

THE COLLEGE OF AERONAUTICSCRANFIELDAn Elementary Study of Gas Injection and
Sublimation into a Simple Shear Layer

- by -

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ADDENDA AND CORRIGENDAPage 3. In eq. 11 $-\lambda \frac{dt}{dy}$ should read $-\lambda \frac{dT}{dy}$ Page 6. Eq. 28 should read $-\dot{q} = \frac{\lambda}{C_p} \left[\frac{dh}{dy} + (Le - 1)T\Delta \frac{dc}{dy} \right]$ Page 11. The phrase following eq. 45 should read "where $h_{w_0} = C_p T_w$."

It is important to note that \dot{q} is the rate at which energy is transported relative to a surface moving with the mass average or flow velocity. Thus $-\dot{q}_w$ is not the rate at which energy is transferred into the wall at $y = 0$. The latter quantity, written as $-\dot{q}_s$ say, is given by :-

$$-\dot{q}_w - \dot{m} h_w = -\dot{q}_s = \left(\lambda_b \frac{dT_b}{dy} \right)_w - \dot{m} h_{iw} .$$

The second form of $-\dot{q}_s$ follows from the energy balance at the interface and is the energy flux within the body. Since $-\dot{q}_w = \left(\lambda \frac{dT}{dy} \right)_w + \dot{m} h_w - \dot{m} h_i$

(for any Lewis No. value) it follows that $\left(\lambda \frac{dT}{dy} \right)_w$ is always equal to $\left(\lambda_b \frac{dT_b}{dy} \right)_w$ in the case of gas injection, i.e. the conduction in gas and solid balance at the interface.

With these definitions eq. 45 is modified so as to read :-

$$-\dot{q}_s \delta / \bar{\mu} \approx (h_{ro} - h_{wo}) - h_{wo} \Delta' b' - (h_{ro} + h_w) b' / 2 \quad (45a)$$

where $h_{wo} = C_p T_w$.

The second sentence in the paragraph following eq. 45 should be modified to read: "Since b' can be written as $\dot{m} \delta / \bar{\mu}$ approximately the last term in eq. 45a gives a reduction in $-\dot{q}_s$ equal to $(h_{ro} + h_w) \dot{m} / 2$ "

Further on in this same paragraph the words ". lead to an increase in heat transfer rate if" should be modified to read ". lead to an increase in $-\dot{q}_w$ if".

The last sentence in the same paragraph should be deleted and replaced by: "Even though $-\dot{q}_w$ may be increased by injection of the 'wrong' gas, it can readily be shown that injection always decreases $-\dot{q}_s$. The amount of this decrease becomes smaller as C_{pi} decreases, however".

It is worth pointing out that Stanton number is in fact more conveniently defined in terms of $-\dot{q}_w$ than in terms of $-\dot{q}_s$. Also that recovery enthalpy refers to the zero of $-\dot{q}_w$ and not to the zero of $-\dot{q}_s$.

Page 14. The paragraph following eq. 56 should be deleted, as should eq. 57, and the following substituted.

"The energy transfer rates at the interface must now be matched, taking account of the fact that the solid material absorbs an amount of latent heat L per unit mass during sublimation, and also accounting for convection of energy in both the solid and the gas. Thus:

$$-\dot{q}_w - \dot{m} h_w = \lambda_b \left. \frac{dT}{dy} \right|_{y=0-} - \dot{m} C_b T_w + \dot{m} L,$$

or, using eqs. 45a and 56,

$$-\dot{q}_s = \dot{m} (L - C_b T_b) = \dot{m} L', \quad (57)$$

say."

In the line following eq. 57, change " $-\dot{q}_w$ " to read " $-\dot{q}_s$ " and "eq.(45)" to read "eq.(45a)". Then eq. 58 should read :-

$$-\dot{q}_s \approx (h_{ro} - h_{wo}) \pi / \delta - \dot{m} \left\{ h_{wo} \Delta' + (h_{ro} + h_{wo})/2 \right\} \quad (58)''$$

Page 15. Eq. 59 should now read :-

$$\dot{m} = (h_{ro} - h_{wo}) (\bar{\mu} / \delta) (L' + h_{wo} \Delta' + (h_{ro} + h_{wo})/2)^{-1} \quad (59)''$$

In paragraph following eq. 59, " $-\dot{q}_w$ " should be altered to read " $-\dot{q}_s$ ".

Eq. 60 should read :-

$$-\dot{q}_s = (h_{ro} - h_{wo}) (\bar{\mu} / \delta) L' (L' + \Delta' h_{wo} + (h_{ro} + h_{wo})/2)^{-1} \quad (60)''$$

The section starting 3 lines above eq. 61 and ending 3 lines below eq. 62 should be deleted and the following substituted :-

"However, $-\dot{q}_s$ is not of primary importance here. Rather are we concerned with the heat flux into the wall, which eq. 56 and 57 show to be given by :-

$$\lambda_b \left. \frac{dT}{dy} \right|_{y=0-} = \dot{m} C_b (T_w - T_b) = -\dot{q}_s - \dot{m} L + \dot{m} C_b T_w \quad (61)$$

It follows that :

$$\lambda_b \left. \frac{dT}{dy} \right|_{y=0-} = \frac{(h_{ro} - h_{wo}) (\bar{\mu} / \delta) C_b (T_w - T_b)}{L' + \Delta' h_{wo} + (h_{ro} + h_{wo})/2} \quad (62)$$

Clearly $\lambda_b dT/dy \big|_{y=0-}$ reduces substantially in magnitude as L (and hence L') rises. As $L \rightarrow \infty$, $\lambda_b dT/dy \big|_{y=0-} \rightarrow 0$; note that although $\dot{m} \rightarrow 0$ as $L \rightarrow \infty$ the product $\dot{m} L$ remains finite".

