chosen to minimize $\int_0^{n\delta} |\eta_{p_D}|\,d\tau$ (or some other effort function).

The optimization procedures for piloted and pilotless aircraft are therefore similar[1] in many respects, although the procedure for pilotless aircraft is essentially approximate, and Step 7 is superfluous.

As an example, let us consider a pilotless aircraft having similar characteristics to that of Example 7 with n_r autostabilization. From Equations 5.5 and 5.9, we have,

$$\text{Ⓒ}\ \frac{\mu_2 n_\zeta}{i_c{}'}(\zeta_D - \zeta_s) = (0,\ -23{\cdot}162 + 5{\cdot}95\,J,\ -11{\cdot}36 + 3{\cdot}445\,J, \ldots) \qquad (6.12)$$

where $J = K - 1$

and

$$\int_0^{1.2} \frac{\mu_2 n_\zeta}{i_c{}'} |\zeta_D - \zeta_s|\,d\tau = |-23{\cdot}162 + 5{\cdot}95\,J| + |-11{\cdot}36 + 3{\cdot}445\,J| + \ldots$$
$$\ldots + |-0{\cdot}782 - 0{\cdot}0536\,J|$$

Each term on the right-hand side of Equation 6.13 is of the form $|a_r + b_r J|$. It can be shown that $\sum_{r=0}^{n} |a_r + b_r J|$ is stationary when $J = \frac{a_i}{b_i}$ where i is a particular r to be found. Thus the possible optimum values of J are,

3·8927, 3·2975, 16·695, 4·2554, 4·642, 7·416, 4·433, -3·432, 5·173, 3·8432, 2·731, 14·5896, For brevity we put $\int_0^{1.2} \frac{\mu_2 n_\zeta}{i_c} \left| \zeta_D - \zeta_s \right| d\tau = E$

Equation 6.13 may then be written in the form

$$E = |a_0 + b_0 J| + |a_1 + b_1 J| + \ldots\ldots + |a_{12} + b_{12} J| \qquad (6.14)$$

$$= \beta_0 |J - \alpha_0| + \beta_1 |J - \alpha_1| + \ldots\ldots + \beta_{12} |J - \alpha_{12}|$$

$$\text{with } \beta = |b|, \quad \alpha = \frac{-a}{b}$$

Rearranging the terms of the right-hand side of Equation 6.13 in order of decreasing α we obtain

$$E = 1.18|J - 16{\cdot}695| + 0{\cdot}0536|J - 14{\cdot}5896| + \ldots + 0{\cdot}0506|J + 3{\cdot}432|$$

The minimum of E is found by examining the sign of $\frac{dE}{dJ}$ for successive values of J

Thus for $J > 16{\cdot}695$,

$$E = J(1{\cdot}18 + 0.0536 + \ldots + 0{\cdot}0506) - (1{\cdot}18 \times 16{\cdot}695 + \ldots - 0{\cdot}506 \times 3{\cdot}432)$$

$$\frac{dE}{dJ} = 18{\cdot}8304, > 0$$

For $16{\cdot}695 > J > 14{\cdot}5896$

$$\frac{dE}{dJ} = 18{\cdot}8304 - 2 \times 1{\cdot}18, > 0$$

For $14{\cdot}5896 > J > 7{\cdot}416$

$$\frac{dE}{dJ} = 18{\cdot}8304 - 2(1{\cdot}18 + 0{\cdot}0536), > 0$$

Continuing this process we find that $\frac{dE}{dJ}$ becomes negative at $J = 3{\cdot}8927$.

This is the optimum J within the accuracy of the calculation; for greater accuracy a smaller spectrum interval should be employed. The corresponding optimum K-1 in Example 7 is 4·18.

6.2. A SUGGESTED PROCEDURE FOR THE OPTIMIZATION OF AUTOSTABILIZER SYSTEMS FOR PILOTLESS AIRCRAFT WITH NON-LINEARITIES.

It will be observed that in Examples 1 to 7, the use of effort functions which are functions of displacement only yields optimum values of the adjustable constants of the autostabilizer system such that the stick-fixed response (with the optimum adjustment) is close to the desired response. The analysis of Chapter 6.1 indicates why this should be so for linear systems, since it has been shown that the approximate optimization procedure for pilotless aircraft is formally similar to the optimization procedure for piloted aircraft with a certain choice of effort function.

For non-linear systems the above-mentioned analysis is inapplicable: nevertheless the stick-fixed responses obtained in Step 8 of the Non-linear examples (Examples 4, 5 and 6) were in each example close to the desired response, and it appears likely that this will frequently be the case for practical non-linear systems. In view of the redicous and complicated nature of non-linear response calculations starting from the equations of motion, and the possibility that these calculations may have to be repeated many times to locate the optimum values of the adjustable parameters a simple (even if approximate) method of optimization for non-linear pilotless systems is highly desirable. It is therefore suggested that before attempting a rigourous optimization procedure for a non-linear pilotless system the procedure for piloted aircraft should be applied (omitting Step 7) with a suitable choice of effort function. Despite the basic unrealism of this artifice, the relative simplicity of the calculations involved make this procedure one of considerable utility. It must be clearly understood however, that this procedure is at best approximate, and that for ill-chosen autostabilization (i.e. autostabilization that is inherently incapable of providing a response close to the desired response even when at its optimum adjustment) the approximation may be poor.

CHAPTER 7.

7.1. SOME ALTERNATIVE METHODS OF OPTIMIZATION

In this report we have treated piloted and pilotless aircraft separately. In most published work on aircraft autostabilization no subh clear distinction is drawn between the two - generally it is tacitly assumed that the aircraft discussed is piloted, although the presence of the pilot is not explicitly taken into account in the calculations. Some difficultytherefore exists in drawing a comparison between the optimization procedures developed herein and relevant published work. It is, howëver, desirable that some such comparison should be made, and in order to provide a basis for comparison it is assumed throughout this section that the aircraft referred to are piloted.

We give below a brief assessment of relative merits and demerits of some published methods of optimizing aircraft autostabilizer systems vis-a-vis the procedure of the present work. The alternative methods are described only briefly, in order to avoid lengthy digressions: reference should be made to the works cited for a fuller description of each method.

7.2. THE METHOD OF VARIATION OF DERIVATIVES

This is probably the most widely used method of optimization. The type of autostabilization to be employed is first selected (Δn_r, Δm_q etc.) and the optimization is performed by trial and error - repeated response or stability calculations being carried out with varying values of the adjustable parameters of the autostabilizer until the desired response is most closely approached. Complete linearity is usually assumed.

Variation of derivatives has the following advantages over our procedure:

(i) 'Stability' (i.e. free motion) calculations may be used, rather than the more complicated 'response' calculations.

(ii) In finding the optimum autostabilization by trial and error the off-optimum performance of the autostabilizer has been investigated.

The disadvantages relative to our procedure are as follows:

(i) The procedure is one of trial and error, and is therefore likely to be tedious, particularly for complicated autostabilizers with several adjustable parameters.

(ii) Non-linearities can only be taken into account by means of calculations starting from the equations of motion. Such calculations are tedious and complicated even for quite simple non-linear systems. For non-linear systems advantage (i) also disappears.

(iii) The presence of the pilot is ignored.

It is also possible to estimate the effects of variation of derivatives on the free motion of the aircraft by

(a) constructing relative damping diagrams (Ref. 12) and

(b) The approximate method of Mitchell (Ref. 17)

Both these techniques are applicable only to linear systems and would appear to demand rather more tedious calculations than the procedure of this report.

(Disadvantage (iii) also applies).

7.3. OPTIMIZATION OF FREQUENCY RESPONSE

This procedure consists of adjusting the aircraft frequency response by means of trial and error variations in one or more derivatives until a satisfactorily close approximation to the desired frequency response is attained. Compared with our procedure, this has the advantage that the result of the procedure is in graphical form, which consideration will assist rapid convergence on the optimum values of the autostabilizer adjustable parameters.

The disadvantages are:-

(i) The method is applicable only to frequency response. Since we are primarily concerned with transient response it would seem to be more simple and realistic to work in terms of transient response throughout rather than in terms of the frequency response associated with the desired transient response.

(ii) It is difficult to include non-linear effects in the analysis.

(iii) The presence of the pilot can only be taken into account by assuming a form of transfer function for the pilot. This is, in fact, attempted in Ref. 13. However, it appears that the human pilot is sufficiently adaptable to be able to vary his transfer function to suit the demands made upon him; the choice of transfer function is, therefore, somewhat arbitrary and possibly unrealistic. Whilst (as has been shown) it is quite possible to choose an unrealistic effort function (within reason) and yet achieve a satisfactory autostabilizer system by straightforward application of our procedure, an unrealistic choice of transfer function may lead to unrealistic values for the optimum adjustable parameters.

(In this connection, it appears to the writer that although the human pilot is able to vary his transfer function considerably, the possible variation of effort function would be less marked and it might be possible to successfully determine the true effort function experimentally. A possible experimental procedure would be to measure some physiological parameter of mental and physical effort, such as, perhaps, blink rate, while the pilot is piloting a flight simulator under carefully controlled and repeatable conditions. Extraneous disturbances would be simulated and the control deflections supplied by the pilot recorded and correlated with the selected physiological parameter.)

