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An Assessment of Certain Methods of Stress-Analysis of Rectangular Multi-Web Box Beams *

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SUMMARY

The stress distribution in an unswept multi-web box under shear load applied at the centre of a rigid tip rib is examined, and compared with results obtained by a method which replaces the shear webs by a shear-carrying continuum. For the case considered the agreement between the results from the two methods is satisfactory for boxes whose geometry is typical of normal practice.

A method of performing root constraint calculations on such a box is outlined and results of an elementary analysis of this type are compared with experiment.

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NOTATION

b	half depth of box
c	half width of box
l	length of box
t	skin thickness
t_w	thickness of internal webs
t_o	thickness of end webs
A	area of internal booms
A_o	area of end booms
n	number of cells ($n + 1$ = number of webs)
i	web or skin panel identification number
x, y, z	coordinates defined in Fig.1
Z	applied shear force
T_1, S, T_2	stress-resultants defined in skin $z = b$
P_i	load in boom i
q_i	shear flow in web i.
Q_i	modified web shear flow (eq^n.(2))
u, v	displacement components in plane of skin $z = b$.
w_i	z-wise deflection in web i.
F_i, G_i	functions defining spanwise constraint stress distribution
$\Phi_{1i}, \Psi_i, \Phi_{2i}$	functions defining chordwise constraint stress distribution
H_i	constants defining web shear flow distribution
E	Young's Modulus
σ	Poissons Ratio

$\tau = \sigma/2(1 + \sigma)$

$\lambda = t_o/t_w$

$a = nA/2ct$

$a_o = nA_o/2ct$

$c' = c + (n - 1)A/2t + A_o/t$

$\zeta = cZ/2nbc'$

$\theta = ct_w/nbt$

$\delta = (\theta/2)^{\frac{1}{2}}$

$\gamma = \sinh^{-1} \delta$

$\beta = 1 + \theta - \sqrt{\left(\theta(2 + \theta)\right)}$

$\xi = nx/2c, \ \eta = ny/2c, \ \eta_i = \eta - i + 1, \ l' = nl/2c$

1. Introduction

A form of construction for thin aircraft wings which is widely used at the present time is that which makes use of a number of shear webs to stabilise the compression surface. This multiplicity of shear-carrying material provides a higher order of redundancy than that associated with a single-cell box, and various methods have been proposed for dealing with this complexity (References 2,3).

A common method of dealing with a continuous structure reinforced by a large number of discrete elements (e.g. fuselage reinforced by stringers) is to replace the discrete elements by a continuous distribution of material having the same overall properties. Hemp (Reference 1) has used this technique in dealing with the multi-web box, in that he replaces the discrete shear webs by a shear-carrying continuum. The solution obtained gives a chord-wise variation of shear in this continuum which is exponential, and it is natural to enquire as to the accuracy with which the continuum can reproduce the behaviour of the finite number of webs carrying such a rapidly varying loading. The present investigation deals with a particular case dealt with in Reference 1, namely that of the uniform rectangular multi-web box under shear load applied centrally at a rigid tip rib. This loading case is considered under exactly the same assumptions as those used in Reference 1, except that the continuum does not replace the discrete webs.

2. The Rectangular Multi-Web Box under Tip Shear

The box is shown in Figure 1, which gives the overall dimensions and shows the co-ordinate system used, and the direction of action of the applied force Z which acts through the line $x = l$, $y = c$. The root $x = 0$ of the box is assumed to be rigidly built in. The box has (n+1) webs, and web 'i' is in the plane $y = 2ic/n$. Webs 1,2,3,4 ..., n-1 have thickness t_w and boom area A, while webs 0, n have thickness t_o and boom area A_o. All the webs are assumed to have no bending stiffness, but to carry a uniform shear flow q_i. The contribution of the web material to the bending stiffness is allowed for in the boom areas A and A_o. The skins $z = \pm b$ have thickness t, and we identify panel 'i' of the skin as lying between webs 'i-i' and 'i'. The stress-resultants T_1, S, T_2 illustrated in Figure 1 have the positive directions shown in the skin $z = +b$.

