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An Approximate Solution of the Turbulent Boundary Layer  
Equations in Incompressible and Compressible Flow \*

- by -

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SUMMARY

If over the 'outer region' of the boundary layer, where the mean velocity varies but little from its value outside the shear layer, a virtual eddy viscosity is defined, which is constant over the outer region but varies in the direction of the mainstream, a solution of the turbulent boundary layer equations can be found which satisfies the appropriate boundary conditions. The solution leads to a compatibility condition for the virtual eddy viscosity in terms of the wall shear stress, the boundary layer momentum thickness and the mainstream velocity, at least for the case of a constant external velocity. This compatibility condition, which can be expressed as

$$\frac{u_{\tau} \delta}{\nu_T} \approx \text{constant}$$

for moderate to high Reynolds numbers, where  $u_{\tau}$  is the shear velocity,  $\delta$  is the boundary layer thickness and  $\nu_T$  is the virtual eddy (kinematic) viscosity, is just the condition Townsend (1956) found for the equilibrium of the large eddies. The numerical value of the constant derived by Townsend agrees with ours for Reynolds numbers (based on  $x$ ) of about  $10^7$ . With this relation for  $\nu_T$  an equation, analogous to the momentum integral equation solution, can be found for  $R\delta_2$  as a function of  $U_1(x)$ , where  $R\delta_2$  is the Reynolds number based on momentum thickness and  $U_1$  is the local freestream velocity, with one disposable parameter. Provided that this

## Summary (Continued)

parameter is suitably chosen the present method gives results in a negative pressure gradient which are in fair agreement with similar results obtained from the well known methods of Spence and Maskell. It is shown however that the compatibility condition for  $\nu_T$  does not apply in the case of an adverse pressure gradient approaching separation. It is shown that this case requires rather different treatment for whereas the relation for  $\nu_T$  found from zero pressure gradient conditions would show a decrease as separation is approached it is known experimentally that just the opposite is true.

The method is extended to compressible flow, and with relatively minor assumptions gives values of the ratio of the compressible to the incompressible skin friction coefficient in approximate agreement with that predicted by Mager, for the case of zero pressure gradient and zero heat transfer.

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## NOTATION

A	function of p, constant in the law of the wall
$a_1, a_0$	speed of sound
$C \equiv \bar{\rho} \mu / \rho_1 \mu_1$	
$C_1, C_2$	constants
$C_p$	pressure coefficient
$c_f$	local skin friction coefficient
$C_F$	overall skin friction coefficient
$C_5, C_6$	Power law constants
G	shear stress function
h	mean specific enthalpy
$h_r$	recovery enthalpy
$h_a$	mean stagnation enthalpy
$h_w$	wall enthalpy
K	Von-Karman constant
$k_T$	eddy conductivity
L	length of plate
M	Mach number
m	velocity profile index
n	shear stress index
p	pressure, Laplace Transform operator
$\dot{q}_w$	rate of heat transfer at the wall
q	index

Notation (Continued)

$r$	$\equiv$	$\mu_T / \mu_0$
$R_c$		Reynolds number based on chord $c$ and velocity $U_0$
$R_{\delta_2}$		local Reynolds number based on momentum thickness
$R_x$		local Reynolds number based on $x$
$t$		independent variable
$s$		index
$S$		$1 - h_s / h_{s_1}$
$\bar{u}, \bar{v}$		mean velocity components in compressible flow
$\bar{u}_1$		external velocity in compressible flow
$U, V$		mean velocity components in incompressible flow
$U_1, U_0$		external velocity in incompressible flow
$\bar{u}^2, \bar{v}^2, \bar{uv}$		Reynolds stresses in incompressible flow
$\overline{u'v'}$		Reynolds stress in compressible flow
$u_T$		shear velocity
$U_T$		shear velocity in pseudo incompressible
$X, Y$		co-ordinates in the pseudo incompressible flow
$Y$		independent variable
$y$	$\equiv$	$x^{\beta+1}$ (Incompressible flow)
$Z$	$\equiv$	$U_1^2 - U^2$
$\alpha$	$\equiv$	$(1 + n)G$
$\beta$		index
$\delta$		boundary layer thickness
$\delta_2$		momentum thickness

### Notation (Continued)

$\mu$	viscosity
$\nu$	kinematic viscosity
$\mu_T$	eddy viscosity
$\nu_T$	(kinematic) eddy viscosity
$\rho_0$	density in incompressible flow
$\bar{\rho}$	mean density in compressible flow
$\sigma$	Prandtl number
$\sigma_T$	turbulent Prandtl number
$\phi(1)$	constant in the log law velocity formula
$\psi$	stream function
$\tau_w$	wall shear stress
$\tau$	independent variable, local shear stress
$\omega$	viscosity-temperature index
$\zeta$	= $U_0/u_\tau$

Other symbols are defined in the text where they occur

### Suffices

i	denotes incompressible
c	denotes compressible
1	denotes freestream conditions
o	denotes constant freestream conditions

An asterisk denotes an 'intermediate' value

## 1. Introduction

Townsend (1956) has shown that the 'outer region' of the boundary layer is dominated by a group of large eddies, similar to those that exist in wake flow. He shows that, on the assumption that the virtual eddy viscosity is constant across the 'outer region', the equilibrium condition for these eddies is given by

$$\frac{u_T \delta}{\nu_T} = \text{constant}$$

Although the more recent work of Townsend (1957) and Grant (1958) has suggested an improved structure to the big eddies, nevertheless a similar equilibrium condition could be found. The refinement imposed by the assumption that  $\nu_T$  is only constant over a region where the fluid is fully turbulent, and is otherwise multiplied by an intermittency factor, would hardly affect the conclusions drawn from the analysis given below.

The present method of attack follows on somewhat similar lines to that used by Lighthill (1950) in his approximate solution of the laminar boundary layer equations with arbitrary pressure gradient. This method has the advantage over the many other methods of solving the laminar boundary layer problem in that it provides a rapid solution when the external flow is such, that similar velocity profiles do not exist, and only the relation between the wall shear stress and the external velocity are required. The main assumption used by Lighthill was to apply the Fage-Falkner approximation  $U \sim y$  near the wall. The resulting linearised equation of motion was then solved by operational methods.

In our problem the Fage-Falkner approximation does not apply, except in the viscous laminar sub-layer. However in the 'outer region' the change in mean velocity is relatively small and in a first approximation to a turbulent boundary layer solution, can be neglected. Such an approximation has already been used by Liepmann (1958) and Lilley (1960) for solving approximately the heat transfer for fluids of small Prandtl number where the velocity boundary layer thickness is many times that of the temperature boundary layer thickness.

With this assumption and the further one that a virtual eddy viscosity can be specified locally constant over the 'outer region' the turbulent boundary layer equations can be solved if the boundary conditions are chosen appropriately. The solution then leads to a compatibility condition for the virtual eddy viscosity,  $\nu_T$  comparable with the equilibrium condition deduced by Townsend and this is the main result of the present paper.

The present paper does not attempt to review the very extensive literature dealing with methods for solving the overall characteristics of the turbulent boundary layer. In addition a detailed comparison between the results quoted below and those obtained by other authors is not given although sufficient is said to indicate when such agreement is likely and when it is not.

## 2. The incompressible turbulent boundary layer

The steady boundary layer equations of motion and continuity for an incompressible turbulent flow are

$$\frac{\partial U^2}{\partial x} + \frac{\partial UV}{\partial y} = U_1 \frac{\partial U_1}{\partial x} + \frac{\partial}{\partial y} \left( \frac{\mu_0}{\rho_0} \frac{\partial U}{\partial y} - \overline{uv} \right) \quad (1)$$

and 
$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \quad (2)$$

where  $\rho_0$  and  $\mu_0$  are the constant density and viscosity respectively.

Equation (2) shows that if  $\psi$  is the stream function

$$\rho_0 U = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \rho_0 V = - \frac{\partial \psi}{\partial x} \quad (3)$$

If the Reynolds stress  $-\rho_0 \overline{uv}$  is given by

$$-\rho_0 \overline{uv} = \mu_T \frac{\partial U}{\partial y}, \quad \text{where} \quad \nu_T = \mu_T / \rho_0 \quad \text{is the virtual eddy}$$

viscosity, which is a function of  $x$  only, at least in the outer region, then throughout that region equation (1) can be written in terms of von Mises' variables  $(x, \psi)$ , if

$$r(x) = \frac{\mu_T(x)}{\mu_0} = \frac{\nu_T(x)}{\nu_0} \quad (4)$$

and 
$$Z = U_1^2(x) - U^2(x, \psi),$$

as 
$$\frac{\partial Z}{\partial x} = \rho_0 \mu_0 (r(x) + 1) U \frac{\partial^2 Z}{\partial \psi^2} \quad (5)$$

with the boundary conditions  $Z \rightarrow 0$  as  $\psi \rightarrow \infty$  and as  $x \rightarrow 0$ .

But in the 'outer region'  $U \sim U_1$  and  $\mu_T/\mu_0 \gg 1$  so that an approximation to (5) is\*

$$\frac{\partial Z}{\partial x} = \rho_0 \mu_0 r(x) U_1(x) \frac{\partial^2 Z}{\partial \psi^2} \quad (6)$$

at least in the region  $\psi^* < \psi \leq \infty$ , where  $\psi^*$  is the value of  $\psi$  roughly defining the inner limit of the 'outer region'.

If we let

$$t = \int_0^x \rho_0 \mu_0 r(z) U_1(z) dz \quad (7)$$

then from (6)

$$\frac{\partial Z}{\partial t} = \frac{\partial^2 Z}{\partial \psi^2} \quad (8)$$

which is the standard diffusion equation.

Since  $Z \rightarrow 0$  as  $t(x) \rightarrow 0$  the Laplace Transform of (8) is

$$p\bar{Z} = \frac{\partial^2 \bar{Z}}{\partial \psi^2} \quad (9)$$

where  $\bar{Z}(\psi; p) = \int_0^\infty e^{-pt} Z(\psi, t) dt$ .

The solution of (9) which satisfies the boundary condition at  $\psi = \infty$  is

$$\bar{Z} = A(p) \exp(-p^{\frac{1}{2}} \psi) \quad (10)$$

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\* This simple equation does not seem to have been considered previously by authors writing on turbulent boundary layers, including the recent extensive account by Ferrari (1959).

where the unknown function  $A(p)$  has to be determined from the application of appropriate boundary conditions. One obvious method would be to find a solution to the velocity distribution over the inner region and then patch the solution at the junction to both regions by suitably choosing  $A(p)$ . However this approach has the disadvantage that the turbulent structure of the 'inner layer' must then be specified. We prefer however to use as a boundary condition the momentum integral equation found by integrating the exact transformed equation of motion

$$\frac{\partial Z}{\partial x} = \rho_0 \mu_0 U \frac{\partial^2 Z}{\partial \psi^2} + 2U \frac{\partial}{\partial \psi} \rho_0 \overline{uv} \quad (11)$$

over the boundary layer, leading to

$$\int_0^{\psi_\delta} \frac{\partial Z}{\partial x} / U d\psi = 2 \tau_w(x) \quad (12)$$

since  $\overline{uv} = 0$  at both  $\psi = 0$  and  $\psi = \delta$ , and  $\psi = \psi_\delta$  corresponds to  $y = \delta$ .

Now it can be shown that only a small error is introduced if  $U$  is put equal to  $U_1$  in (12) throughout the entire boundary layer and we find therefore that\*

$$\frac{d}{dx} \int_0^\infty Z d\psi = 2 U_1 \tau_w \quad (13)$$

since  $Z = 0$  when  $\psi \geq \psi_\delta$ .