Page 17. In line 4, replace pm_e / RT by pm_e / kT .

In the paragraph beginning just before eq. 69 the condition should be " $L \gg h_w \Delta' + (h_{ro} + h_{wo})/2 - C_b T_b$ ".

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LIST OF SYMBOLS

b, b'	Defined in eqs. 30 and 33
c	Foreign gas concentration (mass fraction)
C_b	Specific heat of wall material
C_f	Friction coefficient
C_p	Specific heat of external gas
C_{pi}	Specific heat of injected gas
D	Diffusion coefficient
h	Enthalpy per unit mass
k	Boltzmann's Constant
L	Latent heat of sublimation
Le	Lewis Number
\dot{m}	Mass injection rate per unit surface area
p	Pressure
Pr	Prandtl number
\dot{q}	Energy flux
Sc	Schmidt number
T	Temperature
u	Velocity parallel to walls
v	Velocity perpendicular to walls
y	Co-ordinate perpendicular to walls
δ	Distance between walls
Δ, Δ'	Defined in eqs. 12 and 33
ρ	Density
μ	Viscosity
λ	Conductivity
τ	Shear stress

List of Symbols Continued

Suffixes

b	Lower wall material value
e	External gas
i	Injected gas
o	Zero-injection rate value
r	Recovery value
w	Evaluated at $y = 0$
δ	Evaluated at $y = \delta$

1. Introduction

There is a great deal of current interest in the processes of ablation into hypersonic boundary layers and several rigorous theoretical treatments have appeared in the published literature. Notable amongst these we may mention the work of Lees (1958) and Scala. The latter has given a summary of a great volume of his work in this field (Scala, 1960).

The processes involved are quite complex and the number of variables which can affect them is large. The following represents an ultra-simplified treatment of gas injection and gaseous ablation into a simple shear layer (Couette flow) in an attempt to emphasise some of the concepts and parameters involved.

First the injection of a perfect gas into another perfect gas is discussed and later on these results are used to establish ablation, and the corresponding heat transfer rates. Homogeneous chemical reactions are not considered. The variations of sublimation temperature with pressure are examined briefly, via the kinetic theory. An attempt is made to confine the treatment to the barest essentials, and no attempt made to give a copious list of references. As far as possible the work concerns the general situation.

2. Gas Injection at the Lower Wall

The assumption is made that both gases, namely that of the external flow and the injected gas, are ideal. Then the respective specific enthalpies, h_e and h_i can be written as

$$h_e = C_p T \quad ; \quad h_i = C_{pi} T , \quad (1)$$

where the specific heats at constant pressure, C_p and C_{pi} , are both constants. T is the absolute temperature. Since we exclude the possibility of chemical reactions between the two species there is no need to refer energies to a common zero level. If the injected gas is present at concentration (mass fraction) c , the specific enthalpy of the mixture, h , is given by

$$h = (1 - c)h_e + c h_i = \left\{ C_p + (C_{pi} - C_p)c \right\} T . \quad (2)$$

For plane parallel Couette flow with no x-wise pressure gradient, the usual conservation equations become

$$\frac{d(\rho v)}{dy} = 0, \quad (3)$$

$$\rho v \frac{du}{dy} = \frac{d}{dy} \left(\mu \frac{du}{dy} \right), \quad (4)$$

$$\rho v \frac{dv}{dy} = -\frac{dp}{dy} + \frac{d}{dy} \left(\frac{4\mu}{3} \frac{dv}{dy} \right), \quad (5)$$

$$\rho v \frac{dh}{dy} - v \frac{dp}{dy} = -\frac{dq}{dy} + \frac{4\mu}{3} \left(\frac{dv}{dy} \right)^2 + \mu \left(\frac{du}{dy} \right)^2. \quad (6)$$

ρ , μ and p are the mixture density, viscosity and thermodynamic pressure respectively, and u and v are the gas velocities parallel and perpendicular to the plates. \dot{q} is the y -component of the energy flux vector which, in the presence of interdiffusion of the external and injected gases, must be written

$$\dot{q} = -\lambda \frac{dT}{dy} + \rho c v_i h_i + \rho(1-c)v_e h_e. \quad (7)$$

v_i and v_e are the appropriate diffusion velocities and λ is the coefficient of thermal conductivity. The diffusion velocities in the simple binary mixture under consideration are simply related to the concentration gradients ;

$$c v_i = -D \frac{dc}{dy} = -(1-c)v_e, \quad (8)$$

where D is the binary diffusion coefficient for the particular mixture.

In addition to the overall mass conservation equation (eq. 3), continuity equations for each separate species can be written down. These are

$$\frac{d}{dy} \left\{ \rho c v - \rho D \frac{dc}{dy} \right\} = 0, \quad (9)$$

$$\frac{d}{dy} \left\{ \rho(1-c)v + \rho D \frac{dc}{dy} \right\} = 0, \quad (10)$$

for the injected and external gases respectively, and use has been made of eqs. (8) to eliminate the diffusion velocities.

