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An approximate solution to the Swept Wing Root Constraint problem

-by-

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SUMMARY

This report presents an approximate solution to the problem associated with the root constraint in a swept wing.

The structure considered is a uniform rectangular single cell box having closely spaced rigid ribs. Approximate allowance is made for the effect of boom area at the spar-skin joints.

The stress distribution in the skin is represented by a polynomial in the chordwise ordinate, oblique coordinate notation being used. The final equations are derived by use of the theory of minimum strain energy.

Comparison with experimental results has indicated that the theory gives a satisfactory estimation of the constraint effects, and especially the influence of the spar booms.

Consideration of the validity of the boom approximation shows it to be justified in two particular instances having relatively large boom area.

MEP

CONTENTS

SYMBOLS

A	Spar boom area
A_{ij}	Matrix inversions in definition of strains
B	$T_w + T'_w$
$C_i (i=0-3)$	Coefficients in expressions for stress resultants
D	$T_w - T'_w$
E	Youngs Modulus
F_{ij}	Coefficients in strain energy equation
G	Shear Modulus
K_{ij}	Coefficients of equations for C_2 and C_3
L_1, M_1	Oblique couple components
Oxy	System of oblique axes
S	Shear stress resultant in skin
S_w, S'_w	Shear flows in rear and front spar webs respectively
T_1, T_2	Direct stress resultants in skin
T_w, T'_w	Loads in rear and front spar booms respectively
U	Strain energy
Z	Normal shear force
a_R	Rib pitch parallel to x axis
a_S	Stringer pitch parallel to y axis
b	Half depth of box
c	Half width of box
e_{xx} e_{xy}	Direct and shear strains in skin
l	Span of box parallel to x axis
t_w, t'_w	Thickness of rear and front spar webs respectively
α	Included angle between axes Ox and Oy
β	$\dfrac{EA\ A_{11}}{c}$
γ	$\dfrac{EA\ A_{13}}{c}$
λ	$1 + \beta$
ω	$1 + 3\beta$

1. Introduction

The root constraint problem of a swept box is complex. Based on the oblique coordinate theory of Hemp (ref. 1), the solution presented is an approximate one for the case of a uniform rectangular swept box having closely spaced rigid ribs.

The type of solution, which is a strain energy analysis, has been used by Hemp (ref. 2) for a box having zero boom area. In this report an approximate extension to cover the case of discrete booms is given.

2. Description of Box

The box is shown in Fig. 1. The notation used is that of Ref. 1. Skin reinforcement is by stringers and ribs which are both assumed to be distributed and contribute to the skin thickness. The spar webs are capable of carrying only shear loads, and are of unequal thickness. The spar booms are of equal area. The root end is built in.

3. Theory - Initial Assumptions

It is assumed that the skin spanwise direct stress resultant, T_1, can be expressed as a polynomial in the chordwise ordinate, y. Terms containing powers of y greater than two are considered to be negligible. The skin chordwise direct stress resultant, T_2, is assumed to be zero, shear equilibrium being maintained by the ribs.

Stress Distribution

$$\left.\begin{aligned} T_1 &= C_1(x) + 2C_2(x)\frac{y}{c} + 3C_3(x)\left(\frac{y}{c}\right)^2 \\ T_2 &= 0 \end{aligned}\right\} \quad \text{.........(1)}$$

where $C_1(x)$, $C_2(x)$ and $C_3(x)$ are arbitrary functions of x.

Since for equilibrium :- $\dfrac{\partial T_1}{\partial x} + \dfrac{\partial S}{\partial y} = 0$

$$S = -C\left(C_o + \frac{y}{c}\frac{dC_1}{dx} + \frac{y^2}{c^2}\frac{dC_2}{dx} + \frac{y^3}{c^3}\frac{dC_3}{dx}\right) \quad \text{............(2)}$$

where C_o is a function of x.

Equilibrium at the spar boom joints requires :-

$$S_w = -(S)_{y=c} + \frac{\partial T_w}{\partial x}$$

$$S'_w = (S)_{y=-c} + \frac{\partial T'_w}{\partial x}$$

$$\therefore \quad S_w = \frac{\partial T_w}{\partial x} + c\left(C_o + \frac{dC_1}{d\,x} + \frac{dC_2}{d\,x} + \frac{dC_3}{d\,x}\right)$$

$$S'_w = \frac{\partial T'_w}{\partial x} - c\left(C_o - \frac{dC_1}{d\,x} + \frac{dC_2}{d\,x} - \frac{dC_3}{d\,x}\right) \qquad \ldots\ldots\ldots\ldots(3)$$