The analysis of Appendix A leads us to the following expression for the distribution of the applied shear force Z between the webs.

$$Q_i = \zeta\left[(1+a-\tau)+\theta\left(\frac{2\lambda-1+2\lambda a-2a_o-(n+2\lambda-1)\tau}{\beta+\beta^{n-1}-(1+2\lambda\theta)(1+\beta^n)}\right)(\beta^i+\beta^{n-i})\right] \tag{1}$$

where Q_i is defined as

$$Q_i = \begin{cases} q_i & (i = 1,2,3,\ldots,n-1) \\ \dfrac{t_w}{t_o}\, q_i & (i = 0,n) \end{cases} \tag{2}$$

The various parameters are defined in the list of notation, but we may examine the important parameter θ, which defines

$$\beta = 1 + \theta - \sqrt{\left(\theta(2+\theta)\right)}$$

θ is defined by $\theta = ct_w/nbt = (a_w/2b)(t_w/t)$ where a_w is the web pitch. θ is therefore essentially a "shape" parameter, describing the geometry of the internal cells, and we shall find that the distribution of web shear flow given by (1) is noticeably sensitive to θ. Another important parameter in (1) is $\tau = \sigma/2(1+\sigma)$. If the web shear distribution is evaluated by the usual "Batho" method of equating the twist in each cell to zero, as in Refs. 2,3, the solution is given by equation (1) with τ zero. As will be seen later the term in τ has an important effect on the magnitude of the departure of the shear distribution from the uniform. The term in τ arises because the skin shear strain, which has to correspond with the stress obtained using equilibrium considerations, contains a term arising from the chordwise Poisson contraction due to the linearly varying bending stress. It is clear that if the box is built in to a completely rigid root then this Poisson contraction will be restrained and so the importance of the ern in τ will be reduced. This is a matter for experimental investigation.

Comparison with the 'Continuum' Theory

If we write $\gamma = \frac{1}{2}\left|\log \beta\right|$, (2) may be thrown into the form

$$Q_i = \zeta\left[A - B\cosh(2i-n)\gamma \Big/ \left(\cosh n\gamma(2\lambda-1+\tanh n\gamma/\tanh\gamma)\right)\right] \qquad (3)$$

where $A = (1+a-\tau)$, $B = \left(2\lambda-1+2\lambda a-2a_0 - (n+2\lambda-1)\tau\right)$

We may transform the expression (in equation (28) of Reference 1) for the shear stress in the continuum into our notation (using i as a running co-ordinate $= ny/2c$) to give

$$q = (n\zeta/2c)\left[A - B\cosh(2i-n)\delta \Big/ \left(\cosh n\delta(2\lambda-1+\tanh n\delta/\delta)\right)\right] \qquad (4)$$

where $\delta = (\theta/2)^{\frac{1}{2}} = \sinh\gamma$

The close relationship in form between equations (3) and (4) may be noted, and we may immediately compare γ and δ for varying θ to find the difference in the 'shape' of the shear distribution given by the two methods.

The six-cell box which provided the experimental results noted in Reference 1 had $b = 4$ ins., $c = 18$ ins., $t = 0.159$ ins., $t_w = t_o = 0.0575$ ins., $a = a_0 = 0.1406$. We thus have $\theta = 0.2712$, and so $\gamma = 0.3604$, $\delta = 0.3682$. In this case, then, the 'continuum' method gives a two per cent variation in the exponential term in the web shear distribution, which is certainly satisfactory.

It will be seen that in both (3) and (4) the term in B indicates the departure of the web shear distribution from the uniform and that this departure is greatest when $i = 0$ or n. We may compare the difference in this term between the two expressions for the extreme cases $n = 2$ and n large.

For $n = 2$ $\quad 1/(1 + \tanh 2\gamma / \tanh\gamma) = 0.3589$

$\quad 1/(1 + \tanh 2\delta/\delta) = 0.3700$ (3.1% difference)

For n large $\quad 1/(1 + 1/\tanh\gamma) = 0.2568$

$\quad 1/(1 + 1/\delta) = 0.2691$ (4.8% difference)

Thus for this value of θ the continuum theory gives a reasonably close approximation to the departure from uniform shear in the webs, which does not vary greatly in accuracy with number of cells.

In the above discussion we have compared web shear flow given by this paper with (one web pitch) × (local value of continuum shear stress at web line) given by Reference 1. Strictly speaking, one should integrate the result of equation (4) over one half web-pitch either side of the web line, adding in the case of webs 0 and n the contribution of the 'special' edge webs. If this is done the differences noted above are increased to 13.3% and 6.0% respectively. Thus it may be said that in the continuum theory it is numerically favourable, though logically incorrect, to multiply the <u>local</u> value of the continuum shear stress by the appropriate web pitch. This artifice, however, will lead to a distribution of web shear flow which does not exactly balance the applied shear force Z.

