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On the sudden contact between a hot gas and a cold solid

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SUMMARY

The flow induced by the sudden contact between a semi-infinite expanse of gas and a solid, initially at different temperatures, is examined on the basis of a linear continuum theory. For times large compared with the mean time between molecular collisions in the gas, the velocity and pressure disturbances are found to be concentrated around a wave front propagating out from the interface at the ambient isentropic sound speed, whilst, near to the interface, these disturbances are small and the gas temperatures are nearly equal to those predicted by the classical constant pressure heat conduction theory.

The possible significance of these results in connection with reflected shock wave techniques to measure high temperature gas properties is commented upon.

CONTENTS

LIST OF SYMBOLS

a	Isentropic sound speed
c	Isothermal sound speed
C_p	Specific heat at constant pressure
D_n	Weber parabolic cylinder function of order n
erf	Error function
h	Specific enthalpy
p	Pressure
$\mathring{q}$	Energy flux vector
Q	Ratio of thermal properties
R	Gas constant per unit mass
s	Specific entropy or Laplace transform variable
t	Time
T	Gas temperature
u	Velocity
x	Distance
γ	Ratio of specific heats
θ	Temperature difference, $T - T_\infty$
κ	Diffusivity
λ	Conductivity
μ	Viscosity
ν	Kinematic viscosity
ρ	Density
σ	Prandtl number
τ	Shear stress tensor

Suffixes

∞	Initial conditions in the gas
m	Refers to value in the solid

Other symbols are defined in the text

1. Introduction

The conduction of heat in a compressible gas will in general be accompanied by changes in the gas pressure, density and velocity. It is the purpose of the present work to study these changes for the particular case of the sudden contact between a semi-infinite solid and semi-infinite gas, initially at different uniform temperatures.

This simple theoretical model would be difficult to achieve in practice, but something approaching this situation is found when a shock wave reflects from the closed end wall of a conventional shock tube, and it is hoped to gain some idea of what may happen in this more complicated case from the present study. The interest in reflected shock wave zones arises from the ease with which a sample of gas at a high temperature can be produced by this process, and the resulting possibility that measurements of the gas properties under these conditions can then be made.

The question of the compressibility effect on heat conduction has been examined previously by Cole and Wu (1952) for the case of the Dirac heat pulse, but these writers have made the assumption that gas viscosity can be neglected. It will be shown below that viscosity can be included, however, provided that Prandtl number equals 3/4. Like Cole's and Wu's, the present treatment is based on the assumption of small disturbances, so that linear equations can be derived. To avoid over-complication at this stage the gas is assumed to consist of structureless particles, i.e. to be monatomic and unexcited electronically, and to be perfect both thermally and calorically. Although Prandtl number equal to 3/4 is a most practical state of affairs, and the solutions are quite readily obtained in that event, the zero Prandtl number case is also examined here in an attempt to assess how drastic the zero viscosity assumption will be. The gas is treated as a continuum, and, since the characteristic time for the processes to be studied turns out to be comparable with the mean time between molecular collisions, we are effectively limited to a consideration of "large time" solutions only.

Reference will be made in later sections to the "classical solution" of the sudden contact problem. This solution treats the gas as a solid and assumes at the outset that only its temperature will change subsequent to the initial instant, the pressure (or density) remaining constant. Gas velocities are also assumed to be zero throughout. It is one of the results of the present analysis that the classical constant pressure solution is approached asymptotically with increasing time in the regions near to the interface. To assist in the interpretation of these results, a sketch of the classical solid-to-solid contact temperature profiles is given in Fig. 6.

2. The Equations

The gas is assumed to be thermally and calorically perfect so that its pressure p, density ρ temperature T and specific enthalpy h are related as follows,

$$p = \rho RT \quad ; \quad h = C_p T . \tag{1}$$

R is the gas constant for unit mass and C_p the (constant) specific heat at constant pressure.

The continuity equation is

$$\frac{D\rho}{Dt} + \rho \frac{\partial u}{\partial x} = 0 , \tag{2}$$

(where $D/Dt = \partial/\partial t + u\partial/\partial x$ for the one-dimensional unsteady problem), but a more convenient form for present purposes can be derived by writing first of all

$$\frac{D\rho}{Dt} = \left(\frac{\partial \rho}{\partial p}\right)_s \frac{Dp}{Dt} + \left(\frac{\partial \rho}{\partial s}\right)_p \frac{Ds}{Dt} . \tag{3}$$

s is the specific entropy and

$$\left(\frac{\partial \rho}{\partial p}\right)_s = a^{-2} , \tag{4}$$

where a is the usual isentropic sound speed. The derivative $(\partial\rho/\partial s)_p$ is readily evaluated for a gas with the simple thermodynamics described in eq.(1) and we find that

$$\left(\frac{\partial \rho}{\partial s}\right)_p = -(\gamma - 1)\rho T a^{-2} . \tag{5}$$

(γ is the (constant) ratio of specific heats). Writing $\dot{q}$ for the heat flux and τ for the viscous part of the stress tensor, the energy equation is

$$\rho C_p \frac{DT}{Dt} - \frac{Dp}{Dt} = - \frac{\partial \dot{q}}{\partial x} + \tau \frac{\partial u}{\partial x} . \tag{6}$$

Combined with the thermodynamic equation $Tds = dh - \rho^{-1}dp$, eq. (6) shows that

$$\rho T \frac{Ds}{Dt} = - \frac{\partial \dot{q}}{\partial x} + \tau \frac{\partial u}{\partial x} . \tag{7}$$

It follows at once that the continuity equation can be written as

$$\frac{Dp}{Dt} + \rho a^2 \frac{\partial u}{\partial x} + (\gamma - 1)\left(\frac{\partial \dot{q}}{\partial x} - \tau\frac{\partial u}{\partial x}\right) = 0 . \qquad (8)$$

The heat flux and viscous stress are assumed to have their usual values

$$-\dot{q} = \lambda \frac{\partial T}{\partial x} \quad ; \quad \tau = \frac{4}{3}\mu \frac{\partial u}{\partial x} , \qquad (9)$$

so that eqs. 6 and 8, coupled with the momentum equation

$$\rho \frac{Du}{Dt} + \frac{\partial p}{\partial x} - \frac{\partial \tau}{\partial x} = 0 , \qquad (10)$$

constitute three equations for the unknowns p, u and T.

We shall now assume that all of these three unknown quantities differ but little from their undisturbed values, the undisturbed state being defined as one of uniform pressure p_∞ and temperature T_∞, and zero velocity, over the whole of the region of interest. Then the equations can be linearised by neglecting all terms involving squares or products of disturbance quantities, leading to the following three equations.

$$\frac{\partial p}{\partial t} + \rho_\infty a_\infty^2 \frac{\partial u}{\partial x} - (\gamma - 1)\lambda_\infty \frac{\partial^2 T}{\partial x^2} = 0 , \qquad (11)$$

$$\rho_\infty \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - \frac{4}{3}\mu_\infty \frac{\partial^2 u}{\partial x^2} = 0 , \qquad (12)$$

$$\rho_\infty C_p \frac{\partial T}{\partial t} - \frac{\partial p}{\partial t} - \lambda_\infty \frac{\partial^2 T}{\partial x^2} = 0 . \qquad (13)$$

Since we shall be interested in problems for which boundary values are expressed mainly in terms of temperature, a single equation satisfied by T alone will be derived from eqs. 11 to 13. The thermal diffusivity κ and kinematic viscosity ν are defined as

$$\kappa = \frac{\lambda_\infty}{\rho_\infty C_p} \quad ; \quad \nu = \frac{\mu_\infty}{\rho_\infty} , \qquad (14)$$

and the Prandtl number σ as

$$\nu = \sigma \kappa . \qquad (15)$$

The equation satisfied by T is then

$$\frac{1}{\kappa}\frac{\partial}{\partial t}\left\{\frac{1}{a_\infty^2}\frac{\partial^2 T}{\partial t^2}-\frac{\partial^2 T}{\partial x^2}+\frac{4\gamma\sigma\kappa^2}{3a_\infty^2}\frac{\partial^4 T}{\partial x^4}\right\}$$

$$-\frac{\partial^2}{\partial x^2}\left\{\frac{4\sigma/3+\gamma}{a_\infty^2}\frac{\partial^2 T}{\partial t^2}-\frac{\partial^2 T}{\partial x^2}\right\}=0 \,. \quad (16)$$

3. The Problem

At this stage it is convenient to formulate the actual problem to be tackled. The gas, whose temperature is to satisfy eq. 16, is assumed to occupy the half-plane $x > 0$ and to be at rest at uniform pressure p_∞ and temperature T_∞ for all $t < 0$. At time $t = 0$ a semi-infinite solid, which has been at a uniform temperature $T_{-\infty}$ for all $t < 0$, is placed in contact with the gas along the plane $x = 0$. The solid then occupies the half-plane $x < 0$. Without any loss of generality $T_{-\infty}$ can be set equal to zero. Subsequent to time $t = 0$ the temperature T_m of the solid is assumed to satisfy the classical heat conduction equation

$$\frac{\partial T_m}{\partial t} - \kappa_m \frac{\partial^2 T_m}{\partial x^2} = 0 \,, \quad (17)$$

where κ_m is the appropriate diffusivity, (assumed constant here).

A new temperature θ is defined for the gas such that

$$\theta = T - T_\infty \,. \quad (18)$$

The initial conditions then become

$$T_m = 0,\ t < 0,\ x < 0 \ ; \qquad \theta = 0,\ t < 0,\ x > 0 \,. \quad (19)$$

Compatibility of temperature and heat flux at the interface require

$$T_m = \theta + T_\infty \,,\ t > 0,\ x = 0 \ ; \qquad \lambda_m \frac{\partial T_m}{\partial x} = \lambda_\infty \frac{\partial \theta}{\partial x} \,,\ t > 0,\ x = 0, \quad .....(20)$$

where λ_m is the (constant) thermal conductivity of the solid, and two further conditions are

$$T_m \to 0,\ x \to -\infty \,,\ t > 0 \ ; \quad \theta \to 0,\ x \to +\infty \,,\ t > 0. \quad (21)$$

A further requirement is that the gas velocity **u** shall be zero at $x = 0$ for all time, since the solid is impermeable. This condition can be translated into a temperature condition at $x = 0$ via eqs. 11 to 13 by eliminating p and all derivatives of u which contain operations involving $\partial/\partial x$ in terms of T (or what amounts to the same thing, θ), leaving an expression for $\partial^2 u/\partial t^2$ in terms of derivatives of θ. Then, since $u = 0$ when $x = 0$ and $t > 0$ we have also $\partial^2 u/\partial t^2 = 0$, $x = 0$, $t > 0$ and it follows that

$$\frac{\partial^2\theta}{\partial x\partial t} - \frac{4\gamma\sigma\kappa^2}{3a_\infty^2}\frac{\partial^4\theta}{\partial x^3\partial t} - \kappa\frac{\partial^3\theta}{\partial x^3} + \frac{4\sigma\kappa}{3a_\infty^2}\frac{\partial^3\theta}{\partial x\partial t^2} = 0,\ x = 0,\ t > 0 . \qquad \ldots\ldots (22)$$

The conditions 19 to 22 inclusive are sufficient to specify the problem. Before proceeding with a solution, however, the equations will be written in dimensionless form according to the definitions

$$t = \frac{\kappa t'}{a_\infty^2} \quad ; \quad x = \frac{\kappa x'}{a_\infty} . \qquad (23)$$