\* From the definition of momentum thickness,  $\delta_2$ , it can be shown that equ.(13) is approximately given by

$$\frac{d}{dx} (2\rho_0 U_1^3 \delta_2) \doteq 2 U_1 \tau_w$$

or

$$\frac{d \delta_2}{dx} + 3 \delta_2 \frac{dU_1/dx}{U_1} \doteq \frac{\tau_w}{\rho_0 U_1^2}$$

Thus (13) is consistent with the momentum integral equation when  $H=0(1)$  which is certainly the case for zero pressure gradient at least.

But from (7)

$$\frac{d}{dx} \equiv \rho_0 \mu_0 r(x) U_1(x) \frac{d}{dt}$$

so that

$$\frac{d}{dt} \int_0^{\infty} Z(\psi, t) d\psi = \frac{2 \tau_w(t)}{\rho_0 \mu_0 r(t)} \quad (14)$$

or

$$p \int_0^{\infty} \bar{Z}(\psi; p) d\psi = \frac{2}{\rho_0 \mu_0} \int_0^{\infty} e^{-pt} \frac{\tau_w(t)}{r(t)} dt \quad (15)$$

If we now substitute for  $\bar{Z}$  from (10)

$$p^{\frac{1}{2}} A(p) = \frac{2}{\rho_0 \mu_0} \int_0^{\infty} e^{-pt} \frac{\tau_w(t)}{r(t)} dt \quad (16)$$

then

$$\bar{Z}(\psi; p) = \frac{2}{\rho_0 \mu_0} \int_0^{\infty} e^{-pt} \frac{\tau_w(t)}{r(t)} dt \frac{\exp(-p^{\frac{1}{2}} \psi)}{p^{\frac{1}{2}}} \quad (17)$$

and

$$Z(\psi, t) = \frac{2/\sqrt{\pi}}{\rho_0 \mu_0} \int_0^t \frac{\tau_w(\tau)}{r(\tau)} \frac{\exp(-\psi^2/4(t-\tau))}{\sqrt{t-\tau}} d\tau \quad (18)$$

Although we have specified above that  $Z(\psi, t)$  can only apply to the region  $\psi^* < \psi \leq \infty$  where  $\psi^*/\psi_0$  is a small quantity compared with unity, we note that (18) describes values for  $Z(\psi^*, t) \approx Z(0, t)$ . If we therefore put  $\psi = 0$  in (18)

$$U_1^2(x) = \frac{2}{\rho_0 \mu_0 \sqrt{\pi}} \left( \int_0^x \frac{\tau_w(x') U_1(x') dx'}{\left( \int_{x'}^x r(z) U_1(z) dz \right)^{\frac{1}{2}}} \right) \quad (19)$$

$$= \frac{2}{\sqrt{\pi}} \int_0^x \frac{u_T^2(x') U_1(x') dx'}{\left( \int_{x'}^x v_T(z) U_1(z) dz \right)^{\frac{1}{2}}} \quad (19')$$

\* It should be noted that equ. 18 cannot strictly apply at  $\psi = 0$  because from (18)

$$-\frac{\rho_0 \mu_0}{2} \left( \frac{\partial Z}{\partial \psi} \right)_{\psi=0} = \mu_0 \left( \frac{\partial U}{\partial y} \right)_{y=0} = 0,$$

a relation which only applies at separation.

since  $u_T(x) = \sqrt{\tau_w(x) / \rho}$  and  $r(x) = \frac{\mu_T}{\mu_0} = \frac{\nu_T}{\nu_0}$ .

If we put

$$Y(x) = \int_0^x \nu_T(z) dz \quad (20)$$

and we consider the special case  $U_1 = \text{const} = U_0$ ,

$$\frac{\sqrt{\pi}}{2} U_0^{3/2} = \int_0^{Y(x)} u_T^2(x') \frac{dx'}{dY(x')} \frac{dY(x')}{\sqrt{Y(x) - Y(x')}} \quad (21)$$

which is Abel's integral equation.

It has the solution

$$u_T^2(x) \frac{dx}{dY(x)} = \frac{1}{2\sqrt{\pi}} \frac{U_0^{3/2}}{\sqrt{Y(x)}} \quad (22)$$

so that if we integrate over a length  $L$  of the plate

$$\int_0^L u_T^2(x) dx = \frac{U_0^{3/2}}{\sqrt{\pi}} \sqrt{\int_0^L \nu_T(x) dx} \quad (23)$$

Now by definition the overall value of skin friction coefficient is

$$\begin{aligned} C_F &= \frac{1}{L} \int_0^L c_f dx = \frac{2}{U_0^2 L} \int_0^L u_T^2(x) dx \\ &= \frac{2}{\sqrt{\pi} L U_0^{3/2}} \sqrt{\int_0^L \nu_T(x) dx} \end{aligned} \quad (24)$$

by virtue of equation (23). This relation between  $C_F$  and the integrated value of  $\nu_T$  effectively displays the relation between the overall skin friction and the Reynolds stresses. A more significant relation is found from (22) by noting that

$$\frac{dY}{dx} = \nu_T(x)$$

giving 
$$u_T^2(x) = \frac{U_0^{3/2}}{2\sqrt{\pi}} \frac{\nu_T(x)}{\sqrt{\int_0^x \nu_T(z) dz}} \quad (25)$$

which shows that the local shear stress depends not only on the local value of the eddy viscosity but also on its past values. This is another way of saying that local shear stress depends on both the local characteristics of the turbulent structure and the past history of the turbulence. This is more or less a significant departure from the older mixing length theories which relate only the local structure of the turbulence to the mean characteristics of the flow in the boundary layer. Our equation (25) agrees qualitatively with the physical description of the transport processes existing in the outer region as described by Townsend (1956). Although this state of affairs is highly satisfactory, we note that we cannot proceed without knowing the numerical values for  $\nu_T(x)$ .

### 3. The momentum integral equation in zero pressure gradient.

If the momentum thickness is defined by

$$\delta_2 = \int_0^{\delta} \frac{U}{U_1} \left(1 - \frac{U}{U_1}\right) dy \quad (26)$$

then the momentum integral equation becomes when  $U_1 = \text{const} = U_0$

$$\frac{d\delta_2}{dx} = \frac{\tau_w}{\rho_0 U_0^2} \quad (27)$$

or 
$$\frac{2 d R_{\delta_2}}{d R_x} = c_f(x) \quad (28)$$

where 
$$R_{\delta_2} = \frac{U_0 \delta_2}{\nu} ; R_x = \frac{U_0 x}{\nu} ; c_f(x) = \frac{2 u_T^2(x)}{U_0^2} .$$

The integral of (28) over the length L of the plate is

$$C_F = \frac{2}{\left(\frac{U_0 L}{\nu}\right)} R_{\delta_2}(L) \quad (29)$$

where  $R_{\delta_2}(L)$  is the value of the Reynolds number based on the momentum thickness at  $x = L$ .

4. The compatibility condition for  $\nu_T$

If we now eliminate  $C_F$  between equations (24) and (29) we see that

$$R_{\delta_2}(L)^2 = \frac{1}{\pi} \int_0^{R_{\delta_2}(L)} \frac{\nu_T(x)}{\nu_0} \frac{U_0^2}{u_T^2(x)} dR_{\delta_2}(x) \quad (30)$$

since 
$$\frac{U_0 dx}{\nu_0} = \frac{U_0^2}{u_T^2(x)} dR_{\delta_2}(x).$$

Now (30) can only be satisfied if

$$\frac{\nu_T(x) U_0^2}{\pi \nu_0 u_T^2(x)} = 2 R_{\delta_2}(x) \quad (31)$$

provided that  $R_{\delta_2}(0) = 0$ .

Thus the compatibility condition on  $\nu_T(x)$  is given by

$$\frac{u_T(x) \delta_2(x)}{\nu_T(x)} = \frac{U_0}{2\pi u_T(x)} \quad (32)$$

where the right hand side is a slowly varying function of  $x$ .

This result is new\* for the only previous suggestion for the value of  $\nu_T$  is that given by Townsend (1956) who showed that from considerations of the equilibrium of the large eddies

$$\frac{u_T(x) \delta(x)}{\nu_T(x)} = \text{constant} \quad (33)$$

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\* A somewhat different relation, obtained on dimensional grounds, was suggested by Clauser (1956).

A check on Townsend's constant in equation (33) can be found by using Coles' relations between  $u_T$ ,  $\delta$  and  $\delta_2$ .

$$\text{Thus } \frac{\delta u_T}{\nu_0} = \exp(K \zeta - A) \quad (34)$$

where  $\zeta = U_0/u_T$  and  $K$  is the von-Karman constant, and

$$R_{\delta_2} = (C_1 - C_2/\zeta) \exp(K\zeta - A) \quad (35)$$

$$\text{or } \delta_2/\delta = \frac{1}{\zeta} (C_1 - C_2/\zeta) \quad (36)$$

From equations (34), (36) and (32) we find that

$$\frac{u_T \delta}{\nu_T} = \frac{U_0}{2\pi u_T} \frac{\delta}{\delta_2} = \frac{\zeta}{2\pi} \cdot \frac{\zeta}{(C_1 - C_2/\zeta)} = \frac{\zeta^2}{2\pi C_1 (1 - \frac{C_2}{C_1 \zeta})} \dots\dots (37)$$

If we use Coles' values for the constants  $C_1$  and  $C_2$  (i.e.  $C_1 = 4.05$ ;  $C_2/C_1 = 7.16$ ) then

$$\frac{u_T \delta}{\nu_T} = \frac{0.0393 \zeta^2}{(1 - 7.16/\zeta)} \quad (38)$$

whereas Townsend suggests that at moderately high Reynolds numbers  $\frac{u_T \delta}{\nu_T} \approx 55$ .

Table 1 and Fig. 1 give values of  $u_T \delta / \nu_T$  for a wide range of Reynolds numbers and shows that around  $R_x \approx 10^7$  our relation is in good agreement with that of Townsend. However at higher Reynolds numbers our relation gives higher, and at lower Reynolds numbers lower than Townsend's values.

TABLE 1

$10^3 c_f$	$R_x$	$u_T \delta / \nu_T$
8	$1.02 \times 10^4$	18.0
6	$4.07 \times 10^4$	21.6
4	$3.64 \times 10^5$	29.0
2	$3.55 \times 10^7$	51.0
1	$1.52 \times 10^{10}$	93.5

5. Flow with a pressure gradient

If we now make the assumption that our relation (32) should hold locally in a self-preserving flow having a pressure gradient we have that

$$c_f(x) = \frac{2 \tau_w}{\rho U_1^2} = \frac{\nu_T / \nu_0}{\pi R_{\delta_2}} \quad (39)$$

This would indicate that in a region of adverse pressure gradient the virtual eddy viscosity falls in the downstream direction. This cannot however apply to a non self preserving flow in a region of adverse pressure gradient when separation is approached as will be shown in Section 9. If now we eliminate  $\nu_T$  between equations (39) and (19')

$$U_1^2(x) = \frac{1}{\pi} \sqrt{\frac{2}{\nu_0}} \int_0^x \frac{u_T^2 / U_1^2 \cdot U_1^3 dx'}{\left( \int_{x'}^x R_{\delta_2} \frac{u_T^2}{U_1^2} \cdot U_1 dz \right)^{\frac{1}{2}}} \quad (40)$$

An approximate inversion of (40) can be found by putting

$$R_{\delta_2} \frac{u_T^2}{U_1^2} U_1 \sim x^\beta \quad (41)$$

so that from (40), with  $y = x^{\beta+1}$

$$U_1^2(y) = \frac{1}{\pi} \sqrt{\frac{2}{\nu_0(\beta+1)}} \int_0^y \frac{\frac{u_T}{U_1} \cdot U_1^{5/2} dy'}{R_{\delta_2}^{\frac{1}{2}} y'^{\frac{\beta}{2(\beta+1)}} \sqrt{y - y'}} \quad (42)$$

Equation (42) gives on inversion

$$\frac{u_T}{U_1} \frac{U_1^{5/2}}{R_{\delta_2}^{1/2}} \cdot \frac{1}{x^{\beta/2}} = \sqrt{\frac{\nu_0(\beta+1)}{2}} \left[ \frac{U_1^2(0)}{x^{\frac{\beta+1}{2}}} + \int_0^y \frac{d U_1^2(y')}{\sqrt{y - y'}} \right] \quad (43)$$

where the latter integral is a Stieltjes integral.

