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A Method for Numerical Evaluation of the Integral

$$\int_0^1 \int_0^1 F(x,y) \log \left| \frac{1}{x-y} \right| dx\, dy$$

-by-

T. Nonweiler, B.Sc.

SUMMARY

This report presents formulae and data for the numerical evaluation of the double integral named in the title in the form $\sum_{\mu} \sum_{\nu} c_{\mu\nu} F(s_\mu, s_\nu)$, where numerical values for the weighting coefficients $c_{\mu\nu}$ and the lattice points (s_μ, s_ν) are given. The method depends on a double Fourier series representation of $F(x,y)$ in terms $\theta = \cos^{-1}(1-2x)$ and $\phi = \cos^{-1}(1-2y)$; the lattice points s_μ, s_ν are in fact at equally spaced intervals in θ and ϕ.

The method is particularly applicable to integration where F has half-order singularities (or zeros) along the borders of the region of integration; but it may also be adequate for the treatment of continuous functions F which are finite at these borders, (an accuracy of within less than 1 per cent being then expected from the use of a 11 x 11 lattice). Comments on applications of the formula to the evaluation of wave drag are given.

CONTENTS

LIST OF SYMBOLS

$F \equiv F(x,y)$	function of x,y in the double integral to be evaluated
$F_{\mu\nu}$	coefficient defined by equation (1) of Appendix
$I_{\mu}(\phi)$	integral defined by equation (8) of Appendix
I	the double integral to be evaluated
$S(x)$	cross-sectional area of body at fraction x of its length from nose
$b_{\mu} = 1 - \frac{1}{2}\left(\delta_{n\mu} + \delta_{(-n)\mu}\right)$	
$c_{\mu\nu}$	weighting coefficients of integration formulae tabulated in section 2.1 for various μ,ν and n.
${}_{m}c_{\mu\nu}$	defined in section 3(m)
$d_{\mu\nu}$	weighting coefficients of integration formulae tabulated in section 2.2

n number specifying lattice spacing in integration formula

$s_\mu = \frac{1}{2}\left(1 + \sin\frac{\mu\pi}{2n}\right)$

x,y variables of integration

$\delta_{\mu\nu} = 1$ if $\mu = \nu$; $= 0$ if $\mu \neq \nu$. The Kronecker delta

$\zeta(x)$ non-dimensional wing section ordinate at fraction x of its chord from nose

$\theta = \cos^{-1}(1-2x)$

μ, ν numbers characterising point of evaluation in integration or interpolation scheme.

$\phi = \cos^{-1}(1-2y)$

$\psi_\mu = \frac{\pi}{2} + \frac{\pi\mu}{2n}$.

1. Introduction

A double integral of the type named in the title appears for instance in expressions for the wave drag of slender bodies and of swept wings and presents some difficulty in its evaluation, as the function $F(x,y)$ is often only known either by its numerical values at discrete points, or else by a closed algebraic expression rendering formal evaluation tedious, if indeed possible. For this reason numerical quadrature is often a convenience, but the well-known formulae for the evaluation of double integrals (such as Weddle's Rule) are inapplicable owing to the logarithmic singularity in the integrand. We present here a method for its approximate numerical evaluation which depends essentially on the representation of the function $F(x,y)$ in terms of a finite double Fourier series, which can be shown to be equivalent to its expression as a polynomial in x and y divided by $\sqrt{xy(1-x)(1-y)}$.

This form of representation is evidently particularly suitable where $F(x,y)$ has half-order singularities (or zeros) at the boundaries of the region of integration, which is sometimes so, but admittedly would introduce unnecessary errors where such conditions are not met. For this reason, the quadrature formulae quoted are tested for a particular example of the latter category, where the function F is a constant over the range of integration. This probably represents the most severe test possible, implying as it does the representation of $(\sin \theta)$ over a half period $0 < \theta < \pi$ by a finite cosine series.

The integration formulae are quoted in the form

$$\int_0^1 \int_0^1 F(x,y)\ ln \left|\frac{1}{x-y}\right| dx\ dy \doteq \sum_{\mu=-n+1}^{n-1} \sum_{\nu=-n+1}^{n-1} c_{\mu\nu}\ F(s_\mu, s_\nu)$$

where $\quad s_\mu = \frac{1}{2}\left(1 + \sin \frac{\mu\pi}{2n}\right)$

Using an 11 x 11 -point lattice (i.e. $n = 6$), and putting F equal to a constant, the error is less than one per cent of the exact value. The numerical value is less than the true one: the precise error with this and other lattice spacings is tabulated below.

n	lattice points	error as percentage of exact value
1	1 x 1	32.71
2	3 x 3	6.45
3	5 x 5	2.91
4	7 x 7	1.66
5	9 x 9	1.07
6	11 x 11	0.75

For the reasons stated above, it is to be expected that accuracy in relation to functions F which have half-order

/singularities ...

singularities or zeros on the borders of the region of integration would be considerably better.