Then we have to solve

$$\frac{\partial}{\partial t'}\left\{\frac{\partial^2\theta}{\partial t'^2} - \frac{\partial^2\theta}{\partial x'^2} + \frac{4\gamma\sigma}{3}\frac{\partial^4\theta}{\partial x'^4}\right\} - \frac{\partial^2}{\partial x'^2}\left\{(\gamma + 4\sigma/3)\frac{\partial^2\theta}{\partial t'^2} - \frac{\partial^2\theta}{\partial x'^2}\right\} = 0 \qquad (24)$$

and

$$\frac{\partial T_m}{\partial t'} - \left(\frac{\kappa_m}{\kappa}\right)\frac{\partial^2 T_m}{\partial x'^2} = 0 \qquad (25)$$

subject to the conditions

$$T_m = 0,\ t' < 0,\ x' < 0 \quad ; \quad \theta = 0,\ t' < 0,\ x' > 0, \qquad (26)$$

$$T_m = \theta + T_\infty,\ t' > 0,\ x' = 0 \quad ; \quad \left(\frac{\lambda_m}{\lambda}\right)\frac{\partial T_m}{\partial x'} = \frac{\partial\theta}{\partial x'},\ t' > 0,\ x' = 0, \qquad \ldots\ldots (27)$$

$$T_m \to 0,\ x' \to -\infty,\ t' > 0 \ ; \ \theta \to 0,\ x' \to +\infty,\ t' > 0, \qquad (28)$$

$$\frac{\partial^2\theta}{\partial x'\partial t'} - \frac{4\gamma\sigma}{3}\frac{\partial^4\theta}{\partial x'^3\partial t'} - \frac{\partial^3\theta}{\partial x'^3} + \frac{4\sigma}{3}\frac{\partial^3\theta}{\partial x'\partial t'^2} = 0,\ x' = 0,\ t' > 0 . \qquad \ldots\ldots (29)$$

In subsequent sections we shall omit the primes from x and t for brevity, since from now on we shall work exclusively in the dimensionless co-ordinates.

4. Laplace Transform Solutions

The Laplace transform with respect to the dimensionless time t will be denoted by a bar (-) over the appropriate symbol, e.g.

$$\bar{\theta}\,(x\ ;\ s) = \int_0^\infty \theta(x,\ t)\ \exp(-st)\,dt.$$

(Entropy will not be needed in subsequent discussions so that from now on s refers to the transform variable).

With conditions 26 (expressing initial quiescence) the operational forms of eqs. 24 and 25 are

$$\bar{\theta}^{(iv)}\left(1+\frac{4\gamma\sigma}{3}s\right) - \bar{\theta}''\left(s + \left(\gamma+\frac{4\sigma}{3}\right)s^2\right) + s^3\,\bar{\theta} = 0\ , \qquad (30)$$

$$\bar{T}''_m - (s\,\kappa/\kappa_m)\bar{T}_m = 0\ . \qquad (31)$$

(Primes denote differentiation with respect to x).

Eq. 31 can be solved at once, with the appropriate condition from eq.28, to give

$$\bar{T}_m = A(s)\ \exp\left((s\kappa/\kappa_m)^{\frac{1}{2}}x\right), \qquad (32)$$

where A(s) is a function of s to be found from the boundary conditions.

Conditions 27 and the second of 28 in transform form are

$$\bar{\theta}\,(0\ ;\ s) = A(s) - T_\infty\, s^{-1}\ \ ;\ \ \Lambda\, Q\ \sqrt{s} = \bar{\theta}'(0\ ;\ s)\ \ , \qquad (33)$$

where we have written

$$\frac{\lambda_m}{\lambda}\sqrt{\frac{\kappa}{\kappa_m}} = Q\ . \qquad (34)$$

The transform versions of the remaining conditions 28 and 29 are

$$\bar{\theta}\,(x\ ;\ s) \to 0\ ,\ x \to \infty\ , \qquad (35)$$

$$s\left(1+\frac{4\sigma}{3}s\right)\bar{\theta}' - \left(1+\frac{4\gamma\sigma}{3}s\right)\bar{\theta}''' = 0,\ x = 0\ , \qquad (36)$$

and the problem is now reduced to that of finding a solution of eq. 30 subject to conditions 33, 35 and 36.

An appropriate solution of eq. 30 is $\bar{\theta} \propto \exp(\alpha_n x)$ where α_n is any one of the four roots of the auxiliary biquadratic equation

$$(1 + 4\gamma\sigma s/3)\alpha^4 - s(1 + (\gamma + 4\sigma/3)s)\alpha^2 + s^3 = 0 . \qquad (37)$$

The general solution of this equation could be written down, but would give formidable values for the α_n. Instead we shall consider two special cases.

(i) $\sigma = \frac{3}{4}$.

When $\sigma = \frac{3}{4}$, eq. 37 factorises quite simply and gives the four solutions

$$\alpha = \pm \sqrt{s} \; ; \; \pm \; s(1 + \gamma s)^{-\frac{1}{2}} \qquad (38)$$

Condition 35 excludes the solutions with positive signs and it follows that the most general solution of eq. 30 subject to this requirement is

$$\bar{\theta}\,(x \; ; \; s) = B(s)\,\exp(-\,s^{\frac{1}{2}}x) + C(s)\,\exp(-\,s(1 + \gamma s)^{-\frac{1}{2}}x). \quad (39)$$

The value $\frac{3}{4}$ for σ is not far from the accepted value for a number of interesting gases, air for example for which $\sigma = 0.72$ is quoted, so that the solution 39 should give a plausible description of the physical picture.

(ii) $\sigma = 0.$

This not very practical value of the Prandtl number corresponds to the solution for which κ is assumed to have a suitable finite, non-zero value whilst the viscosity μ is set equal to zero. Physically, of course, this is quite inadmissable but it is argued that the effects of heat conduction and viscosity are similar, so that a reasonable physical picture should be obtained by ignoring one of them altogether. This is rather like saying that Prandtl number is of order unity so that we shall approximate to its effect by putting it equal to zero ! - but there does seem to be an intuitive feeling that the physical picture should be retained despite this. Accordingly we shall examine the $\sigma = 0$ case with this in mind. As remarked in the Introduction, Cole and Wu have studied the Dirac heat pulse problem for $\sigma = 0$ and Lagerstrom, Cole and Trilling (1949) have studied a variety of essentially <u>viscous</u> problems under the assumption $\lambda = 0$ while retaining μ finite and non-zero. (As can be seen from the equations 11, 12 and 13, $\lambda = 0$ uncouples the 'p, u' problem from the energy equation, so that the present theory is not directly comparable with Lagerstrom's. One might say that we are interested in problems primarily of heat conduction).

When $\sigma = 0$ then, eq. 37 has the solutions

$$\alpha = \pm\sqrt{s}\left\{ (1 + \gamma s)/2 \pm \left|(1 + \gamma s)^2/4 - s\right|^{\frac{1}{2}} \right\}^{\frac{1}{2}}. \quad (40)$$

The two solutions starting $+\sqrt{s}$ etc. must be abandoned to conform with eq. 35 and so, for our purposes, we have

$$\bar{\theta}(x\,;\,s) = B'(s)\exp\left[-\sqrt{s}\left\{(1 + \gamma s)/2 + \left|(1 + \gamma s)^2/4 - s\right|^{\frac{1}{2}}\right\}^{\frac{1}{2}} x\right] + C'(s)\exp\left[-\sqrt{s}\left\{(1 + \gamma s)/2 - \left|(1 + \gamma s)^2/4 - s\right|^{\frac{1}{2}}\right\}^{\frac{1}{2}} x\right]. \quad \ldots\ldots (41)$$

(The constants B' and C' are different from B and C in eq. 39). Cole and Wu remark that setting $\mu = 0$ simplifies the equations to be studied. Examination of eq. 16 would certainly tend to suggest that this is true, but comparison of the solutions 39 and 41 indicate that the reverse is the case, certainly when μ is retained and σ put equal to $3/4$.
Fortunately Cole and Wu were able to find a transformation which renders an attack on the $\sigma = 0$ case possible, but, as will become evident below, it must be applied with some care and greater labour is involved in the $\sigma = 0$ problem than when $\sigma = 3/4$.

Neither case produces a particularly simple solution owing to the appearance of the complicated exponential functions, so it may be advisable to examine briefly the physics of the situation in order to decide just what kind of solutions it would be best to aim for. From equations 23 it can be seen that the characteristic time and length for the system are κ/a_∞^2 and κ/a_∞ respectively. Simple kinetic theory indicates that $\lambda \simeq (1/3)\ \rho\bar{c}\ell C_v$, where $\bar{c}$ is the mean molecular speed, ℓ the mean free path and C_v the constant volume specific heat. Consequently $\kappa \simeq \bar{c}\,\ell/3\gamma$ and, since $a^2 = \gamma p/\rho \simeq \gamma\,\bar{c}^2/3$, it follows that $\kappa/a_\infty^2 \simeq (\ell/\bar{c})$ and $\kappa/a_\infty \simeq \ell$, apart from multiplying factors of order unity. The characteristic time and length are therefore comparable with the mean time between collisions of the molecules and the mean free path respectively.

Thus for $t \sim 1$ or less a continuum theory such as that formulated here can hardly be valid and we should direct attention primarily towards the case $t >> 1$, where it is plausible to use such a theory. For the sake of completeness some results for $t = 0+$ will be given, however.

5. Solutions for $\sigma = 3/4$.

The functions $B(s)$ and $C(s)$ of eq. 39 can be related via the zero-velocity-at-the-wall condition, eq. 36. We find that

$$C(s) = (\gamma - 1)\sqrt{s}\sqrt{1+\gamma s}\,B(s),$$

whence the solution 39 can be written

$$\bar{\theta}(x\,;\,s) = [\exp(-x\sqrt{s}) + (\gamma - 1)\sqrt{s}\sqrt{1+\gamma s}\exp(-sx(1+\gamma s)^{-\frac{1}{2}})]B(s). \quad \ldots\ldots \quad (42)$$

It should be noted that eq. 42 is a valid solution for the gas temperature in the half-plane $x > 0$ when $u = 0$ at $x = 0$ for <u>any</u> variation of θ at this interface, only the function $B(s)$ changing in accordance with the specified behaviour of $\theta(0, t)$. For the present problem one readily infers from conditions 33, that

$$B(s) = -(T_\infty Q/s)\left[1 + Q + (\gamma - 1)s + Q(\gamma - 1)\sqrt{s}\sqrt{1+\gamma s}\right]^{-1}. \qquad (43)$$

Using the inversion theorem for Laplace transforms it follows that

$$-\frac{\theta(x,t)}{Q\,T_\infty} = \frac{1}{2\pi i}\int_L \frac{\exp(ts - x\sqrt{s})}{1 + Q + (\gamma - 1)s + Q(\gamma - 1)\sqrt{s}\sqrt{1+\gamma s}}\frac{ds}{s}$$

$$+ \frac{1}{2\pi i}\int_L \frac{(\gamma - 1)\sqrt{1+\gamma s}\exp(ts - s(1+\gamma s)^{-\frac{1}{2}}x)}{1 + Q + (\gamma - 1)s + Q(\gamma - 1)\sqrt{s}\sqrt{1+\gamma s}}\frac{ds}{\sqrt{s}}, \quad \ldots\ldots \quad (44)$$

L being the usual inversion contour.

The first and second integrals in eq. 44 will be denoted by I_1 and I_2 respectively and each one treated separately below.