6. The determination of  $\beta$

When  $U_1 = \text{const.} = U_0$  equation (43) becomes

$$\frac{U_0^2 R_{\delta_2}}{u_T^2} = \frac{2 R_X}{(\beta + 1)} \quad (44)$$

where  $R_X = \frac{U_0 x}{\nu_0}$  and  $R_{\delta_2} = \frac{U_0 \delta_2}{\nu_0}$  .

If we use the empirical power law variation of the velocity distribution

$$u_T^2 / U_0^2 = G R_{\delta_2}^{-n} \quad (45)$$

where G is a function of n only.

From (44) and (45) we obtain

$$R_{\delta_2} = \left( \frac{2G}{\beta + 1} \right)^{1/1+n} R_X^{1/1+n} \quad (46)$$

Now it can be shown that if  $n = 2m/1+m$

$$\frac{u_T^2}{U_0^2} = \frac{C_6}{R_{\delta_2}^{2m/1+m}} \quad (47)$$

and  $\frac{R_{\delta_2}}{R_X} = \frac{C_5}{R_X^{2m/3m+1}}$  (48)

where  $C_5, C_6$  are constants, so that

$$\frac{u_T^2}{U_0^2} \frac{R_X}{R_{\delta_2}} = \frac{C_6}{\frac{C_5}{R_X^{2m/3m+1}} \frac{2m/1+m}{m+1}} = \left( \frac{m+1}{3m+1} \right) = \frac{\beta+1}{2} \quad (49)$$

giving

$$\beta = \frac{1 - m}{3 + 3m} \quad (50)$$

$$= \frac{1 - n}{1 + n} \quad (50')$$

From (46) and (50') we find

$$R_{\delta_2} = \left( G(1+n) R_x \right)^{1/1+n} \quad (51)$$

and if

$$R_{\delta_2} = (C_1 - C_2/\zeta) \exp(K\zeta - K\phi(1)) \quad (52)$$

then from (44) with (50')

$$R_x = \frac{C_1}{1+n} \left( \zeta^2 - \frac{C_2\zeta}{C_1} \right) \exp(K\zeta - K\phi(1)) \quad (53)$$

which agrees moderately well with Coles (1954) relation.

If following Young (1953) we choose  $m = 1/9$  then  $\beta = \frac{2}{3}$ , and in table II below we see that the error in our relation (44) compared with the more exact formula of Coles is quite small at moderately large Reynolds numbers.

TABLE II

Coles		Equ.(44)	
$10^3 c_f$	$R_x$	$10^3 c_f = \frac{5}{3} \frac{R_{\delta_2}}{R_x} \cdot 10^3$	Error %
8	$1.02 \times 10^4$	8.6	+ 7.5
6	$4.07 \times 10^4$	6.35	+ 6
4	$3.64 \times 10^5$	4.10	+ 2.5
2	$3.55 \times 10^7$	1.95	- 3
1	$1.52 \times 10^{10}$	0.93	- 7

7. The equation for  $R_{\delta_2}$  and a comparison with Spence's (1956) method

If in equation (43) we put  $u_r^2 / U_1^2 = \frac{G}{R_{\delta_2}^n}$  as found at zero pressure gradient from the power law relation (equ. 45), with  $\bar{y} = (x/c)^{\beta+1}$

$$R_{\delta_2}(\bar{x})^{1+n} = \frac{2\alpha R_c \bar{x} \bar{U}_1(\bar{x})^5}{(1+\beta)(1+n) \left[ \bar{U}_1(0)^2 + \sqrt{\bar{y}} \int_0^{\bar{y}} \frac{d\bar{U}_1^2(\bar{y}')}{\sqrt{\bar{y} - \bar{y}'}} \right]^2} \quad (54)^*$$

compared with Spence's formula

$$R_{\delta_2}(\bar{x})^{1+n} = \frac{\alpha R_c}{\bar{U}_1(\bar{x})^3} \int_0^{\bar{x}} \bar{U}_1(z)^4 dz \quad (55)$$

where in both (54) and (55)

$$\alpha = (1+n)G$$

$$R_c = \frac{U_0 c}{\nu_0}$$

$$\bar{U}_1 = U_1 / U_0$$

$$\bar{x} = x/c .$$

---

\* If  $\bar{U}_1(\bar{y})$  is constant up to  $\bar{y} = \bar{y}_1$  and then varies with  $\bar{y}$ , the lower limit of integration in the integral in the denominator is  $\bar{y}_1$  in place of zero.

If as an example we put  $\bar{U}_1(\bar{x}) = \bar{x}^{-q}$ , with  $\bar{U}_1(0) = 0$ , i.e.  $q > 0$  we can obtain a numerical comparison between equations (54) and (55). In this case (54) reduces to  $(s = \frac{2q}{\beta+1} - 1)$

$$\begin{aligned} \frac{R_{\delta_2}^{1+n}}{\alpha R_c \bar{x}^{q+1}} &= \frac{(1+\beta)}{2q^2(1+n) \left( \int_0^1 \frac{\bar{y}^s d\bar{y}}{\sqrt{1-\bar{y}}} \right)^2} \\ &= \frac{2}{\pi(1+n)(1+\beta)} \left[ \frac{\left( \frac{2q}{1+\beta} - \frac{1}{2} \right)!}{\left( \frac{2q}{1+\beta} \right)!} \right]^2 \end{aligned} \quad (56)$$

while (55) reduces to

$$\frac{R_{\delta_2}^{1+n}}{\alpha R_c \bar{x}^{q+1}} = \frac{1}{4q+1} \quad (57)$$

It is important in (56) to choose a suitable value for  $\beta$ . As noted above (see equ. 41)

$$R_{\delta_2} u_T^2 / U_1^2 \cdot U_1 \sim x^\beta$$

which with  $u_T^2 / U_1^2 = G / R_{\delta_2}^n$ , gives

$$\beta = \frac{2q+1-n}{1+n} \quad (58)$$

Thus equations (56) and (57) reduce to a comparison between

$$\frac{1}{4q+1} \quad \text{and} \quad \frac{1}{\pi(q+1)} \left[ \frac{\left( \frac{q(1+n)}{1+q} - \frac{1}{2} \right)!}{\left( \frac{q(1+n)}{1+q} \right)!} \right]^2$$

These values are tabulated in table III below for  $n = 0.2$ , the value used by Spence. The agreement between our two results is in this case reasonable.

TABLE III

q	$\frac{1}{4q+1}$	$\frac{1}{\pi(q+1)}$	$\left[ \frac{\left\{ \frac{7q-5}{10(1+q)} \right\} !}{\left\{ \frac{6q}{5(1+q)} \right\} !} \right]$
0.2	0.556		0.530
0.4	0.385		0.358
0.6	0.294		0.270
0.8	0.238		0.216
1.0	0.200		0.180
1.2	0.172		0.155
1.4	0.152		0.135
1.6	0.135		0.120
1.8	0.122		0.108
2.0	0.111		0.099
3.0	0.077		0.068
4.0	0.059		0.052

8. Conditions near separation

From equation (43) separation occurs ( $u_\tau = 0$ ) when

$$\bar{U}_1^2(o) = \sqrt{\bar{y}} \int_0^{\bar{y}} \frac{d\bar{U}_1^2(\bar{y}')}{\sqrt{\bar{y} - \bar{y}'}} \quad (59)$$

where  $\bar{y} = \bar{x}^{\beta+1}$  and  $\beta$  must be suitably chosen.

If  $\bar{U}_1(\bar{x}) = 1 - \bar{x}$ , with  $\bar{U}_1(o) = 1$ , then from (59), writing  $\bar{x}_{sep}$  as the distance to separation,

$$\frac{1+\beta}{2} = \bar{x}_{sep} \int_0^1 \frac{d\bar{y}}{\bar{y}^{\frac{\beta}{1+\beta}} \sqrt{1-\bar{y}}} - \bar{x}_{sep}^2 \int_0^1 \frac{\bar{y}^{\frac{1-\beta}{1+\beta}} d\bar{y}}{\sqrt{1-\bar{y}}}$$

or

$$\bar{x}_{sep}^2 \frac{\sqrt{\pi} \left(\frac{2}{1+\beta}\right)!}{\left(\frac{2}{1+\beta} - \frac{1}{2}\right)!} - 2\bar{x}_{sep} \frac{\sqrt{\pi} \left(\frac{1}{1+\beta}\right)!}{\left(\frac{1}{1+\beta} - \frac{1}{2}\right)!} + 1 = 0 \quad (60)$$

Approaching separation it can be shown that  $\beta \approx 0$  and with this value for  $\beta$  equation (60) becomes

$$\frac{8}{3} \bar{x}_{sep}^2 - 4 \bar{x}_{sep} + 1 = 0 \quad (61)$$

having the roots  $\bar{x}_{sep} = 0.317$  and 1.183, where only the smaller root can have a physical significance. With the value of  $\bar{x}_{sep} = 0.317$  we find that

$$\left(\bar{U}_1\right)_{sep} = 1 - 0.317 = 0.683$$

Although at first sight this result seems not unreasonable it can be shown to be fundamentally incorrect. In the first place it cannot strictly be compared with the experimental results of Schubauer and Klebanoff (1950) for in our case the boundary layer thickness is zero at the commencement of the region of adverse pressure gradient. In the experimental set up of Schubauer and Klebanoff there existed about a 17ft length of boundary layer having approximately zero pressure gradient upstream of the region of adverse pressure gradient. If these conditions are inserted into our equations we find the results are nonsensical. The reason for this discrepancy is not hard to find. In deriving the equations for a pressure gradient we have used the compatibility condition on  $\nu_T$  which was derived for the case of zero pressure gradient and can possibly be justified to hold also in a region of self-preserving turbulent flow. However if we look closely at this relation we see that the eddy viscosity decreases in an adverse pressure gradient, which is certainly not the case when separation is approached. We must therefore seek a new relation for the eddy viscosity in a region of adverse pressure gradient if the flow is not self-preserving. This case is explored rather tentatively in Appendix B. It is shown that if the eddy viscosity varies as  $\bar{x}^4$  the position of separation and the variation of skin friction coefficient with distance are in fair agreement with the results of Schubauer and Klebanoff. However more comparisons with other experimental data are necessary before the derived relations can be taken to apply to general positive pressure gradients.

### 9. The compressible turbulent boundary layer

The steady boundary layer equations of motion, continuity and energy for a compressible turbulent flow are

$$\bar{\rho} \bar{u} \frac{\partial \bar{u}}{\partial x} + (\bar{\rho} \bar{v} + \overline{\rho' v'}) \frac{\partial \bar{u}}{\partial y} = \bar{\rho} \bar{u} \frac{d\bar{u}_1}{dx} + \frac{\partial \tau}{\partial y} \quad (62)$$

$$\tau = \bar{\mu} \frac{\partial \bar{u}}{\partial y} - \bar{\rho} \overline{u'v'} \quad (63)$$

$$\frac{\partial \bar{\rho} \bar{u}}{\partial x} + \frac{\partial}{\partial y} (\bar{\rho} \bar{v} + \overline{\rho' v'}) = 0 \quad (64)$$

$$\bar{\rho} \bar{u} \frac{\partial \bar{h}_s}{\partial x} + (\bar{\rho} \bar{v} + \overline{\rho' v'}) \frac{\partial \bar{h}_s}{\partial y} = \frac{\partial}{\partial y} \left[ \frac{\bar{\mu}}{\sigma} \frac{\partial}{\partial y} \left( \bar{h}_s + (\sigma-1) \frac{\bar{u}^2}{2} \right) - \bar{\rho} \overline{v' h'_s} \right] \quad (65)$$

where  $h_s = h + u^2/2$  is the stagnation enthalpy,  $h$  is the specific enthalpy,  $\sigma$  is the Prandtl number and

$$\overline{v' h'_s} \approx \overline{v' h'} \quad (66)$$

In this section  $(\bar{u}, \bar{v})$  denote the mean velocities,  $(u', v')$  the fluctuating velocities and  $(\bar{\rho}, \rho')$  are the mean and fluctuating densities respectively. Equation (64) shows that if  $\psi$  is the stream function

$$\frac{\bar{\rho} \bar{u}}{\rho_o} = \frac{\partial \psi}{\partial y} ; \quad \frac{\bar{\rho} \bar{v} + \overline{\rho' v'}}{\rho_o} = - \frac{\partial \psi}{\partial x} \quad (67)$$

where  $\rho_o$  is the stagnation density.