The energy flux is better expressed in terms of concentration gradients, namely, (using eqs.(7), (8) and (1)),

$$\dot{q} = -\lambda \frac{dT}{dy} - \rho D \Delta T \frac{dc}{dy} , \quad (11)$$

where we have written

$$C_{pi} - C_p = \Delta . \quad (12)$$

The species continuity equations (9 and 10) can be integrated at once to give

$$\rho c v - \rho D \frac{dc}{dy} = \dot{m} , \quad (13)$$

$$\rho(1 - c)v + \rho D \frac{dc}{dy} = 0 . \quad (14)$$

Here we have made use of the boundary condition at the lower wall to evaluate the integration constants. The left hand side of eq.13 is equal to $\rho c(v + v_i)$ and is therefore equal to the mass flow rate of injected gas per unit area, which is written as \dot{m} at the lower wall. Similarly the left hand side of eq. 14 is $\rho(1 - c)(v + v_o)$, and since no external gas is allowed to enter the lower wall the appropriate value for this quantity at $y = 0$ is zero. It immediately follows that

$$\rho v = \dot{m} , \quad (15)$$

(a result which could also be deduced directly from eq. 3.), and the equation satisfied by c now becomes

$$\rho D \frac{dc}{dy} = -\dot{m}(1 - c) . \quad (16)$$

The momentum equation parallel to the wall can also be integrated directly and, using the condition $\mu(du/dy) = \tau_w$ when $y = 0$ (τ_w = shear stress at the wall), we have

$$\mu \frac{du}{dy} = \dot{m} u + \tau_w . \quad (17)$$

Using eq. 17, all d/dy derivatives can now be written in terms of d/du derivatives, (this is the Couette flow version of Crocco's laminar boundary layer transformation). In particular this transformation applied to eq.16 yields

$$\frac{dc}{du} = -\frac{\dot{m} Sc (1 - c)}{(\dot{m} u + \tau_w)} , \quad (18)$$

for the relation between c and u . The quantity Sc is the Schmidt number,

$$Sc = \mu / \rho D, \quad (19)$$

and in what follows we shall assume it to be a constant throughout the gas layer. It is not likely to vary too much for most gas mixtures of interest and, in any case, we can assume it to have some suitable mean value for the purposes of the present, heuristic, analysis. This being so, eq. 18 integrates to give

$$1 - c = (1 - c_w)(1 + \dot{m}u/\tau_w)^{Sc}, \quad (20)$$

where c_w is the foreign gas concentration at the lower wall. Note that if $\dot{m} = 0$, $c = c_w$ everywhere; if $\dot{m} > 0$ (injection) then $c < c_w$ everywhere and vice versa if $\dot{m} < 0$ (suction). Some simplification will result later if we take c_δ , the foreign gas concentration at the upper wall, to approach zero. This is consistent with the $\dot{m} > 0$ requirement for injection and also corresponds most closely with the practical, boundary layer, problem. The assumption $c_\delta \rightarrow 0$ precludes consideration of the suction case however. It can be seen from eq. 8 that as $c_\delta \rightarrow 0$ the product $(c v_i)_\delta$ must remain finite and non-zero (in other words the diffusion velocity v_i at $y = \delta$ increases without limit). The mean vertical velocity of the injected gas is $v + v_i$ and its mass flow rate per unit area is therefore $\rho c(v + v_i)$, as remarked previously. The limiting case $c_\delta \rightarrow 0$ then implies that the injected gas is carried away through the upper wall purely by diffusion (since $\rho v = \dot{m} = \text{const.}$ and $\rho c v \rightarrow 0$ as $c \rightarrow 0$). Eq. 20 shows that c_w is directly related to \dot{m} once c_δ is fixed, because,

$$1 - c_w = (1 - c_\delta)(1 + \dot{m} U/\tau_w)^{-Sc}. \quad (21)$$

Furthermore, even though we let $c_\delta \rightarrow 0$, $c_w \neq 0$ as long as $\dot{m} > 0$. The injected gas therefore enters into the flow region by a mixture of diffusion and convection.

The transport of foreign gas across the shear layer by a mixture of convection and diffusion is precisely what happens in the practical case of a boundary layer flow, and it is for this reason that we study it here. As is already apparent the inclusion of diffusion introduces the Schmidt number into the analysis, a number which must be important also in boundary layer flows since it determines the relative thickness of the momentum and concentration layers (i.e. $Sc = \nu / D$).

We must now turn to a consideration of the energy equation (eq. 6). With eq. 11 for the energy flux, this can be written

$$\dot{m} \frac{dh}{dy} = v \frac{dp}{dy} + \frac{d}{dy} \left(\lambda \frac{dT}{dy} + \rho D T \frac{dc}{dy} \right) + \frac{4\mu}{3} \left(\frac{dv}{dy} \right)^2 + \mu \left(\frac{du}{dy} \right)^2, \quad (22)$$

where ρv has been written as \dot{m} . With the result given in eq. 11 the term $\mu (du/dy)^2$ can be written as $\tau_w (du/dy) + (\dot{m}/2) du^2/dy$, so that the only hindrance to a direct integration of eq. 22 is the appearance of the terms in v . Eliminating $v dp/dy$ from eq. 22 with the aid of eq. 5 we have

$$\dot{m} \frac{d}{dy} \left(h + v^2/2 \right) = + \frac{d}{dy} \left(\lambda \frac{dT}{dy} + \frac{4\mu}{3} \frac{d(v^2/2)}{dy} + \rho D \Delta T \frac{dc}{dy} \right) + \mu \left(\frac{du}{dy} \right)^2,$$

and writing

$$\bar{C}_p = C_p + (C_{pi} - C_p) c, \quad (23)$$

eq. 2 shows that

$$\frac{dT}{dy} = \frac{1}{\bar{C}_p} \frac{dh}{dy} - \frac{T\Delta}{\bar{C}_p} \frac{dc}{dy}.$$