7.4. THE METHOD OF STANDARD FORMS

A full discussion of this technique is given in Ref.10 In the present context, a standard form is a particular numerical form of a given aircraft transfer function. Thus in Example 7, from Equation 5.3, Aircraft Transfer Function $\frac{\bar{v}(s)}{\bar{v}_o} = \frac{Q(s)}{\Delta(s)}$

where $Q(s)$ is a quadratic expression in s

$\Delta(s)$ is a quartic expression in s

Since each possible form of response is associated with a given form for $Q(s)/\Delta(s)$ it follows that the 'optimum' response is associated with a certain standard form of the aircraft transfer function. In this context the 'optimum' response is that for which a certain specified response parameter (for example $\int_0^\infty \bar{v}^2 \, d\tau$) is a minimum. Lists of coefficients of $Q(s)$ and $\Delta(s)$ for various 'optimum' forms of response are available, and are usually referred to as 'standard form coefficients'.

Compared with our technique the method of standard forms has the advantage of greater simplicity and ease of working. The relative disadvantages are:-

(i) A prohibitively complicated autostabilizer system may be demanded to attain the standard form exactly, e.g. simultaneous variation of a large number of derivatives may be demanded. With a practical autostabilizer system it may well be impossible to attain the standard form exactly; in such circumstances it is difficult to formulate a systematic procedure for optimizing the available autostabilizer system, since the relation between the standard form coefficients and the time history of the response is generally complicated.

(ii) The desired response must be a (published) 'optimum' form.

(iii) The method is not applicable to non-linear systems.

(iv) The presence of the pilot can only be taken into account by assuming a pilot's transfer function. The disadvantages of such an assumption are similar to those discussed in Section 7.3.

7.5. OTHER METHODS.

It will be found that (with two exceptions) most of the remaining published methods of optimization are variants of one of the methods described above. The exceptions are:-

(i) Phase-plane methods of optimization

and

(ii) Methods appropriate to statistically-described inputs.

Neither of these methods are readily comparable with the procedure of this report. Phase-plane methods of optimization are at present virtually restricted to systems of one degree of freedom.[1] Small-perturbation aircraft motions usually possess two or three degrees of freedom and the representation resulting from removal of one or more degrees of freedom is generally of too limited realism to be suitable for optimization purposes. Optimization for statistically described inputs has not been attempted in the present work and no comparison can therefore be made.

From the foregoing comparisons, the procedure of this report is seen to possess some important advantages over those hitherto available, and the author believes that it will be found to be of considerable utility in practical calculations.

7.6. CONCLUSIONS

1. A novel procedure for the optimization of aircraft autostabilizer systems has been developed.

2. The procedure is straightforward and its application does not result in demands for autostabilizer systems of prohibitive complexity.

3. Many important non-linear effects may be taken into account, with only slight extra complication in the calculation required.

4. The procedure is applicable to piloted aircraft, but may be modified to form an approximate optimization procedure of good accuracy for pilotless aircraft without non-linearities.

5. The results of some examples presented herein support a suggestion that this approximate procedure may frequently be applied with success to pilotless aircraft having certain non-linearities, either in the autostabilizer, or in the aircraft dynamics.

7.7. ACKNOWLEDGMENTS

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NOTES ON CHAPTERS

CHAPTER 1

1. (Page 3). For a full discussion of some of the problems of simulator presentation see Ref.16.

2. (Page 3). See (for example) Ref.3.

CHAPTER 2

1. (Page 13). Simultaneous variation of m_q, m_w and z_w so that the manoeuvre margin is kept constant whilst the damping of the longitudinal oscillation is increased is possible, but hardly practicable.

CHAPTER 3.

1. (Page 29) Note that although, in this example, the maximum value of the desired response (in $\hat{q}$) occurs at $\tau = 0$, this will not generally be the case. In general $\hat{q}_D$ max may be assigned a value slightly higher than the true value with negligible loss of accuracy. No special significance attaches to the value of 1.4 chosen here.

2. (Page 34) This was, in fact, necessary in Examples 5 and 6 due to the large β and a of the desired response.

CHAPTER 6

1. (Page 52) Although the demonstration of this fact has been effected by means of Cardinal Spectrum Analysis it is generally true for linear systems since the approximation

inherent in Cardinal Spectrum Analysis can be removed by allowing the spectrum interval to tend to zero.

CHAPTER 7.

1. (Page 60) Our authority for this statement is Ref.11.

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APPENDIX 1 - LIST OF DERIVATIVES.

The derivatives are calculated for a Light Fighter type of aircraft, flying at M = 0·9, 50,000'.

Span = 22·0'   Wing Area = 136·6 '

Tail Moment Arm = 10·39'   A.U.W. = 6,000 lb.

$m_w = -\cdot 108$	$m_{\dot{w}} = -\cdot 0895$	$m_q = -\cdot 2263$	$i_B = \cdot 298$
$z_w = -2\cdot 35$	$m_{\dot{\eta}} = -\cdot 205$	$\mu \equiv \mu_1 = 365\cdot 0$	
$\ell_v = -\cdot 0783$	$n_v = + .0825$	$y_v = -.393$	$C_{L_o} = 0.32$
$\ell_p = -.40$	$n_p = -.0148$	$y_p = 0$ (assumed)	
$\ell_r = +.1085$	$n_r = -.214$	$y_r = 0$ (assumed)	
$\ell_\zeta = + .0128$	$n_\zeta = -.071$	$y_\zeta = 0$ (assumed)	
$\ell_\xi = -.14$	$n_\xi = -.0172$	$y_\xi = 0$ (assumed)	
$i_A' = +.0446$	$i_C' = +.284$	$i_E = -.0164$	$\mu_2 = 343.5$

TABLE I. TABULATED VALUES OF THE COEFFICIENTS $P_o, P_2, \ldots\ldots S_6, S_8$.

Coefficient	Fractional value	Decimal value	Coefficient	Fractional value	Decimal value
P_o	$+\frac{35.45.7}{32.64}$	+ 5.3833	R_o	$+\frac{49.99.13}{32.64}$	+ 30.7925
P_2	$-\frac{25.49.9}{16.32}$	−21.5332	R_2	$-\frac{15.63.143}{16.32}$	−263.9355
P_4	$+\frac{49.81.11}{32.32}$	+42.6357	R_4	$+\frac{49.121.117}{32.32}$	+677.4346
P_6	$-\frac{15.99.13}{16.32}$	−37.7051	R_6	$-\frac{27.77.169}{16.32}$	−686.2324
P_8	$+\frac{5.143.35}{32.64}$	+12.2192	R_8	$+\frac{21.121.13.15}{32.64}$	+241.9409
Σ P	+1	+ 0.9999	Σ R	0	+ 0.0001
Q_o	$-\frac{21.55.35}{32.64}$	−19.7388	S_o	$-\frac{33.15.65}{32.64}$	− 15.7104
Q_2	$+\frac{15.49.99}{16.32}$	+142.1191	S_2	$+\frac{35.143.15}{16.32}$	+146.6309
Q_4	$-\frac{35.81.121}{32.32}$	−334.9951	S_4	$-\frac{21.135.143}{32.32}$	−395.9033
Q_6	$+\frac{35.39.121}{16.32}$	+322.5879	S_6	$+\frac{121.117.15}{16.32}$	+414.7559
Q_8	$-\frac{35.33.13.15}{32.64}$	−109.9731	S_8	$-\frac{121.169.15}{32.64}$	−149.7729
Σ Q	0	0.0000	Σ S	0	+ 0.0002

TABLE II

COEFFICIENTS OF b^2

	constant	h	h^2	h^3	h^4	h^5	h^6	h^7	h^8
z^2	28.97992	-231.83936	922.72022	-2,242.12204	3,573.18507	-3,741.40362	+2,463.62287	-921.45232	149.30885
z^4	-212.51976	2,380.21910	-11,410.47136	31,507.40934	-54,841.85154	60,978.87828	-41,890.72252	16,176.62560	-2,687.56660
z^6	721.15077	-9,778.33106	54,708.74390	-169,344.67772	318,835.18010	-373,860.44870	266,678.83201	-105,967.21632	18,006.73122
z^8	-1,384.76160	21,427.25150	-134,311.76210	462,211.16108	-900,211.96170	1,098,909.41662	-807,561.14910	328,458.99734	-56,874.19162
z^{10}	1,568.38754	-26,508.59960	179,214.98722	-637,141.34688	1,323,171.37802	-1,665,600.43308	1,253,727.45165	-519,909.20492	91,477.37931
z^{12}	-967.52498	17,323.32854	-123,069.44164	454,756.27416	-973,406.28854	1,255,316.13220	-963,730.96396	406,250.86470	-72,472.38044
z^{14}	246.81667	-4,607.26018	33,940.21923	-129,135.27656	283,077.46909	-372,329.12924	290,613.62730	-124,238.38788	22,431.92157