5.1. Evaluation of I_1

The singularities of the integrand in I_1 are branch points at $s = 0$ and $-1/\gamma$. It can be verified that L is equivalent to a dumbell contour proceeding parallel to the Re s axis just below and just above the branch cut between $s = 0$ and $-1/\gamma$ and encircling the branch points at either end. On the straight line parts of this contour we put $s = y\exp(\pm i\pi)$, y real and positive, taking upper and lower signs on the upper and lower paths. Then $\sqrt{s} = \sqrt{y}\exp(\pm i\pi/2)$ accordingly and $\sqrt{1+\gamma s} = \sqrt{1-\gamma y}$ on either path. On the circles surrounding the branch points we put

$s = \epsilon \exp(i\theta)$ and $s = -1/\gamma + \epsilon \exp(i\theta)$ respectively and then allow ϵ to approach zero. The contribution from the circle around $s = -1/\gamma$ vanishes in the limit and that from the circle round $s = 0$ is $(1 + Q)^{-1}$, as may be easily verified. Rearranging the 'straight line' integrations it follows that

$$I_1 = \frac{1}{1+Q} - \frac{1}{\pi}\int_0^{1/\gamma} \frac{(1+Q-(\gamma-1)y)\sin(x\sqrt{y}) + Q(\gamma-1)\sqrt{y}\sqrt{1-\gamma y}\cos(x\sqrt{y})}{(1+Q-(\gamma-1)y)^2 + Q^2(\gamma-1)^2 y(1-\gamma y)} \frac{e^{-yt}}{y}\,dy. \qquad \ldots\ldots \quad (45)$$

When $t \gg 1$ the principal contribution to the integral here comes from the region near $y = 0$. Accordingly we can expand the integrand in ascending powers of y, the first term in each of the two integrals in eq. 45 being as follows

$$I_1 \sim \frac{1}{1+Q} - \frac{1}{1+Q}\cdot\frac{1}{\pi}\int_0^{1/\gamma} \sin(x\sqrt{y})e^{-yt}\frac{dy}{y} - \frac{Q(\gamma-1)}{(1+Q)^2}\frac{1}{\pi}\int_0^{1/\gamma}\cos(x\sqrt{y})e^{-yt}\frac{dy}{\sqrt{y}} \qquad \ldots\ldots \quad (46)$$

Replacing the upper limit $1/\gamma$ by ∞, I_1 can be written in terms of well known functions to give

$$(1+Q)I_1 \sim 1 - \operatorname{erf}(x/2\sqrt{t}) - \frac{Q(\gamma-1)}{1+Q}\cdot\frac{\exp(-x^2/4t)}{\sqrt{\pi t}}, \qquad (47)^*$$

We must now investigate the errors in the result 47. First of all, in replacing $1/\gamma$ by ∞ in the limits of the integrals in eq. 46 we imply errors of the order of

$$\frac{1}{\pi}\int_{1/\gamma}^{\infty} \sin(x\sqrt{y})e^{-yt}\frac{dy}{y} + \frac{Q(\gamma-1)}{1+Q}\frac{1}{\pi}\int_{1/\gamma}^{\infty}\cos(x\sqrt{y})e^{-yt}\frac{dy}{\sqrt{y}}$$

on the right hand side of eq. 47. By a mean value theorem we can write

$$\frac{1}{\pi}\int_{1/\gamma}^{\infty}\sin(x\sqrt{y})e^{-yt}\frac{dy}{y} = \frac{\sin(xa)}{\pi}\int_{1/\gamma}^{\infty} e^{-yt}\frac{dy}{y} = \frac{\sin(xa)}{\pi}E_1(t/\gamma)$$

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* $\operatorname{erf}(\alpha)$ is the error function, $= (2/\sqrt{\pi})\int_0^{\alpha} e^{-y^2}dy$.

where a is a suitable mean value of $\sqrt{y}$ and $E_1(t/\gamma)$ is the exponential integral defined in Bateman (1953 p.143 eq. 9.7(1)). When $t/\gamma >> 1$, $E_1(t/\gamma) \sim (\gamma/t) \exp(-t/\gamma)$, (vide Bateman, 1953 p.144 eq. 9.7.(7)). Likewise we can write

$$\frac{1}{\pi} \int_{1/\gamma}^{\infty} \cos(x\sqrt{y}) e^{-yt} \frac{dy}{\sqrt{y}} = \frac{\cos(xb)}{\pi} \int_{1/\gamma}^{\infty} e^{-yt} \frac{dy}{\sqrt{y}} = \frac{\cos(xb)}{\sqrt{\pi t}} (1 - \mathrm{erf}(\sqrt{t/\gamma}))$$

where b is a suitable mean value of $\sqrt{y}$. When $t/\gamma >> 1$, $\mathrm{erf}(\sqrt{t/\gamma}) \sim 1 - \sqrt{\gamma / \pi t} \exp(-t/\gamma)$. It follows therefore that the error implied in neglecting the difference between $1/\gamma$ and ∞ is of order

$$\sin(xa) \frac{\gamma}{\pi t} e^{-t/\gamma} + \frac{Q(\gamma-1)}{1+Q} \frac{\sqrt{\gamma}}{\pi t} e^{-t/\gamma} \cos(xb)$$

We notice that even when x approaches t quite closely in eq. 47 the errors here are at most of order $1/\sqrt{t}$ times this result, and are correspondingly less significant as $x \to 0$.

The terms neglected in expanding the part of the integrand in braces in eq. 45 give rise to errors $1 + O(y)$ in the integrals in eq. 46. It can be seen that the resulting error terms are obtained as second and third derivatives of the first integral in eq. 46 (ignoring constant multiplying factors) with respect to x, i.e. successive derivatives of $\mathrm{erf}(x/2\sqrt{t})$ with respect to x as a not unreasonable estimate. These derivatives of the error function contain a factor $(1/\sqrt{\pi t})\exp(-x^2/4t)$ times either x/t, or $1/t$ and x^2/t^2. Provided x is not of order t the errors are small compared with the terms written in eq. 47.

Provided x is not of order t, then, eq. 47 is a reasonable representation of the integral I_1. When x is of order t or greater it is clear that I_1 contributes a small amount only to the overall value of θ, by reason of the exponential terms. We shall see now that most of the contribution to θ in the region $x \sim t$ comes from the second integral I_2 when t is large. We remark that the complete solution of the 'classical' heat conduction problem (for which pressure is assumed constant throughout) is given exactly by the first two terms on the right hand side of eq. 47, the solution in that case being valid for all x and t. The integral I_1 gives solutions of a purely diffusive nature, as seems reasonable from the presence of the $\exp(- x \sqrt{s})$ factor there.

5.2. Evaluation of I_2

To complete the solution for $\theta(x,t)$ it is now necessary to examine the second integral of eq. 44, namely I_2. We have just seen how I_1 leads to diffusion-type solutions and examination of I_2 may lead us to suspect that this integral will produce a combination of diffusion type and wave-like solutions, from the presence of the exponential factor $\exp(ts - s(1 + \gamma s)^{-\frac{1}{2}}x)$, which has a character somewhere between these two.

In fact just this exponential term arises in the study of purely viscous phenomena, mentioned previously as having been examined by Lagerstrom, Cole and Trilling. It is the complete transform solution of the Dirac velocity pulse problem in a fluid for which $\mu \neq 0$, $\lambda = 0$ and, when multiplied by s^{-1}, gives the solution for a unit step function of velocity applied at $t = 0$, $x = 0$. The above named authors have found solutions valid for large and small times by a subtle choice of contour, followed by some lengthy sifting of various contributions to the whole integral from different parts of the contour in order to extract the most significant terms. Later Hanin (1957) treated the Dirac pulse problem at great length, finding solutions in series, as real integral representations and as asymptotic series, covering various ranges of x and t. Morrison (1957) also discovered the real integral representations for the impulse solution, during the course of his investigations of wave propagation in visco-elastic materials, by using certain theorems on Laplace transforms.

The present problem is more difficult than any of these, however, by reason of the complicated algebraic factor which multiplies the exponential term in I_2.

It is clear that the two integrals I_1 and I_2 express the "combined" nature of our problem quite well. The sudden changes of temperature occurring first at the interface between gas and solid are bound to produce changes of pressure, density and velocity in the gas, and one would expect such changes to propagate out into the gas as some kind of wave motion. Such wave motion is necessarily going to be of a somewhat complicated nature, since it is an essential part of the whole problem that the dispersive and absorptive mechanisms of conduction and momentum diffusion shall be present. These will act to change the form of the wave motion, and these changes will themselves react back on the diffusive and convective processes which are responsible for the changing energy balance between solid and gas. Thus we may say that the algebraic factor in I_2 represents the type of "input" to the wave motion in the gas as a result of diffusion whilst that in I_1 represents the "input" to the diffusion processes as a result of the primary heat conduction processes plus the feedback from the wave motion. It is characteristic of the $\sigma = 3/4$ case that these two types of process separate in the way found in eq. 39. It is clear from the form of the auxiliary equation 37 that such separation will not occur so obviously for other Prandtl numbers, and indeed the solution 41 for $\sigma = 0$ provides a specific example.

To return to the problem in hand, namely the evaluation of I_2, we shall concentrate as before on solutions valid for large times, following very closely the methods used by Hanin. First of all we examine the region around $x = t$ by defining

$$\delta = x - t \tag{48}$$

and making the substitution $(1 + \gamma s)^{\frac{1}{2}} = \omega$. Then

$$I_2 = \frac{(\gamma - 1)\sqrt{\gamma}}{\pi i} \int_{C_\omega} \frac{\omega^2 \exp\left[(t/\gamma)(\omega^2-1)(1-1/\omega)\right] \exp\left[(-\delta/\gamma)(\omega - 1/\omega)\right]}{\gamma(1+Q) + (\gamma-1)(\omega^2-1) + Q(\gamma-1)\sqrt{\gamma(\omega^2-1)}\,\omega} \frac{d\omega}{\sqrt{\omega^2-1}} \quad \ldots\ldots \tag{49}$$

The contour $C\omega$ is asymptotic to $\infty \exp(\pm i\pi/4)$ at its ends and crosses the real axis somewhere to the right of $\omega = 1$. We shall imagine that t in eq. 49 is large. δ will be assumed small and we shall see later just what this must imply about the actual allowable magnitude of δ. Using the method of steepest descents, it is now necessary to find, first a suitable saddle point for the function

$$f(\omega) = (\omega^2 - 1)(1 - 1/\omega), \tag{50}$$

and second to ensure that the steepest path through this point can be reconciled with the contour C_ω. The condition $df/d\omega = 0$, which defines the saddle points of $f(\omega)$ is satisfied by setting $\omega = 1$ (i.e. $df/d\omega = 2\omega - 1 - 1/\omega^2 = 0$) and the steepest path of descent from the col at $\omega = 1$ proceeds from $1/2 - i\infty$, through $\omega = 1$, to $1/2 + i\infty$. $\omega = 1$ happens to be a singular point of the integrand in eq. 49, however, so that clearly the steepest path for the exponential function cannot be reconciled directly with C_ω. By indenting the steepest path so as to pass to the right of $\omega = 1$ around an arc of a small circle given by $\omega = 1 + \epsilon \exp(i\theta)$ we may still make use of it though, and it quickly follows that the contribution to I_2 made by integration around this arc approaches zero as $\epsilon \to 0$. Consequently we can now proceed in the usual way by writing

$$f(\omega) = -\phi^2/2, \tag{51}$$

thereby defining ϕ, the real variable of integration on the steepest path. It follows from 50 and 51 that ω expressed as a series in ϕ begins

$$\omega = 1 \pm i\phi/2 \quad \ldots\ldots \tag{52}$$

and that to a first order

$$\frac{d\omega}{d\phi} = \pm i/2 \quad . \tag{53}$$

The upper and lower signs in 52 and 53 are to be taken on the upper ($\omega = 1$ to $\frac{1}{2} + i\infty$) and lower ($\frac{1}{2} - i\infty$ to 1) halves of the steepest path respectively.