The energy equation can now be written as

$$\dot{m} \frac{d}{dy} \left(h + v^2/2 \right) = \frac{d}{dy} \left\{ \frac{\lambda}{\bar{C}_p} \left[\frac{d}{dy} \left(h + v^2/2 \right) + \left(\frac{4Pr}{3} - 1 \right) \frac{d(v^2/2)}{dy} + (Le - 1) T\Delta \frac{dc}{dy} \right] \right\} + \mu \left(\frac{du}{dy} \right)^2. \quad (24)$$

The quantities Pr and Le are the Prandtl and Lewis numbers respectively, defined here as

$$Pr = \frac{\mu \bar{C}_p}{\lambda}; \quad Le = \frac{\rho \bar{C}_p D}{\lambda} \quad (25)$$

Eq. 24 only contains terms in $v^2/2$ and for small amounts of injection these may plausibly be neglected in relation to the enthalpy terms. With this approximation eq. 24 becomes

$$\dot{m} \frac{dh}{dy} \approx \frac{d}{dy} \left\{ \frac{\lambda}{\bar{C}_p} \left[\frac{dh}{dy} + (Le - 1) T\Delta \frac{dc}{dy} \right] \right\} + \tau_w \frac{du}{dy} + \frac{\dot{m}}{2} \frac{du^2}{dy}, \quad (26)$$

having made use of eq. 17 to eliminate $\mu du/dy$.

Eq. 26 can now be integrated to give

$$\dot{m} (h - h_w) = \frac{\lambda}{C_p} \left[\frac{dh}{dy} + (Le - 1) T \Delta \frac{dc}{dy} \right] + u r_w + \frac{\dot{m} u^2}{2} + \dot{q}_w \quad (27)$$

Suffix w refers to quantities evaluated at the lower wall, $y = 0$ and in obtaining eq. 27 we have made use of the fact that

$$-\dot{q} = \frac{\lambda}{C_p} \left[\frac{dh}{dy} + (Le - 1) T \Delta \frac{dc}{dy} \right]. \quad (28)$$

This can be seen from eqs. 11, 1, 23 and 25. $-\dot{q}_w$ is the energy transfer rate into the lower wall.

The Crocco-type transformation is now applied to eq. 27 with the result that

$$(\dot{m} u + \tau_w) \left\{ \frac{dh}{du} + (Le - 1) T \Delta \frac{dc}{du} \right\} - \dot{m} Pr (h - h_w) + Pr (u r_w + \dot{m} u^2 / 2 + \dot{q}_w) = 0 \quad \dots (29)$$

The term involving T in this equation can be eliminated via eqs. 1 and 20*, i.e.

$$(Le - 1) T \Delta \frac{dc}{du} = - \frac{(Le - 1) (\Delta Sc \dot{m} / \tau_w) (1 - c_w) (1 + \dot{m} u / \tau_w)^{Sc-1}}{C_p + \Delta \left[1 - (1 - c_w) (1 + \dot{m} u / \tau_w)^{Sc} \right]} \cdot h ,$$

and if we now write

$$\dot{m} / \tau_w = b \quad (30)$$

* It is assumed from here on that Le and Pr are constant across the layer. Since $Sc = Pr/Le$ this is consistent with the previous assumption that Sc is also constant. In practice all of these dimensionless groups will vary to some extent with composition and we must regard the values used here as appropriate mean values.

for brevity, eq. 29, can be written

$$\frac{dh}{du} = \left\{ \frac{\text{Pr}}{(u + 1/b)} + \frac{(Le - 1) \Delta (1 - c_w) b Sc (ub + 1)^{Sc-1}}{C_p + \Delta [1 - (1 - c_w)(ub + 1)^{Sc}]} \right\} \cdot h$$

$$= -\text{Pr} (u + 1/b)^{-1} \left\{ (u + 1/b)^2 / 2 - 1/2b^2 + \dot{q}_w / \dot{m} + h_w \right\} \quad (31)$$

Eq. 31 can be integrated to yield ,

$$f(u) h - f(0) h_w = -\text{Pr} \int_0^u (\dot{q}_w / \dot{m} + h_w + g(u)) f(u) (u + 1/b)^{-1} du \quad (32)$$

where

$$f(u) = (b^{\text{Pr}}) (ub + 1)^{-\text{Pr}} \left\{ C_p + \Delta \left[1 - (1 - c_0)(ub + 1)^{Sc} / (Ub + 1)^{Sc} \right] \right\}^{Le-1}$$

$$g(u) = (1/2b^2) \left[(ub + 1)^2 - 1 \right] \quad \dots \dots \quad \begin{matrix} (32a) \\ (32b) \end{matrix}$$

(Eq. 21 has been used to eliminate $1 - c_w$ in terms of c_0 , U and b in eq. 32a, and $f(0)$ in eq. 32 is $f(u)$ evaluated when $u = 0$).