Full details of the method of calculation and the values of the weighting coefficients $c_{\mu\nu}$ are given in the next section. The mathematical derivation of the formulae is treated in the Appendix. The third section concerns the evaluation of such integrals where the limits are not between zero and unity, and the next section concerns some simplifications resulting from symmetries in the function $F(x,y)$. Finally some comments are made concerning the application of the formulae to the evaluation of wave drag, by way of example.

2. Formulae for Numerical Integration

The analysis of Appendix I leads us to distinguish between two conditions n the behaviour of $F(x,y)$, which is assumed bounded everywhere inside the region of integration. The distinctions arise from the presence or absence of singularities in $F(x,y)$ on the boundaries of the region of integration.

2.1. The function $F(x,y)$ bounded over the complete region of integration

Equations (10) and (12) of Appendix I show that

$$\int_0^1 \int_0^1 F(x,y) \ln \left|\frac{1}{x-y}\right| dx\, dy \simeq \sum_{\mu=-n+1}^{n-1} \sum_{\nu=-n+1}^{n-1} c_{\mu\nu} F(s_\mu, s_\nu)$$

where $s_\mu = \frac{1}{2}\left(1 + \sin \frac{\mu\pi}{2n}\right)$. The coefficients $c_{\mu\nu}$ (together with numerical values of s_μ, s_ν) are given below for $n = 2(1)\,6$. It will be observed that

$$c_{\mu\nu} = c_{\nu\mu}$$

and

$$c_{\mu\nu} = c_{(-\mu)(-\nu)}$$

so that out of each array of $(2n-1)^2$ coefficients, arranged in a square, only n^2 are different, and these appear in each of the four triangular parts of the array formed by and including the diagonal elements.

It will also be seen that the lattice points s_μ, s_ν are at equally spaced in terms of θ and ϕ where

$$\theta = \cos^{-1}(1-2x), \qquad \phi = \cos^{-1}(1-2y), \qquad (0 \leqslant \theta,\phi \leqslant \pi) \;;$$

so that it is convenient to suppose that $F(x,y) \equiv f(\theta,\phi)$, say, in which event

$$F(s_\mu,s_\nu) = f\left(\frac{\pi}{2} + \frac{\mu\pi}{2n}\,,\ \frac{\pi}{2} + \frac{\nu\pi}{2n}\right)$$

The lattice points corresponding to the values $\mu = \pm\, n$, or $\nu = \pm\, n$ lie on the boundaries of the region of integration, and the values of F (and so of f) at these points are not used in the integration schemes. Given F in terms of numerical data at discrete points which do not coincide with the lattice points, its values required in the integration scheme would have to be found by interpolation; whether or not this can more easily be accomplished using θ and ϕ as independent variables, or by interpolating with respect to x and y, depends of course upon the nature of the derivation of the function F.

Coefficients $c_{\mu\nu}$ for n=2				
	s_μ	0.1464	0.5	0.8536
s_ν	ν \ μ	-1	0	1
0.8536	1	0.0137	0.1375	0.2193
0.5	0	0.1375	0.3873	0.1375
0.1464	-1	0.2193	0.1375	0.0137

Coefficients $c_{\mu\nu}$ for n=3						
	s_μ	0.0670	0.25	0.5	0.75	0.9330
s_ν	ν \ μ	-2	-1	0	1	2
0.9330	2	0.0008	0.0144	0.0247	0.0555	0.0625
0.75	1	0.0144	0.0297	0.0922	0.1599	0.0555
0.5	0	0.0247	0.0922	0.2035	0.0922	0.0247
0.25	-1	0.0555	0.1599	0.0922	0.0297	0.0144
0.0670	-2	0.0625	0.0555	0.0247	0.0144	0.0008

Coefficients $c_{\mu\nu}$ for n=4								
	s_μ	0.0381	0.1464	0.3087	0.5	0.6913	0.8536	0.9619
s_ν	ν \ μ	-3	-2	-1	0	1	2	3
0.9619	3	0.00016	0.00266	0.00507	0.01227	0.01682	0.02452	0.02396
0.8536	2	0.00266	0.00566	0.01674	0.02643	0.04888	0.07028	0.02452
0.6913	1	0.00507	0.01674	0.02933	0.06320	0.11128	0.04888	0.01682
0.5	0	0.01227	0.02643	0.06320	0.12734	0.06320	0.02643	0.01227
0.3087	-1	0.01682	0.04888	0.11128	0.06320	0.02933	0.01674	0.00507
0.1464	-2	0.02452	0.07028	0.04888	0.02643	0.01674	0.00566	0.00266
0.0381	-3	0.02396	0.02452	0.01682	0.01227	0.00507	0.00266	0.00016