Taking the first two terms in the expansion of the integrand in eq. 49 in terms of ϕ we have, after some manipulation

$$\frac{\sqrt{\gamma}\,(1 \pm Q)}{(\gamma - 1)} \cdot I_2 \sim \frac{1}{\pi} \int_0^\infty \exp\left(-t\,\phi^2/2\,\gamma\right) \cos\left(\delta\phi/\gamma + \pi/4\right)\phi^{-\frac{1}{2}}\, d\phi$$

$$- \frac{Q(\gamma - 1)}{\sqrt{\gamma}\,(1 + Q)} \cdot \frac{1}{\pi} \int_0^\infty \exp\left(-t\,\phi^2/2\,\gamma\right) \cos\left(\delta\phi/\gamma\right)\, d\phi \,. \qquad (54)$$

The derivation of result 54 follows the standard procedure of the steepest descents method, namely series expansion of the function of ω, which multiplies the exponential $\exp(\; t\, f(\omega)/\gamma\;)$, in ascending powers of ϕ, except that we have written the term $\exp(-(\delta/\gamma)(\omega - 1/\omega))$ as approximately equal to $\exp(\mp i\phi\delta/\gamma)$, taking signs appropriate to the particular half of the steepest path being considered. This approximation gives rise to the cosine functions in 54. The first integral can be reduced to a recognizable form on substituting $\phi = \sqrt{\gamma/t}\; y$, namely,

$$\left(\frac{\gamma}{t}\right)^{\frac{1}{4}} \frac{1}{\pi} \int_0^\infty e^{-y^2/2} \cos\left[(\delta/\sqrt{\gamma t})y + \pi/4\right] y^{-\frac{1}{2}}\, dy \,,$$

Bateman (1953, p.120 eq.8.3(4)) showing that this is related to Weber's parabolic cylinder function of order $-\frac{1}{2}$ (written as $D_{-\frac{1}{2}}$). The second integral in 54 is a well known one and we can write

$$\frac{\sqrt{\gamma}\,(1+Q)}{(\gamma - 1)}\; I_2 \sim \frac{(\gamma/t)^{\frac{1}{4}}}{\sqrt{2\pi}}\; e^{-\delta^2/4\gamma t}\; D_{-\frac{1}{2}}\left(\delta/\sqrt{\gamma t}\right)$$

$$- \frac{Q(\gamma - 1)}{(1 + Q)}\; \frac{e^{-\delta^2/2\gamma t}}{\sqrt{2\pi t}} \,. \qquad (55)$$

A careful investigation of the errors in 54 or 55 as approximations to I_2 for $t >> 1$ indicates that we must restrict $\delta/\sqrt{\gamma t}$ to be $< O(1)$ to prevent them becoming comparable with the terms retained there. When this is done the next term in 55 is $O(t^{-\frac{3}{4}})$. Owing to the complicated nature of the integral it is impossible to give any general term for an asymptotic expansion, even in the present relatively manageable region of x and t. We observe, incidentally, that when $\delta = 0$, $D_{-\frac{1}{2}}$ is $O(1)$, (the exact value will be given later on in Section 5.3.).

The result 55 fails as a general approximation, valid when x is small for example, because of the behaviour of the exp $[-(\delta/\gamma)(\omega - 1/\omega)]$ term in this region, δ becoming large like the time t.
However, if one assumes that δ is large and negative, as when x is small, it can be shown (using the results of Miller (1955)) that the first term in 55 behaves like $\delta^{-\frac{1}{2}}$ and that the first error term from the exponential just mentioned behaves like $\delta^{\frac{1}{2}}/t$. This suggests that at small values of x (i.e. near the interface), there is a part of the disturbance which is of a wave-like character and that it may be of a comparable order of magnitude to the third term of eq. 47. Physically this state of affairs appears highly plausible and accordingly we will attempt to evaluate the contribution which the integral I_2 makes to θ in the regions of x near to the interface.

To do this it is observed that the exponential term in the integral 49 can be re-written as

$$\exp\left[(\omega^2 - 1)(1 - \alpha/\omega)(t/\gamma)\right] \tag{56}$$

where

$$\alpha = x/t \,. \tag{57}$$

We now seek solutions for I_2 which are valid when $t \gg 1$ and α is small. Using steepest descents, the col for the function

$$g(\omega) = (\omega^2 - 1)(1 - \alpha/\omega) \tag{58}$$

is found as a solution of

$$g'(\omega) = 2\omega - \alpha - \alpha/\omega^2 = 0 \,, \tag{59}$$

α is real and positive and it follows that there must be one real and two complex roots of eq. 59. Of these we choose the real root, noting that when α is small this root is approximately

$$\omega_0 \simeq \left(\frac{\alpha}{2}\right)^{\frac{1}{3}} \left[1 + \frac{1}{3}\left(\frac{\alpha}{2}\right)^{\frac{2}{3}} + \ldots\right] . \tag{60}$$

The steepest path of descent passes through $\omega = \omega_0$ and is asymptotic to $(\omega_0 - \alpha/2\omega_0^2) \pm i\infty$ at its ends. However, the algebraic factor in eq. 49 contains a branch point at $\omega = 1$ and the original contour C_ω cannot be reduced to this steepest path directly. Instead we shall use the contour illustrated in Fig. 1 which consists of the upper and lower halves of the steepest path connected by a loop around the branch cut from $\omega = 1$.

The integration around the small circle of radius ϵ contributes an amount to I_2 which is proportional to $\sqrt{\epsilon}$, and consequently yields zero in the limit as $\epsilon \to 0$. Writing Re $\omega = v$, the parts AB, CD of the contour contribute an amount

$$\frac{2(\gamma-1)\sqrt{\gamma}}{\pi}\int_{\omega_0}^{1}\frac{\left[\gamma(1+Q)-(\gamma-1)(1-v^2)\right]\exp\left[-(1-v^2)(1-\alpha/v)(t/\gamma)\right]}{\left[(1+Q)\gamma-(\gamma-1)(1-v^2)\right]^2+Q^2(\gamma-1)^2\gamma v^2(1-v^2)}\cdot\frac{v^2dv}{\sqrt{1-v^2}}\,.$$

$$\ldots\ldots \quad (61)$$

The index of the exponential term here is zero at the upper limit $v = 1$ and is negative everywhere else within the range of integration. Whence it follows that when $t \gg 1$ the principal contribution to the integral will come from the region near $v = 1$. Changing the variable from v to y via the relation $(1-v^2)(1-\alpha/v) = y$, the integrand can now be expanded as a power series in y, the most important term being

$$\frac{(\gamma-1)}{\sqrt{\gamma}(1+Q)}\;\frac{1}{\sqrt{1-\alpha}}\;\frac{1}{\pi}\int_0^{(1-\omega_0^2)(1-\alpha/\omega_0)} e^{-yt/\gamma}\,.\,y^{-\frac{1}{2}}\,dy$$

$$= \frac{\gamma-1}{1+Q}\cdot\frac{1}{\sqrt{\pi(t-x)}}\cdot\operatorname{erf}\left[(1-\omega_0^2)(1-\alpha/\omega_0)(t/\gamma)\right]. \quad (62)$$

The errors in writing 62 for the integral 61 are very small compared with the result 62 provided α is less than about 1/2 and t is large. Owing to the complicated nature of the integral it is not practicable to give a general result for the error terms, but they are roughly of order $t^{-\frac{1}{2}}\exp\left[-(1-\omega_0)(1-\alpha/\omega_0)(t/\gamma)\right]$. Since the argument of the error function in 62 is very large if α is small we may reasonably approximate to the expression there by writing it as

$$\simeq \frac{\gamma-1}{1+Q}\cdot\frac{1}{\sqrt{\pi(t-x)}} \quad (63)$$

To carry out the integration along the steepest path part of the contour in Fig. 1 we define the real variable of integration y as follows,

$$g(\omega) - g(\omega_0) = -y^2\,.$$

The usual procedure for evaluating integrals by the method of steepest descents then leads to the major term arising from this part of the contour and this is found to be

$$\frac{2\sqrt{\gamma}(\gamma - 1)\ A\,\omega_0\,(\gamma/\pi t)\ \exp\,(g(\omega_0)t/\gamma)}{\gamma(1 + Q) - (\gamma - 1)(1 - \omega_0^2) + Q(\gamma - 1)\sqrt{\gamma}\,\omega_0\sqrt{1 - \omega_0^2}},$$

where A is written for $1/(1 + \alpha/\omega_0^3)$. Since $g(\omega_0) \simeq -1$ when α, and hence ω_0, is small, this term is very much less than the result 63 and we conclude that, provided $t >> 1$ and α is reasonably small compared with unity, a plausible estimate of the integral I_2 is

$$I_2 \sim \frac{\gamma - 1}{1 + Q} \cdot \frac{1}{\sqrt{\pi (t - x)}} \tag{64}$$

This result confirms the view, expressed earlier, that wave-like disturbances exist in the regions of x near to the interface.

As in Hanin's paper, the form of the exponential function in eq. 56 can be used to find approximations for the case $t >> 1$ and α large also. In that event the solution of eq. 59 indicates a saddle point at approximately $\omega_0 = (\alpha/2)$, and this will certainly lie to the right of the branch point at $\omega = 1$. Consequently C_ω and the steepest path are directly reconcilable and it can then be shown that the major contribution to θ from I_2 is roughly

$$\frac{2\gamma\sqrt{t/\pi x^2}}{1+Q\,\sqrt{\gamma}} \cdot e^{-x^2/2\gamma t}, \tag{65}$$

which is a very small quantity. (In deriving 65 we have neglected unity in comparison with $\omega_0^2 \simeq \alpha^2/4$).

5.3. The Temperature at Large Times

Collecting the results from the last three sections enables us to build up a reasonable picture of the behaviour of the gas temperature at times large compared with the mean time between molecular collisions.

Thus, in the region where x is small compared with t

$$\theta(x,t) \sim -\frac{Q\,T_\infty}{1+Q}\left\{ 1 - \operatorname{erf}(x/2\sqrt{t}) - \frac{Q(\gamma-1)}{1+Q} \cdot \frac{\exp(-x^2/4t)}{\sqrt{\pi t}} + \frac{\gamma - 1}{\sqrt{\pi (t - x)}} \right\}, \tag{66}$$

(using the results 44, 47 and 63). When x is comparable with t,

$$\theta(x,t) \sim -\frac{QT_\infty}{1+Q} \cdot (\gamma-1)\left\{ \frac{\exp(-\delta^2/4\gamma t)D_{-\frac{1}{2}}(\delta/\sqrt{\gamma t})}{\sqrt{2\pi}\quad(\gamma t)^{\frac{1}{4}}} - \frac{Q(\gamma-1)}{(1+Q)}\ \frac{\exp(-\delta^2/2\gamma t)}{\sqrt{2\pi\gamma t}} \right\} . \qquad (67)$$

For the latter case we have used the results 44 and 55 and it is recalled that $\delta = x - t$. When x is greater than t by an appreciable amount the value of θ has been shown to be practically zero.