It should be noted that the percentage 'errors' given above are only differences in the term multiplied by the constant B. If we return to the example of Reference 1, and evaluate A and B numerically ($n = 6$) we find that if $\sigma = 1/3$,

$B = 0.1250 \ll A = 1.0156$ and we shall expect the differences in the final results of equations (3) and (4) to be proportionately less. Remembering that in this case $n = 6$, equation (3) gives $q_o = 0.9827\,\zeta$. The local value of shear stress in (4) leads to $q_o = 0.9811\zeta$ (0.1% difference), while the 'integrated' value gives $q_o = 0.9800\zeta$ (0.3% difference). Thus the overall difference between the two methods is much reduced in this case. It may be remarked that this masking of the difference between the two methods is most apparent when $n \simeq 7$ and that in the torsion case (equation (25), Reference 1) there is no constant term present and thus the original figures given are more appropriate. We may note that if $\tau = 0$ ("Batho Theory") $B = 1.000$ and so A and B are strictly comparable in magnitude and the above reduction does not occur.

The above example provides quite close agreement between the results of the two methods. A value of θ equal to unity is not at all unreasonable structurally, and if we take a ten cell box for which $\theta = 1$, $a = a_0 = 0.10$, $\lambda = 1$ we have $\gamma = 0.6585$, $\delta = 0.7071$ ($\therefore \delta = 1.0738\gamma$) and

$$\begin{aligned} q_o &= 1.1132\zeta \text{ by (10)} \\ &= 1.1303\zeta \text{ by (11) (local)} \qquad (1.5\%) \\ &= 1.1378\zeta \text{ by (11) (integrated)} \qquad (2.2\%) \end{aligned}$$

For a four cell box of the same geometry

$$\begin{aligned} q_o &= 0.8368\zeta \text{ by (10)} \\ &= 0.8190\zeta \text{ by (11) (local)} \qquad (2.1\%) \\ &= 0.8081\zeta \text{ by (11) (integrated)} \qquad (3.4\%) \end{aligned}$$

For the torsion case the above differences would be increased to the order of 15% . We may thus conclude that the accuracy of the continuum in following the behaviour of the discrete shear webs of a multi-web box is quite reasonable for a box having practically significant geometry. It may be said that the number of webs is less significant than the geometry of the cells as reflected in the parameter θ.

All the foregoing may be said to be concerned with the 'beam theory' solution of the multi-web box built in at its root under a shear load applied at a rigid tip rib. The solution could have been obtained by finding the vertical deflection at the tip of each web with the tip rib absent and some simple distribution of the applied shear among the webs, and then seeking the statically zero web shear distribution necessary to make all the tip deflections equal. This approach to the problem will be found useful when considering constraint effects at the root.

It is clear from the expression for u in Equation (A.9) that the solution we have obtained involves an equal warping of all cross-sections and that the distribution of shear flow between the webs will affect this warping. Since the bending stresses corresponding to T_1 must be reacted at the root of any box, it is clear that there must be a greater or lesser amount of restraint of the warping of the root section. This restraint is incompatible with the results of equation (A.9) and so a statically zero modification to the elementary stress distribution must be added in order to make the root deformations consistent with the boundary conditions. Hemp in Reference 1 suggests a modification to T_1 which varies parabolically in the chordwise direction and has an arbitrary spanwise variation which may be determined by variational methods. This solution involves no shear in the webs, and is correct for a box where l is large. However, since there is now variation of bending stress across the chord there must be a corresponding differential bending of the webs (equation (A.9) with $q_i = 0$), and this is incompatible with the assumption of a rigid tip rib.

4. Root Constraint in Shear

Following a remark made earlier, it would seem that a reasonable approach to the problem of the root constraint of a multi-web box under tip shear is to solve first the root constraint problem for an arbitrary distribution of the applied shear between the webs. The web shear distribution is then chosen to make the web tip deflections <u>including those due to the constraint stresses</u> compatible with the conditions applied by the tip rib. The method chosen has been the familiar one of specifying a distribution of bending stress across the cross-section with spanwise variation determined variationally using the Theorem of Minimum Strain Energy. It is clear that the choice of the cross-sectional variation of stress must be sufficiently general to allow for effects arising from the as yet unknown web shears.

Extending the method of Reference 1 to allow T_1 to vary with y as an even order polynomial brings considerable algebraic difficulty because of the effect of the web booms in disturbing the simplicity of the skin shear stresses. It was decided to allow T_1 to vary linearly between arbitrary values at the web lines. For the n-cell box this will mean that the constraint stresses are specified by either $n/2$ or $(n-1)/2$ unknown functions of x, the equations governing which must be obtained by minimising the strain energy under appropriate conditions. This will lead to linear differential equations each involving all of the unknown functions. It is therefore profitable to endeavour to minimise the coupling between these functions. We may do this by building up our constraint stresses from distributions which are themselves each statically zero and which cover a minimum number of skin bays.