Coefficients $c_{\mu\nu}$ for $n=5$

s_ν	ν \ μ (s_μ)	0.0245	0.0955	0.2061	0.3455	0.5	0.6545	0.7939	0.9045	0.9755
		-4	-3	-2	-1	0	1	2	3	4
0.9755	4	0.00007	0.00067	0.00148	0.00353	0.00546	0.00848	0.01023	0.01231	0.01109
0.9045	3	0.00067	0.00161	0.00449	0.00766	0.01358	0.01852	0.02696	0.03478	0.01231
0.7939	2	0.00148	0.00449	0.00816	0.01581	0.02359	0.03920	0.06078	0.02696	0.01023
0.6545	1	0.00353	0.00766	0.01581	0.02526	0.04602	0.08041	0.03920	0.01852	0.00848
0.5	0	0.00546	0.01358	0.02359	0.04602	0.08767	0.04602	0.02359	0.01358	0.00546
0.3455	-1	0.00848	0.01852	0.03920	0.08041	0.04602	0.02526	0.01581	0.00766	0.00353
0.2061	-2	0.01023	0.02696	0.06078	0.03920	0.02359	0.01581	0.00816	0.00449	0.00148
0.0955	-3	0.01231	0.03478	0.02696	0.01852	0.01358	0.00766	0.00449	0.00161	0.00067
0.0245	-4	0.01109	0.01231	0.01023	0.00848	0.00546	0.00353	0.00148	0.00067	0.00007

Coefficients $c_{\mu\nu}$ for $n = 6$

s_ν	s_μ	0.0170	0.0670	0.1464	0.25	0.3706	0.5	0.6294	0.75	0.8536	0.9330	0.9830
	ν \ μ	-5	-4	-3	-2	-1	0	1	2	3	4	5
0.9830	5	0.00017	0.00005	0.00070	0.00113	0.00258	0.00344	0.00355	0.00606	0.00590	0.00717	0.00561
0.9330	4	0.00005	0.00042	0.00174	0.00248	0.00458	0.00674	0.01091	0.01209	0.01607	0.01885	0.00717
0.8536	3	0.00070	0.00174	0.00257	0.00580	0.00391	0.01327	0.02073	0.02504	0.03475	0.01607	0.00590
0.75	2	0.00113	0.00248	0.00580	0.00826	0.01526	0.01958	0.03143	0.04954	0.02504	0.01209	0.00606
0.6294	1	0.00258	0.00458	0.00391	0.01526	0.03003	0.03579	0.05039	0.03143	0.02073	0.01091	0.00355
0.5	0	0.00344	0.00674	0.01327	0.01958	0.03579	0.06359	0.03579	0.01958	0.01327	0.00674	0.00344
0.3706	-1	0.00355	0.01091	0.02073	0.03143	0.05039	0.03579	0.03003	0.01526	0.00391	0.00458	0.00258
0.25	-2	0.00606	0.01209	0.02504	0.04954	0.03143	0.01958	0.01526	0.00826	0.00580	0.00248	0.00113
0.1464	-3	0.00590	0.01607	0.03475	0.02504	0.02073	0.01327	0.00391	0.00580	0.00257	0.00174	0.00070
0.0670	-4	0.00717	0.01885	0.01607	0.01209	0.01091	0.00674	0.00458	0.00248	0.00174	.00042 0.0042	0.00005
0.0170	-5	0.00561	0.00717	0.00590	0.00606	0.00355	0.00344	0.00258	0.00113	0.00070	0.00005	0.00017

2.2 <u>The function $F(x,y)$ with half-order singularities at the boundaries of the region of integration, but bounded over the interior of the region</u>

Equation (11) of Appendix I shows that, under the stated conditions of the heading,

$$\int_0^1 \int_0^1 F(x,y)\ ln \left|\frac{1}{x-y}\right| dx\, dy \doteq \sum_{\mu=-n}^{n} \sum_{\nu=-n}^{n} c_{\mu\nu}\ F(s_\mu, s_\nu)$$

where s_μ and the values of $c_{\mu\nu}$ for $-n < \mu,\ \nu < +n$ are as given in the previous section. But if μ or $\nu = \pm n$, we must interpret

$$c_{\mu\nu}\ F(s_\mu, s_\nu) \equiv d_{\mu\nu} \lim_{x \to s_\mu} \lim_{y \to s_\nu} \left[\sqrt{xy(1-x)(1-y)}\ F(x,y)\right]$$

and the coefficients $d_{\mu\nu}$ are tabulated below for $n = 6$. Again it is found that a symmetry exists for the coefficients $d_{\mu\nu}$ such that

$$d_{\mu\nu} = d_{\nu\mu}$$

and $$d_{\mu\nu} = d_{(-\mu)(-\nu)}$$

so that it is necessary only to quote $d_{n\nu}$ for $-n \leqslant \nu \leqslant n$; evidently

$$d_{\mu n} = d_{n\mu}$$

$$d_{(-n)\nu} = d_{n(-\nu)}$$

and $$d_{\mu(-n)} = d_{n(-\mu)}$$

Table of values of $d_{n\nu}$ for $n=6$							
ν	-6	-5	-4	-3	-2	-1	0
$d_{n\nu}$	0.12796	0.15048	0.08962	0.06804	0.04562	0.03300	0.02210
ν	1	2	3	4	5	6	
$d_{n\nu}$	0.02026	0.00828	0.00700	0.00082	-0.00226	-0.00077	