The temperature at the interface $x = 0$ follows from eq. 66, namely,

$$\theta(0,t) \sim -\frac{Q\,T_\infty}{1+Q}\left\{ 1 + \frac{(\gamma-1)}{1+Q}\ (\pi t)^{-\frac{1}{2}} \right\} , \qquad (68)$$

and, remembering that the gas temperature $T = \theta + T_\infty$, it can be seen that the interface temperature is increasing with time, if T_∞ is positive (i.e. gas hotter than solid). The classical constant pressure solution for $\theta(0,t)$ indicates that it jumps abruptly to the value $- Q\,T_\infty/(1 + Q)$ and remains constant for all later times. Thus in the practical case the classical solution is approached asymptotically. This statement is also true of the whole solution for θ in the region near the wall, since as t increases the last two terms in eq. 66 become small compared with the first two (which represent the classical solution). As distance from the wall increases at given time, however, the solution 66 indicates that deviations from the classical solution increase and it seems plausible to suggest that such deviations tend to become of a predominantly wave-like character. The behaviour of the last two terms in 66 as x increases is such as to cause the gas temperature to fall below the classical value. Turning now to the regions where x and t are of comparable magnitude we find (from eq. 67) some notable deviations from the classical value of $\theta(x,t)$. This latter solution would indicate that θ has fallen to an almost negligible size when $x = t$, for example, because $1 - \mathrm{erf}\,\sqrt{t/2} \sim e^{-t/2}/\sqrt{\pi t/2}$ when t is large. Eq. 67, however, shows that

$$\theta(t,t) \sim -\frac{Q\,T_\infty}{1+Q}\ (\gamma-1)\left\{ \frac{\Gamma(\frac{1}{4})}{2\pi\,(2\gamma t)^{\frac{1}{4}}} - \frac{Q(\gamma-1)}{\gamma(1+Q)}\ \frac{1}{\sqrt{2\pi t}} \right\} \qquad (69)$$

(NB. $D_{-\frac{1}{2}}(0) = \Gamma(\frac{1}{4})/2^{\frac{3}{4}}\pi^{\frac{1}{2}}$), which, although small because t must be large, is certainly of a greater order of magnitude than the classical solution.

A sketch of the complete temperature distribution is given in Fig. 2 the full line curves being calculated from equations 66 and 67, whilst the dotted lines represent a plausible estimate of the behaviour of the temperature in the regions where these asymptotic solutions fail. The classical solution is shown for comparison, and it can be seen how the deviations from this solution become more marked as x increases. Fig. 3 is a sketch of the wave front for two values of t (50 and 100), and indicates how its amplitude diminishes and how it becomes more diffuse as time increases. These two effects arise from the dissipative actions of viscosity and heat conduction. In an actual case the non-linear terms in the equations describing the motion (which have been neglected in our linearised treatment) would act to flatten the wave front even further.* Both Fig. 2 and Fig. 3 have been drawn for a value of Q = 100, which is roughly the magnitude of this quantity for an air to pyrex-glass contact. This is the set-up generally encountered in the use of thin-film platinum resistance thermometers in shock-tube work. The variation of interface temperature from the classical value is far too small to appear on Fig. 2 with this particular value of Q (vide eq. 68), so that Fig. 4 shows a sketch of $T\ (= \theta + T_\infty)$ at the interface plotted against time. A rather more accurate estimate of this value is made in Section 5.5. below and Fig. 4 is a plot of eq. 85 appearing there. It can be seen that for $t > 100$ the differences between actual and classical values of T are insignificant for all practical purposes. For conditions around N.T.P. the mean collision time is of order 10^{-10} secs., so that no difference from the classical solution would be observed for times greater than about 1/100th of a microsecond, which implies that the practical effects of compressibility in heat transfer at the interface cannot be resolved experimentally.

5.4. The Pressure and Velocity Perturbations.

Further comment on the significance of the results obtained above will be given in the final section: we proceed now to consider the pressure and velocity perturbations which must arise in the gas. The linearised energy equation (13) in dimensionless form is

$$\frac{\partial p}{\partial t} = \rho_\infty\ C_p \left\{ \frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} \right\} \tag{70}$$

which gives

$$s\,\bar{p} = \rho_\infty\ C_p \left\{ s\,\bar{\theta} - \bar{\theta}'' \right\} \tag{71}$$

in the transform plane, provided $\bar{p}$ represents the transform of $p - p_\infty$. Eq. 42 then shows that

$$\bar{p} = \rho_\infty\ C_p\ B(s)\ (\gamma - 1) \sqrt{s}\ \frac{1 + (\gamma - 1)s}{\sqrt{1 + \gamma s}}\ \exp\left\{ - sx/\sqrt{1 + \gamma s} \right\}, \qquad \ldots\ldots (72)$$

B(s) having been given in eq. 43. It is observed that $\bar{p}$ can be written as

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* In the event that the gas is initially colder than the solid ($T_\infty < 0$) the wave front is one through which temperature increases. In that event the non-linear convective effects will counteract the dispersive effects of viscosity and heat conduction and the wave front will tend to remain steep, i.e. it will be a shock wave whose strength will depend on T_∞ and Q.

$$\bar{p} = \rho_\infty C_p \left\{ \frac{1 + (\gamma - 1)s}{1 + \gamma s} \right\} \bar{\theta}_2$$

where we write $\bar{\theta}_2$ for the second term in eq. 42.
Consequently

$$p - p_\infty = - \rho_\infty C_p Q T_\infty . I_2'$$

where I_2' is an integral exactly like I_2 in eq. 44, except that its integrand is multiplied by $(1 + (\gamma - 1)s)(1 + \gamma s)^{-1}$, i.e. an integral like eq. 49 whose integrand is multiplied by $(\gamma - 1)/\gamma + 1/\gamma\omega^2$.
It follows that, using the steepest descents approach when x is of order t (i.e. for δ "small"), the first two terms of I_2' will be identical with the first two terms of I_2 and we can write directly

$$p - p_\infty \simeq \rho_\infty C_p \theta(x,t) \qquad (73)$$

where $\theta(x,t)$ is given by eq. 67.

By very similar arguments we can infer that a first order estimate of $p - p_\infty$ for x small is

$$p - p_\infty \sim - \rho_\infty C_p (\gamma - 1) \frac{Q T_\infty}{1 + Q} \frac{1}{\sqrt{\pi (t - x)}} . \qquad (74)$$

The velocity induced by the heat conduction processes can be found from the non-dimensional version of eq. 11, namely

$$\rho_\infty a_\infty \frac{\partial u}{\partial x} = \rho_\infty C_p (\gamma - 1) \frac{\partial^2 \theta}{\partial x^2} - \frac{\partial p}{\partial t} . \qquad (75)$$

It follows that the transform of the velocity $\bar{u}$ is given by

$$\rho_\infty a_\infty \bar{u} = \rho_\infty C_p (\gamma - 1)(\bar{\theta}')_o^x - s \int_o^x \bar{p} \, dx. \qquad (76)$$

Making use of the previous results for $\bar{\theta}$ and $\bar{p}$ (eqs. 42 and 72 respectively) we find that

$$\rho_\infty a_\infty \bar{u} = -\rho_\infty C_p (\gamma - 1) \sqrt{s} B(s) \left\{ \exp(-x\sqrt{s}) - \exp(-sx/\sqrt{1 + \gamma s}) \right\} \quad ..... \qquad (77)$$

The first integral which must be evaluated to find $u(x,t)$ is very like I_1, in fact it is I_1 with the integrand multiplied by $\sqrt{s}$, and can be treated in a similar fashion. It is found to contribute an amount

$$\rho_\infty C_p (\gamma - 1) \frac{T_\infty Q}{1 + Q} . \frac{e^{-x^2/4t}}{\sqrt{\pi t}}$$

to the whole value of $\rho_\infty a_\infty u$.

Inspection of the second term in eq. 77 shows it to be similar to the I_2 form of integral, the only difference being that its integrand will be that of I_2 divided by $\sqrt{1+\gamma s}$, and the previous remarks made about I_2' follow here too. Whence we can write at once,

$$\rho_\infty a_\infty u \sim \rho_\infty C_p (\gamma - 1) \frac{T_\infty Q}{1+Q} \left\{ \frac{e^{-x^2/4t}}{\sqrt{\pi t}} - \frac{1}{\sqrt{\pi (t-x)}} \right\} \qquad (78)$$

when x is small, and when x is comparable with t

$$\rho_\infty a_\infty u \simeq \rho_\infty C_p \, \theta(x,t) \simeq p - p_\infty , \qquad (79)$$

$\theta(x,t)$ being given by eq. 67.

It is of interest to observe that the pressure, velocity and temperature perturbations in the region around the wave front are, to the accuracy of the solutions presented here, exactly those for an infinitesimal isentropic simple wave, (vide eqs. 73 and 79). This is not surprising, since, in the linearised solution, the irreversible effects of heat conduction and viscosity are neglected and, at the large times for which our solutions are valid, the actual quantities of heat conducted into and out of the wave front are small. This latter statement draws some support from the sketch of $\theta(x,t)$ in Fig. 2, which shows how flat the distribution of temperature is in this region. When x is small, however, the isentropic character of the disturbances vanishes, as is evident from eqs. 74 and 78.

Eq. 73 shows that the pressure perturbation is an expansion across the region $x \sim t$ followed by a gradual recompression as the interface is approached. The velocity disturbance is consistent with this pressure distribution (see eqs. 78 and 79), and it is clear that as time increases the system approaches the classical heat conduction conditions of constant pressure and zero velocity.

5.5. Interface Temperature and Conditions at Zero Time

Before going on to consider the $\sigma = 0$ case we shall briefly examine the temperature at the interface in a little greater detail and also the conditions at time $t = 0+$.

When $x = 0$, the transform solution for the temperature reduces to

$$\bar{\theta}(0;s) = -\frac{T_\infty Q}{s} \frac{1 + (\gamma - 1)\sqrt{s}\sqrt{1+\gamma s}}{1 + Q + (\gamma - 1)s + Q(\gamma - 1)\sqrt{s}\sqrt{1+\gamma s}} \qquad \ldots\ldots \quad (80)$$

(see eqs. 42 and 43). Whence it follows that the wall temperature can be written as

$$T(0,t) = \frac{T_\infty}{2\pi i}\int_L \frac{[1 + (\gamma - 1)s]\, e^{ts}}{1 + Q + (\gamma - 1)s + Q(\gamma - 1)\sqrt{s}\sqrt{1 + \gamma s}}\,\frac{ds}{s} \quad \ldots\ldots (81)$$

L is the usual inversion contour, but it can be deformed into a dumb-bell contour surrounding the branch points at $s = 0$ and $-1/\gamma$. When $t \gg 1$ the integrand in 81 can be expanded in ascending powers of s, the first three significant terms giving

$$T(0,t) = \frac{T_\infty}{1+Q}\left\{1 - \frac{A}{\pi\sqrt{t}}\int_0^{t/\gamma} e^{-y}\frac{dy}{\sqrt{y}} + \frac{B}{\pi t^{3/2}}\int_0^{t/\gamma} e^{-y}\sqrt{y}\,dy\right\} \quad \ldots\ldots (82)$$

where

$$A = \frac{Q(\gamma - 1)}{1 + Q} \quad (83)$$

$$B = \frac{Q^3(\gamma - 1)^3}{(1 + Q)^3} + \frac{Q(\gamma - 1)(3\gamma - 2)}{2(1 + Q)} - \frac{2Q(\gamma - 1)}{(1 + Q)^2} \quad \ldots\ldots (84)$$

The integrals in 82 are incomplete gamma functions which, however, differ a negligible amount from the complete values when $t \gg 1$ (vide Bateman, 1953 p.135). Consequently we can write

$$T(0,t) \simeq \frac{T_\infty}{1 + Q}\left\{1 - \frac{A}{\sqrt{\pi t}} + \frac{B}{2\sqrt{\pi}\, t^{3/2}}\right\}, \quad (85)$$

the next term being of order $t^{-5/2}$. It can be seen how complicated the coefficients are becoming, even in such a simple case as the present one.