Overall equilibrium with the applied loading at any given sections requires :-

$$L_1 = 2bc\,(S_w - S'_w) - 2b\int_{-c}^{c} S\;dy\,.$$

and

$$M_1 = 2b\,(T_w + T'_w) + 2b\int_{-c}^{c} T_1\;dy$$

Writing :-

$$\left.\begin{aligned} B &= T_w + T'_w \\ D &= T_w - T'_w \end{aligned}\right. \qquad \ldots\ldots\ldots\ldots(4)$$

and substituting from Equations (1) - (3) :-

$$L_1 = 8bc^2\left(C_o + \frac{2}{3}\,\frac{dC_2}{d\,x}\right) + 2bc\,\frac{dD}{dx} \qquad \ldots\ldots\ldots\ldots(5)$$

$$M_1 = 2b\,\left\{B + 2c\,(C_1 + C_3)\right\} \qquad \ldots\ldots\ldots\ldots(6)$$

Equations (5) and (6) yield:-

$$\left.\begin{aligned} C_o &= \frac{L_1}{8bc^2} - \frac{2}{3}\cdot\frac{dC_2}{d\,x} - \frac{1}{4c}\cdot\frac{dD}{dx} \\ C_1 &= \frac{M_1}{4bc} - C_3 - \frac{B}{2c} \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots(7)$$

Using Equation (7) to eliminate C_o, C_1 in Equations (1)-(3):-

$$\left.\begin{aligned}
T_1 &= \frac{M_1}{4bc} - \frac{B}{2c} + 2C_2 . \frac{y}{c} + C_3 \left(3\frac{y^2}{c^2} - 1\right) \\
S &= -\frac{L_1}{8bc} + \frac{1}{4}\frac{dD}{dx} - \frac{y}{c}\left(\frac{Z}{4b} - \frac{1}{2}\frac{dB}{dx}\right) + c\left(\frac{2}{3} - \frac{y^2}{c^2}\right)\frac{dC_2}{d\,x} + y\left(1 - \frac{y^2}{c^2}\right)\frac{dC_3}{d\,x} \\
S_W &= \frac{L_1}{8bc} + \frac{1}{4}\frac{dD}{dx} + \frac{Z}{4b} + \frac{c}{3}\frac{dC_2}{d\,x} \\
S'_W &= -\frac{L_1}{8bc} - \frac{1}{4}\frac{dD}{dx} + \frac{Z}{4b} - \frac{c}{3}\frac{dC_2}{d\,x}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(8)$$

since $\frac{dM_1}{d\,x} = Z.$

The spar boom load:- $T_W = EA(e_{xx})_{y=c}$

$$T'_W = EA(e_{xx})_{y=-c}$$

and since $e_{xx} = A_{11}T_1 + A_{13}S$ (see Ref. 1):-

$$\frac{B}{EA} = A_{11}\left\{(T_1)_{y=c} + (T_1)_{y=-c}\right\} + A_{13}\left\{(S)_{y=c} + (S)_{y=-c}\right\}$$

$$\frac{D}{EA} = A_{11}\left\{(T_1)_{y=c} - (T_1)_{y=-c}\right\} + A_{13}\left\{(S)_{y=c} - (S)_{y=-c}\right\}$$

Writing
$$\left.\begin{aligned}
\beta &= \frac{EA\ A_{11}}{c} \\
\gamma &= \frac{EA\ A_{13}}{c}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(9)$$

and substituting from Equation (8):-

$$\left.\begin{aligned}
\frac{\gamma^2c^2}{2}\frac{d^2B}{dx^2} - (1+\beta)\,B &= \frac{\gamma L_1}{4\,b} - \frac{\beta M_1}{2\,b} + \frac{\gamma^2c^2}{4b}\cdot\frac{dZ}{dx} - 2\gamma c^2\left(\beta - \frac{1}{3}\right)\frac{dC_2}{d\,x} - 4\beta c C_3 \\
D &= 4\,\beta c\, C_2 + \gamma . c\,\frac{dB}{dx} - \frac{\gamma c . Z}{2b}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(10)$$

Equation (10) reveals the coupling between B and D which is a function of sweep (A_{13}) and boom area (A). Since the presence of this coupling term has the effect of doubling the order of the final equations, and as γ will be small for small boom area, the following assumption will be made in calculating the strain energy:-

$$\gamma^2 . \frac{d^2B}{dx^2} = 0 \qquad \text{.............(11)}$$

The validity of Equation (11) when γ is not small is discussed in §5.

$$\left.\begin{aligned} \text{Thus:-}\quad B &= \frac{\beta M_1}{2b\lambda} - \frac{\gamma L_1}{4b\lambda} + \frac{2\gamma c^2}{3\lambda}(3\beta-1)\frac{dC_2}{dx} + \frac{4\beta c C_3}{\lambda} \\ D &= 4\beta c\, C_2 + \frac{\gamma c Z}{2b}\left(\frac{\beta}{\lambda} - 1\right) + \frac{4\beta\gamma c^2}{\lambda}\frac{dC_3}{dx} \\ &\text{where } \lambda = (1+\beta) \end{aligned}\right\} \text{......(12)}$$