3. Modification where integration limits are altered

A simple substitution will show that, if $b > a$,

$$\int_a^b \int_a^b G(x,y)\, ln \left|\frac{1}{x-y}\right| dx\, dy = \frac{1}{(b-a)^2} \int_o^1 \int_o^1 F(x,y)\, ln \left|\frac{1}{x-y}\right| dx\, dy$$

$$- \frac{ln(b-a)}{(b-a)^2} \int_o^1 \int_o^1 F(x,y)\, dx\, dy$$

where $\quad F(x,y) = G\left[(b-a)x+a,\ (b-a)y+a\right]$

The first integral on the right-hand-side may be evaluated by the method of the previous section. An analysis similar to that used in the Appendix I shows that the second double integral

$$\int_o^1 \int_o^1 F(x,y)dx\, dy = \frac{\pi^2}{16n^2} \sum_{\mu=-n}^{+n} \sum_{\nu=-n}^{+n} b_{\mu\nu} \lim_{x\to s_\mu} \lim_{y\to s_\nu} \left[\sqrt{xy(1-x)(1-y)}\ F(x,y)\right]$$

where $b_{\mu\nu} = 4$ if μ and $\nu \neq \pm n$; $b_{\mu\nu} = 2$ if either μ or $\nu = \pm n$; and $b_{\mu\nu} = 1$ if both μ and $\nu = \pm n$. (The limits in this double sum only require interpretation if μ or $\nu = \pm n$). Use of this formula can be convenient particularly if F has half-order singularities or zeros at the edges of the region of integration, as it involves data which is required to evaluate the double integral containing the logarithmic singularity. Other, and more accurate, methods may of course be used to evaluate the second double integral if it is non-singular.

4. Simplification resulting from particular symmetries in $F(x,y)$

(i) If $F(x,y) = F(y,x)$ we see from the previous sections that as $c_{\mu\nu} = c_{\nu\mu}$,

$$\sum_{\mu=-n}^{+n} \sum_{\nu=-n}^{+n} c_{\mu\nu}\, F(s_\mu, s_\nu) = \sum_{\mu=-n}^{+n} \sum_{\nu=-n}^{\mu} {}_1c_{\mu\nu}\, F(s_\mu, s_\nu)$$

where $\quad {}_1c_{\mu\nu} = (2-\delta_{\mu\nu})c_{\mu\nu}$

and where

$$\delta_{\mu\nu} = 1 \text{ if } \mu = \nu, = 0 \text{ if } \mu \neq \nu \text{ is the Kronecker delta.}$$

Thus instead of a square lattice of points at which the function $F(s_\mu, s_\nu)$ has to be evaluated, only the terms in a triangular lattice formed by and including the diagonal elements on $\mu = \nu$ need be used. Thus for $n = 3$, and a function F bounded over the entire region of integration, we find that instead of the array of 5×5 terms given in §2.1, the triangular array for the coefficients $c_{\mu\nu}$:

$\nu \backslash \mu$	-2	-1	0	1	2
2					0.0625
1				0.1599	0.1110
0			0.2035	0.1844	0.0494
-1		0.1599	0.1844	0.0594	0.0288
-2	0.0625	0.1110	0.0494	0.0288	0.0016

of 15 terms suffices. Similar tabulations may easily be constructed for other values of n.

(ii) If $F(x,y) = F(1-x,y)$, then as $s_\mu = 1 - s_{(-\mu)}$

$$\sum_{\mu=-n}^{+n} \sum_{\nu=-n}^{+n} c_{\mu\nu}\, F(s_\mu, s_\nu) = \sum_{\mu=0}^{n} \sum_{\nu=-n}^{n} {}_2c_{\mu\nu}\, F(s_\mu, s_\nu)$$

where $\quad {}_2c_{\mu\nu} = c_{\mu\nu} + (1-\delta_{\mu o})\, c_{(-\mu)\nu}$.

Here the square lattice is replaced by a rectangular lattice formed by ahd including the elements along $\mu = 0$. Thus again, for $n = 3$, and a function F bounded over the entire range of integration, the rectangular array for ${}_2c_{\mu\nu}$ is

ν \ μ	0	+1	+2
2	0.0247	0.0699	0.0633
1	0.0922	0.1896	0.0699
0	0.2035	0.1844	0.0494
-1	0.0922	0.1896	0.0699
-2	0.0247	0.0699	0.0633

which is again of 15 instead of 25 terms.

(iii) If $F(x,y) = F(x,1-y)$, then similarly

$$\sum_{\mu=-n}^{+n} \sum_{\nu=-n}^{+n} c_{\mu\nu}\, F(s_\mu, s_\nu) = \sum_{\mu=-n}^{n} \sum_{\nu=o}^{n} {}_3c_{\mu\nu}\, F(s_\mu, s_\nu)$$

where $\quad {}_3c_{\mu\nu} = c_{\mu\nu} + (1-\delta_{o\nu})c_{\mu(-\nu)}$

and again the square lattice is replaced by a rectangular one divided now along the elements $\nu=o$.