3. Recovery Enthalpy and Heat Transfer Rate

Some general results can now be derived from eq. 32. First we note that when $u = U$, $h = h_\delta$ and if, in addition, $\dot{q}_w = 0$ then $h_w = h_r$, the recovery enthalpy. It follows therefore that

$$h_r = \left\{ f(U)h_\delta + \sqrt{\text{Pr}} \int_0^U g(u) f(u) (u + 1/b)^{-1} du \right\} / \left\{ f(0) - \text{Pr} \int_0^U f(u)(u + 1/b)^{-1} du \right\}$$

and in terms of this quantity,

$$-\dot{q}_w = (1/\text{Pr}) \left\{ \left(f(0) / \int_0^U f(u)(u + 1/b)^{-1} du \right) - 1 \right\} (h_r - h_w) b \tau_w$$

These two quantities can be rewritten in a more convenient form by defining dimensionless functions f' and g' which are related to f and g in eqs. 32a and 32b. Thus we write

$$f'(u') = (u' b' + 1)^{-\text{Pr}} \left\{ 1 + \Delta' \left[1 - (1 - c_\delta)(u' b' + 1)^{\text{Sc}} / (b' + 1)^{\text{Sc}} \right] \right\} \text{Le}^{-1} \\ = f(u) / \left(b^{\text{Pr}} C_p \text{Le}^{-1} \right) \quad (33a)$$

$$g'(u') = (u' b' + 1)^2 - 1 = 2b^2 g(u) / U^2 \quad (33b)$$

where $u' = u/U$, $b' = bU$, $\Delta' = \Delta/C_p$. Whence it follows that

$$h_r = \left\{ f'(1)h_\delta + (\text{Pr} U^2 / 2) \int_0^1 (g' f') b'^{-1} (1 + u' b')^{-1} du' \right\} / \left\{ f'(0) - \text{Pr} b' \int_0^1 f' (1 + u' b')^{-1} du' \right\} \quad (34)$$

$$-\dot{q}_w = (1/\text{Pr}) \left\{ \left(f'(0) / \int_0^1 f' (1 + u' b')^{-1} du' \right) - b' \right\} (h_r - h_w) (\tau_w / U) \quad (35)$$

These expressions for h_r and q_w are very unwieldy and little can be gained from an examination of them as they stand. Consequently we must resort to approximation in order to gain some insight into the physical picture and it seems natural to attempt solutions for $b' \ll 1$. Referring to the definition of b' , this assumption is equivalent to setting

$\dot{m}U/\tau_w \ll 1$. Physically this group of variables has a simple interpretation since $\dot{m}U$ is proportional to the flux of x-wise momentum induced in a direction away from the wall by the act of blowing into the shear layer, whilst τ_w indicates the magnitude of the same flux taking place towards $y = 0$ as a result of the microscopic, molecular motions. Setting $b' \ll 1$ is equivalent to assuming that the amount of blowing is such as to cause only small decreases of skin friction.

It is worth noting that b' is almost always multiplied by u' in the integrals where it creates difficulties, so that a reasonable approximation should be found for quite large values of b' ; at least the physics of the situation should be preserved in such circumstances.

After a certain amount of algebra it can be shown that

$$h_r \approx h_{ro} - b' \left[(Le - 1)Sc \Delta' h_\delta + (Pr U^2/6)(2(Le - 1)Sc \Delta' + 1 - Pr) \right] \dots \dots \dots (36)$$

$$- \dot{q}_w UPr/\tau_w \approx (h_{ro} - h_w) \left[1 - (b'/2)((Le - 1)Sc \Delta' + 1 - Pr) \right] - b' \left[(Le - 1)Sc \Delta' h_w + (Pr U^2/6)(1 - Pr - (Le - 1)Sc \Delta') \right] \quad (37)$$

where h_{ro} is the recovery enthalpy with zero mass injection rate,

$$h_{ro} = h_\delta + Pr U^2/2. \quad \text{Handwritten: } h_{ro} = 1 + \frac{1}{2} Pr U^2 \quad (38)$$

When both Pr and Le (and hence Sc also) are equal to unity the results 37 and 38 simplify to

$$h_r = h_{ro} \quad , \quad (39)$$

$$- \dot{q}_w UPr = (h_{ro} - h_w) \tau_w \quad (40)$$

In this case then, injection of a foreign gas through the lower wall has no influence on the recovery enthalpy (since $c_\delta = 0$) and the form of the "Reynolds analogy" expression in eq. 40 is exactly like the no-injection one. However, it is clear from eq. 2 that at a given wall temperature, h_w will become greater or less than the no-injection value of $C_p T_w$ depending on whether $C_{pi} > C_p$ or $< C_p$ (i.e. whether $\Delta' > 0$ or < 0).

Eq. 21 shows that an approximation to c_w which is equivalent to those made in deriving eqs. 36 and 37 is

$$c_w \approx Sc \cdot b' \quad (42)$$

Then it follows that

$$h_w \approx C_p \tau_w (1 + \Delta' \cdot Sc \cdot b'), \quad (43)$$

or, in the case of $Le = Pr = Sc = 1$, the fractional increment in the wall enthalpy as a result of foreign gas injection is $\Delta' b'$.