The compressible flow equations above can be transformed into pseudo-incompressible flow equations by use of a modified Stewartson-illingworth transformation. The essential features of such a transformation are that by a stretching of the physical co-ordinates the variable density in the inertia terms can be eliminated. If in the pseudo-incompressible system  $(X, Y)$  the stagnation pressure and enthalpy are the same as in the compressible flow  $(x, y)$  and the stream function, which determines the rate of mass flow, is unchanged in the transformation, the transformation formulae are (where the suffix (o) denotes quantities evaluated at stagnation conditions in the physical flow)

$$X = \int_0^x \frac{\rho_1 \mu_1 a_1}{\rho_o \mu_o a_o} dx ; \quad Y = \int_0^y \frac{a_1 \rho}{a_o \rho_o} dy \quad (68)$$

Equations (62) and (65) then become

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = \frac{h_s}{h_{s_1}} U_1 \frac{dU_1}{dX} + v_o \frac{\partial}{\partial Y} \left( \bar{C} \frac{\partial U}{\partial Y} \right) - \frac{\partial}{\partial Y} (\overline{u'_1 v'_1}) \quad (69)$$

and

$$U \frac{\partial h_s}{\partial X} + V \frac{\partial h_s}{\partial Y} = \frac{v_o}{\sigma} \frac{\partial}{\partial Y} \left\{ \bar{C} \frac{\partial}{\partial Y} \left( h_s - \left( \frac{1-\sigma}{2} \right) \frac{a_1^2 U^2}{a_o} \right) \right\} - \frac{\partial}{\partial Y} \left( \frac{\mu_o a_o}{\mu_1 a_1} \frac{\bar{\rho}}{\rho_1} \overline{v' h'_s} \right) \quad (70)$$

where

$$\begin{aligned}
 U &= a_0 \bar{u} / a_1 \quad ; \quad U_1 = a_0 \bar{u}_1 / a_1 \\
 \sigma &= \frac{\mu C_p}{k} \\
 \bar{C} &= \frac{\bar{\rho} \mu}{\rho_1 \mu_1}
 \end{aligned}
 \tag{71}$$

and

$$-\bar{\rho} \overline{(u'v')}_{\text{comp}} = -\rho_0 \overline{(u'_1 v'_1)}_{\text{inc.}} \frac{a_1^2 \rho_1 \mu_1}{a_0^2 \rho_0 \mu_0}
 \tag{72}$$

The last relation assumes that the turbulent mixing processes are not changed by compressibility effects. Thus following Mager (1958) we assume that the turbulent stress acting on each elemental mass of fluid is unchanged by the transformation. This leads to equation (72) which relates the Reynolds stress  $-\bar{\rho} \overline{u'v'}$  in the compressible flow to  $-\rho_0 \overline{u'_1 v'_1}$  in the pseudo-incompressible flow. This relation can only be justified by performing the appropriate experiments, but since data of this kind are not yet available all we can do is to compare the corresponding skin friction coefficient ratio  $c_{f_c} / c_{f_i}$ , as evaluated according to the above assumption, with experiment.

It is interesting to note that an alternative form of (72) is

$$\frac{-\bar{\rho} \overline{(u'v')}_{\text{comp.}}}{\rho_1 \mu_1 \bar{u}_1^2} = \frac{-\rho_0 \overline{(u'_1 v'_1)}_{\text{inc.}}}{\rho_0 \mu_0 U_1^2}
 \tag{72'}$$

showing the dependence of the Reynolds stress on the ratio of  $\rho_1 \mu_1 / \rho_0 \mu_0$ .

If further we transform equations (69) and (70) from (X, Y) co-ordinates to (X,  $\psi$ ) co-ordinates using von Mises transformation we obtain respectively

$$\frac{\partial Z}{\partial X} = S \frac{d U_1^2}{d X} + \nu_0 U \frac{\partial}{\partial \psi} \left( \bar{C} \frac{\partial Z}{\partial \psi} \right) + 2U \frac{\partial}{\partial \psi} \overline{u'_1 v'_1}
 \dots\dots \tag{73}$$

and

$$\begin{aligned}
 \frac{\partial S}{\partial X} = \frac{\nu_0}{\sigma} \frac{\partial}{\partial \psi} \left\{ U \bar{C} \frac{\partial}{\partial \psi} \left( S + \left( \frac{1-\sigma}{2} \right) \frac{U^2 a_1^2}{a_0^2 h_{s_1}} \right) \right\} \\
 + \frac{\partial}{\partial \psi} \left( \frac{\mu_0 a_0}{\mu_1 a_1} \frac{\bar{\rho}}{\rho_1} \overline{v'_1 h'_1} \right)
 \end{aligned}
 \tag{74}$$

where  $Z = (U_1^2 - U^2)$  and  $S = (1 - h_s / h_{s_1})$ .

Now we will require later the integral forms of these equations as boundary conditions for our velocity and stagnation enthalpy solutions in the outer region. Thus on integrating (73) and (74) respectively with respect to  $\psi$  from 0 to  $\psi_\delta$  (corresponding to  $y = \delta$  in the physical coordinates  $(x, y)$ ),

$$\int_0^{\psi_\delta} \frac{\partial Z / \partial X}{U} d\psi = \frac{dU_1^2}{dX} \int_0^{\psi_\delta} \frac{S}{U} d\psi + 2\bar{C}_w \frac{\tau_w}{\rho_0} \quad (75)$$

where  $\tau_w = \mu_0 \left( \frac{\partial U}{\partial Y} \right)_{Y=0}$  and  $u_i' v_i' = 0$  at  $Y = 0$ ,

and 
$$\int_0^{\psi_\delta} \frac{\partial S}{\partial X} d\psi = - \frac{\bar{C}_w \dot{q}_w}{\rho_0 h_{s_1}} \quad (76)$$

where 
$$\dot{q}_w = - k_0 \left( \frac{\partial T}{\partial Y} \right)_{Y=0} \quad \text{and} \quad \int_0^\infty S d\psi = - \int_0^X \frac{\dot{q}_w \bar{C}_w}{\rho_0 h_{s_1}} dX \quad \dots \quad (77)$$

since  $S = 0$  at  $X = 0$  for all values of  $\psi$ .

The value of  $\bar{C}_w$  in equations (75) and (76) is very significant as will be shown later. According to equation (71)

$$\bar{C}_w = \frac{\rho_w \mu_w}{\rho_1 \mu_1} \quad (78)$$

but because  $\bar{C}_w$  changes most in the inner region, including the viscous layer, we might expect, by comparison with the solution of the compressible laminar boundary layer problem, that it is more appropriate to assume in (73) and (74) that  $\bar{C}$  is a constant evaluated at some 'intermediate' enthalpy conditions. If we denote these latter values by starred quantities then

$$\bar{C}^* = \frac{\rho^* \mu^*}{\rho_1 \mu_1} \quad (79)$$

It is shown later that using Eckert's formula for the 'intermediate' enthalpy the ratio of  $c_{f_c} / c_{f_i}$  as evaluated by our method agrees moderately well

with the available experimental data for the case of zero pressure gradient and both insulated and slightly cooled walls.

On the other hand Eckert's formula (based on the laminar flow analysis) seems singularly inappropriate for the highly cooled wall case.\*

In equation (75) a suitable approximation to the first term is found by replacing  $U$  by  $U_1$  so that

$$\frac{d}{dX} \int_0^{\infty} Z d\psi = \frac{2U_1}{\rho_0} \left( \tau_w \bar{C}_w + \rho_0 U_1 \frac{dU_1}{dX} \int_0^{\infty} \frac{S d\psi}{U} \right) \quad (80)$$

where the additional approximation  $\frac{1}{U_1} \int_0^{\infty} S d\psi \approx \int_0^{\infty} \frac{S d\psi}{U}$

may be adequate in some cases.

This completes our formal presentation of the transformed equations and we are now in a position to attempt a solution of these equations in the outer region of the boundary layer on similar lines to the corresponding solution in the incompressible case given in section 3 above.

In the outer region let us assume that a virtual eddy viscosity,  $\mu_T$ , and a virtual eddy thermal conductivity,  $k_T$ , exist such that in the compressible flow

$$\left. \begin{aligned} - \bar{\rho} \overline{u'v'} &= \mu_T \frac{\partial \bar{u}}{\partial y} \\ - \bar{\rho} \overline{v'h'_s} &= \frac{k_T}{C_p} \frac{\partial h_s}{\partial y} \end{aligned} \right\} \quad (81)$$

---

\* For the special case of zero pressure gradient when the wall is highly cooled it would seem more appropriate to base the reference conditions, in the Stewartson-Illingworth transformation, on the freestream conditions rather than on stagnation conditions, since nowhere in the layer does the enthalpy approach stagnation conditions as it will for the case of zero heat transfer. This leads to a different result for the skin friction (see para. 11).

In this region we have also that  $h_s \approx h_{s_1}$ , the constant external stagnation enthalpy,  $\rho \approx \rho_1$ ,  $U \approx U_1$ , and  $\mu_T \gg \mu$ ;  $k_T \gg k$ . If  $\mu_T$  and  $k_T$  are assumed to be functions of  $x$  only then the transformed equations (73) and (74) reduce to

$$\left. \begin{aligned} \frac{\partial Z}{\partial X} &= \nu_0 U_1 \bar{C}_T \frac{\partial^2 Z}{\partial \psi^2} \\ \text{and} \quad \frac{\partial S}{\partial X} &= \frac{\nu_0}{\sigma_T} U_1 \bar{C}_T \frac{\partial^2 S}{\partial \psi^2} \end{aligned} \right\} \quad (82)$$

where  $\sigma_T = \frac{\mu_T C_p}{k_T}$  is the turbulent Prandtl number and is assumed to be a constant, and  $\bar{C}_T = \mu_T / \mu_1$ .

These equations are uncoupled because we have put  $S = 0$ , in the equation of motion, corresponding to the assumption that  $h_s \approx h_{s_1}$  throughout the outer region. If we retain this term in the first of (82) then we must first solve the second of (82) using (77) as one of the boundary conditions, and afterwards solve the first of (82) using (80) as one of the boundary conditions.

Thus the relations between the wall shear stress, rate of heat transfer and pressure gradient in the pseudo-incompressible flow can be obtained. The equations however simplify considerably in the case of zero pressure gradient and this case will be fully treated in the next section.