(iv) If $F(x,y) = F(1-x,y) = F(x,1-y)$, which is a combination of the conditions (ii) and (iii) above, then

$$\sum_{\mu=-n}^{+n} \sum_{\nu=-n}^{n} c_{\mu\nu}\, F(s_\mu, s_\nu) = \sum_{\mu=o}^{n} \sum_{\nu=o}^{n} {}_4c_{\mu\nu}\, F(s_\mu, s_\nu)$$

where $\quad {}_4c_{\mu\nu} = \left[1 + (1-\delta_{\mu o})(1-\delta_{o\nu})\right] c_{\mu\nu} + (2-\delta_{\mu o} - \delta_{o\nu})\, c_{(-\mu)\nu}$

The lattice used is now the square formed by and including the elements on $\mu = 0$ and $\nu = 0$. Taking the same example as before, the nine coefficients ${}_4c_{\mu\nu}$ for $n = 3$, and with a function F bounded everywhere, are:

ν \ μ	0	1	2
2	0.0494	0.1398	0.1266
1	0.1844	0.3792	0.1398
0	0.2035	0.1844	0.0494

Other tabulations for different n can quickly be assembled.

(v) If the conditions (i) and (iv) are both satisfied, further reduction is possible. For then

$$\sum_{\mu=-n}^{n} \sum_{\nu=-n}^{n} c_{\mu\nu} F(s_\mu, s_\nu) = \sum_{\mu=0}^{n} \sum_{\nu=0}^{\mu} {}_5c_{\mu\nu} F(s_\mu, s_\nu)$$

where $\quad {}_5c_{\mu\nu} = (2-\delta_{\mu\nu})\,{}_4c_{\mu\nu}$

Thus in the previous example only 6 coefficients ${}_4c_{\mu\nu}$ are now needed

ν \ μ	0	1	2
2			0.1266
1		0.3792	0.2796
0	0.2035	0.3688	0.0988

Taking n = 6, and the function F bounded over the range of integration the square array of 11 x 11 coefficients $c_{\mu\nu}$ can be reduced to a triangular array of 21 coefficients ${}_5c_{\mu\nu}$ subject to the stated conditions; these are tabulated below.

μ / ν	0	1	2	3	4	5
5						0.01156
4					0.03854	0.02888
3				0.07464	0.07124	0.02640
2			0.11560	0.12336	0.05828	0.02876
1		0.16084	0.18676	0.09856	0.06196	0.02452
0	0.06359	0.14316	0.07832	0.05308	0.02696	0.01376

(vi) If $F(x,y) = - F(1-x,y)$, then

$$\sum_{\mu=-n}^{n} \sum_{\nu=-n}^{+n} c_{\mu\nu} F(s_\mu, s_\nu) = \sum_{\mu=1}^{n} \sum_{\nu=-n}^{n} {}_6c_{\mu\nu} F(s_\mu, s_\nu)$$

where $\quad {}_6c_{\mu\nu} = c_{\mu\nu} - c_{(-\mu)\nu}$

Here the square lattice is replaced by a rectangular lattice formed by, but not including the elements along $\mu = 0$. Thus for $n = 3$ and a function F bounded over the entire region of integration the non-vanishing coefficients ${}_6c_{\mu\nu}$ are 8 in number, and can be calculated as:

$\nu =$		2	1	0	-1	2
$\mu =$	2	0.0617	0.0411	0	-0.0411	-0.0617
	1	0.0411	0.1302	0	-0.1302	-0.0411

(vii) Similarly if $F(x,y) = - F(x,1-y)$, then

$$\sum_{\mu=-n}^{n} \sum_{\mu=-n}^{n} c_{\mu\nu} F(s_\mu, s_\nu) = \sum_{\mu=-n}^{n} \sum_{\nu=1}^{n} {}_7c_{\mu\nu} F(s_\mu, s_\nu)$$

where $\quad {}_7c_{\mu\nu} = c_{\mu\nu} - c_{\mu(-\nu)}$

(viii) If conditions (vi) and (vii) both apply then as
$c_{\mu\nu} = c_{(-\mu)(-\nu)}$

$$\sum_{\mu=-n}^{n} \sum_{\mu=-n}^{n} c_{\mu\nu} F(s_\mu, s_\nu) = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} {}_8c_{\mu\nu} \; F(s_\mu, s_\nu), \text{ where}$$

${}_8c_{\mu\nu} = 2 \left({}_6c_{\mu\nu}\right)$. For the example quoted before the 5×5 square lattice of coefficients is now reduced simply to a 2×2 lattice.

(ix) If conditions (i) and (viii) both apply, then:

$$\sum_{\mu=-n}^{n} \sum_{\nu=-n}^{n} c_{\mu\nu} \; F(s_\mu, s_\nu) = \sum_{\mu=1}^{n} \sum_{\nu=1}^{\mu} {}_9c_{\mu\nu} \; F(s_\mu, s_\nu)$$

where $\quad c_{\mu\nu} = 2(2-\delta_{\mu\nu}) \; {}_6c_{\mu\nu}$.