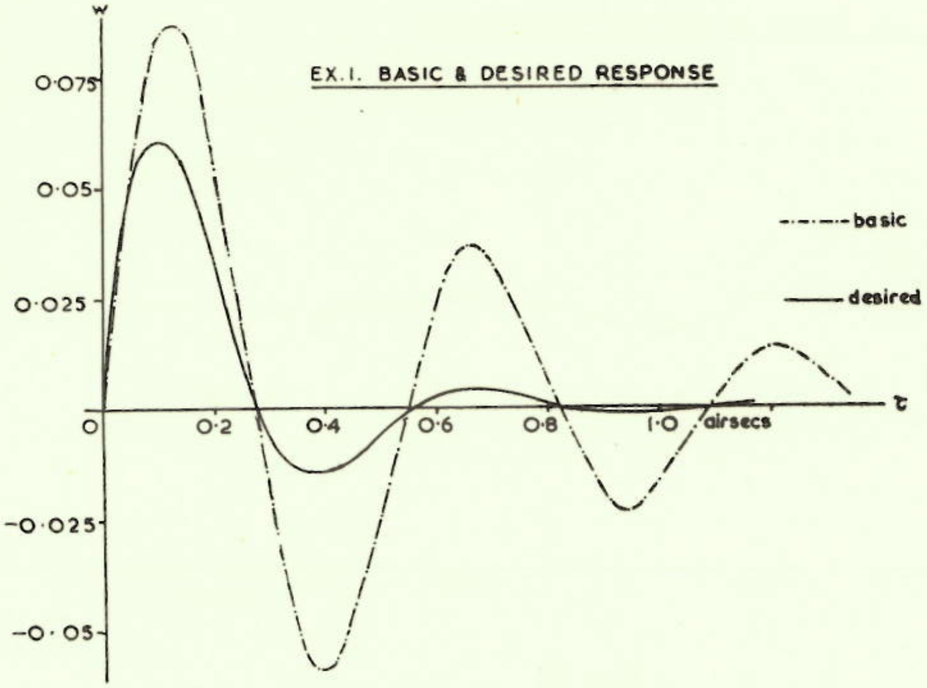

EX.1. BASIC & DESIRED RESPONSE
w
0·075
0·05
0·025
−0·025
−0·05
0
0·2
0·4
0·6
0·8
1·0
airsecs
τ
basic
desired

FIG. 1

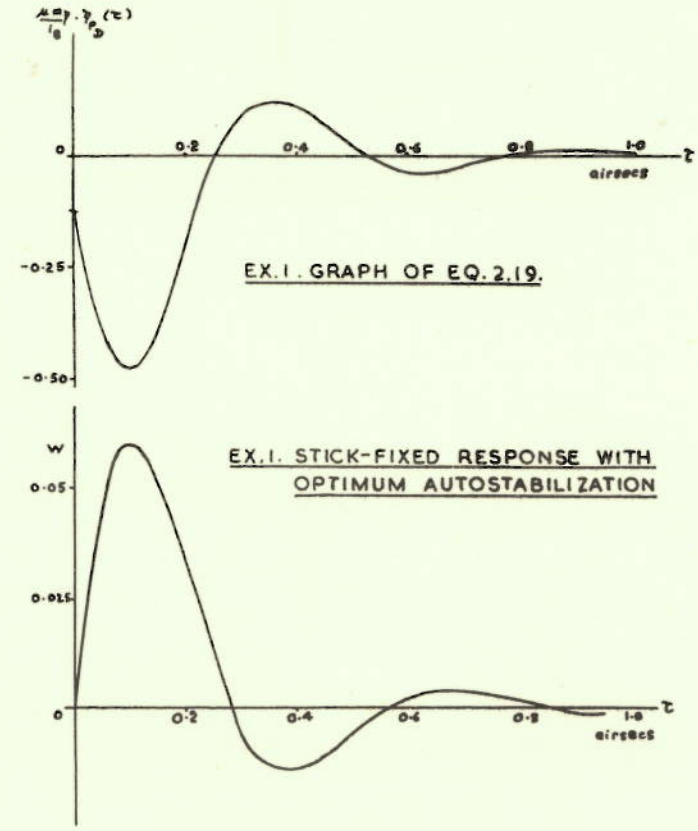

EX.1. GRAPH OF EQ. 2.19.
0
−0·25
−0·50
0·2
0·4
0·6
0·8
1·0
τ
airsecs
EX.1. STICK-FIXED RESPONSE WITH OPTIMUM AUTOSTABILIZATION
w
0·05
0·025
0
0·2
0·4
0·6
0·8
1·0
τ
airsecs

FIG. 2

EX.3. BASIC & DESIRED RESPONSE

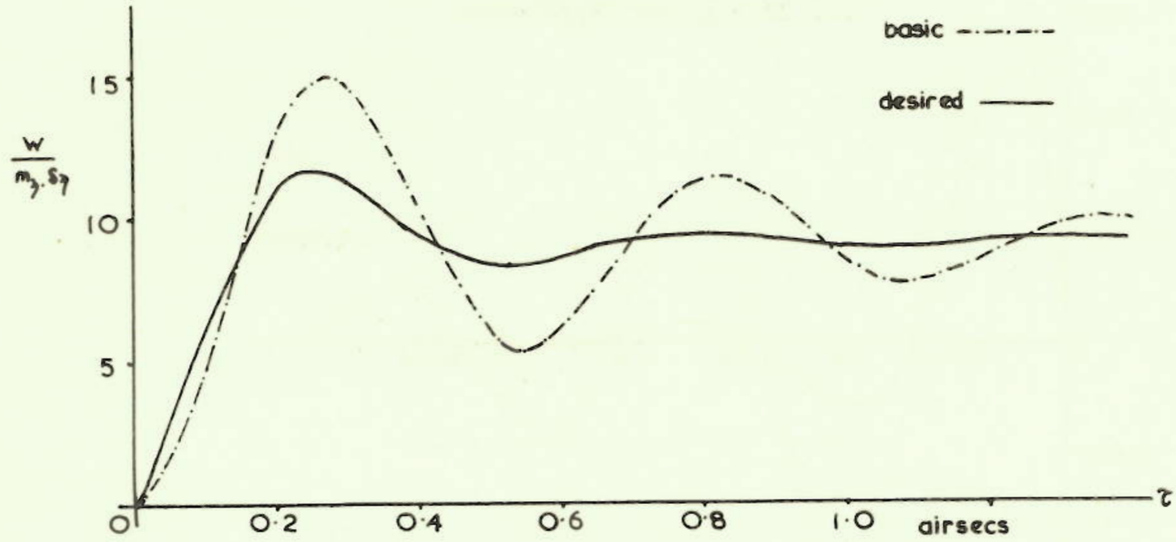


FIG. 3

EX.3. STICK-FIXED RESPONSE WITH OPTIMUM AUTOSTABILIZATION

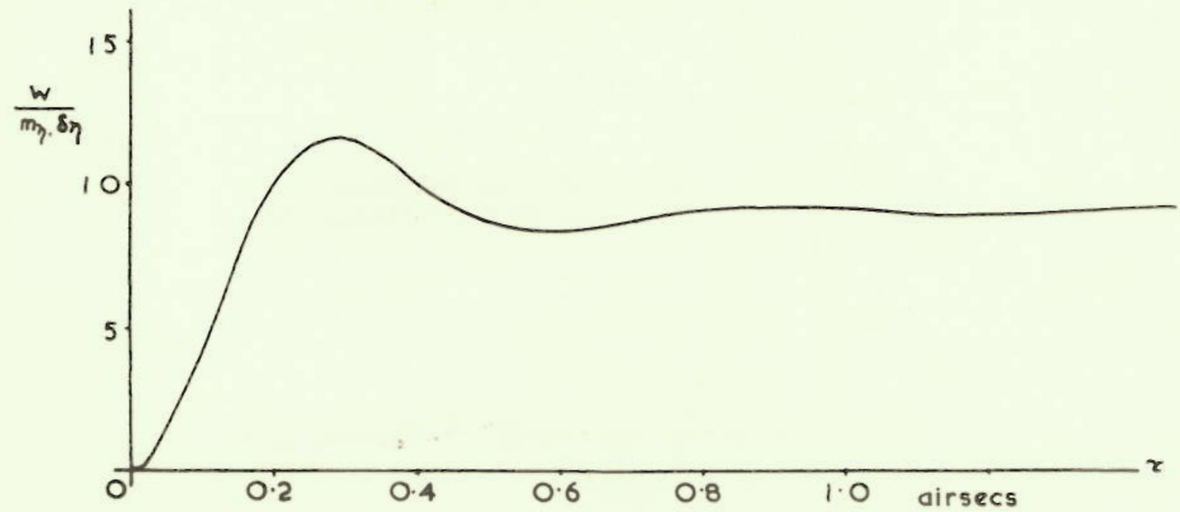
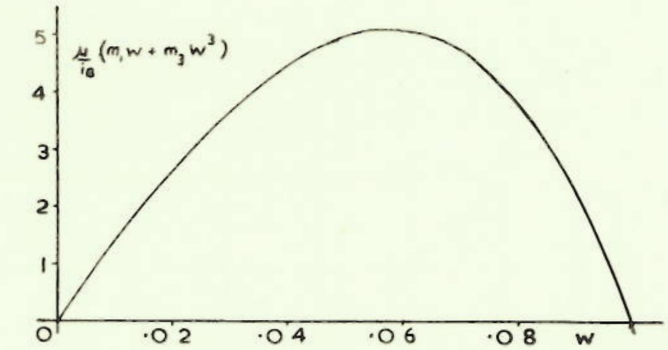


FIG. 4.