10. The relation between the wall shear stress in incompressible and compressible turbulent flow\*

When  $U_1 = \text{const} = U_0$  equation (82) becomes

$$\frac{\partial Z}{\partial X} = \nu_0 U_0 \bar{C}_T \frac{\partial^2 Z}{\partial \psi^2} \quad (83)^{**}$$

with the boundary conditions

$$Z \rightarrow 0 \text{ as } \psi \rightarrow \infty$$

and from (80)

$$\frac{d}{dX} \int_0^\infty Z d\psi = \frac{2 U_0 \tau_w}{\rho_0} \bar{C}_w \quad (84)$$

If we put

$$t = \int_0^X \nu_0 U_0 \bar{C}_T dX \quad (85)$$

then (83) becomes

$$\frac{\partial Z}{\partial t} = \frac{\partial^2 Z}{\partial \psi^2} \quad (86)$$

having the solution

$$\bar{Z}(\psi; p) = A(p) \exp(-p^{1/2} \psi) \quad (87)$$

as in incompressible flow. But from (84) and (87) we easily find that

$$p^{1/2} A(p) = \frac{2}{\mu_0} \int_0^\infty e^{-pt} \frac{\bar{C}_w \tau_w(t)}{\bar{C}_T(t)} dt \quad (88)$$

\* An alternative treatment is given in Appendix A.

\*\* Equations (83) and (84) can be put into an incompressible form by introducing a new variable  $x$  in place of  $X$ , where

$$x = \int_0^X \bar{C}_w dX', \text{ and replacing } \bar{C}_T \text{ by } \bar{C}'_T = \bar{C}_T / \bar{C}_w.$$

giving

$$Z(\psi, t) = \frac{2\sqrt{\pi}}{\mu_0} \int_0^t \frac{\bar{C}_w \tau_w(t)}{\bar{C}_T(t)} \frac{\exp(-\psi^2 / 4(t - \tau))}{\sqrt{t - \tau}} d\tau \dots \dots \quad (89)$$

But in the inner part of the outer region  $Z(\psi^*, t) \approx Z(0, t)$  so that

$$U_0^{3/2} = \frac{2}{\sqrt{\pi}\sqrt{v_0}} \int_0^X \frac{\bar{C}_w U_T^2(x') dx' / dY(x')}{\sqrt{Y(X) - Y(x')}} \quad (90)$$

where

$$U_T^2 = \tau_w / \rho_0 \quad \text{and} \quad Y(X) = \int_0^X \bar{C}_T d(x') \quad (91)$$

(and is not the pseudo-incompressible normal co-ordinate).

Equation (90) has the solution

$$\bar{C}_w U_T^2 \frac{dX}{dY} = \frac{1}{2\sqrt{\pi}} \frac{U_0^{3/2} \sqrt{v_0}}{\sqrt{Y}} \quad (92)$$

and since

$$\frac{dY}{dX} = \bar{C}_T$$

$$U_T^2(X) = \frac{U_0^{3/2} \sqrt{v_0} \bar{C}_T}{\bar{C}_w 2\sqrt{\pi} \int_0^X \bar{C}_T dX} \quad (93)$$

( see equation (25) )

Alternatively from (92) on integrating over the length L of the plate

$$\int_0^L \bar{C}_w U_T^2(X) dX = \frac{U_0^{3/2} \sqrt{v_0}}{\sqrt{\pi}} \sqrt{\int_0^L \bar{C}_T(X) dX} \quad (94)$$

But by definition, in the pseudo-incompressible flow,

$$C_F = \frac{2}{U_o^2 L} \int_0^X U_T^2 dX \quad (95)$$

and the transformed (X, Y) equation of motion gives on integration

$$\frac{d}{dX} R_{\delta_2} = \bar{C}_W \frac{U_T^2}{U_o^2} \cdot \frac{U_o}{\nu_o} \quad (96)$$

where  $R_{\delta_2} = \delta_2 U_o / \nu_o$  is the Reynolds number based on the momentum thickness in the pseudo-incompressible flow.

Thus

$$R_{\delta_2}(L) = \frac{U_o / \nu_o}{U_o^2} \int_0^L \bar{C}_W U_T^2 dX \quad (97)$$

and when combined with (94), we obtain

$$R_{\delta_2}^2(L) = \frac{1}{\pi} \int_0^{R_{\delta_2}(L)} \frac{\bar{C}_T}{\bar{C}_W} \frac{U_o^2}{U_T^2} dR_{\delta_2}(X) \quad (98)$$

But this equation can only be true if

$$\frac{\bar{C}_T U_o^2}{\pi \bar{C}_W U_T^2} = 2 R_{\delta_2}(X) \quad (99)$$

giving the compatibility condition in the outer region of the pseudo-incompressible flow (of density  $\rho_o$  and kinematic viscosity  $\nu_o$ )

$$\frac{\rho_o U_T \delta_2}{\mu_T} = \frac{\rho_1 \mu_o}{2\pi \rho^* \mu^*} \frac{U_o}{U_T} \quad (100)$$

where we have put  $\bar{C}_T = \mu_T / \mu_1$  and  $\bar{C}_W = \frac{\rho^* \mu^*}{\rho_1 \mu_1}$ . Now  $\delta_2$

in equation (100) is given by

$$\delta_2 = \int_0^\infty \frac{U}{U_o} \left( 1 - \frac{U}{U_o} \right) dY = \delta_2 c \frac{a_1 \rho_1}{a_o \rho_o}$$

where  $\delta_{2c}$  is the momentum thickness in the compressible flow. Also in that equation  $\rho_0 U_T^2 = \tau_{wi}$ . Hence the corresponding

compressible compatibility relation can be written in the form

$$c_{fc} = \frac{\tau_{wc}}{\frac{1}{2} \rho_1 \bar{u}_1^2} = \frac{1}{\pi} \frac{\mu_{Tc}}{\rho_1 \bar{u}_1 \delta_{2c}} = \frac{\mu_{Tc}}{\pi \mu_1 R_{\delta_{2c}}} \quad (101)$$

since in (100)  $\mu_{Tc}$  is the eddy viscosity in the compressible flow. The corresponding result in incompressible flow (see equation (32)) is

$$c_{fi} = \frac{1}{\pi} \frac{\nu_{Ti}}{U_0 \delta_{2i}} = \frac{\nu_{Ti}}{\pi \nu_0 R_{\delta_{2i}}} \quad (102)$$

Equations (101) and (102) show that similar relations exist between the skin friction coefficient, the Reynolds number based on momentum thickness and the eddy viscosity in both incompressible and compressible flows.

Now from the transformation formulae the ratio of the wall shear stress in the compressible to that in the pseudo-incompressible flow is

$$\frac{\tau_{wc}}{\tau_{wi}^+} = \frac{\mu_w \left( \frac{\partial \bar{u}}{\partial y} \right)_w}{\mu_0 \left( \frac{\partial U}{\partial Y} \right)_w} = \frac{\rho_w \mu_w}{\rho_0 \mu_0} \frac{a_1^2}{a_0^2} \quad (103)$$

where  $\tau_{wi}^+$  is the wall shear stress in the pseudo-incompressible flow and

$$\frac{\tau_{wc} / \frac{1}{2} \rho_1 \bar{u}_1^2}{\tau_{wi}^+ / \frac{1}{2} \rho_0 U_0^2} = \left( \frac{\rho_w \mu_w}{\rho_1 \mu_1} \right) \cdot \frac{\mu_1}{\mu_0} \quad (104)$$

But this neglects the fact that the level of the wall shear stress is set by the Reynolds stress in the outer region. Now the ratio of the eddy viscosities in the compressible and pseudo-incompressible flows is

$$\mu_{Tc} / \mu_{Ti} = \frac{\rho^* \mu^*}{\rho_1 \mu_1} \cdot \frac{\mu_1}{\mu_0}$$

and it follows that

$$\frac{\tau_{wc} / \frac{1}{2} \bar{\rho}_1 \bar{u}_1^2}{\tau_{wi}^+ / \frac{1}{2} \rho_0 U_0^2} = \left( \frac{\rho^* \mu^*}{\bar{\rho}_1 \mu_1} \right) \frac{\mu_1}{\mu_0} \quad (104')$$

In general the ratio between the skin friction coefficients in compressible and incompressible flow is required at the same Reynolds number. But it can easily be shown that the values of  $\tau_{wc}$  and  $\tau_{wi}^+$  in (104') correspond to  $R_{xc}$  and  $R_{xi}^+$  respectively\*\*, where

$$R_{xc} = \frac{\bar{\rho}_1 \bar{u}_1 x}{\mu_1} \quad \text{and} \quad R_{xi}^+ = \frac{\rho_0 U_0 X \bar{C}_w}{\mu_0} \quad \text{and}$$

$$R_{xc} = R_{xi}^+ \left( \frac{\mu_0}{\mu_1} \right)^2 \frac{1}{\bar{C}_w} \quad (105)$$

if the wall enthalpy is constant.

Now it was shown in Section 7 that in incompressible flow when  $m = 1/7$

$$c_{fi} \sim R_{xi}^{-1/5}, \quad \text{and therefore from (105) and (104')}$$

$$\frac{c_{fc} R_{xc}^{1/5}}{c_{fi} R_{xi}^{1/5}} = \left( \frac{\rho^* \mu^*}{\bar{\rho}_1 \mu_1} \right)^{4/5} \left( \frac{\mu_1}{\mu_0} \right)^{3/5} \quad (106)$$

Hence at the same Reynolds number (in the range where  $m = 1/7$ )

$$\frac{c_{fc}}{c_{fi}} = \left( \frac{\rho^* \mu^*}{\bar{\rho}_1 \mu_1} \right)^{4/5} \left( \frac{\mu_1}{\mu_0} \right)^{3/5} \quad (107)$$

This result is of the form given by many authors\*.

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\*\* In the pseudo-incompressible flow the effective length scale is  $X \bar{C}_w$  (see footnote to equation 83).

\* For instance Mager (1958) finds that, in our notation,  $\frac{c_{fc}}{c_{fi}} = \left( \frac{\mu_1}{\mu_0} \right)^{0.6}$  when  $c_{fi} \sim R_{xi}^{-1/5}$ .

Alternatively we can write

$$\frac{c_{fc}}{c_{fi}} = \left( \frac{h_1}{h^*} \right)^{(1-\omega)} \left( \frac{h_1}{h_0} \right)^{\frac{3\omega}{5}} \quad (108)$$

if it is assumed  $\mu \sim h^\omega$ , and  $h^*$ , the 'intermediate' enthalpy, is given by Eckert's relation, which for  $\gamma = 1.4$  is,

$$\frac{h^*}{h_1} = 1 + 0.5 \left( \frac{h_w - h_1}{h_1} \right) + 0.22 \left( \frac{h_r - h_1}{h_1} \right) \quad (109)$$

where  $h_w$  and  $h_r$  are the wall and recovery enthalpies respectively.  $h_1$  and  $h_0$  are the freestream and stagnation enthalpies respectively.

For the special case of zero heat transfer,  $h_w = h_r$ , and

$$\frac{h_r}{h_1} = \left( 1 + \frac{\gamma-1}{2} \sigma^{\frac{1}{3}} M_1^2 \right) \quad (110)$$

Thus Eckert's intermediate enthalpy relation becomes

$$h^*/h_1 = 1 + \frac{0.72}{5} \sigma^{\frac{1}{3}} M_1^2 \quad (111)$$

and if  $\omega = 0.8$

$$\frac{c_{fc}}{c_{fi}} \Big|_{h_w = h_r} = \frac{1}{\left[ 1 + 0.144 \sigma^{\frac{1}{3}} M_1^2 \right]^{0.16} \left[ 1 + M_1^2/5 \right]^{0.48}} \quad \dots \quad (112)$$

which is plotted in Fig. 2, together with some experimental data. It is seen that there is fair agreement between theory and experiment for this special case of zero heat transfer (insulated walls). The theory should also apply to the cases of slightly heated or cooled walls.

As stated above Eckert's intermediate enthalpy relation is inapplicable to the highly cooled wall case, since nowhere in the boundary layer does the enthalpy rise above the freestream value. The case of the highly cooled wall is discussed in Appendix 3. Since in this case the enthalpy in the boundary layer does not rise above its

freestream value it is suggested that the reference conditions in the Stewartson-illingworth transformation are more likely to be appropriate to freestream rather than stagnation conditions. This leads finally to the approximate relation

$$\frac{c_{f_c}}{c_{f_i}} = \left( \frac{1 + \left( h_1/h_w \right)^{1/4}}{2} \right)^{4/5} \quad (113)$$

a result independent of Mach number.