In Appendix B such distributions are derived and a method of solution is indicated. Calculations of this type have been performed in the case of the six-cell box already mentioned and the results are plotted in Figure 2. Section A is at 4.5 in. and section B at 27 in. from the root of the box whose length is 54 in.

The conclusions of this necessarily crude approach should be treated with caution, but they indicate that the effect of web shear distri--bution on the constraint stresses is not negligible.

5. Further Developments

The results of a simplified calculation such as that just described will serve as a lead in the solution of the complete constraint problem involving the functions F_i (ξ) of Appendix B as well as the G_i (ξ). The methods of this paper are eminently suitable for examining the effect of flexibility of the tip rib; and they will also handle without extra difficulty the problem of the multi-web box carrying concentrated loads applied to its webs at stations distant from ribs, a case which has proved unsuit--able for the continuum approach. The methods of the first part of the paper may also be used to investigate the accuracy of the continuum approach for further loading cases (e.g. the box under distributed normal pressure).

6. Conclusions

For the simple loading case discussed, namely the box under uniform shear, the method of replacing uniformly spaced webs by a 'shear continuum' is sufficiently accurate for the purposes of stress analysis. The effect of stresses due to any constraints must, however, be taken into account when evaluating the shear distribution in the webs.

References

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APPENDIX A.

Analysis of Web Shear Distribution

The bending stresses are distributed as in the Engineer's Beam Theory, to give

$$T_1 = -Z(\ell - x)/4bc' \quad ; \quad T_2 = 0 \tag{A 1}$$

where

$$c' = c + (n-1)A/2t + A_o/t \tag{A 2}$$

The (constant) shear flow in web 'i' is q_i, and since we are dealing with a loading case which is symmetrical about $y = c$ we have

$$q_{n-i} = q_i \text{(all i)} \quad ; \quad \sum_{i=0}^{n} q_i = Z/2b$$

We may integrate the equation of x-wise equilibrium in the skins ($\partial T_1/\partial x + \partial S/\partial y = 0$), and allow for the equilibrium conditions applying at the web booms to find that the shear flow in skin panel 'i' is given by

$$S_i(y) = \sum_{j=0}^{i-1} q_j - (Z/bc')\left(y + A_o/t + (i-1)A/t\right) \tag{A 3}$$

It will be noted that this expression satisfies the symmetry conditions $S_{n+1-i}(2c - y) = - S_i(y)$. The stress resultants given by (1), (3) automatically satisfy the requirements of compatibility of displacement. The strains in this skin panel are given by

$$e_{xx} = \frac{-Z(\ell - x)}{4Etbc'} \quad ; \quad e_{yy} = \frac{\sigma Z(\ell - x)}{4Etbc'}$$

$$(e_{xy})_i = \frac{2(1+\sigma)}{Et}\left(\sum_{j=0}^{i-1} q_j\right) - \frac{(1+\sigma)Z}{2Etbc'}\left(y + \frac{A_o}{t} + (i-1)\frac{A}{t}\right) \tag{A 4}$$

Integrating the first two of (A 4) we obtain

$$\left.\begin{aligned} u_i(x,y) &= -Z(\ell x - \tfrac{1}{2}x^2)/4Etbc' + f_i(y) \\ v(x,y) &= \sigma Z(\ell - x)\ (y-c)/Etbc' \end{aligned}\right\} \quad (A\ 5)$$

where $f_i(y)$ is an as yet unknown function of y. We substitute from (A 5) into the last of (A 4).

$$\frac{-\sigma Z(y-c)}{4Etbc'} + f_i'(y) = \frac{2(1+\sigma)}{Et}\left(\sum_{j=0}^{i-1} q_j\right) - \frac{(1+\sigma)Z}{2Etbc'}\left(y + \frac{A_o}{t} + (i-1)\frac{A}{t}\right) \quad (i=1,2,3 \ldots,n) \qquad (A\ 6)$$

Integrating $f_i'(y)$ between $2(i-1)c/n$ and $2ic/n$, and using (A 5) we find that

$$u_i\left(\frac{2ic}{n}\right) - u_i\left(\frac{2(i-1)c}{n}\right)$$

$$= \frac{2(1+\sigma)}{Et}\left[\left(\sum_{j=0}^{i-1} q_j\right) - \frac{Z}{4bc'}\left\{\frac{(2i-1)c}{n} + \frac{A_o}{t} + (i-1)\frac{A}{t} - \tau\ \frac{(2i-n-1)c}{n}\right\}\right] \quad (i = 1,2,3, \ldots, n) \qquad (A\ 7)$$

where $\tau = \sigma/2(1+\sigma)$.