GRAPHS FOR EXAMPLE 4

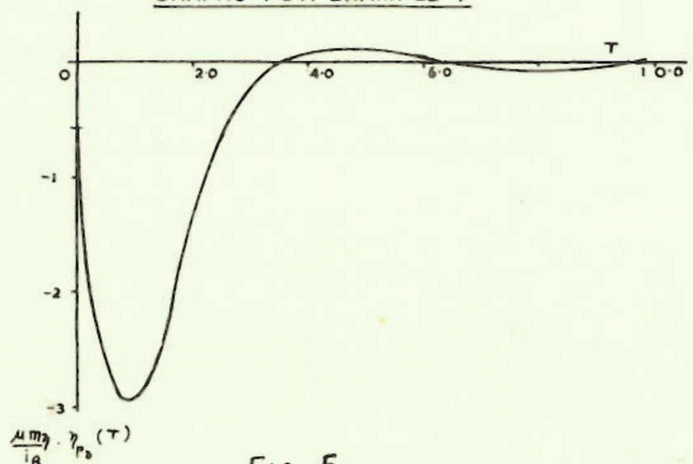


FIG. 5

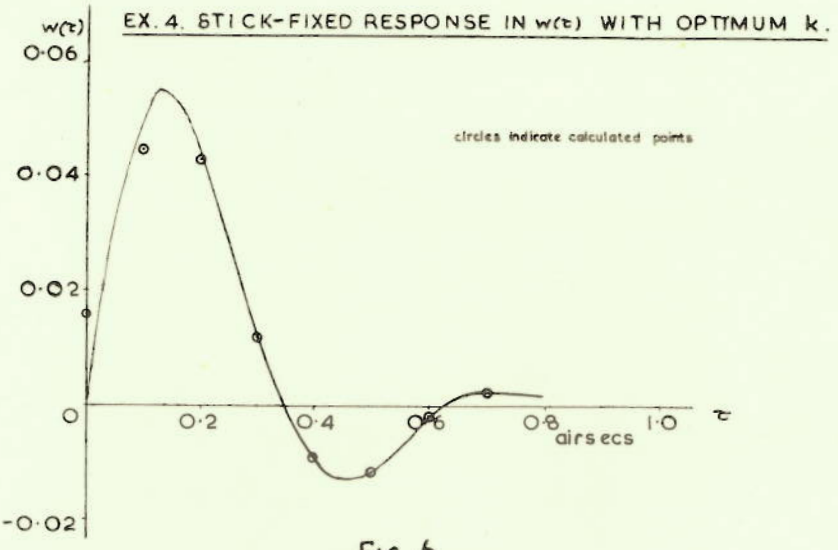
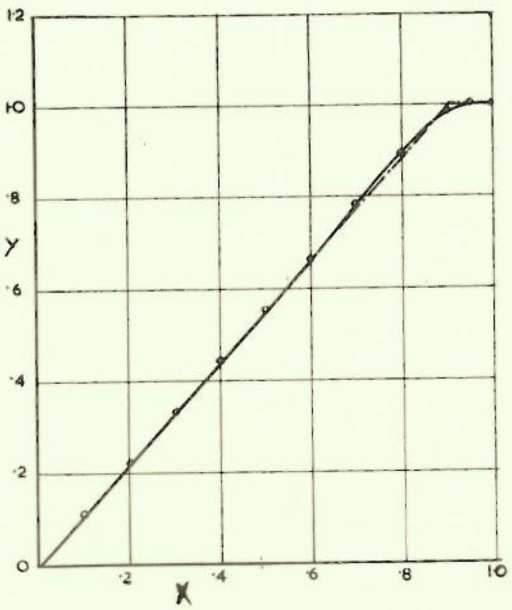