In the example for $n=3$ and a function F bounded everywhere, the number of coefficients ${}_9c_{\mu\nu}$ is now only 3 given by

ν \ μ	1	2
2		0.1234
1	0.2604	0.1644

Taking $n = 6$, and the function F bounded over the range of integration the square array of 11×11 coefficients $c_{\mu\nu}$ can be reduced to a triangular array of 15 coefficients ${}_9c_{\mu\nu}$ subject to the stated conditions: these are tabulated below.

ν \ μ	1	2	3	4	5
5					0.01088
4				0.03686	0.02848
3			0.06436	0.05732	0.02080
2		0.08256	0.07696	0.03844	0.01972
1	0.04072	0.06468	0.06728	0.02532	0.00388

5. Applications in Aerodynamics

As mentioned in the introduction to the paper integrals of the type referred to in this note appear in formulae for the wave drag at supersonic speeds. Two particular applications can be quoted: with

$$F(x,y) = \zeta'(x)\,\zeta'(y)$$

the integral appears in the expression for the wave drag of a swept wing of infinite span with subsonic leading edges, and with a section whose thickness is t and whose ordinates are $z = \frac{1}{2}t\zeta(x)$ at a fraction x of the chord from the nose.[1] The present integration formulae are particularly suited to evaluation of such integrals where the section leading-edge and trailing edge are either rounded or cusped, because the function $z'(x)$ has then half-order singularities at $x = 0$ or 1. Where such conditions are satisfied it is found that the formulae given above yield exact values in comparison with all simple examples worked exactly (by formal integration) merely because the form of the interpolating formula (used in deriving the present method) describes the section shape exactly. Where such conditions are not met only a few examples have been calculated by exact methods: one is for the biconvex wing section, which corresponds to

$$\zeta'(x) = 4(1-2x), \quad \text{and} \quad F(x,y) = 16(1-2x)(1-2y).$$

In this example $F(x,y)$ is finite (and non-zero) at the borders of the region of integration: use of the integration formula with $n = 6$ then provides a result too small by a little less than 1 per cent. Application to sections with round noses and angular trailing-edges (so that ζ' has the half-order singularity at one end of the range of integration only) has been found to provide answers correct within less than 0.1 per cent; the same order of accuracy is also attainable for the treatment of sections with one cusped edge. The method is certainly not applicable to the evaluation of integrals where F (although bounded) has discontinuities: thus, in the application under discussion, if

$$\zeta'(x) = 2 \operatorname{sgn}(1-2x),$$

corresponding to the double wedge section, the integration formula provides too small an answer by some 10 per cent.

The second application which comes to mind is in the evaluation of the expression for the wave drag of certain classes of slender bodies[2] where in our notation an integral with

$$F(x,y) = S''(x)\, S''(y)$$

appears, $S(x)$ denoting the body cross-section area at a fraction x of the length from the nose. The integration formula quoted here are particularly applicable if $S'(x)$ can be represented by a finite sine series in θ, where $\theta = \cos^{-1}(1-2x)$, because then the function $F(x,y)$ has half-order singularities or zeros along the borders of the region of integration (as is assumed in the method for numerical integration). Many shapes of body, which have been derived to provide minimum drag under stated conditions, fall within this category, and likewise those obtained closely approximating to them. On the other hand many body shapes - such as ellipsoids, or those with parabolic meridian sections - have a cross-sectional area distribution $S(x)$ which is represented by a polynomial in x, and which is equivalent to the expression of $S'(x)$ by a cosine series. Applied to such shapes the integration formula are accurate only within about 1 per cent, even with $n = 6$: this is in direct analogy to the results quoted above for wing sections with angular edges. Likewise, too, the formula are inapplicable where $S''(x)$ is discontinuous (i.e. where the body has discontinuities in curvature).

It is understood that work on similar lines to that under present discussion has been undertaken at the Royal Aircraft Establishment, Farnborough, in relation to this application to the wave drag of slender bodies. The accuracy quoted from their preliminary examples does not seem as close as that obtained from the use of the above formulae: this is probably due to the fact that the double differentiation of $S(x)$ to obtain $S''(x)$ is implicitly accomplished in their technique, as a Fourier series representation of $S(x)$ is used to fit its values at stipulated points, and not of $S''(x)$. Numerical differentiation by Fourier series interpolation is known to be a relatively inaccurate device. This source of error does not appear in the examples quoted here, as no attention has been paid to the problem of derivation of $S''(x)$ given only numerical data for $S(x)$, which of course often arises in practice.

To sum up, in aeronautical applications where $F(x,y)$ is separable as $f'(x)\,f'(y)$, say, the integration formulae have been found quite satisfactory where $f'(x)$ is known and has half-order singularities (or zeros) at $x = 0$ and/or 1. Where $f'(x)$ is finite but non-zero at these end points, an accuracy of about 1 per cent (if the formulae relevant to $n = 6$ are used) is all that can be anticipated.