On examination of the above process it will be seen that the term in τ in (A 7) which leads to that in equations (3), (4) referred to in the main text, derives from the shear strain which arises from the Poisson contraction varying with x in the second of (A 5).

Summing equations (A 7) from 1 to i, we find that

$$u_i(x,\ 2ic/n)$$

$$= u_1(x,0) + \frac{4(1+\sigma)c}{Etn}\left[\left(\sum_{j=0}^{i-1}(i-j)q_j - \frac{Zic}{4nbc'}\left\{i + \frac{nA_o}{ct} + \frac{n(i-1)A}{2ct} + (n-i)\ \tau\right\}\right)\right] \quad (i=1,2,3, \ldots, n) \qquad (A\ 8)$$

We may again note that symmetry is satisfied since $u_{n-i}(x,\ 2(n-i)c/n) = u_i(x,\ 2ic/n)$.

We must now consider the shear webs. We assume that they experience no direct strain in the z-direction, and we have already postulated that they carry no direct stress in the x-direction though naturally they must extend in that direction with the skins. Clearly, then, the web shear strains are given by

$$2(1+\sigma)q_i/Et_w = \partial w_i/\partial x + u_i(x, 2ic/n)/b \qquad (i = 1,2,3, \ldots, n-1)$$

$$2(1+\sigma)q_o/Et_o = \partial w_o/\partial x + u_o(x,0)/b \qquad \text{(A 9)}$$

where w_i is the z-wise displacement at web i. Since the root is built in we have $u = w = 0$ at $x = 0$ and since we assume that the load Z is applied at a rigid rib we must have $\partial w/\partial y = 0$ at $x = l$. Substituting from (5) in (6) we find that $\partial w_i/\partial x - \partial w_{i-1}/\partial x$ is independent of x and so the above condition implies that

$$\begin{aligned} 0 &= \partial w_i/\partial x - \partial w_{i-1}/\partial x \\ &= \frac{2(1+\sigma)}{Et_w}(q_i - q_{i-1}) - \frac{4(1+\sigma)c}{Etbn}\left[\sum_{j=0}^{i-1} q_j - \frac{Z}{4bc'}\left\{\frac{(2i-1)c}{n} + \frac{A_o}{t} + \frac{(i-1)A}{t} - \tau\frac{(2i-n-1)c}{n}\right\}\right] \end{aligned}$$

$$(i = 2,3,4, \ldots, n-1)$$

$$0 = \partial w_1/\partial x - \partial w_o/\partial x$$

$$= \frac{2(1+\sigma)}{Et_w}\left(q_1 - \frac{t_w q_o}{t_o}\right) - \frac{4(1+\sigma)c}{Etbn}\left[q_o - \frac{Z}{4bc'}\left\{\frac{c}{n} + \frac{A_o}{t} + \tau(n-1)\frac{c}{n}\right\}\right]$$

If we define $Q_i = q_i \, (i = 1,2,3, \ldots, n-1)$

$$= t_w q_i/t_o \, (i = 0, n)$$

and write $\theta = ct_w/nbt = a_w t_w/2bt$ (a_w = web pitch) and $\zeta = cZ/2nbc'$, $\alpha = nA/2ct$, $\alpha_o = nA_o/2ct$, $\lambda = t_o/t_w$

then

$$\left.\begin{aligned} &Q_{i-1} - 2(1+\theta)Q_i + Q_{i+1} + 2\theta\zeta(1+\alpha-\tau) = 0 \\ &\qquad\qquad (i = 1,2,3,\ldots, n-1) \\ &-(1+2\lambda\theta)Q_o + Q_1 + \theta\zeta(1+2\alpha_o+(n-1)\tau) = 0 \end{aligned}\right\} \quad (A\ 10)$$

We may solve the recurrence relations of (7) to give

$$Q_i = \zeta(1+\alpha-\tau) + C(\beta^i + \beta^{n-i}) \qquad (i = 0,1,2,\ldots,n) \quad (A\ 11)$$

where

$$\beta^2 - 2(1+\theta)\beta + 1 = 0$$

and we choose the root $\beta = 1 + \theta - \sqrt{\left(\theta(2+\theta)\right)} \leq 1$. We find the unknown constant C by substituting in the equation involving Q_o and Q_1 only, and so obtain equation (1).