A further modification to the heat transfer rate arises via the reductions in τ_w brought about by gas injection. Assuming that μ in eq. 17 can be replaced by a properly weighted constant mean value, $\bar{\mu}$ say, it follows that

$$\tau_w = \frac{\dot{m} U}{\exp(\dot{m} \delta / \bar{\mu}) - 1} \approx \bar{\mu} \frac{U}{\delta} - \dot{m} \frac{U}{2}, \quad (44)^*$$

to a reasonable degree of approximation. Aside from any variations which may arise in the value of $\bar{\mu}$ due to the presence of the foreign gas, the term $\bar{\mu} U / \delta$ is the zero injection skin friction. Eq. 44 shows that injection reduces skin friction by an amount equal to the mean rate of upwards transport of x -wise momentum which results from blowing, a plausible-looking first estimate.

* Eq. 44 shows that significant skin friction reductions occur when \dot{m} is comparable with $\bar{\mu} / \delta$. Simple kinetic theory gives $\bar{\mu} \sim \rho \Omega \ell / 3$, $\Omega =$ mean molecular speed ($\approx a$, speed of sound) and $\ell =$ mean free path, whence condition is $\dot{m} / \rho U \sim (\ell / \delta) M^{-1}$ where $M = U / a =$ Mach number at outer edge of shear layer. This illustrates how very small $\dot{m} / \rho U$ values are effective in reducing τ_w , since $\ell / \delta \ll 1$ at reasonable altitudes (pressures). It also suggests that blowing is more effective at higher Mach numbers. The present theory is only valid if $\dot{m} \delta / \bar{\mu} \ll 1$, although this condition need not be too strictly observed, see the remarks preceding eq. 36.

Still considering the simple case where Le and Pr are unity, we can now write

$$-\dot{q}_w \delta / \bar{\mu} \approx (h_{ro} - h_{wo}) - h_{wo} \Delta' b' - (h_{ro} - h_w) \frac{b'}{2} \quad (45)$$

where $h_{wo} = C_p T_w$.

See corrigenda

The first term on the right-hand side corresponds to zero-injection heat transfer, the remaining terms express the effect of injection. Since b' can be written as $\dot{m} \delta / \bar{\mu}$ approximately, the last term in eq. 45 gives a reduction in $-\dot{q}_w$ equal to $(h_{ro} - h_w) \dot{m} / 2$. This term is wholly analogous to the term $\dot{m} U / 2$ in eq. 44, and represents the mean upwards transport of enthalpy as a result of injection. The second term in eq. 45 represents a potentially significant source of heat transfer rate reduction by injection. The reduction in $-\dot{q}_w$ from this source is $(C_{pi} - C_p) \tau_w \dot{m}$, a quantity which expresses the loss of enthalpy difference across the layer which results from the injection of a gas with a large heat capacity, C_{pi} . It is interesting to observe that this quantity may be negative and it follows from eq. 45 that the effect of injection as summarised in the right-hand side of this equation may lead to an increase in heat transfer rate if $2(1 - C_{pi}/C_p) > (T_{ro}/T_w - 1)$, where $C_p T_{ro} = h_{ro}$. (For example, if $C_{pi}/C_p = 1/2$ the heat transfer rate is increased if $T_{ro}/T_w < 2$). That is to say, it seems quite possible that the increase in heat flux driving force arising from injection of the 'wrong' gas can more than counterbalance the effects of convection of energy away from the surface by blowing. *see corrigenda*

For this reason it seems advisable to choose a light gas for heat transfer reduction. Within limits the molar heats of gases do not vary a great deal and a large C_{pi} is perhaps more easily obtained by selecting a gas of small molecular weight, rather than one of complex structure with many modes of communicable energy storage.

So far we have overlooked the possibility that $\bar{\mu}$ may be favourably or unfavourably affected by injection. So long as the amounts of injection are small, so that c remains small throughout the layer it is possible that $\bar{\mu}$ is not much affected, although it is difficult to generalise here. Suffice it to say that light gases have lower viscosities than the heavier ones in the main.

A further point of some importance when attempting an appraisal of boundary layer behaviour via the predictions of the present ultra-simplified analysis is that here the thickness δ remains constant. In a boundary layer δ will increase with increasing \dot{m} and will result in reductions of $-\dot{q}_w$ and τ_w over and above those considered to date.

We note from eqs. 39 and 40 that defining Stanton number and friction coefficient in the usual way, namely

$$\text{St} = \frac{-\dot{q}_w}{\rho_\delta U (h_{r0} - h_w)} \quad ; \quad C_f = \frac{2 \tau_w}{\rho_\delta U^2} \quad , \quad (46)$$

results in the relation

$$\text{St} = \frac{1}{2} C_f \quad (47)$$

One may expect a simple Reynolds analogy expression of this type to hold also in boundary layer flows for which $Le = Pr = 1$. Since $b = \dot{m}U/\tau_w$ it follows from these results that

$$b' = \frac{\dot{m}}{\rho_\delta U \text{St}} \quad (48)$$

and this group of variables may be expected to correlate boundary layer flows too. It should be noted that St is the Stanton number with injection, (see, for example, Lees (1958)).

Eq. 35 shows that eq. 47 can be generalised to include the effects of Le and Pr different from unity. With

$$\text{St} = \frac{-\dot{q}_w}{\rho_\delta U (h_r - h_w)} \quad (49)$$

and C_f as in eq. 46, the approximate form of eq. 35 yields

$$\text{St} \approx \frac{1}{2 Pr} \left[1 + b'/2 (\Delta' Sc(Le - 1) + Pr - 1) \right] \cdot C_f \quad (50)$$

The factor multiplying C_f in eq. 50 is the modified Reynolds analogy factor according to Couette flow theory. Eq. 50 can be rewritten as

$$2 \text{St} Pr \left\{ 1 - \frac{1}{2} \cdot \frac{\dot{m}}{\rho_\delta U \text{St}} \left[\Delta' (1 - Le^{-1}) + 1 - Pr^{-1} \right] \right\} = C_f \quad (51)$$

and it seems that the group $\dot{m}/\rho_\delta U \text{St}$ will still be useful in the more complicated cases where $Le \neq Pr \neq 1$.