FIG. 6

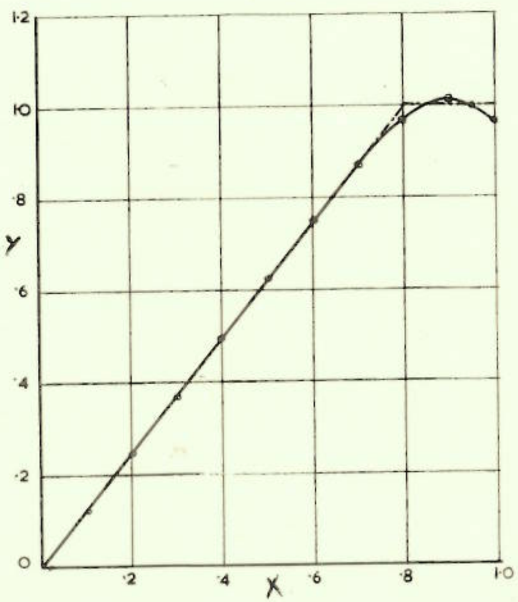


POLYNOMIAL APPROXIMATION

TO AUTOSTABILIZER CHARACTERISTICS

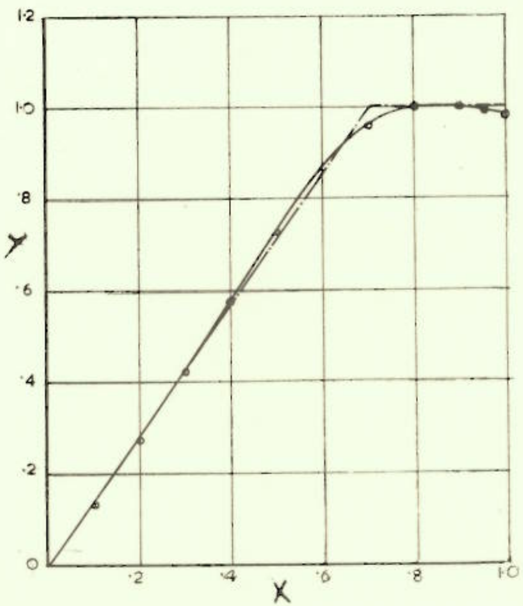
k = 0·9

FIG. 7



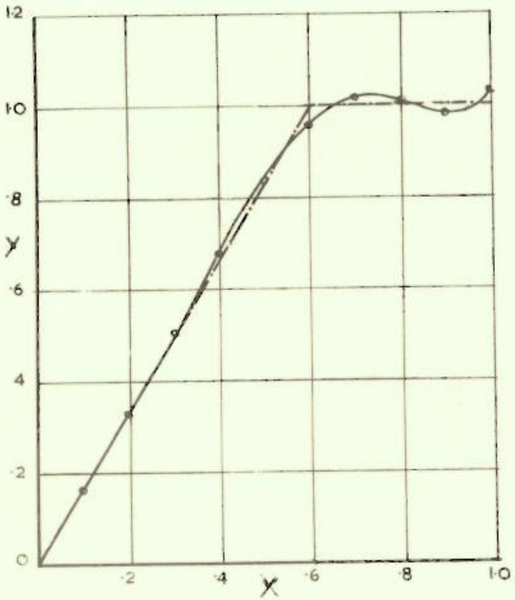
k = 0·8

FIG. 8



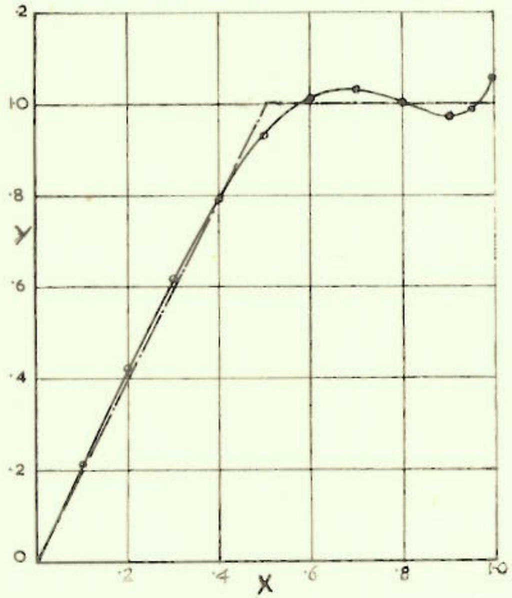
k = 0·7

FIG. 9



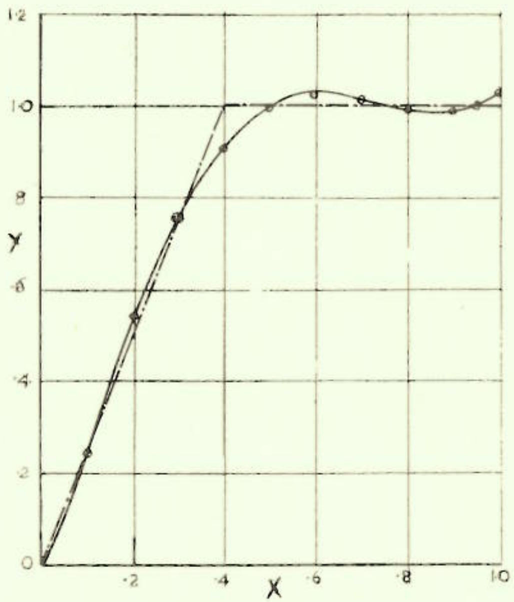
k=0·6

FIG. 10



k=0·5

FIG. 11

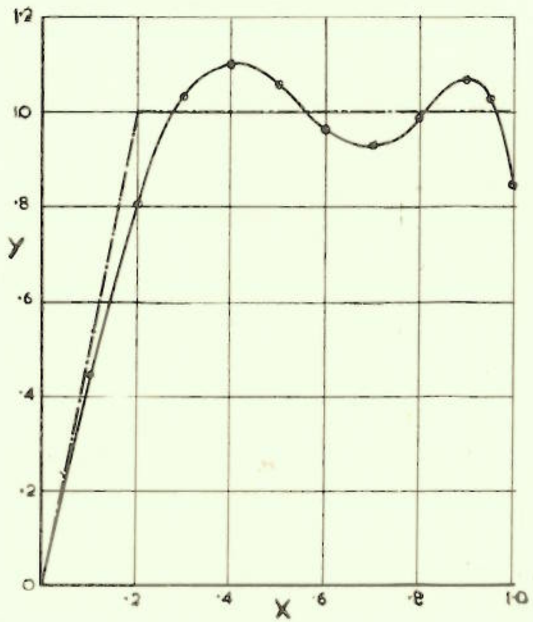


k=0·4

FIG. 12

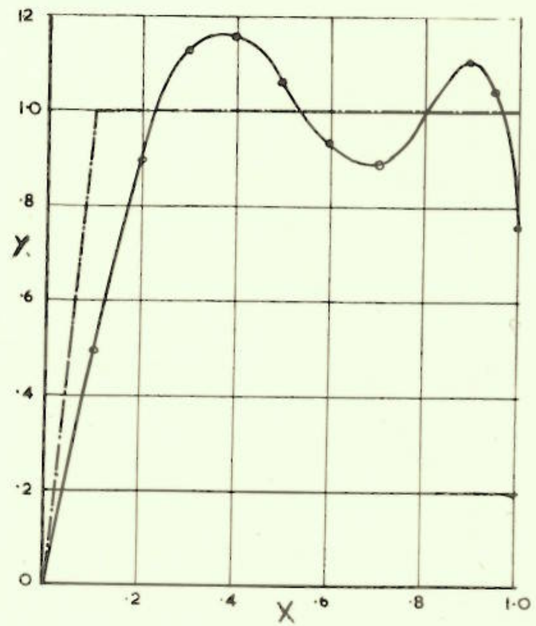
k=0·3

Fig. 13



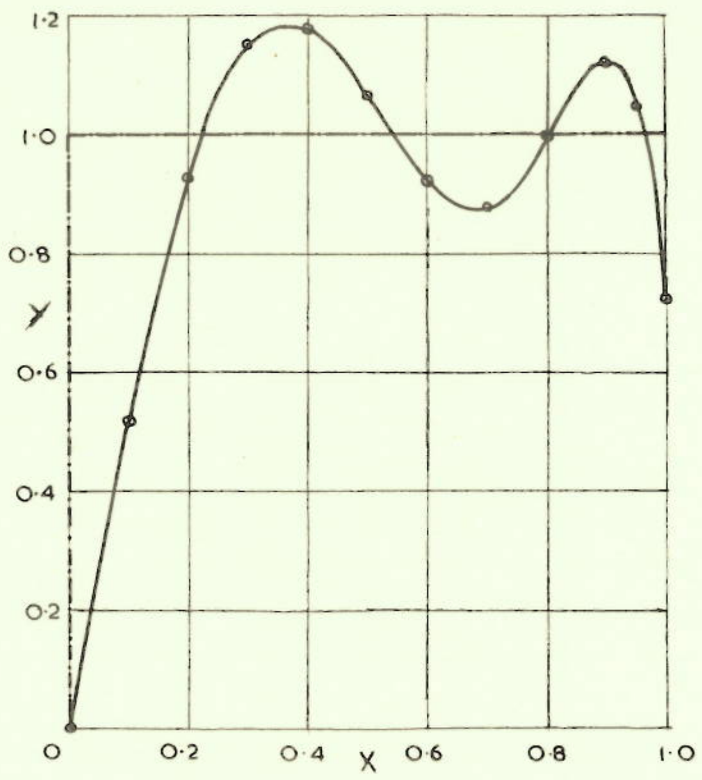
k=0·2

Fig. 14



k=0·1

Fig. 15



k = O

FIG. 16

EX.5. STICK-FIXED RESPONSE IN $w(\tau)$ & $\hat{q}(\tau)$ WITH OPTIMUM FLICKER AUTOSTABILIZATION

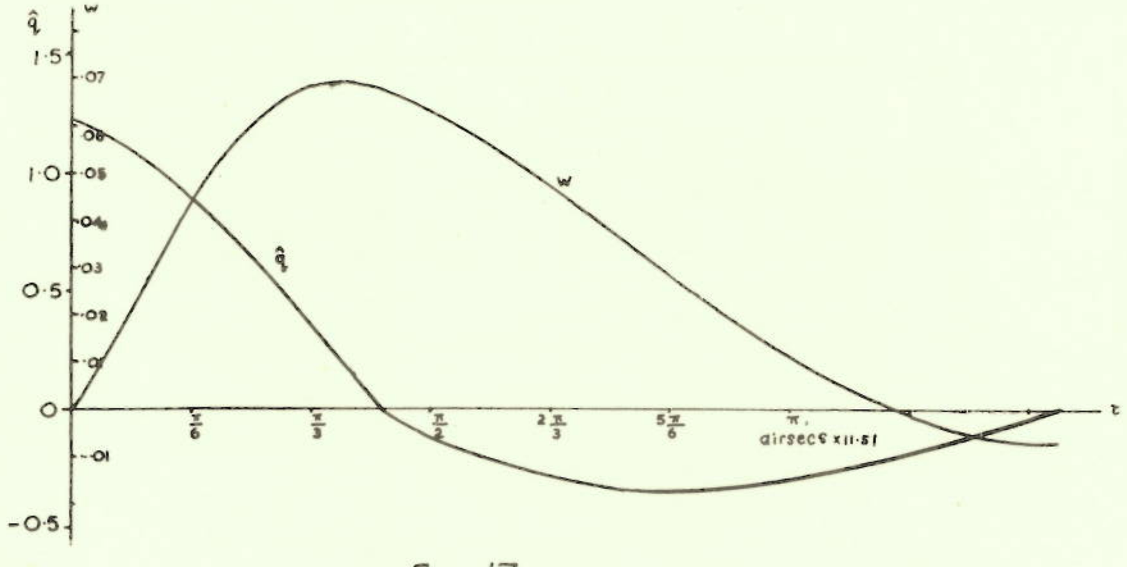


FIG. 17

VARIATION OF I_∞ WITH a, FOR n=1,2,3.