Although the large time condition has been studied exclusively for the reasons stated in Section 4, it is of interest to look briefly at the small time predictions of continuum theory. Since the major departures from ambient conditions will arise when x is small also for small times, we shall content ourselves with the interface values of p and θ at time $t = 0+$. These can easily be found from eqs. 72 and 42 by letting $s \to \infty$, whence

$$p(0,0) = -\rho_\infty\,(C_p/\gamma)(\gamma - 1)T_\infty \cdot \frac{Q\sqrt{\gamma}}{1 + Q\sqrt{\gamma}}, \quad (86)$$

$$\theta(0,0) = -\frac{T_\infty Q\sqrt{\gamma}}{1 + Q\sqrt{\gamma}} \quad . \quad (87)$$

These results show that, initially, the density has not changed (N.B. $(C_p/\gamma)(\gamma - 1) = R$, the gas constant, whence the constancy of density follows from the first of equations 1) and that the whole process begins as if it is to be one taking place at constant volume. This latter fact is apparent on observing the definitions of Q (eq. 34) and κ (eq. 14) when it can be seen that writing $Q\sqrt{\gamma}$ is equivalent to redefining the diffusivity κ in terms of C_p/γ, the specific heat at constant volume.
Eq. 87 then represents the classical heat conduction solution appropriate to this type of process. The continuum solution thus represents a transition between the two processes of heat conduction at constant volume and heat conduction at constant pressure.

These remarks conclude our treatment of the $\sigma = \frac{3}{4}$ case and we shall now examine the case $\sigma = 0$ to see how it varies in its behaviour from the results given above.

6. Solutions for $\sigma = 0$

When $\sigma = 0$ the solution 41 must be used. The constants B' and C' can be evaluated from conditions 33, 34 and 36 and the solution for $\theta(x,t)$ expressed in the form

$$\theta(x,t) = \frac{Q\,T_\infty}{2\pi i}\int_L \left\{\frac{(v_1+v_2)^{\frac{1}{2}}(1-v_1-v_2)e^{\alpha_2 x} - (v_1-v_2)^{\frac{1}{2}}(1-v_1+v_2)e^{\alpha_1 x}}{\left\{(v_1+v_2)^{\frac{1}{2}}-(v_1-v_2)^{\frac{1}{2}}\right\}\left\{(v_1+v_2)^{\frac{1}{2}}+(v_1-v_2)^{\frac{1}{2}}+Q(1+\gamma\sqrt{s})\right\}}\right\} \cdot \frac{e^{st}\,ds}{s\sqrt{s}} \qquad (88)$$

where

$$\alpha_{1,2} = -\sqrt{s}\,(v_1 \pm v_2)^{\frac{1}{2}} \quad ; \qquad v_1 = \frac{1+\gamma s}{2} \quad ; \quad v_2 = \left[\left(\frac{1+\gamma s}{2}\right)^2 - s\right]^{\frac{1}{2}} . \qquad (89)$$

Eq. 88 is a very unwieldy expression and even approximate evaluation of the integrals seems impossible without first attempting some kind of transformation which will simplify the exponential terms there. Fortunately the necessary transformations have been supplied by Cole and Wu (1952), but care must be taken in their application and accordingly some useful general observations about the integrals in eq. 88 will be made here.

Closing the straight line contour L to the right with a semi-circle, whose radius R will be allowed to approach infinity, it is found that the real parts of the exponential terms behave as follows. The term in $\exp(\alpha_2 x)$ behaves like $\exp(tR\cos\phi)$, (N.B. $s = R\exp(i\phi)$), and hence contributes to $\theta(x,t)$ for all $t > 0$, whilst the term in $\exp(\alpha_1 x)$ behaves like $\exp(R\cos\phi(t - \sqrt{\gamma}\,x))$ and will be different from zero only when $x < t/\sqrt{\gamma}$.

Referring to the definitions of the dimensionless co-ordinates x and t, it can be seen that the line $x = t/\sqrt{\gamma}$ indicates a velocity equal in magnitude to the isothermal sound speed $c_\infty = a_\infty/\sqrt{\gamma}$. Across this line, therefore, there is a discontinuity in the representation of the solution $\theta(x,t)$ and in fact Cole and Wu have shown such lines to be characteristics of the "$\mu = 0$ system" of equations which leads to the result 88. Bearing in mind the remarks made above, we now apply Cole's and Wu's transformations to 88. There are two stages of transformation. The first, common to both terms of 88 consists of writing

$$\gamma^2 s = (1 + b\,\omega)(1 - b/\omega) \qquad (90)$$

where

$$b = \sqrt{\gamma - 1},$$

whence eq. 88 becomes

$$\theta(x,t) = -\frac{Qb\sqrt{\gamma}T_\infty}{2\pi i}\int_{C'_\omega} \frac{\exp\{(1-b/\omega)[(1+b\omega)t/\gamma - \sqrt{1+b\omega}\,x/\sqrt{\gamma}]/\gamma\}\,(\omega^2+1)}{\{\omega^2+1+Q\sqrt{\gamma}[\omega^2\sqrt{1-b/\omega}+\sqrt{1+b\omega}]\}(1+b\omega)(1-b/\omega)^{\frac{1}{2}}}\,d\omega \qquad \ldots\ldots (91)$$

$$-\frac{Qb\sqrt{\gamma}T_\infty}{2\pi i}\int_{C'_\omega} \frac{\exp\{(1+b\omega)[(1-b/\omega)t/\gamma - \sqrt{1-b/\omega}\,x/\sqrt{\gamma}]/\gamma\}\,(\omega^2+1)}{\{\omega^2+1+Q\sqrt{\gamma}[\omega^2\sqrt{1-b/\omega}+\sqrt{1+b\omega}]\}\,\omega^2(1-b/\omega)\sqrt{1+b\omega}}\,d\omega\,.$$

A possible contour C'_ω starts from $\omega = -i\infty$ and proceeds towards $\omega = i\infty$ passing to the right of the singularities of the integrands. These latter are branch points at $\omega = 0$, b and $-1/b$ and it can be verified that the C'_ω described above can be replaced by a contour which comes from $\omega = \infty\exp(-i\pi)$, loops around $\omega = b$ and returns to $\infty\exp(+i\pi)$. This second form of C'_ω will shortly be found useful.

Each integral in eq. 91 is now tackled separately. Taking the first of these first, it should be noted that this is the ω- plane version of the "$\exp(\alpha_2 x)$" integral in the s-plane which has been shown to contribute to $\theta(x,t)$ for all $t > 0$. Since the second or "$\exp(\alpha_1 x)$" integral is zero for $x > t/\sqrt{\gamma}$ the first integral in eq. 91 gives the whole solution for this condition. We now write

$$\omega = (\zeta^2 - 1)/b, \qquad (92)$$

which transforms the first integral of eq. 91 into

$$\underset{x > t/\sqrt{\gamma}}{\theta(x,t)} = -\frac{Q\sqrt{\gamma}\,T_\infty}{\pi i}\int_{C_>}\left(\frac{\zeta^2-1}{\zeta^2-\gamma}\right)^{\frac{1}{2}}\left\{1 + Q\sqrt{\gamma}\left[\frac{1+\zeta\left(\frac{\zeta^2-\gamma}{\zeta^2-1}\right)^{\frac{1}{2}}}{\zeta+\left(\frac{\zeta^2-\gamma}{\zeta^2-1}\right)^{\frac{1}{2}}}\right]\right\}^{-1}\exp\left[\frac{\zeta}{\gamma}\left(\frac{\zeta^2-\gamma}{\zeta^2-1}\right)\left(\frac{t\zeta}{\gamma}-\frac{x}{\sqrt{\gamma}}\right)\right]\cdot\frac{d\zeta}{\zeta}\,. \qquad (93)$$

The contour $C_>$ in the ζ-plane is illustrated in Fig. 5, the singularities of the integrand being branch points at $\zeta = \pm\sqrt{\gamma}$ and ± 1 with a simple pole at $\zeta = 0$.

When $x < t/\sqrt{\gamma}$ it is necessary to consider the contribution from the second, or "$\exp(\alpha_1 x)$" integral in eq. 91. Writing

$$\omega = -b/(\zeta^2 - 1) \tag{94}$$

it can be shown that the integrand transforms into precisely the form given in eq. 93. The contour of integration for this second integral will be different from $C_>$, however. Making use of the second form of C'_ω contour described above it can be shown that an appropriate ζ-plane contour joins the points ABCODEF on Fig. 5, in the order written. The points A and F lie on the lower and upper halves of the branch cut between $\zeta = \sqrt{\gamma}$ and 1, just to the right of the singularity at $\zeta = 1$. To find the solution for $x < t/\sqrt{\gamma}$ it is now necessary to add the integrals taken along $C_>$ and the contour A to F. Since their integrands are identical it is clear that the parts of $C_>$ between BC and ED are cancelled so that integration should take place along the new contour $C_<$ which is illustrated in Fig. 5, (i.e. replace the symbol > by < in eq. 93).

The difference between the contours which are necessary here and in the case treated by Cole and Wu is apparent. As will be shortly seen, and as may be inferred from their results, the difference is associated with the presence of wave-like phenomena in the region $x < t/\sqrt{\gamma}$ as well as when $x > t/\sqrt{\gamma}$.

6.1. $\theta(x,t)$ for $x < t/\sqrt{\gamma}$ and t large

The part of $C_<$ to the left of $\zeta = 1$ can be deformed into the Imζ axis, retaining the indentation at $\zeta = 0$, of course. This latter part of the contour is then a semi-circle and contributes an amount $-Q\,T_\infty/(1 + Q)$ to $\theta(x,t)$ for $x < t/\sqrt{\gamma}$. Writing Im $\zeta = \eta$ and

$$\left[(\eta^2 + \gamma)/(\eta^2 + 1)\right]^{\frac{1}{2}} = f \quad ; \quad Q\sqrt{\gamma} = Q'$$

for brevity, the remaining contribution to θ from $C_<$ reduces to the real integrals

$$\frac{2\,Q'T_\infty}{\pi}\int_0^\infty\left\{\frac{f(f + Q') + \eta^2(1 + fQ')}{(f + Q')^2 + \eta^2(1 + fQ')^2}\,\sin\left(\frac{x\eta}{\gamma^{3/2}}\cdot f^2\right)\right.$$
$$\left. - \frac{(f + Q') - f(1 + fQ')}{(f + Q')^2 + \eta^2(1 + fQ')^2}\,\eta\cos\left(\frac{x\eta}{\gamma^{3/2}}\,f^2\right)\right\}\exp\left(-\frac{t\eta^2 f^2}{\gamma^2}\right)\cdot\frac{d\eta}{\eta f}\,. \tag{95}$$

When $t >> 1$ the principal contribution to the integrals 95 comes from the region $\eta \sim 0$. In fact $\zeta = 0$ and $\mathrm{Re}\,\zeta = 0$ are a saddle point and steepest path for the original integral, as can be easily verified. In this case $f \simeq \sqrt{\gamma}$ and the expression 95 is approximately

$$\frac{Q\,T_\infty}{1+Q}\cdot\frac{2}{\pi}\int_0^\infty e^{-t\eta^2/\gamma}\sin(\eta x/\sqrt{\gamma})\,\eta^{-1}\,d\eta$$
$$+\frac{(\gamma-1)Q^2T_\infty}{\sqrt{\gamma}(1+Q)^2}\cdot\frac{2}{\pi}\int_0^\infty e^{-t\eta^2/\gamma}\cos(\eta x/\sqrt{\gamma})\,d\eta\,. \qquad (96)$$

In more familiar terms, the part of $C_<$ to the left of $\zeta = 1$ contributes

$$-\frac{Q\,T_\infty}{1+Q}\left\{1-\mathrm{erf}(x/2\sqrt{t})-\frac{Q(\gamma-1)}{1+Q}\;\frac{\exp(-x^2/4t)}{\sqrt{\pi t}}\right\} \qquad (97)$$

to $\theta\,(x,t)$. The errors in 97 are negligible when $t >> 1$ provided x is not too near $t/\sqrt{\gamma}$.