This shows that for values of  $(h_w/h_1)$  less than unity  $c_{f_c}/c_{f_i}$  will have values close to unity.

This conclusion was previously suggested by Rose, Probstein and Adams (1958) for the case of a partially dissociated boundary layer. However they based their conclusions on the assumed fact that the density was approximately constant over the entire boundary layer outside the sub layer, whereas our result does not depend on such an assumption.

## 11. Conclusions

1. On the assumption that a virtual eddy viscosity exists in the outer region of the boundary layer, it is shown that the turbulent boundary layer equations can be solved approximately to give the velocity distribution in the outer region.
2. For the case of zero pressure gradient it is shown that a compatibility condition for the eddy viscosity exists, which for moderate to high Reynolds numbers takes the form  $\frac{u_T \delta_2}{\nu_T} \approx \text{constant}$ . This relation is just Townsend's equilibrium condition for the large eddies in the outer region.
3. If it is assumed that the compatibility condition for  $\nu_T$ , also applies to the case with negative pressure gradient or at least in a self-preserving flow, an equation, analogous to the momentum equation, is found giving  $R_{\delta_2}$  as a function of  $U_1(x)$ . This equation gives results comparable with those obtained by Spence and Maskell's method.

4. When the pressure gradient is adverse and separation is approached the compatibility condition on  $\nu_T$  is found to be inapplicable. A revised analysis shows how this case can be dealt with and it is shown that if  $\nu_T \sim x^4$  approaching separation the calculated skin friction distribution agrees qualitatively with the measured results of Schubauer and Klebanoff.
  
5. The method is extended to compressible flow by making use of the Stewartson-illingworth transformation, which transforms the compressible flow equations to pseudo-incompressible flow equations. One result that is obtained is the ratio of the skin friction coefficients in compressible and incompressible flow. This ratio is of a form analogous to that found by other workers in the field and in particular that of Mager for the case of zero pressure gradient and zero heat transfer.

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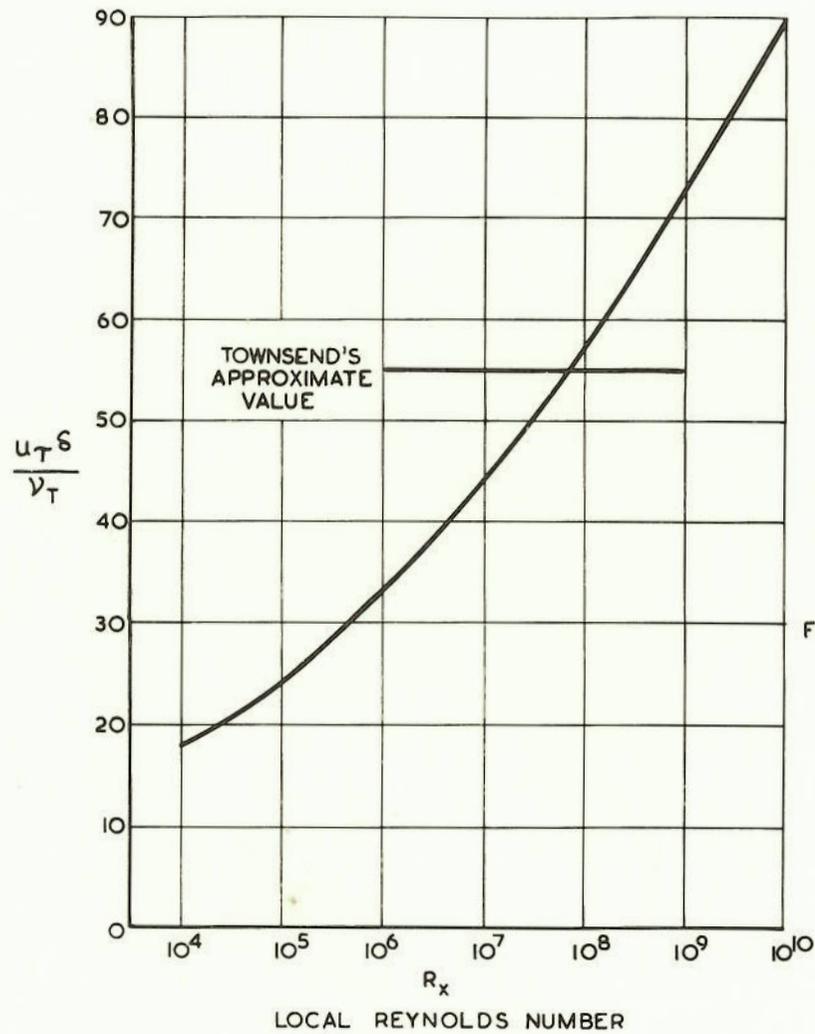


FIG. 1. THE VARIATION OF  $\frac{u_T \delta}{\nu_T}$  WITH REYNOLDS NUMBER IN ZERO PRESSURE GRADIENT

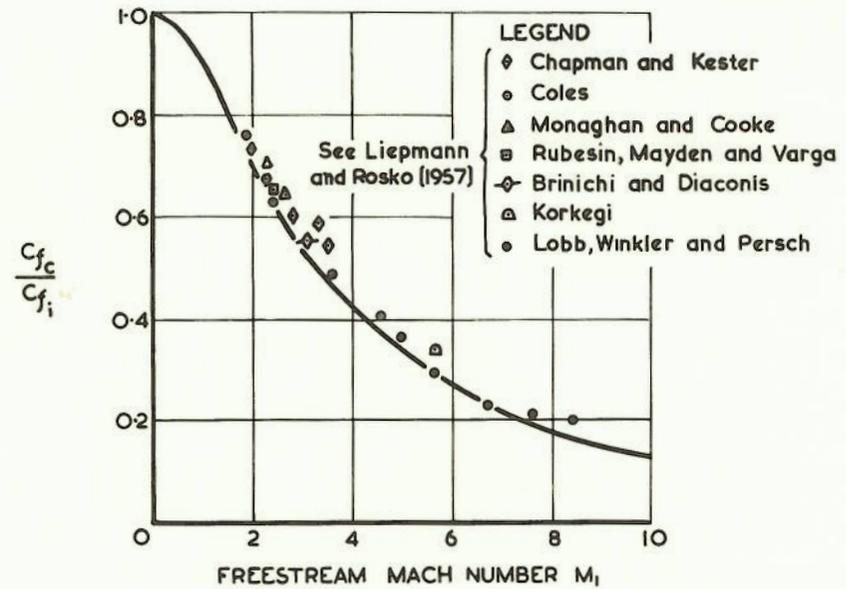


FIG. 2. RATIO OF SKIN FRICTION COEFFICIENTS IN COMPRESSIBLE AND INCOMPRESSIBLE FLOW IN THE ZERO PRESSURE GRADIENT (INSULATED WALLS)

APPENDIX A

The Compressible Turbulent Boundary Layer in Zero Pressure Gradient

The compatibility condition for  $\mu_T$  found in equation (103) can be derived without recourse to the Stewartson-illingworth transformation.

The equations of motion and continuity are

$$\bar{\rho} \bar{u} \frac{\partial \bar{u}}{\partial x} + (\bar{\rho} \bar{v} + \overline{\rho' v'}) \frac{\partial \bar{u}}{\partial y} = \frac{\partial \tau}{\partial y} \quad (\text{A.1})$$

and

$$\frac{\partial \bar{\rho} \bar{u}}{\partial x} + \frac{\partial}{\partial y} (\bar{\rho} \bar{v} + \overline{\rho' v'}) = 0. \quad (\text{A.2})$$

In the outer region of the boundary layer we will assume that

$$\tau \approx \mu_T \frac{\partial \bar{u}}{\partial y} \quad (\text{A.3})$$

where  $\mu_T$  is the virtual eddy viscosity and  $\mu_T \gg \mu$ . If we define the stream function  $\psi$  by

$$\bar{\rho} \bar{u} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad (\bar{\rho} \bar{v} + \overline{\rho' v'}) = - \frac{\partial \psi}{\partial x} \quad (\text{A.4})$$

then on application of von Mises transformation to equation A.1 we find that

$$\frac{\partial Z}{\partial x} = \bar{u} \frac{\partial}{\partial \psi} (\bar{\rho} \mu_T \frac{\partial Z}{\partial \psi}) \quad (\text{A.5})$$

where  $Z = \bar{u}_1^2 - \bar{u}^2$ .

But in the outer region we can put  $\bar{\rho} \mu_T$  as a function of  $x$  only if both quantities are evaluated at some suitable intermediate value. If therefore we put

$$\bar{\rho} \mu_T = \rho^*(x) \mu_T^*(x) \quad (\text{A.6})$$

then

$$\frac{\partial Z}{\partial x} = \rho^* \mu_T^* \bar{u} \frac{\partial^2 Z}{\partial \psi^2} \quad (\text{A.7})$$

If we also assume that in this region  $\bar{u} \approx \bar{u}_1$  equation (A.7) reduces to the incompressible form given by equation (6) with  $\rho_0 \mu_0 r(x)$  replacing  $\rho^* \mu_T^*$ . From the solution to (6) we can therefore find the solution to (A.7). In the region near the wall we then find that

$$\frac{\bar{u}_1^{3/2}}{u_1} = \frac{2}{\sqrt{\pi}} \int_0^x \frac{\tau_w(x') dx'}{\left( \int_{x'}^x \rho^*(z) \mu_T^*(z) dz \right)^{1/2}} \quad (\text{A.8})$$

which on inversion gives

$$\tau_w(x) \frac{dx}{dt(x)} = \frac{1}{2\sqrt{\pi}} \frac{\bar{u}_1^{3/2}}{\sqrt{t(x)}} \quad (\text{A.9})$$

where 
$$t = \int_0^x \rho^*(z) \mu_T^*(z) dz .$$

If we integrate (A.9) over a length L of the plate

$$\int_0^L \frac{\tau_w(x)}{\frac{1}{2} \bar{\rho}_1 \bar{u}_1^2} dx = \frac{2}{\sqrt{\pi} \bar{\rho}_1 \sqrt{\bar{u}_1}} \sqrt{\int_0^L \rho^* \mu_T^* dx} \quad (\text{A.10})$$

But 
$$C_F = \frac{1}{L} \int_0^L c_f(x) dx$$

$$= \frac{2}{\sqrt{\pi} \bar{\rho}_1} L \sqrt{\bar{u}_1} \sqrt{\int_0^L \rho^* \mu_T^* dx} \quad (\text{A.11})$$

by virtue of equation (A.10).

The corresponding relation obtained from the momentum integral equation is

$$C_F = \frac{2 R_{\delta_2}(L)}{\frac{\bar{\rho}_1 \bar{u}_1 L}{\mu_1}} \quad (\text{A.12})$$

where

$$\frac{dR_{\delta_2}}{dx} = \frac{\tau_w}{\rho_1 \bar{u}_1^2} \cdot \frac{\bar{\rho}_1 \bar{u}_1}{\mu_1} = \frac{\tau_w}{\bar{u}_1 \mu_1} \quad (\text{A.13})$$

Hence from A.11, A.12 and A.13,

$$R_{\delta_2}(L)^2 = \frac{1}{\pi} \int_0^{R_{\delta_2}(L)} \frac{\rho^* \mu_T^*}{\mu_1} \frac{\bar{u}_1^2}{\tau_w} dR_{\delta_2} \quad (\text{A.14})$$

which can only be satisfied if

$$\frac{1}{\pi} \frac{\rho^* \mu_T^*}{\mu_1} \frac{\bar{u}_1^2}{\tau_w} = 2 R_{\delta_2} \quad (\text{A.15})$$

Thus

$$\frac{\bar{\rho}_1 \delta_2 \sqrt{\tau_w / \rho_1}}{\mu_T^*} = \frac{1}{2\pi} \frac{\rho^*}{\rho_1} \frac{\bar{u}_1}{\sqrt{\tau_w / \rho_1}} \quad (\text{A.15}')$$

is the compatibility condition for  $\mu_T^*$ .