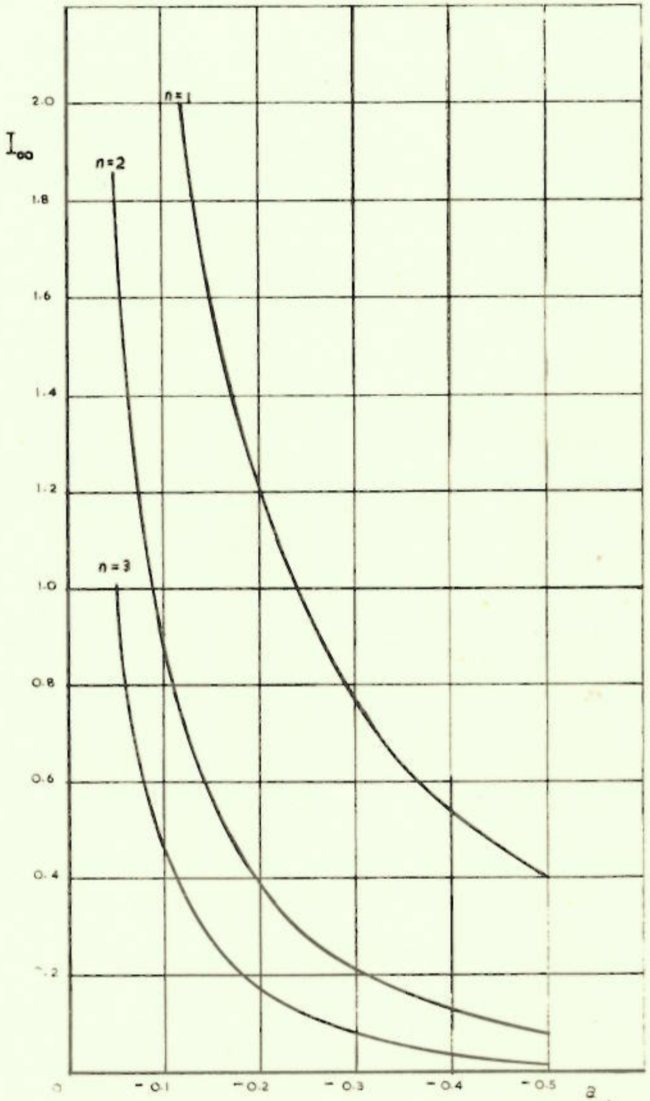
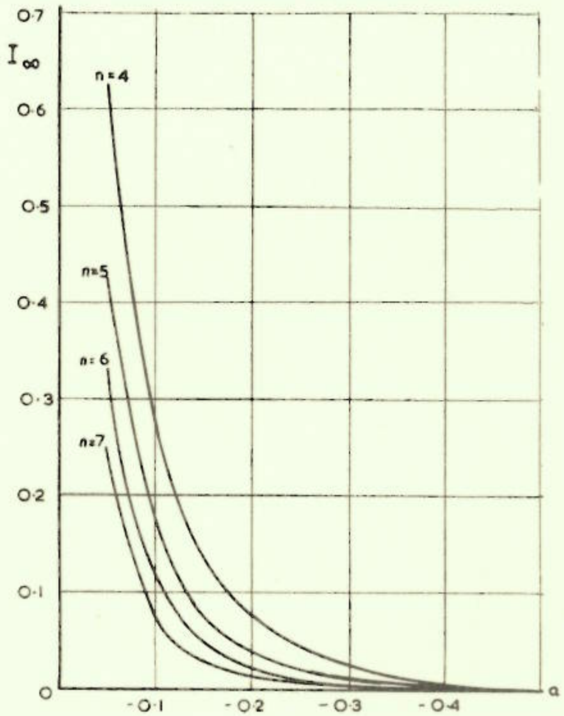
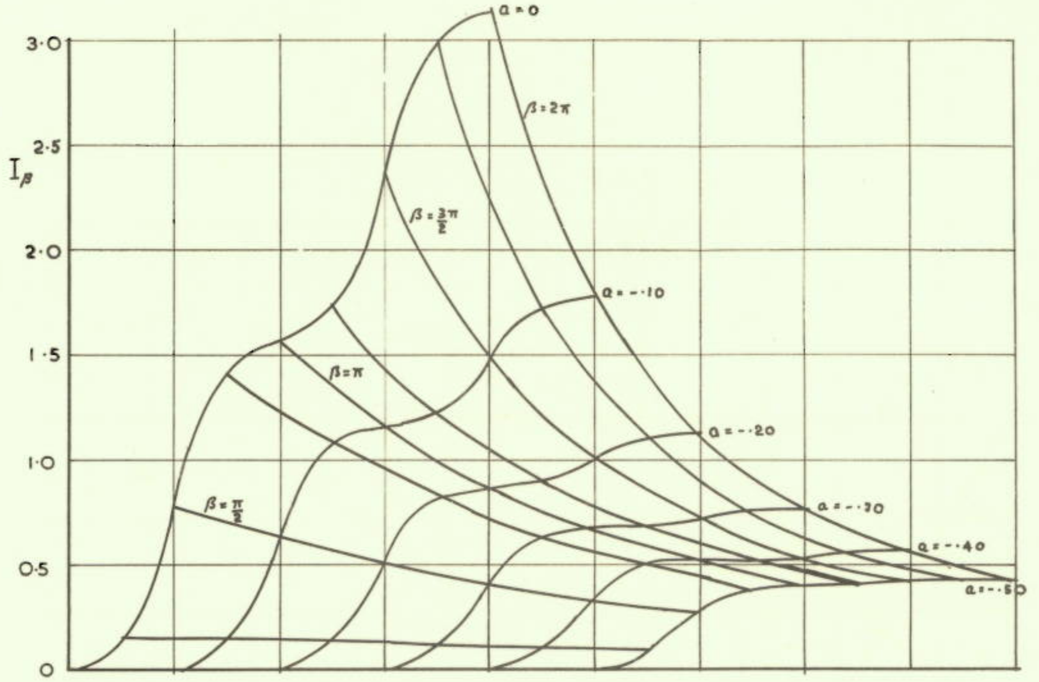


FIG 18



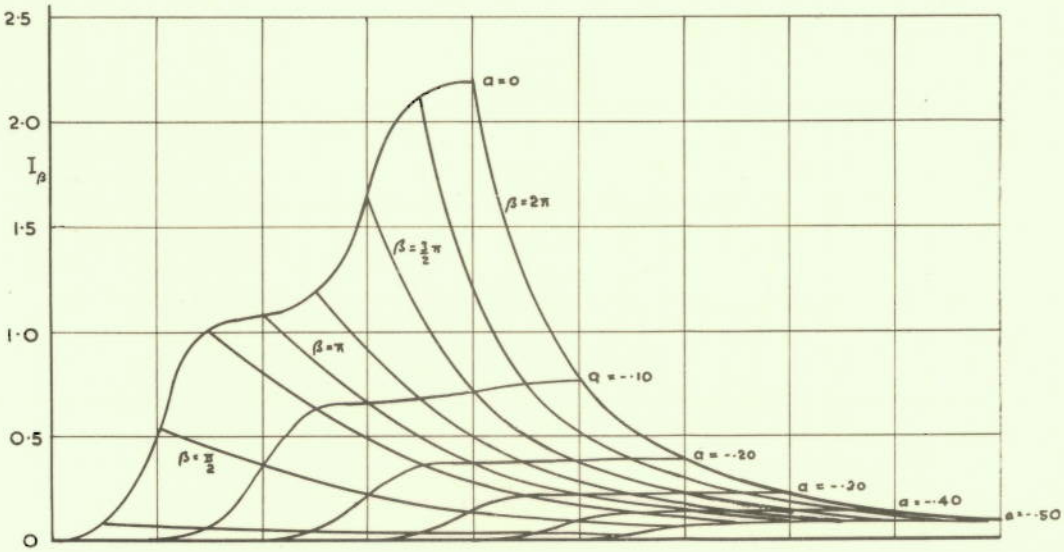
VARIATION OF I_∞ WITH a, FOR n=4,5,6,7.

FIG. 19



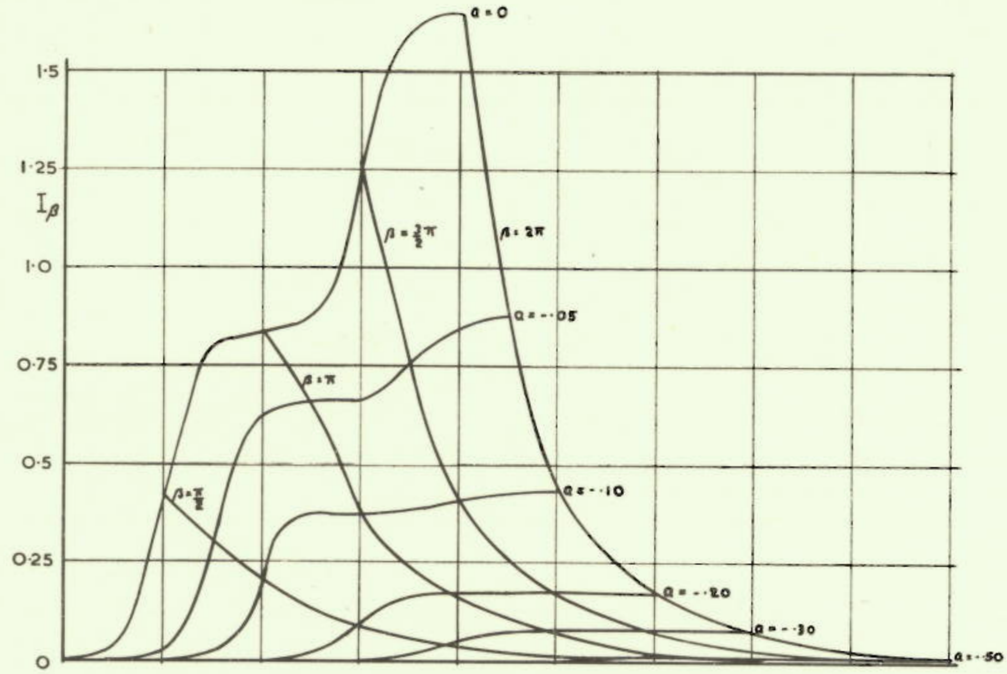
VARIATION OF I_β WITH a AND β, FOR n = 1.