Returning to eqs. 49 and 50 we find that

$$\frac{\dot{q}_w}{\dot{q}_{wo}} = \frac{h_r - h_w}{h_{ro} - h_{wo}} \cdot \frac{C_f}{C_{fo}} \cdot \left(1 + \frac{b'}{2} (\Delta' Sc(Le - 1) + Pr - 1)\right)$$

where suffix o indicates zero injection value. A reasonable estimate for C_f/C_{fo} is $1 - b'/2$ and using the results in eqs. 36 and 37 we find after some manipulation that

$$\begin{aligned} \frac{\dot{q}_w}{\dot{q}_{wo}} &= 1 - b' \frac{Pr \Delta' h_{wo} + (Pr U^2/6)(1 - Pr - (Le - 1) \Delta' Sc)}{h_{ro} - h_{wo}} \\ &- \frac{b'}{2} (2 - Pr + \Delta' Sc(Le - 1)) . \end{aligned} \quad (52)$$

This result reduces to the equivalent of eq. 45 when $Le = Pr = 1$. The effects of Le and Pr different from unity are difficult to assess in general, certainly the last term in eq. 52 is favourably affected by the situation $Le > 1$, $Pr < 1$ provided $\Delta' > 0$. (Note $Sc(Le - 1) = Pr - Sc$ and $Pr > Sc$ in this case). This does not follow in the case of the second term however and the situation depends on the relative magnitudes of all the parameters present.

4. Sublimation

If the conditions are right, it may happen that the lower wall reaches a temperature at which sublimation of the wall material takes place, resulting in the "injection" of a foreign gas into the shear layer. In this event the quantity \dot{m} is no longer a free, or independent, variable but must be related in some way to the heat transfer rate and the latent heat of the sublimating material.

To illustrate this effect in as simple a fashion as possible we will assume that Pr and Le for the resulting gas mixture are unity and use eqs. 39 and 40. To achieve a steady state it is necessary to allow the lower wall material to move upwards at a rate just sufficient to keep the sublimating interface at the position $y = 0$. Allowing for this "convection" of solid material upwards towards $y = 0$, the energy balance for the solid is

$$\dot{m} C_b \frac{dT}{dy} = \lambda_b \frac{d^2 T}{dy^2} \quad (53)$$

(Specific heat C_b and conductivity λ_b are assumed constant). Then

$$\lambda_b \frac{dT}{dy} = \dot{m} C_b (T - T_b) \quad (54)$$

where T_b is the solid's temperature at a position so far inside the wall that dT/dy is sensibly zero. It follows at once that

$$T - T_b = (T_w - T_b) \exp(\dot{m} C_b y / \lambda_b), \quad (55)$$

and the heat transfer rate into the solid just below the sublimating interface is

$$\lambda_b \left(\frac{dT}{dy} \right)_{y=0-} = \dot{m} C_b (T_w - T_b) \quad (56)$$

This rate must be matched to the value of $-\dot{q}_w$ at $y = 0+$, taking account of the fact that the material absorbs an amount of latent heat L whilst undergoing the change of phase from solid to gas. In other words

$$-\dot{q}_w = \dot{m} (L + C_b (T_w - T_b)) = \dot{m} L', \quad (57)$$

say.

With the constant mean viscosity assumption $-\dot{q}_w$ can be written (see eq. 45) as

$$-\dot{q}_w \approx (h_{ro} - h_{wo}) \bar{\mu} / \delta = \dot{m} \left\{ h_{wo} \Delta' + (h_{ro} - h_{wo}) / 2 \right\}. \quad (58)$$

It follows at once from eq. 57 that

$$\dot{m} = (h_{ro} - h_{wo})(\bar{\mu}/\delta)(L' + h_{wo} \Delta' + (h_{ro} - h_{wo})/2)^{-1}. \quad (59)$$

This result exemplifies the self-regulating character of the sublimation process. The mass lost through sublimation is seen to decrease as L' and Δ' increase, other things being equal. From a structural viewpoint then, it is best to choose an ablating material whose latent heat of sublimation is high and which degrades into a gas with as high a specific heat as possible. This would suggest that the reductions in $-\dot{q}_w$ arising from ablation may be come small, since \dot{m} itself is small. Putting 59 into 58 gives

$$-\dot{q}_w = (h_{ro} - h_{wo})(\bar{\mu}/\delta)L' (L' + h_{wo} \Delta' + (h_{ro} - h_{wo})/2)^{-1} \quad (60)$$

So long as $\Delta' > 0$ this value is always less than the no-ablation value, but it will not be much below it if L' becomes very large.