The solution for $x < t/\sqrt{\gamma}$ is completed by evaluating the integral 93 along the contour ABEF. The circle surrounding $\zeta = \sqrt{\gamma}$ contributes zero to θ in the limit as its radius approaches zero and writing $\mathrm{Re}\,\zeta = \xi$, the straight line parts of the contour yield

$$-\frac{2Q\sqrt{\gamma}\,T_\infty}{\pi}\int_1^{\sqrt{\gamma}}\left(\frac{\xi^2-1}{\gamma-\xi^2}\right)^{\frac{1}{2}}\left\{\frac{\xi(\xi+Q\sqrt{\gamma})(\xi^2-1)+(\gamma-\xi^2)(1+\xi Q\sqrt{\gamma})}{(\xi^2-1)(\xi+Q\sqrt{\gamma})^2+(\gamma-\xi^2)(1+\xi Q\sqrt{\gamma})^2}\right\}$$
$$\exp\left[\left(\frac{\gamma-\xi^2}{\xi^2-1}\right)\left(-\frac{t\xi^2}{\gamma^2}+\frac{x\xi}{\gamma^{3/2}}\right)\right]\cdot\frac{d\xi}{\xi} \qquad (98)$$

This integral can be transformed to an infinite one by substituting

$$\xi^2 = \frac{y^2+\gamma}{y^2+1}\,,$$

whence, writing

$$\left[(y^2+\gamma)/(y^2+1)\right]^{\frac{1}{2}} = g \quad ; \quad Q\sqrt{\gamma} = Q'$$

for brevity, we have

$$-\frac{2Q(\gamma-1)T_\infty}{\pi}\int_0^\infty\left\{\frac{g(g+Q')+y^2(1+gQ')}{(g+Q')^2+y^2(1+gQ')^2}\right\}\frac{\exp\left[-y^2\left(\frac{tg^2}{\gamma^2}-\frac{xg}{\gamma^{3/2}}\right)\right]}{(y^2+1)(y^2+\gamma)}\,dy$$
$$\ldots\ldots(99)$$

Once again this integral has a significant contribution mainly near $y = 0$ when $t \gg 1$ and this, to a first order, is

$$-\frac{(\gamma-1)}{\sqrt{\gamma}}\;\frac{Q\,T_\infty}{1+Q}\;\frac{2}{\pi}\int_0^\infty \exp\left[-\left(\frac{t}{\gamma}-\frac{x}{\gamma}\right)y^2\right]dy\,.$$

$$= -\frac{(\gamma-1)Q\,T_\infty}{1+Q}\;\cdot\;\frac{1}{\sqrt{\pi(t-x)}}\;. \qquad (100)$$

The errors are small so long as x is not too near $t/\sqrt{\gamma}$, for $t \gg 1$.

The whole solution for $x < t/\sqrt{\gamma}$ is made up from the sum of the expressions 97 and 100, namely

$$\underset{x\,<\,t/\sqrt{\gamma}}{\theta(x,t)} \sim -\frac{Q\,T_\infty}{1+Q}\left\{1-\text{erf}(x/2\sqrt{t})-\frac{Q(\gamma-1)}{1+Q}\;\frac{\exp(-x^2/4t)}{\sqrt{\pi\;t}}\;\frac{(\gamma-1)}{\sqrt{\pi(t-x)}}\right\}.$$

..... (101)

To a first order of approximation then, this solution is identical with the "small x" solution for $\sigma = \frac{3}{4}$, as can be seen from eqs. 101 and 66, despite the apparently very different exact solutions (eqs. 44 and 88). It is worth noting that there does exist a certain similarity between the contours which have been used to obtain these results, however. Differences between the $\sigma = 0$ and $\frac{3}{4}$ cases would arise in higher order terms than those presented here, but these differences are clearly of no great physical significance at large times.

It is perhaps a little surprising that the agreement between the two seemingly so different cases should be as good as has just been demonstrated here; in fact we shall find that it is not quite so good when the $x \sim t$ regions are compared. The "small x" region is one in which diffusion effects predominate, however, and presumably Prandtl number is of less significance in these circumstances.

6.2. $\theta(x,t)$ for $x > t/\sqrt{\gamma}$ and t large

To examine $\theta(x,t)$ in the region $x \sim t$ write

$$x = t + \delta$$

so that the exponential term in eq. 93 becomes

$$\exp\left[\left(\frac{\zeta^2-\gamma}{\zeta^2-1}\right)\cdot\frac{\zeta t}{\gamma^2}(\zeta-\sqrt{\gamma})\right]\cdot\exp\left[-\frac{\zeta\delta}{\gamma^{3/2}}\left(\frac{\zeta^2-\gamma}{\zeta^2-1}\right)\right] \qquad (102)$$

When $t \gg 1$ and δ is small the method of steepest descents can be used to find an appropriate integration contour which has a rapidly decreasing value of the first exponential factor in in expression 102 along its length. A saddle point for the function of ζ in this term is $\zeta = \sqrt{\gamma}$ and the steepest path of descent lies between this point and the points $\zeta = \sqrt{\gamma}/2 \pm i\infty$. A small semi-circular indentation of this path is necessary to avoid the singularity at $\zeta = \sqrt{\gamma}$ which occurs in eq. 93, but this contributes zero to the final value of $\theta(x,t)$. It can be verified that the steepest path with this indentation is equivalent to $C_>$. Writing

$$\zeta^2 \left(\zeta^2 - \gamma\right)\left(\zeta - \sqrt{\gamma}\right)\left(\zeta^2 - 1\right)^{-1} = - \phi^2, \tag{103}$$

and thereby defining the real variable of integration ϕ on the steepest path, it is observed that expansion of ζ in ascending powers of ϕ begins

$$\zeta = \sqrt{\gamma} \pm i \sqrt{\frac{\gamma - 1}{2\gamma}}\ \phi + \ldots . \tag{104}$$

The upper sign in 104 refers to the upper half of the path and vice versa.

The first order estimate of eq. 93 when $x \sim t$ and $t \gg 1$ can now be written as

$$\theta(x,t) \sim -\frac{Q\, T_\infty}{1+Q} \cdot \frac{2^{\frac{1}{4}}(\gamma - 1)^{\frac{3}{4}}}{\sqrt{\gamma}}\ \frac{1}{\pi} \int_0^\infty e^{-t\phi^2/\gamma^2} \cos\left(\frac{\sqrt{2}\,\delta\phi}{\gamma(\gamma-1)^{\frac{1}{2}}} + \frac{\pi}{4}\right) \frac{d\phi}{\phi^{\frac{1}{2}}} \quad \ldots\ldots \tag{105}$$

Putting $t\phi^2/\gamma^2 = y^2/2$ this becomes

$$\theta(x,t) \sim -\frac{Q\, T_\infty}{1+Q} \cdot \frac{(\gamma - 1)^{\frac{3}{4}}}{t^{\frac{1}{4}}} \cdot \frac{1}{\pi} \int_0^\infty e^{-y^2/2} \cos\left(\frac{\delta\, y}{\sqrt{(\gamma-1)t}} + \frac{\pi}{4}\right) \frac{dy}{\sqrt{y}} \quad \ldots\ldots \tag{106}$$

which is recognizable as the parabolic cylinder function form found in Section 5.2. In other words

$$\theta(x,t) \sim -\frac{Q\, T_\infty}{1+Q} \cdot \frac{(\gamma-1)^{\frac{3}{4}}}{\sqrt{2\pi}\ t^{\frac{1}{4}}}\ \exp(-\delta^2/4(\gamma-1)t)\, D_{-1/2}\left(\delta/\sqrt{(\gamma-1)t}\right), \tag{107}$$

a result which should be compared with the first term of eq. 67 in Section 5.3. They are seen to be identical if γt in eq. 67 is replaced by $(\gamma-1)t$.

When $\mu = 0$ then, the form of the disturbance in the regions around the wave front is the same as in the more practical case for which μ is retained, but the shape of the disturbance is much sharper at a given δ and t. This is hardly surprising, since in the $\sigma = 0$ case the dispersive effects of viscosity are absent. It appears from the results 107 and 101, when compared with the corresponding $\sigma = \frac{3}{4}$ solutions, that the $\mu = 0$ approximation is indeed not as drastic as might be supposed in the first place, provided t is large enough. Also the effects of Prandtl number seem to be of most importance in the wave-front zone and of little significance near to the interface, certainly as far as temperature is concerned.

6.3. Pressure and Velocity Disturbances

The disturbance to the initially uniform pressure and velocity fields can be found in a similar manner to the $\sigma = \frac{3}{4}$ case, although by reason of the condition $\mu = 0$ here, the velocity problem is in fact somewhat simpler to solve. We will not give the details here to avoid wearisome repetition, but merely quote the results.

Thus it is found that, when $x < t/\sqrt{\gamma}$ and x is not too nearly equal to $t/\sqrt{\gamma}$

$$p - p_\infty = - \rho_\infty \; C_p \frac{Q \; T_\infty}{1 + Q} \; \frac{(\gamma - 1)}{\sqrt{\pi} \; (t-x)} \; , \qquad (108)$$

whilst when $x \sim t$

$$p - p_\infty \simeq \rho_\infty \; C_p \; \theta(x,t) \; , \qquad (109)$$

$\theta(x,t)$ being given by eq. 107.

Eq. 108 is identical with eq. 74 and 109 is the equivalent of the result 73, to which it reduces if $(\gamma - 1)t$ is replaced by γt.

Likewise, the velocity u is exactly the same as eq. 78 when $x < t/\sqrt{\gamma}$ and a result equivalent to 79 is obtained when $x \sim t$. To the order of accuracy of these results, the similarity between the $\theta(x,t)$ values for $\sigma = 0$ and $\frac{3}{4}$ is retained for the other flow variables therefore.

It can also be shown that the $t = 0+$ values of θ and p are identical in the two cases, so that the transition from constant volume to constant pressure heat transfer occurs when $\sigma = 0$, too. Some idea of the extent of the difference between the two solutions at the interface for large t can be gained by comparing eq. 85 with the corresponding result for $\sigma = 0$. In the latter case $T(0,t)$ is given by an equation exactly like 85 in which the coefficient B is now

$$B = \frac{Q(\gamma - 1)}{2(1 + Q)} \left\{ \gamma \left(\frac{3 - \gamma}{\gamma - 1} \right) + 3 + \frac{3(\gamma - 1)Q}{1 + Q} \right\} .$$

For typical values of γ and Q of 1.4 and 100, B above equals 1.94 whilst B in eq. 85 equals 0.49, but, since the result is only valid when t >>1, the effect of this difference on T(0,t) is very small.