FIG. 20



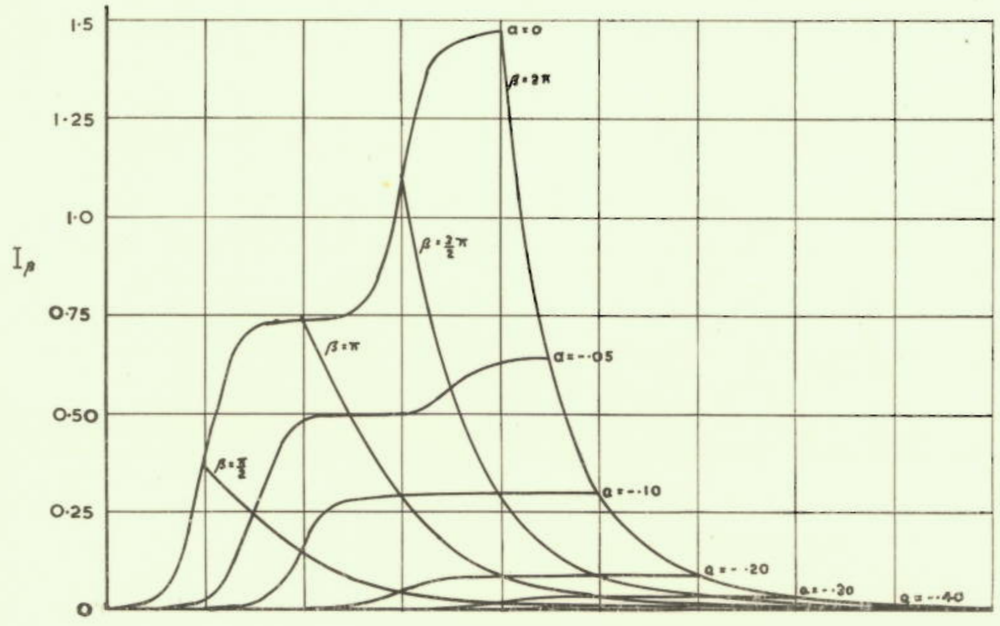
VARIATION OF I_β WITH β AND a FOR n = 2

FIG 21



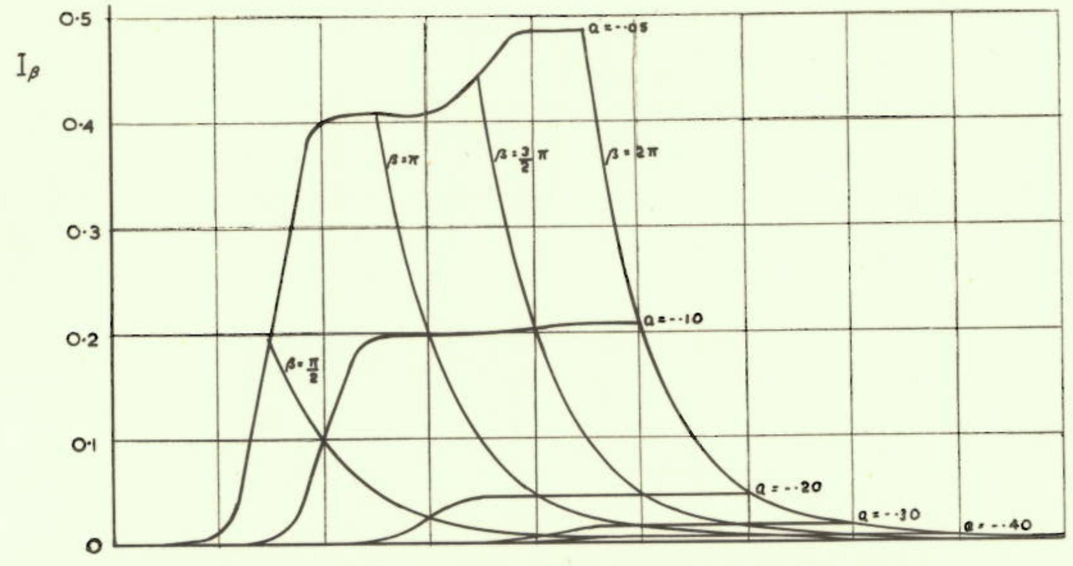
VARIATION OF I_β WITH β AND a FOR n = 3

FIG 22



VARIATION OF I_β WITH β AND a, n = 4

FIG. 23



VARIATION OF I_β WITH β AND a, n=5

FIG. 24

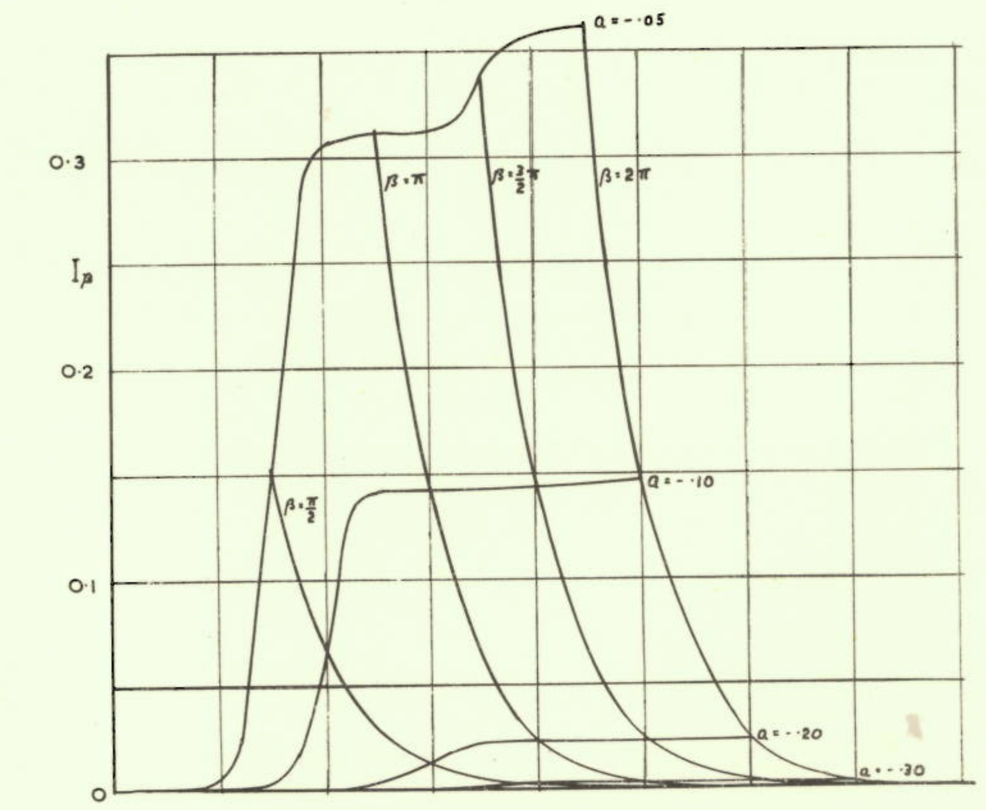
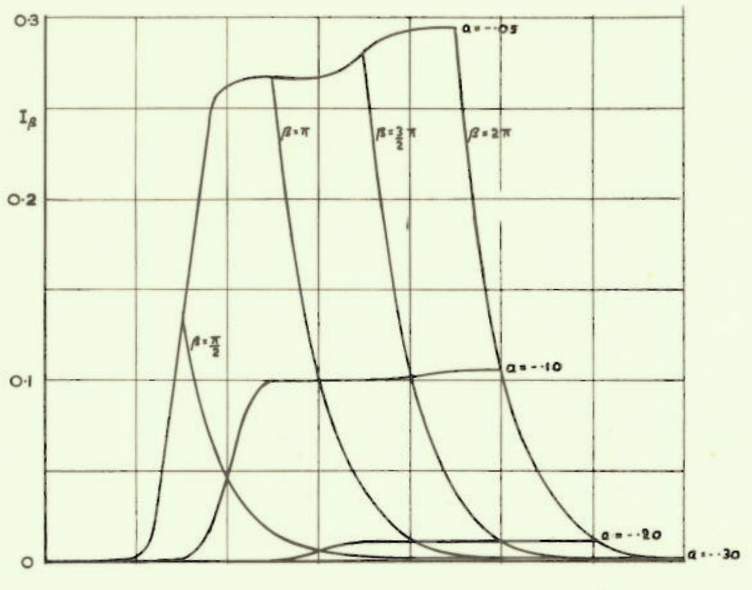


FIG. 25 VARIATION OF I_β WITH β & a, n=6.



VARIATION OF I_β WITH β AND a, $n=7$.