In the failing cases, which certainly include those where $f'(x)$ is discontinuous at interior points of $0 < x < 1$, the work of Legendre[3] may be of particular use. In this he reduces the double integral of the type under consideration to one with a bounded integrand, together with a line integral (and other terms if $f(x)$ is discontinuous); this new double integral may be treated by Weddle's Rule, or another similar formula. The flexibility of this Rule, in allowing arbitrarily close spacing of lattice points at which the integrand is evaluated, together with the fact that it is $f(x)$, and not $f'(x)$, which appears in the finite integrand derived by Legendre,* commend this method. Weddle's Rule is strictly speaking not in this connection applicable if $f'(x)$ has half-order singularities or zeros at the end points of the range of integration, but as has been pointed out, the method herein described is then quite adequate and probably rather simpler to apply.

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References

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2. G.N. Ward — Supersonic flow past Slender Pointed Bodies. Quart.J.Mech. and App.Maths.Vol.II,Pt.1 (1949)

3. R. Legendre — Limite sonique de la resistance d'ondes d'un aeronef. Comptes Rendus, Tome 236, No.26,p.2479 (1953).

* The result is that $\int_0^1 \int_0^1 f'(x)f'(y)\,ln \left|\frac{1}{x-y}\right| dxdy = \frac{1}{2}\int_0^1 \int_0^1 \left[\frac{f(x)-f(y)}{x-y}\right]^2 dxdy$ and other terms, so that in fact the value of $f'(x)$ is required for the value of the integrand along $x = y$.

APPENDIX

Derivation of Integration Formula

We suppose that the function $F(x,y)$ is expressible in the form:

$$F(x,y) = \sum_{\mu=-n}^{n} \sum_{\nu=-n}^{n} F_{\mu\nu} \frac{\sin 2n\theta \sin 2n\phi}{(\cos\psi_\mu - \cos\theta)(\cos\psi_\nu - \cos\phi)} \quad \ldots\ldots\ldots\ldots(1)$$

where n, μ, and ν are integral, $F_{\mu\nu}$ constant, and

$$\theta = \cos^{-1}(1-2x), \quad \phi = \cos^{-1}(1-2y), \quad \psi_\mu = \frac{\pi}{2} + \mu\frac{\pi}{2n}. \quad \ldots\ldots\ldots\ldots(2)$$

Using the identity:

$$\frac{\sin 2n\theta}{\cos\psi_\mu - \cos\theta} = \frac{2(-1)^{n+\mu}}{\sin\theta} \sum_{k=o}^{2n} b_{k-n} \cos k\psi_\mu \cos k\theta \quad \ldots\ldots\ldots\ldots(3)$$

where $b_\mu = 1$ for $\mu \neq \pm n$, and $b_\mu = \frac{1}{2}$ for $\mu = \pm n$, it will be seen that $\left[\sin\theta \sin\phi\, F(x,y)\right]$ is represented as a finite Fourier double cosine series over the half-periods $(0,\pi)$ in θ and ϕ. By taking n as unbounded the well-known methods of Fourier analysis may be used to show that a large class of functions $F(x,y)$ can be expressed in the chosen form. However in what follows we take n as bounded, and in general any arbitrary function $F(x,y)$ can only be expressed approximately by the expression (1).

Now it can be shown that, if p is any integer

$$\lim_{\theta\to\psi_p} \left(\frac{\sin 2n\theta}{\cos\psi_\mu - \cos\theta}\right) \left.\begin{aligned} &= 2(-1)^{n+\mu}\, n \operatorname{cosec}\psi_\mu, && \text{if } p = \mu \neq \pm n; \\ &= 0, && \text{if } p \neq \mu. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4)$$

Also, if $\mu = \pm n$

$$\left(\frac{\sin 2n\theta}{\cos\psi_\mu - \cos\theta}\right) \sim \frac{4n}{\sin\theta}, \text{ as } \theta \to \psi_\mu. \quad \ldots\ldots\ldots(5)$$

It follows, from (4) and (5) in (1), that

$$F_{\mu\nu} = \frac{(-1)^{\mu+\nu}}{4n^2} \lim_{\theta\to\psi_\mu} \lim_{\theta\to\psi_\nu} \left[b_\mu b_\nu \sin\theta \sin\phi\, F(x,y) \right], \quad \ldots\ldots\ldots\ldots(6)$$

where, as before, $b_\mu = 1$ if $\mu \neq \pm n$, and $b_\mu = \frac{1}{2}$ if $\mu = \pm n$. The limits are of course easily written down unless μ or ν equals $\pm n$. It will be seen that the advantage of the representation (1) lies in the ease with which the coefficients $F_{\mu\nu}$ of the series are expressible in terms of the values of the function it represents. If the value of $F(x,y)$ is known only at the lattice points (s_μ, s_ν), where $\mu,\nu = -n, -n+1, \ldots n-1, n,$ and

$$s_\mu = \tfrac{1}{2}\left(1 + \sin\frac{\mu\pi}{2n}\right), \text{ i.e. } \psi_\mu = \cos^{-1}(1-2s_\mu), \quad \ldots(7)$$

then the right-hand side of (1) may be regarded as an interpolation formula for F, giving its value at all intermediate points.