Alternatively we can write \*

$$c_{f(x)} = \frac{\tau_w}{\frac{1}{2} \bar{\rho}_1 \bar{u}_1^2} = \frac{1}{\pi} \frac{\bar{\rho}^* \bar{\mu}_T^*}{\bar{\rho}_1 \bar{\mu}_1} \cdot \frac{1}{R_{\delta_2}} \quad (\text{A.16})$$

This is the generalised relation for a compressible turbulent boundary layer corresponding to

$$c_{f_i(x)} = \frac{1}{\pi} \frac{\nu_{Ti}}{\mu_o R_{\delta_2}} \quad (\text{A.17})$$

in incompressible flow.

Thus the ratio of the compressible to the incompressible skin friction coefficient is

$$\frac{c_{f_c}}{c_{f_i}} = \frac{\bar{\rho}^* \bar{\mu}^*}{\bar{\rho}_1 \bar{\mu}_1} \frac{\bar{\mu}_T^* / \bar{\mu}^*}{\mu_{Ti} / \mu_o} \cdot \frac{R_{\delta_2 i}}{R_{\delta_2 c}} \quad (\text{A.18})$$

and at the same Reynolds number  $\frac{u_o L}{\nu_o} = \frac{\bar{\rho}_1 \bar{u}_1 L}{\mu_1}$

$$\frac{C_{F_c}}{C_{F_i}} = \sqrt{\frac{\int_0^1 \frac{\bar{\rho}^* \bar{\mu}_T^*}{\bar{\rho}_1 \bar{\mu}_1} d\bar{x}_c}{\int_0^1 \frac{\mu_{Ti}}{\mu_o} d\bar{x}_i}} \quad \text{where } \bar{x} = x/L \quad (\text{A.19})$$

and suffixes (i) and (c) refer to incompressible and compressible flow conditions respectively.

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\* This relation is similar to that given in equ. (103) except that in the latter equation we have put  $\bar{\rho}^* = \bar{\rho}_1$ .

However we cannot proceed further without knowing the relation between  $\mu_{T_i}$  and  $\mu_{T_c}^*$ . We note that in order to keep our various approximations consistent we should put  $\rho^*$  and  $\mu^*$  equal to  $\rho_1$  and  $\mu_1$  respectively in equations (A.18 and (A.19), but this does not help us find the desired relation between  $c_{f_c}$  and  $c_{f_i}$ .

Thus we see that the formal compressible flow analysis, without recourse to the Stewartson-Illingworth transformation, is necessarily incomplete and justifies the more elaborate treatment given in Sections 10 and 11.

APPENDIX B

The incompressible turbulent boundary layer in an adverse pressure gradient

The compatibility condition for the eddy viscosity as derived in equation (32) is true only in the case of zero pressure gradient and possibly in self preserving flow. It is not a priori justified in a turbulent boundary layer approaching separation. Indeed (32) suggests that the eddy viscosity falls in value in a region of adverse pressure gradient whereas experiments suggest that it increases rapidly as separation is approached. The rapid increase in the form parameter H near separation is analogous to this corresponding variation in the eddy viscosity. It would therefore seem desirable to rederive our equations to see, if with this variation in eddy viscosity, the position of separation can be predicted in agreement with measured results.

The equations of motion and continuity are respectively, for an incompressible flow in a turbulent boundary layer,

$$\frac{\partial U^2}{\partial x} + \frac{\partial UV}{\partial y} = - \frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu_0 \frac{\partial^2 U}{\partial y^2} - \frac{\partial}{\partial y} \overline{uv} - \frac{\partial}{\partial x} \overline{u^2} \dots \dots \quad (B.1)$$

$$0 = - \frac{1}{\rho_0} \frac{\partial p}{\partial y} - \frac{\partial \overline{v^2}}{\partial y} \quad (B.2)$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \quad (B.3)$$

From (B.2) we see that  $p + \rho_0 \overline{v^2}$  is a function of x only and we can therefore rewrite (B.1) as

$$\frac{\partial U^2}{\partial x} + \frac{\partial UV}{\partial y} = U_1 \frac{dU_1}{dx} + \nu_0 \frac{\partial^2 U}{\partial y^2} - \frac{\partial}{\partial y} \overline{uv} - \frac{\partial}{\partial x} (\overline{u^2} - \overline{v^2}) \dots \dots \quad (B.4)$$

If we define the stream function in terms of

$$\rho_0 U = \frac{\partial \psi}{\partial y} \quad \rho_0 V = - \frac{\partial \psi}{\partial x} \quad (B.5)$$

then on transformation from (x,y) co-ordinates to (x,ψ) equation B.4 becomes

$$\begin{aligned} \frac{\partial Z}{\partial x} = \rho_0 \mu_0 U \frac{\partial^2 Z}{\partial \psi^2} + 2 \rho_0 U \frac{\partial}{\partial \psi} \overline{uv} + 2 \frac{\partial}{\partial x} (\overline{u^2} - \overline{v^2}) \\ - 2 \rho_0 V \frac{\partial}{\partial \psi} (\overline{u^2} - \overline{v^2}) \end{aligned} \quad (B.6)$$

where  $Z = U_1^2 - U^2$ .

But because  $V \ll U$  and  $\overline{uv}$ ,  $\overline{v^2}$ ,  $\overline{u^2}$  are all of the same order of magnitude we can neglect  $V \frac{\partial}{\partial \psi} (\overline{u^2} - \overline{v^2})$  compared with

$U \frac{\partial}{\partial \psi} \overline{uv}$ . With this approximation (B.6) becomes

$$\frac{\partial}{\partial x} (Z - 2(\overline{u^2} - \overline{v^2})) = \rho_0 \mu_0 U \frac{\partial^2 Z}{\partial \psi^2} + 2 \rho_0 U \frac{\partial}{\partial \psi} \overline{uv} \quad (\text{B.7})$$

The integral of this equation over the boundary layer is

$$\int_0^\infty \frac{\partial}{\partial x} (Z - 2(\overline{u^2} - \overline{v^2})) \frac{d\psi}{U} = 2 \tau_w \quad (\text{B.8})$$

where the wall shear stress  $\tau_w = \left( \mu_0 \frac{\partial U}{\partial y} \right)_w = \rho_0 \mu_0 \left( U \frac{\partial U}{\partial \psi} \right)_{\psi=0}$ .

Equations (B.7) and (B.8) are exact according to normal boundary layer approximations.

In the outer region we assume an eddy viscosity,  $\nu_T$ , exists given by

$$-\overline{uv} = \nu_T \frac{\partial U}{\partial y} = -\frac{\rho_0 \nu_T}{2} \frac{\partial Z}{\partial \psi} \quad (\text{B.9})$$

where  $\nu_T$  is a function of  $x$  only. Also if structural similarity of the turbulence exists in the outer region  $-\overline{uv} / \sqrt{\overline{u^2}} \sqrt{\overline{v^2}}$  is roughly constant and so

$$\frac{\overline{u^2} - \overline{v^2}}{-\overline{uv}} = \alpha \quad (\text{B.10})$$

where  $\alpha$  is a numerical constant of order 2.5. Hence in the outer region equation (B.7) becomes, if the viscous term is neglected,

$$\frac{\partial Z}{\partial x} = \rho_0 \mu_T U \frac{\partial^2 Z}{\partial \psi^2} - \alpha \frac{\partial}{\partial x} \left( \mu_T \frac{\partial Z}{\partial \psi} \right) \quad (\text{B.11})$$

where  $\mu_T = \rho_0 \nu_T$ .

But in the outer region  $U$  is approximately constant and if the average value is  $U^+$ , we find on applying the further transformation

$$t = \int_0^x \rho_0 \mu_T U^+ dx' \quad (\text{B.12})$$

that

$$\frac{\partial Z}{\partial t} = \frac{\partial^2 Z}{\partial \psi^2} - \alpha \frac{\partial}{\partial t} \left( \mu_T \frac{\partial Z}{\partial \psi} \right) \quad (\text{B.13})$$

with the boundary conditions  $Z \rightarrow 0$  as  $\psi \rightarrow \infty$  and as  $t(x) \rightarrow 0$ . (This assumes that the turbulent boundary layer commences from  $x = 0$ . If a region of laminar flow exists upstream of the turbulent boundary layer the analysis needs a small modification.)

If a similar approximation is applied to the integral form of the momentum equation (B. 8) we can by suitable choice of  $U^+$  put

$$\frac{d}{dt} \int_0^\infty Z d\psi = \frac{2 \tau_w}{\rho_0 \mu_T} + \alpha \frac{d}{dt} \left( \mu_T U_1^2 \right) \quad (\text{B.14})$$

since 
$$- \int_0^\infty 2(\bar{u}^2 - \bar{v}^2) d\psi \approx \alpha \mu_T Z \Big|_0^\infty = -\alpha \mu_T U_1^2 .$$

The last term in (B.14) arises from the inclusion of the Reynolds stresses  $\rho_0 \bar{u}^2$  and  $\rho_0 \bar{v}^2$  in the equations of motion. An order

of magnitude analysis shows that it can be neglected in (B.14) except near to separation, and then only when the wall shear stress is vanishingly small.

For convenience in what follows we will write

$$T(t) = \frac{2 \tau_w}{\rho_0 \mu_T} + \alpha \frac{d}{dt} \left( \mu_T U_1^2 \right) \quad (\text{B.15})$$

so that

$$\frac{d}{dt} \int_0^\infty Z d\psi = T(t) \quad (\text{B.16})$$

and its Laplace Transform is

$$p \int_0^\infty \bar{Z}(\psi; p) d\psi = \int_0^\infty e^{-pt} T(t) dt \quad (\text{B.17})$$

If we now return to equation (B.13) we can show that the inclusion of the last term, since it is of smaller order than the remaining terms, does not materially change the form of  $Z$  as a function of  $\psi$  and  $t$ . We will therefore omit it, although still retaining it in the boundary condition B.16 or B.17. The solution of the modified B.13 then follows from equation (10) in Section 3 and is

$$Z(\psi, t) = \frac{1}{\sqrt{\pi}} \int_0^t T(t') \frac{\exp(-\psi^2/4(t-t'))}{\sqrt{t-t'}} dt' \quad (\text{B.18})$$

If we now assume that (B.19) holds for values just in the inner region also, where  $\psi \rightarrow 0$ , then approximately

$$U_1^2(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{T(t') dt'}{\sqrt{t-t'}} \quad (\text{B.19})$$

If now  $T(t)$  is a continuous function of  $t$  then on inversion

$$T(t) = \frac{1}{\sqrt{\pi}} \left[ \frac{U_1^2(0)}{\sqrt{t}} + \int_0^t \frac{dU_1^2(t')}{\sqrt{t-t'}} \right] \quad (\text{B.20})$$

where, as above,  $t = \int_0^x \rho_0 \mu_T(x') U^+(x') dx'$ .