FIG. 26

EX. 6. STICK-FIXED RESPONSE IN $w(\tau)$ WITH OPTIMUM AUTOSTABILIZATION

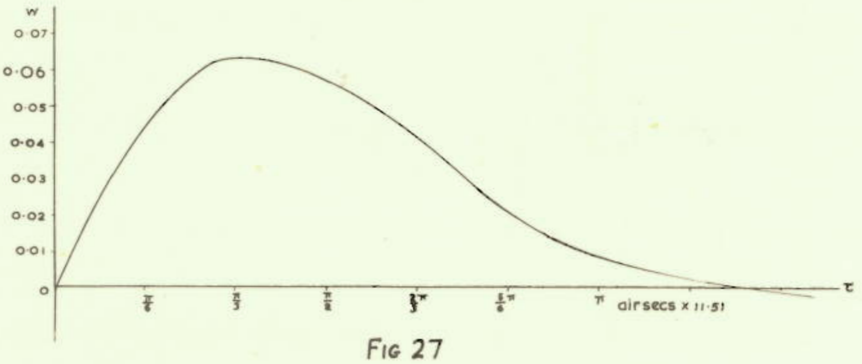


FIG 27

EX. 6. STICK-FIXED RESPONSE IN $\hat{q}(\tau)$ WITH OPTIMUM AUTOSTABILIZATION

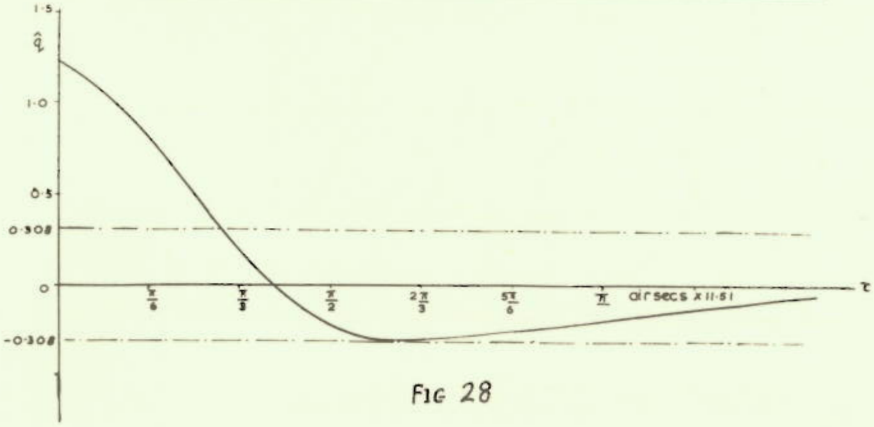


FIG 28

EX.7. BASIC & DESIRED RESPONSES

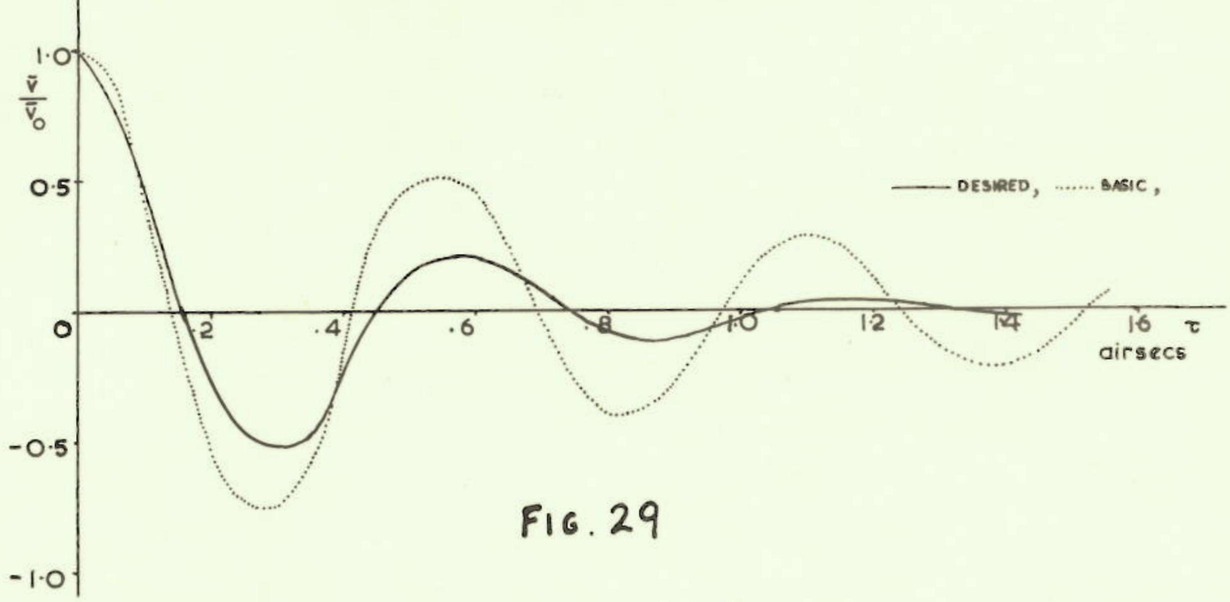


FIG. 29

EX.7 STICK-FIXED RESPONSE WITH OPTIMUM AUTOSTABILIZATION

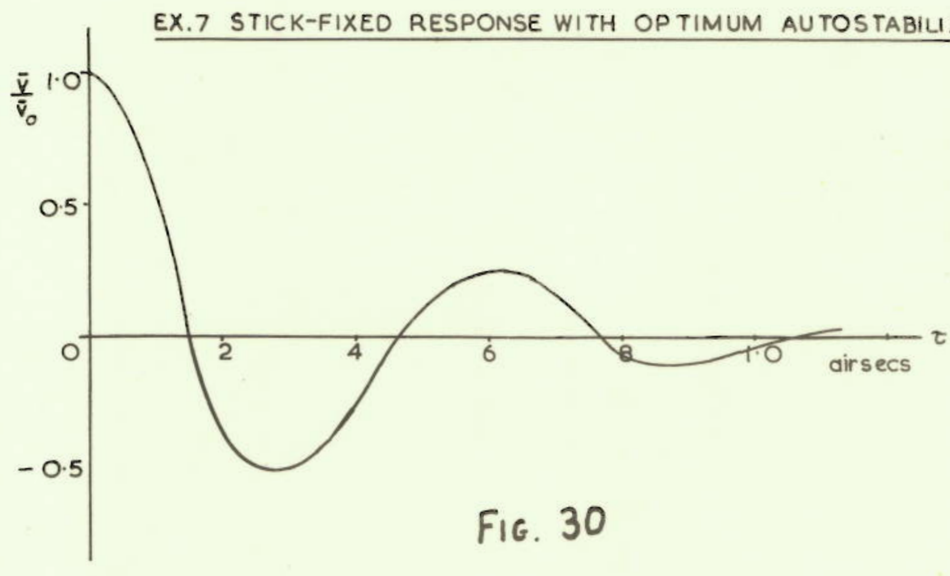
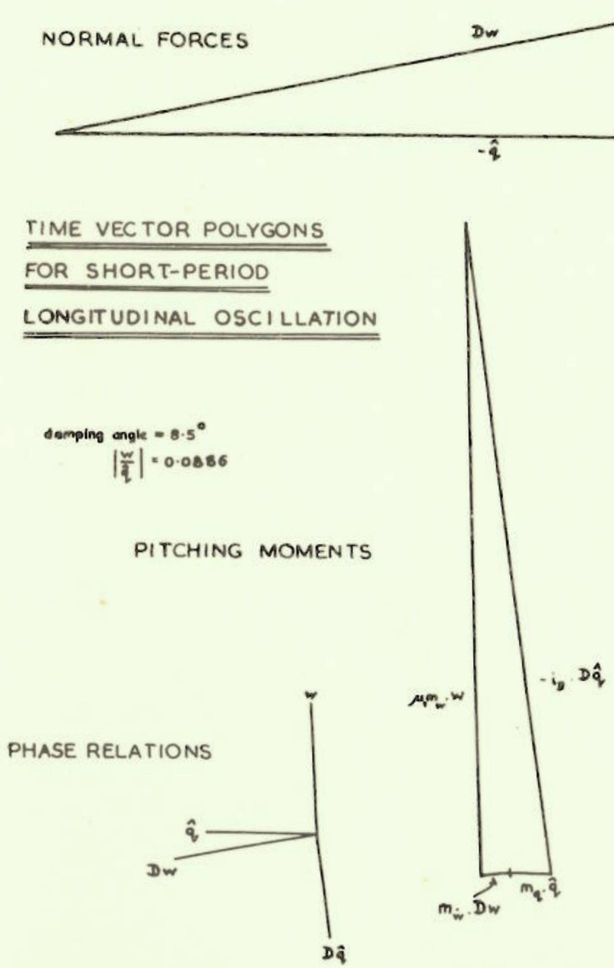
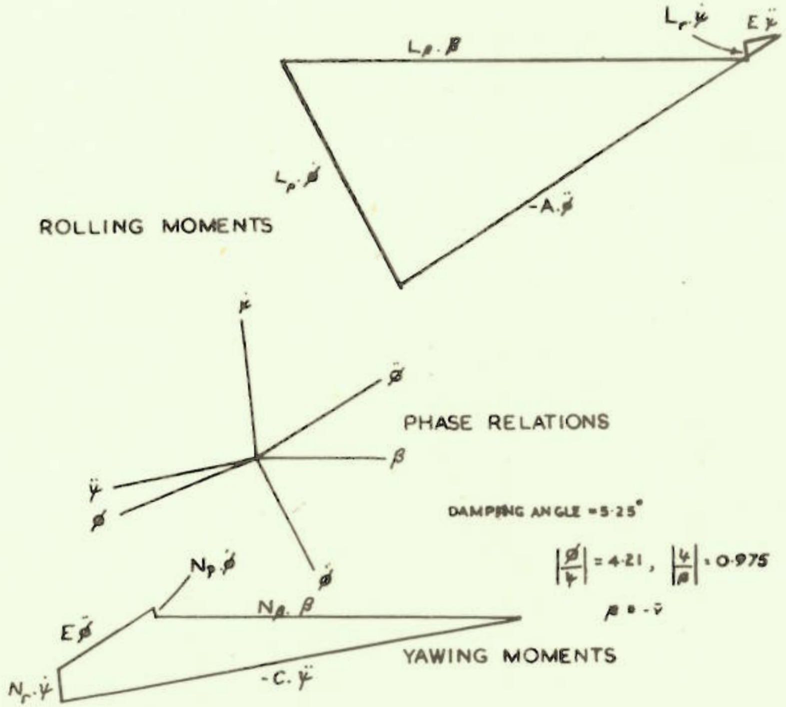


FIG. 30



TIME VECTOR POLYGONS
FOR SHORT-PERIOD
LONGITUDINAL OSCILLATION

FIG. 31



EX.7. TIME VECTOR POLYGONS

(SIDE FORCE POLYGON NOT SHEWN)

FIG. 32