However, $-\dot{q}_w$ is not of primary importance here. Rather are we concerned with the heat flux into the wall, which eqs. 56 and 57 show to be given by

$$-\dot{q}_w - m L = -\dot{q}_s \quad (61)$$

say. Thus

$$-\dot{q}_s = (h_{ro} - h_{wo}) \frac{\bar{\mu}}{\delta} \left\{ 1 - \frac{L' + h_{wo} \Delta' + (h_{ro} - h_{wo})/2}{L' + h_{wo} \Delta' + (h_{ro} - h_{wo})/2} \right\} \quad (62)$$

Clearly $-\dot{q}_s$ reduces substantially in magnitude as L rises. (As $L \rightarrow \infty$, $-\dot{q}_s \rightarrow 0$; note that although $\dot{m} \rightarrow 0$ as $L \rightarrow \infty$ the product $\dot{m}L$ remains finite and non-zero).

It seems reasonable to suggest that a high L and a high C_{pi} will result in the most effective and least wasteful type of surface sublimation.

The simple results just presented are not the whole story, however, because it has been tacitly assumed throughout that T_w is known. In order to find $-\dot{q}_s$ for example it is, of course, necessary to know T_w and this must be a function of the pressure in the shear layer.

In other words, the final solution of the ablation problem must depend on a knowledge of the variation of the vapour pressure and latent heat of the wall material with temperature, as we shall see.

The kinetic theory of the vapour pressure* assumes, (i) that the process of condensation requires no activation energy (so that every molecule of the condensing substance which strikes the wall returns to the solid phase) and, (ii) that the molecules in the solid phase behave as a number of multiple oscillators, n_0 per unit area of surface, vibrating with a frequency ν . The number of molecules vapourising (or subliming) from the solid phase is then given by the product of $n_0 \nu$ and the probability that they possess sufficient energy, λ_s . If $2s$ is the number of quadratic momentum or displacement co-ordinates which contribute to this energy, this latter probability is given by Berthoud's relation

$$(\lambda_s/kT)^{s-1} e^{-\lambda_s/kT} / (s-1)!$$

(see Moelwyn - Hughes, loc.cit). Then if we write n for the number of molecules of the subliming material which exist in unit volume of the vapour phase at the interface, Ω for their mean velocity and m_g for their mass, it follows that the rate of sublimation, \dot{m} , is given by

$$\dot{m} = m_g n_0 \nu \frac{(\lambda_s/kT)^{s-1}}{(s-1)!} e^{-\lambda_s/kT} - m_g n \frac{\Omega}{4} \quad (63)$$

The velocity Ω is given by

$$\Omega = (8 kT/\pi m_g)^{1/2}, \quad (64)$$

or, since the vapour pressure p_v is given by $n k T$, we can write

$$n \frac{\Omega}{4} = p_v (2\pi m_g kT)^{-1/2}. \quad (65)$$

At equilibrium $\dot{m} = 0$ and the equilibrium vapour pressure p_{veq} follows directly from eqs. 63 and 65. If n_{eq} is the corresponding number density in the equilibrium state, the expression for \dot{m} can be written simply as

$$\dot{m} = m_g \frac{\Omega}{4} (n_{eq} - n) = \rho \frac{\Omega}{4} (c_{eq} - c) \quad (66)$$

* The treatment presented here is a precis of the account given by Moelwyn-Hughes (1957) of vapourisation from the liquid phase.

where c_{eq} and c are the corresponding mass fractions. A relation like eq. 66 has been given by Scala (1958). It should be noted that ρ is itself a function of c at given p and T , but if $c \ll 1$ we may reasonably estimate ρ as $p m_e / RT$ where m_e is the mass of an external gas molecule.

Specialising the result 66 to apply at the lower wall, we know that $c_w \approx b'$ (when $Sc = 1$, see eq. 42) and for present purposes we can write

$$b' \approx \frac{\dot{m} U}{T_{wo}} \approx \frac{\dot{m}}{\rho \delta} U St_o, \quad (67)$$

(suffix o = no-injection case). Then

$$\dot{m} \approx \frac{(p m_e / k T_w) (\Omega_w / 4) c_{weq}}{1 + (p m_e / k T_w) (\Omega_w / 4 \rho \delta U St_o)}. \quad (68)$$

One can now equate 68 and 59, yielding a relation for T_w in terms of p , c_{weq} (which is a function of p and T_w) and the latent heats etc. (Clearly λ_s must be related to the latent heat L at the temperature T_w).

The resulting equation is obviously not one for which an analytical solution can be obtained, but by stripping it down to the essentials some estimate of the trend of T_w with p can be made.

Assuming that $\lambda_s \approx L m_g$ very roughly, and that $L \gg C_b (T_w - T_b) + h_{wo} \Delta' + (h_{ro} - h_{wo})/2$ (which is consistent with the analysis taking $b \ll 1$) we can write via eqs. 68 and 59 etc.

$$A e^{-L m_g / k T_w} \approx (h_{ro} - C_p T_w) T_w^{s-3/2} (1 + B p T_w^{-1/2}) \quad (69)$$

where A and B are constants which depend on the materials involved. Then any rise in p can be counterbalanced by a relatively small rise in T_w in general (via the exponential term). The number of oscillators in each molecule which take part in the vapourisation processes is usually greater than 3, the value for monatomic molecules (i.e. $s > 3$).

In practice eq. 69 implies that T_w will increase (at a stagnation point, say) with increasing Mach number and decreasing altitude. By how much must depend on the particular material which is involved.

Finally, it must be remembered that a wide group of materials pass through the liquid phase before vapourising, and that a liquid layer will exist between gas and solid. It goes without saying that the presence of the liquid film will influence the final heat transfer rate to the interior of the wall, the rate of mass loss, etc. However a number of the broad conclusions reached above will apply to this type of vapourisation process too.

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