APPENDIX B

Constraint Stress Distributions

In S4 it was suggested that a constraint stress distribution could be conveniently built up as a combination of "local" statically zero distributions varying with x. If we assume that T_1 varies linearly with y between web lines we may proceed as follows.

$$\text{If } \xi = nx/2c, \quad \eta = ny/2c, \qquad \eta_i = \eta - i+1 \qquad \text{(B 1)}$$

we may write

$$T_1(x,y) = \sum_{i=1}^{n-1} F_i(\xi)\, \Phi_{1i}(\eta)$$

$$S(x,y) = \sum_{i=1}^{n-1} F_i'(\xi)\, \Psi_i(\eta) \qquad \text{(B 2)}$$

$$T_2(x,y) = \sum_{i=1}^{n-1} F_i''(\xi)\, \Phi_{2i}(\eta)$$

$$P_i(x) = 0 \;; \quad q_i = 0$$

where $F_{n-i}(\xi) = F_i(\xi)$ and $\Phi_{1i}(\eta)$ etc. are defined by

$$\Phi_{11}(\eta_j) = \frac{1}{5}\left[-\delta_j^1(6-11\eta_1) + \delta_j^2(5-7\eta_2) - 2\,\delta_j^3(1-\eta_3)\right]$$

$$\Phi_{1i}(\eta_j) = \frac{1}{2}\left[-\delta_j^{i-1}\,\eta_{i-1} - \delta_j^i(1-3\eta_i) + \delta_j^{i+1}(2-3\eta_{i+1}) - \delta_j^{i+2}(1-\eta_{i+2})\right] \quad (i = 2,3,4,\ldots,n-2)$$

$$\Psi_1(\eta_j) = \frac{1}{10}\left[\delta_j^1\,\eta_1(12-11\eta_1) + \delta_j^2(1-10\eta_2+7\eta_2^2) - 2\,\delta_j^3(1-\eta_3)^2\right]$$

$$\Psi_i(\eta_j) = \frac{1}{4}\left[\delta_j^{i-1}\eta_{i-1}^2 + \delta_j^i(1-\eta_i)(1+3\eta_i) - \delta_j^{i+1}\eta_{i+1}(4-3\eta_{i+1}) - \delta_j^{i+1}(1-\eta_{i+2})^2\right]$$

$$(i = 2,3,4,\ldots,n-2)$$

$$\bar{\sigma}_{21}(\eta_j) = \frac{1}{30}\left[-\delta_j^1\,\eta_1^2(18-11\eta_1)-\delta_j^2(7+3\eta_2-15\eta_2^2+7\eta_2^3)-2\delta_j^3(1-\eta_3)^3\right]$$

$$\bar{\sigma}_{2i}(\eta_j) = \frac{1}{12}\left[-\delta_j^{i-1}\eta_{i-1}^3 - \delta_j^i(1+3\eta_i+3\eta_i^2-3\eta_i^3)-\delta_j^{i+1}(4-6\eta_{i+1}+3\eta_{i+1}^2)-\delta_j^{i+2}(1-\eta_{i+2})^3\right]$$

$(i = 2,3,4, \ldots, n-2)$

(B 3)

where $\delta_l^k = \begin{cases} 1 \text{ when } k = l \\ 0 \text{ when } k \neq l \end{cases}$

The above distributions are illustrated in Fig.3. Such a distribution of T_1 satisfies equilibrium conditions, is statically zero, and covers at the most four skin bays. It thus is suitable for our purpose and reduces the coupling between the functions $F_i(\xi)$ which describe its spanwise variation.

No account has so far been taken of the constraint stresses in the web-booms. Since these derive from the strains due to the above stress-distributions, the relation between them is not simple. It is more convenient to specify the boom loads $P_i(x)$ independently in terms of unknown functions $G_i(\xi)$ $\left(G_{n-i}(\xi) = G_i(\xi)\right)$ as follows

$$P_c(x) = -\frac{c}{n}\,G_1(\xi),\quad P_1(x) = \frac{c}{n}(-G_o(\xi)+2G_1(\xi)),$$

$$P_i(x) = \frac{c}{n}\left(-G_{i-1}(\xi)+2G_i(\xi)-G_{i+1}(\xi)\right)$$

$(i = 2,3,\ldots, n-2)$

$$T_{1i}(x,y) = 0 \qquad \text{(B 4)}$$

$$S_1(x,y) = \tfrac{1}{2}G_1'(\xi)\ ,\ S_i(x,y) = \tfrac{1}{2}\left(-G_{i-1}'(\xi)+G_i'(\xi)\right)\ (i = 2,3, \ldots, n-2)$$

$$T_{21}(x,y) = -\tfrac{1}{2}\eta_1 G_1''(\xi),\quad T_{2i}(x,y) = -\tfrac{1}{2}\left((1-\eta_i)G_{i-1}''(\xi) + \eta_i G_i''(\xi)\right),$$

$(i = 2,3, \ldots, n-2)$

$$q_i = 0$$

The condition of minimum energy may be used to enforce the necessary compatibility relationship between the F_i and the G_i.