Now as explained above, except very close to separation, we can put  $T(t) = \frac{2 \tau_w(t)}{\rho_0 \mu_T(t)}$  and so

$$c_f(t) = \frac{\mu_T}{U_1^2 \sqrt{\pi}} \left[ \frac{U_1^2(0)}{\sqrt{t}} + \int_0^t \frac{dU_1^2(t')}{\sqrt{t-t'}} \right] \quad (\text{B.21})$$

where the local skin friction coefficient

$$c_f = \frac{2 \tau_w}{\rho_0 U_1^2} .$$

As an example let us consider the case where in the range  $0 \leq x \leq x_1$ ,  $U_1 = U_1(o)$  and  $U_1 = U_1(x)$  for  $x > x_1$ . If we now write

$$R_1 = \frac{U_1(o) x_1}{\nu_o} \quad ; \quad \bar{U}_1 = U_1/U_1(o) \quad ; \quad \bar{x} = x/x_1$$

$$\bar{\nu}_T = \nu_T/\nu_o \quad \text{and} \quad \bar{t} = \int_0^{\bar{x}} \frac{\bar{\nu}_T}{\bar{\nu}_T(1)} \bar{U}_1 \frac{U^+}{U_1} d\bar{x}' \quad \text{with } k = \frac{U^+}{U_1}$$

then for  $x \geq x_1$

$$c_f(\bar{x}) = \frac{(\bar{\nu}_T \bar{U}_1)/\bar{\nu}_T(1)}{\bar{U}_1^3 \sqrt{\frac{\pi R_1 \bar{t}}{\bar{\nu}_T(1)}}} \left[ 1 + \sqrt{\bar{t}} \int_{\bar{t}_1}^{\bar{t}} \frac{d\bar{U}_1^2(\bar{t}')}{\sqrt{\bar{t} - \bar{t}'}} \right] \quad (\text{B. 22})$$

with  $c_f(\bar{x}_1) = \frac{1}{\sqrt{\left( \frac{\pi R_1}{\bar{\nu}_T(1)} \int_0^1 \frac{\bar{\nu}_T(\bar{x}')}{\bar{\nu}_T(1)} \frac{U^+}{U_1(o)} d\bar{x}' \right)}} \quad (\text{B. 23}),$

since  $\bar{x}_1 = 1$ , and  $U_1(x_1)/U_1(o) = 1$ .

If we assume that  $\frac{d\bar{U}_1^2}{d\bar{t}}$  is constant in the region of adverse pressure gradient then from (B. 22), the skin friction distribution is (with  $U^+/U_1 = 1$ )

$$c_f(\bar{x}) = \frac{2}{\bar{U}_1^3} \sqrt{\frac{\bar{\nu}_T(1)}{\pi R_1}} \left[ \frac{d\sqrt{\bar{t}}}{d\bar{x}} - \frac{dC_p}{d\bar{x}} \sqrt{\bar{t} - \bar{t}_1} \right] \quad (\text{B. 24})$$

where  $C_p$  is the pressure coefficient given by  $C_p = 1 - \bar{U}_1^2$  and  $\frac{dC_p}{d\bar{x}}$  is positive.

Equation (B. 24) is in a convenient form to compare with the experimental results of Schubauer and Klebanoff (1950), and so enables us to determine empirically the variation of  $\nu_{\Gamma}$  with  $(x)$ . In effect we want to know how the eddy viscosity "stretches" the transformed co-ordinate  $\bar{t}$ . Of course (B. 24) is only an approximate form of (B. 22) but unless  $d\bar{U}_1^2/d\bar{t}$  is known with great accuracy little advantage is gained in using the latter equation. (In fact the use of inaccurate values of  $d\bar{U}_1^2/d\bar{t}$  in (B. 22) usually leads to misleading results).

Equation (B. 24) is most important for it shows how the local skin friction coefficient depends on the local external conditions (the external velocity and the external pressure gradient), the local conditions in the boundary layer (the eddy viscosity in the outer region) and the past history of the boundary layer (the integral of the eddy viscosity up to the local position). It does not depend on some assumed relation between skin friction and momentum thickness, and apart from the relatively minor approximations in the analysis, the only major assumption made is that the Reynolds stress  $-\overline{uv}$  can be defined in terms of a virtual eddy viscosity.

It must be stressed that it is only applicable to the flow in a region of adverse pressure gradient approaching separation following a region of near zero pressure gradient, which is just the case treated experimentally by Schubauer and Klebanoff.

In the evaluation of (B. 24)  $R_1$  is put equal to  $18 \times 10^6$ , and  $\bar{v}_T(1)$ , from the above results for zero pressure gradient, has a value of about 125. This gives a value for  $c_f(1)$  of about 0.0023 compared with an experimental value of about 0.0022 as deduced from the measured velocity profiles.

It would be inappropriate at this stage to gloss over the difficulty in determining the required relation between  $\bar{t}$  and  $\bar{x}$  from a comparison between (B. 24) and the experimental results. Also the relation as determined may be special to this pressure distribution and so at the best the results that follow are only very tentative. Since we are therefore interested only in an order of magnitude result we will make the rather bold assumption that the integrand in the evaluation of  $\bar{t}$  is proportional to  $\bar{x}^\beta$  right from  $\bar{x} = 0$ . With  $k = 1$  the integrand has the value of unity at  $\bar{x} = 1$  for all values of  $\beta$ . From our previous analysis we would expect  $\beta$  to take on different values in the region of near zero pressure gradient from those in the region of adverse pressure gradient. However to include this variation would appear to be an unnecessary elaboration at this stage, apart from making the final results most unwieldy.

Thus with  $\frac{\bar{v}_T}{\bar{v}_T(1)} \bar{U}_1 = \bar{x}^\beta$  we find

that (B. 24) reduces to

$$c_f(x) = \frac{1}{\bar{U}_1^3} \sqrt{\frac{(\beta+1) \bar{v}_T(1)}{\pi R_1}} \left[ \bar{x}^{\frac{\beta-1}{2}} - \frac{2}{(\beta+1)} \frac{dC_p}{dx} (\bar{x}^{\beta+1} - 1)^{\frac{1}{2}} \right] \quad \text{.....} \quad \text{(B. 25)}$$

and with  $\beta = 3$

$$c_f(\bar{x}) = \frac{2\bar{x}}{\bar{U}_1^3} \sqrt{\frac{\bar{v}_T(1)}{\pi R_1}} \left[ 1 - \frac{1}{2\bar{x}} \frac{dC_p}{d\bar{x}} (\bar{x}^4 - 1)^{\frac{1}{2}} \right] \quad \text{(B. 26)}$$

leading to

$$\frac{c_f(\bar{x})}{c_f(1)} = \frac{\bar{x}}{\bar{U}_1^3} \left[ 1 - \frac{1}{2\bar{x}} \frac{dC_p}{d\bar{x}} (\bar{x}^4 - 1)^{\frac{1}{2}} \right] \quad \text{(B. 27)}$$

If we take  $\frac{dC_p}{d\bar{x}}$  as equal to the mean slope between  $\bar{x}$  and  $\bar{x} = 1$  for each value of  $\bar{x}$  (noting that  $\frac{dC_p}{d\bar{x}}$  must equal zero at  $\bar{x} = 1$ )

we find that the following values fit the experimental results for both the pressure gradient and skin friction coefficient exactly.

x ft.	x/x <sub>1</sub>	dC <sub>p</sub> /d $\bar{x}$	c <sub>f</sub> ( $\bar{x}$ )/c <sub>f</sub> (1)
18	1.0	0	1.0
19.8	1.1	1.44	.82
21.6	1.2	1.6	.60 <sup>5</sup>
23.4	1.3	1.66	.38 <sup>5</sup>
25.2	1.4	1.62	.10
25.8	1.43	1.60	0

(Note the tabulated values of  $\frac{dC_p}{d\bar{x}}$  are not the local values but the average values between  $\bar{x} = 1$  and the local value of  $\bar{x}$ ).

We note also that the separation position is predicted exactly, but this is probably fortuitous, and in view of the approximations made in leading up to equation B.27, we cannot claim more than an order of magnitude agreement.

Since in the region of the positive pressure gradient  $U_1 \sim \frac{1}{x}$  we see that in the same region  $\bar{v}_T \sim \bar{x}^4$ , that is its value increases fourfold in its passage from  $\bar{x} = 1$  to separation. This result is also qualitatively in agreement with the variation in the measured values of the Reynolds stress  $-\overline{uv}$  over the same region.

In conclusion it is again necessary to stress the tentative nature of our results especially the rather arbitrary choice of the value  $\beta = 3$ . However the present results are sufficiently promising for us to continue with more detailed comparisons with other known experimental data, and this work is in hand.

APPENDIX C

The compressible turbulent boundary layer in zero pressure gradient  
when the wall is highly cooled

When the walls are highly cooled the enthalpy across the boundary layer does not rise above its freestream value. It is therefore more appropriate to use freestream conditions, as reference conditions in the Stewartson-illingworth transformation, rather than stagnation conditions as used under nearly zero heat transfer. In this case the transformation formulae are

$$X = x \quad \text{and} \quad Y = \int_0^y \frac{\rho(x, y')}{\rho_1} dy' \quad (C.1)$$

$$\text{with} \quad U = u \quad \text{and} \quad U_1 = u_1 \quad (C.2)$$

where capitals denote transformed values while small letters refer to the conditions in the compressible flow. In the transformed flow the constant density and viscosity are  $\rho_1$  and  $\mu_1$  respectively.

The transformed equation of motion in Von Mises form becomes

$$\frac{\partial Z}{\partial X} = \nu_1 U \frac{\partial}{\partial \psi} \left( \bar{C} \frac{\partial Z}{\partial \psi} \right) + 2U \frac{\partial}{\partial \psi} \overline{u'_i v'_i} \quad (C.3)$$

$$\text{where} \quad Z = U_1^2 - U^2 \quad ; \quad \bar{C} = \frac{\rho \mu}{\rho_1 \mu_1} \quad ; \quad \rho \overline{u'_i v'_i} = \rho_1 \overline{u'_i v'_i} \quad .$$

If we replace  $\bar{C}$  by a suitable average value\*  $\bar{C}^*$  and put  $X^* = \int_0^X \bar{C}^* dX'$ , equation (C.3) reduces to the incompressible flow equation

$$\frac{\partial Z}{\partial X^*} = \nu_1 U \frac{\partial^2 Z}{\partial \psi^2} + 2U \frac{\partial}{\partial \psi} \left( \frac{\overline{u'_i v'_i}}{\bar{C}^*} \right) \quad (C.4)$$

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\* the average should be taken over the viscous sub-layer.

Its integral form is

$$\int_0^{\infty} \frac{\partial Z / \partial X^*}{U} d\psi = 2 \frac{\tau_w^*}{\rho_1} \quad (C.5)$$

where  $\tau_w^* = \mu_1 \left( \frac{\partial U}{\partial Y} \right)_w$  is the wall shear stress in the transformed incompressible like flow. In a highly cooled wall case  $\bar{C}^*$  will be greater or equal to unity. Thus we see that, according to our transformation, cooling a wall reduces the effective Reynolds stress and increases the length scale of the boundary layer.

The solution of these incompressible flow equations has been obtained above. The result for the ratio of the skin friction coefficients in compressible and incompressible flow at the same Reynolds number,

if  $c_{f_i} \sim R_{x_i}^{-1/5}$ , is

$$\frac{c_{f_c}}{c_{f_i}} = \bar{C}^{*4/5} = \left( h_1 / h^* \right)^{4/5} \quad (C.6)$$

if  $\omega = \frac{3}{4}$ . If now we put, failing comparable experimental data,

$$\rho^* \mu^* = \frac{\rho_1 \mu_1 + \rho_w \mu_w}{2} \quad (C.7)$$

then

$$\frac{c_{f_c}}{c_{f_i}} = \left[ \frac{1 + \left( h_1 / h_w \right)^{1/2}}{2} \right]^{4/5} \quad (C.8)$$

a result independent of Mach number.

Even if the relation (C.7) is found in practice to be far from accurate we will always find that the relation  $c_{f_c} / c_{f_i}$  has values near unity when  $h_w / h_1$  is less than unity, i. e. the highly cooled wall case. This conclusion that the skin coefficient in a compressible flow having a highly cooled wall is roughly equal to the incompressible skin friction coefficient was suggested by Rose, Probstein and Adams (1958) at least for the case of a partly dissociated boundary layer. However they based their conclusion on the assumed fact that the density was approximately constant over the entire boundary layer outside the laminar sub-layer, whereas our result does not depend on such an assumption.