Let us now put

$$I_\mu(\phi) = \int_0^\pi \frac{\sin\theta \sin 2n\theta}{\cos\psi_\mu - \cos\theta} \ln\left|\frac{2}{\cos\theta - \cos\phi}\right| d\theta \quad (8)$$

Then substituting from (3) and integrating by parts

$$I_\mu(\phi) = 2(-1)^{n+\mu} \int_0^\pi \sum_{k=0}^{2n} b_{k-n} \cos k\psi_\mu \cos k\theta \ln\left|\frac{2}{\cos\theta - \cos\phi}\right| d\theta$$

$$= 2(-1)^{n+\mu} \left[\frac{1}{2}\int_0^\pi \ln\left|\frac{2}{\cos\theta - \cos\phi}\right| d\theta - \right.$$

$$\left. - \sum_{k=1}^{2n} \frac{1}{k} b_{k-n} \cos k\psi_\mu \int_0^\pi \frac{\sin k\theta \sin\theta}{\cos\theta - \cos\phi}\, d\theta \right]$$

i.e. $$I_\mu(\phi) = 2\pi(-1)^{n+\mu}\left[\log 2 + \sum_{k=1}^{2n} \frac{1}{k} b_{k-n} \cos k\psi_\mu \cos k\phi\right] \qquad \ldots\ldots\ldots\ldots(9)$$

But the double integral of which we require the value is

$$I = \int_0^1 \int_0^1 F(x,y)\ ln \left|\frac{1}{x-y}\right| \ dx\, dy \qquad \ldots\ldots\ldots\ldots(10)$$

which on substitution from (1) and (2) becomes

$$I = \frac{1}{4}\int_0^\pi\int_0^\pi \sum_{\mu=-n}^{n}\ \sum_{\nu=-n}^{n} F_{\mu\nu}\ \frac{\sin 2n\theta \sin 2n\phi \sin\theta \sin\phi}{(\cos\psi_\mu - \cos\theta)(\cos\psi_\nu - \cos\phi)} \log\left|\frac{2}{\cos\theta - \cos\phi}\right| d\theta d\phi$$

and from (8)

$$I = \frac{1}{4}\sum_{\mu=-n}^{n}\ \sum_{\nu=-n}^{n} F_{\mu\nu} \int_0^\pi \frac{\sin 2n\phi \sin\phi}{\cos\psi_\nu - \cos\phi}\ I_\mu(\phi)\ d\phi$$

Using the result (9) it follows that

$$I = \frac{\pi}{2}\sum_{\mu=-n}^{n}\ \sum_{\nu=-n}^{n}(-1)^{n+\mu} F_{\mu\nu}\int_0^\pi \left[\log 2 + \sum_{k=1}^{2n}\frac{1}{k} b_{k-n}\cos k\psi_\mu \cos k\phi\right] \times \frac{\sin 2n\phi \sin\phi}{\cos\psi_\nu - \cos\phi}\ d\phi$$

$$= \frac{\pi}{2}\sum_{\mu=-n}^{n}\ \sum_{\nu=-n}^{n}(-1)^{n+\mu} F_{\mu\nu}\int_0^\pi \Big[\log 2.\sin 2n\phi + \sum_{k=1}^{2n}\frac{1}{2k} b_{k-n}\cos k\psi_\mu \qquad (\sin \overline{2n+k}\phi + \sin \overline{2n-k}\phi)\Big] \times \frac{\sin\phi}{\cos\psi_\nu - \cos\phi}\ d\phi$$

i.e. $$I = \frac{\pi^2}{2}\sum_{\pi=-n}^{n}\ \sum_{\nu=-n}^{n}(-1)^{\mu+\nu} F_{\mu\nu}\left[\log 2 + \sum_{k=1}^{2n}\frac{1}{k} b_{k-n}^2 \cos k\psi_\mu \cos k\psi_\nu\right]$$

Using (6), this equation can be written as

$$\left.\begin{aligned} I &= \frac{\pi^2}{8n^2} \sum_{\mu=-n}^{n} \sum_{\nu=-n}^{n} a_{\mu\nu} \lim_{\theta \to \psi_\mu} \lim_{\phi \to \psi_\nu} \left[b_\mu b_\nu \sin\theta \sin\phi \, F(x,y) \right] \\ \text{where} \quad a_{\mu\nu} &= \log 2 + \sum_{k=1}^{2n} \frac{1}{k} b_{k-n}^2 \cos k\psi_\mu \cos k\psi_\nu . \end{aligned}\right\} \quad (11)$$

In particular if $F(x,y)$ is bounded at the edges of the region of integration, from (7),

$$\left.\begin{aligned} I &= \sum_{\mu=-n+1}^{n-1} \sum_{\nu=-n+1}^{n-1} c_{\mu\nu} \, F(s_\mu, s_\nu), \\ \text{where} \quad c_{\mu\nu} &= \frac{\pi^2}{8n^2} a_{\mu\nu} \sin\psi_\mu \sin\psi_\nu \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(12)$$