Our constraint system is now specified by the functions $F_i(\xi)$, $G_i(\xi)$ which automatically ensure that the constraint stresses are self-balancing. Unfortunately, although much has been done to reduce the coupling between the F_i and G_i, a typical differential equation resulting still contains seven F_i and five G_i. We may reduce this complexity by ignoring the F_i, concentrate the bending-stress carrying properties of the skins (so far as the constraint stresses are concerned) into the suitably enlarged web booms, and allow the skins to carry only shear (S) and chordwise direct stress (T_2).

A statically correct distribution of stress which reacts the tip shear Z and which is expresses in terms of unknown constants $H_1, H_2, \ldots, H_{n-1}$ ($H_{n-i} = H_i$) is as follows

$$
\begin{aligned}
&nP_o(x)/2c = -\alpha_o \zeta(l'-\xi), \quad nP_i/2c = -\alpha\zeta(l'-\xi) \quad (i = 1,2,3,\ldots,n-1) \\
&T_{1i}(x,y) = -\zeta(l'-\xi) \\
&S_i(x,y) = -\zeta(\eta_i - \tfrac{1}{2}) \qquad (i = 1,2,3, \ldots, n) \\
&T_{2i}(x,y) = 0 \\
&q_o = \zeta(\tfrac{1}{2}+\alpha_o), \quad q_i = \zeta(1+\alpha) \quad (i = 1,2,3, \ldots, n-1) \\
&\text{where } l' = nl/2c
\end{aligned}
\qquad \text{(B 5)}
$$

Combining (B 4) and (B 5), writing down the strain energy and minimising it with respect to the G_i we obtain equations of the type

$$
\begin{aligned}
&G''''_{i-1}(\xi) + 4G''''_i(\xi) + G''''_{i+1}(\xi) - 12(1+\sigma)\Big(-G''_{i-1}(\xi) + 2G''_i(\xi) - G''_{i+1}(\xi)\Big) \\
&+ (6/\alpha)\Big(G_{i-2}(\xi) - 4G_{i-1}(\xi) + 6G_i(\xi) - 4G_{i+1}(\xi) + G_{i+2}(\xi)\Big) = 0 \\
&\qquad (i = 3,4, \ldots, n-3)
\end{aligned}
\qquad \text{(B 6)}
$$

with typical boundary conditions

$$G_{i-1}'''(0) + 4G_i'''(0) + G_{i+1}'''(0) - 12(1+\sigma)\Big(-G_{i-1}'(0) + 2G_i'(0) - G_{i+1}'(0) + H_{i-1} - 2H_i + H_{i+1}\Big) = 0 \qquad \text{(B 7)}$$

$$G_i''(0) = G_i(l') = G_i'(l') = 0$$

It should be noted that in equations such as (B 6), though not in (B 5), α and α_o should be augmented to include the effective skin area. The determinant which must vanish to make these equations consistent is a regular one, and a recurrence relation may be found which leads to its evaluation for general n. The $F_i(\xi)$ may then be found in terms of the as yet unknown H_i, and finally the tip deflections of the webs found in terms of the H_i. The H_i are then chosen to make these deflections consistent with the conditions at the tip rib.

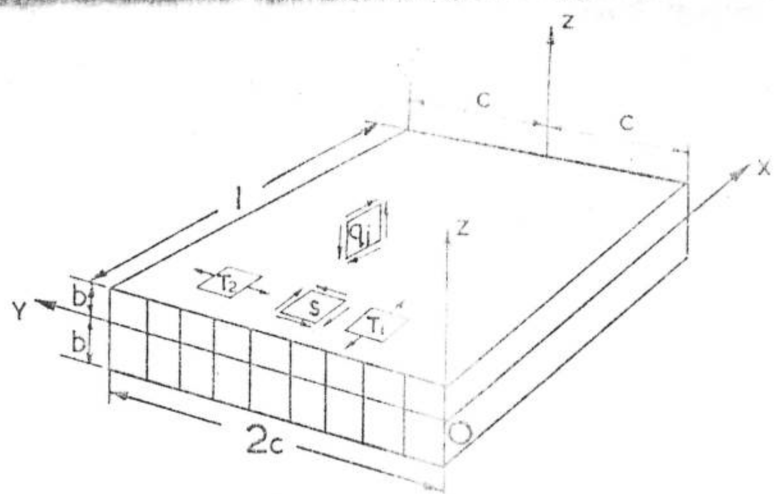

FIG. 1.