The difference between the $\sigma = 0$ and $\sigma = \frac{3}{4}$ cases has been found to be very small at large times, the principal effects arising in the wave front region where viscosity acts to disperse the disturbance more rapidly than can be accomplished by conduction alone. Consequently the intuitive feeling, expressed in Section 4 (ii), that the omission of viscosity will still lead to a reasonable picture of the flow field has received some support in this case. Putting $\sigma = 0$ does not simplify the problem, however.

7. Conclusions

The processes which take place in a gas, initially in a uniform state, when it is placed in sudden contact with a solid at a different temperature have been examined for two values of Prandtl number, namely $\frac{3}{4}$ and zero. The characteristic time for the establishment of the resulting flow has been found to be of the order of the mean time between collisions of the gas molecules whence, since the formulation treats the gas as a continuum, the main effort has been concentrated on finding solutions valid for large times.

In these circumstances it has been found that the flow field divides into two regions in a rough sense. Some distance from the interface the disturbances propagate out into the gas as a wave motion travelling at the ambient isentropic sound speed (in this linearised treatment), upon which are superimposed the dispersive and dissipative effects of viscosity and heat conduction. When the gas is intially hotter than the solid the wave front is of an expansive character which tends to flatten as time proceeds. If the gas should be colder than the solid initially, the wave front will represent a compression. This front will still flatten and decay with increasing time in the linear theory presented here, but in practice non-linear convective effects will oppose these processes and a shock wave may be expected to appear.

In the regions near to the wall the heat diffusion processes dominate, but, as the temperature of the gas changes, corresponding changes of pressure and velocity will occur and these give rise to a wave motion which is superimposed on the main process of diffusion.

The build up of the flow field can be explained as follows. Assuming the gas to be hotter than the solid, at the initial instant, the layer of gas molecules immediately adjacent to the wall lose some of their energy and momentum to the solid, but as yet there has been no time for any appreciable mass motion of the gas to occur and the density remains at its ambient value. The pressure pulse so produced then begins to propagate out into the gas, dropping the gas temperature below that which could be attained as a result of heat diffusion alone, and accelerating the gas towards the interface. Since the solid is impermeable, this motion must be resisted and the gas will recompress more and more as the interface is approached. This recompression will give rise to some reheating of the gas and the temperatures near the wall will begin to rise back towards the values appropriate to pure heat conduction at constant pressure. Molecular velocities cover the whole range of magnitude from zero to infinity, so that changes such as those arising at the interface can be signalled to the furthest corners of the flow field instantaneously. However, the strength of this signal and the extent to which the gas will react to it at any given point depends on the number of molecules which reach the point with the necessary information. Significant changes in gas properties are expected to occur only when a bulk of the molecules from an affected region reach the point in question therefore. In other words the bulk of the disturbance must travel at some average molecular speed, which is the meaning of the isentropic sound speed here.

It seems clear from the results derived above that the compressibility effects are very small near to the wall for times of order, say, 100 collision intervals from the start of the processes. Experimental observation of wall temperatures could hardly be expected to reveal them, therefore, since it is difficult to resolve times of less than about 0.01 microseconds, in which interval most of the effects have vanished. If an experimental set-up equivalent to the theoretical model studied here could be devised, however, it may be possible to see the wave front. Reflection of a shock wave from the closed end of a conventional shock tube would not be an adequate experimental model with which to examine the predictions of the present theory, since the shock wave itself is a front extending over a few molecular mean free paths, across which the gas properties change to their new values. The instantaneous initiation condition demanded by the present theory would therefore be lost, and furthermore interaction between the reflected shock wave and the heat-transfer-induced pressure disturbances would arise to complicate the picture.

It seems plausible to suggest , however, that even in this more complicated problem the "ideal" inviscid shock reflection state would be pretty well established in about the same time interval as it takes for the constant pressure heat transfer state to arrive in our simple theoretical model. This conclusion is not without significance, since it follows that the reflected shock technique may perhaps be used to produce a slug of hot gas which can then be employed to study other important properties of gases. By these we mean the effects of molecular structure, which have been explicitly excluded from the present work. Thus, for example, excitation or de-excitation of the vibrational modes of diatomic molecules is known to take several thousand collisions, so that any effects which may arise as a result of this relatively long relaxation time could be examined in the reflected shock region on the assumption that compressibility effects behind the shock wave are negligible. The simple theoretical model of a semi-infinite gas and solid may then be adequate for a study of, say, interface temperatures in such a relaxing gas, the pressure being substantially constant.

The writer would like to thank Mr. G. M. Lilley for numerous helpful suggestions during the course of the work described above and, indeed, for pointing out that the $\sigma = \frac{3}{4}$ case could be solved.

8. References

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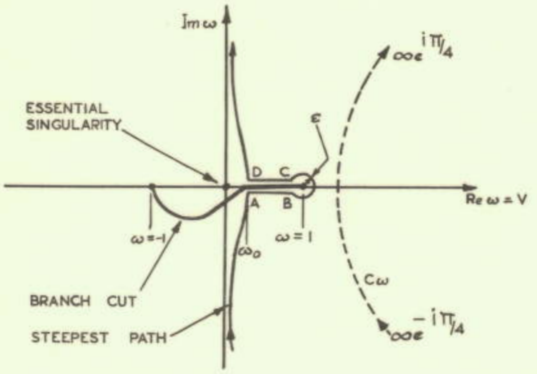

FIG. 1. CONTOUR USED TO EVALUATE I_2 WHEN t IS LARGE AND x/t SMALL

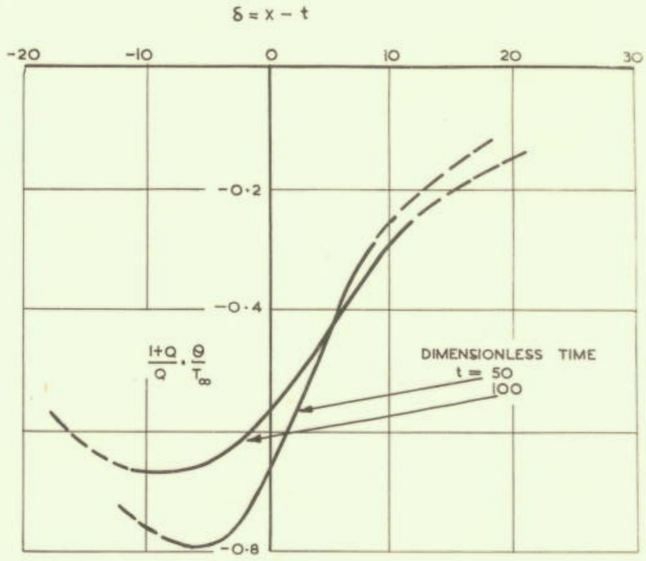

FIG. 3. SHAPE OF THE WAVE FRONT; TEMPERATURE vs. δ AT TWO VALUES OF t

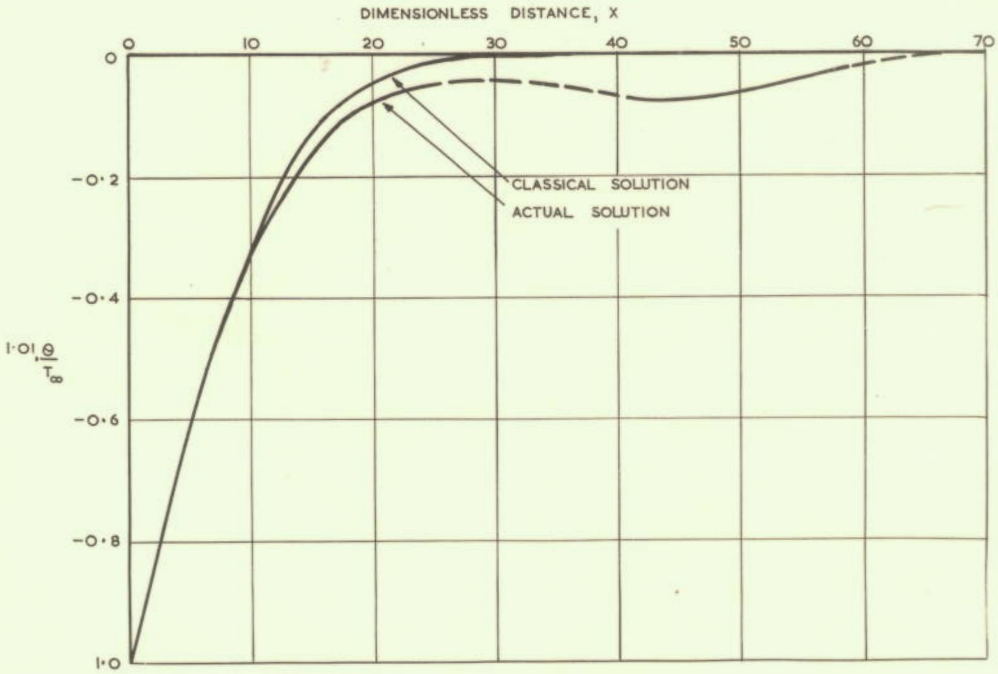

FIG. 2. COMPARISON OF CLASSICAL AND ACTUAL TEMPERATURE DISTRIBUTIONS AT t = 50 Q = 100

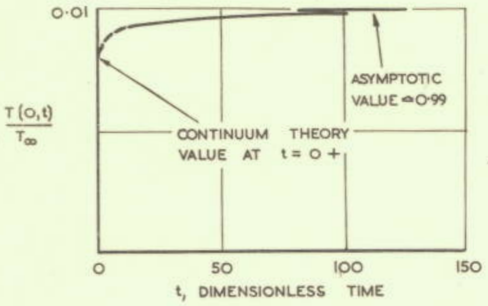

FIG. 4. VARIATION OF INTERFACE TEMPERATURE WITH TIME. Q=100

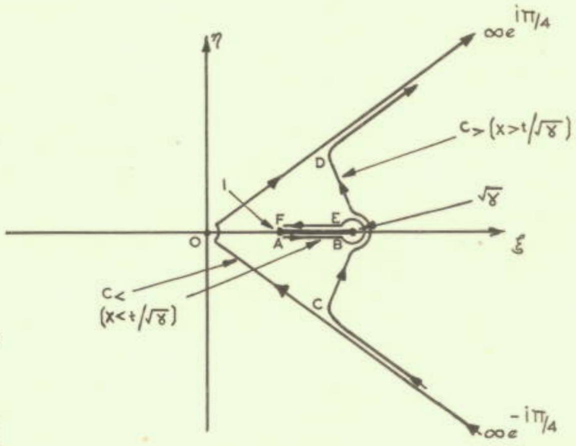

FIG. 5. CONTOURS IN THE ζ - PLANE

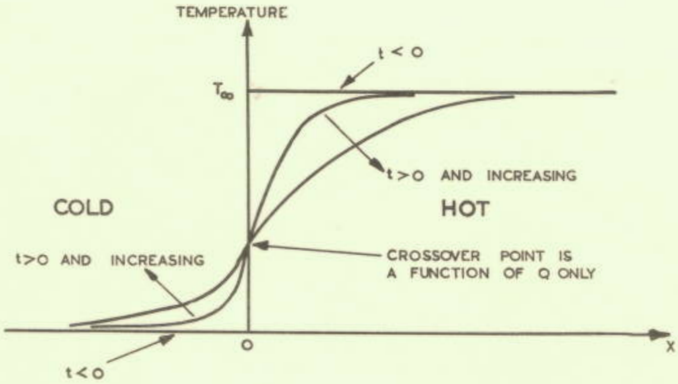

FIG. 6. SKETCH OF THE CONSTANT - PRESSURE TEMPERATURE PROFILES (Not to Scale)