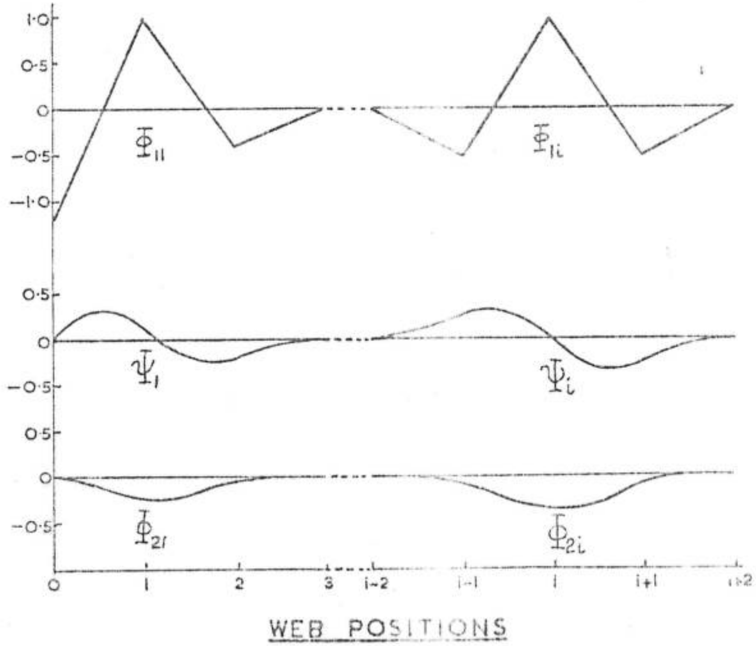

FIG. 3. DISTRIBUTIONS OF CONSTRAINT STRESS

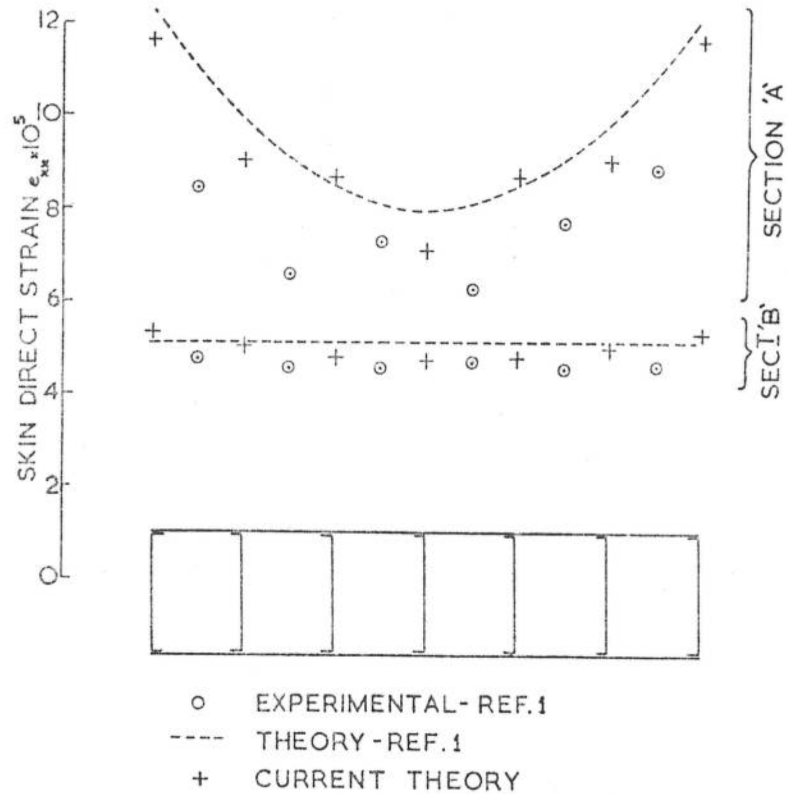

FIG. 2
SKIN DIRECT STRAIN
(1000LB. TIP SHEAR).