Substitution from Equation (11) into Equation (8) gives:-

$$\left.\begin{aligned} T_1 &= \frac{M_1}{4bc\lambda} + \frac{\gamma L_1}{8bc\lambda} + \frac{2y}{c} . C_2 - \gamma c\frac{(3\beta-1)}{3\lambda} . \frac{dC_2}{dx} + \left\{\frac{3y^2}{c^2} - \frac{\omega}{\lambda}\right\} C_3 \\ S &= -\frac{L_1}{8bc} - \frac{Zy}{4bc\lambda} + \left(\frac{2}{3} + \beta - \frac{y^2}{c^2}\right) . c . \frac{dC_2}{dx} + \left\{\frac{\omega}{\lambda} - \frac{y^2}{c^2}\right\} . y . \frac{dC_3}{dx} \\ S_W &= \frac{L_1}{8bc} + \frac{Z}{4b} + \frac{c}{3} . \omega . \frac{dC_2}{dx} \\ S'_W &= -\frac{L_1}{8bc} + \frac{Z}{4b} - \frac{c}{3} . \omega . \frac{dC_2}{dx} \\ &\text{where } \omega = (3\beta + 1) \end{aligned}\right\} (13)$$

<u>Strain Energy</u>

The strain energy in the skins is given by:-

$$U_S = 2\int_0^l \int_{-c}^{c} \left(\tfrac{1}{2}\, T_1 . e_{xx} + \tfrac{1}{2}\, S e_{xy}\right) dx . dy = \int_0^l \int_{-c}^{c} (A_{11}T_1^2 + 2A_{13}T_1 S + A_{33}S^2)\, dx dy$$

since $e_{xy} = A_{13}T_1 + A_{33}\, S$ (see Ref. 1).

The strain energy in the spar webs:-

$$U_W = 2b \int_0^l \frac{1}{2}\left(\frac{S_W}{t_W}\cdot e_{xz} + \frac{S'_W}{t'_W}\cdot e'_{xz}\right) dx = \frac{b}{G}\int_0^l \left(\frac{S_W^2}{t_W} + \frac{S'^2_W}{t'_W}\right) dx$$

The strain energy in the spar booms:-

$$U_B = 2\int_0^l \frac{1}{2}\left(T_W\cdot(e_{xx})_{y=c} + T'_W(e_{xx})_{y=-c}\right) dx = \frac{1}{EA}\int_0^l (T_W^2 + T'^2_W)\, dx$$

The strain energy in the ribs is zero as the ribs are considered to be rigid.

Since, by the Principle of Minimum Strain Energy, a small arbitrary increase in the stress resultants and internal loads must result in zero change of strain energy for given applied loads.

$$\left.\begin{aligned} \delta U &= \delta(U_S + U_W + U_B) = 0 \\ &= \int_0^l \int_{-c}^{c} \left\{2A_{11}T_1\delta T_1 + 2A_{13}(T_1\delta S + S\delta T_1) + 2A_{33}\, S\delta S\right\} dx.dy \\ &+ \int_0^l \frac{2b}{G}\left(\frac{S_W\delta S_W}{t_W} + \frac{S'_W\delta S'_W}{t'_W}\right) dx + \int_0^l \frac{2}{EA}(T_W\delta T_W + T'_W\delta T'_W)dx \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(14)$$

From Equation (13):-

$$\left.\begin{aligned} \delta T_1 &= \frac{2y}{c}\,\delta C_2 - \gamma c\,\frac{(3\beta-1)}{3\lambda}\,\delta\left(\frac{dC_2}{d\,x}\right) + \left\{\frac{3y^2}{c^2} - \frac{\omega}{\lambda}\right\}\delta C_3 \\ \delta S &= \left(\frac{2}{3} + \beta - \frac{y^2}{c^2}\right)c.\delta\left(\frac{dC_2}{d\,x}\right) + \left\{\frac{\omega}{\lambda} - \frac{y^2}{c^2}\right\} y.\delta\left(\frac{dC_3}{d\,x}\right) \\ \delta S_W &= \frac{c.\omega}{3}\cdot\delta\left(\frac{dC_2}{d\,x}\right) \\ \delta S'_W &= -\frac{c.\omega}{3}\cdot\delta\left(\frac{dC_2}{d\,x}\right) \end{aligned}\right\} \quad \ldots\ldots(15)$$

And from Equation (12):-

$$\left.\begin{aligned}\delta B &= \frac{2\gamma c^2}{3\lambda}(3\beta-1)\,\delta\left(\frac{dC_2}{d\,x}\right) + \frac{4\beta c}{\lambda}\,\delta C_3 \\ \delta D &= 4\beta c.\delta C_2 + \frac{4\beta c^2\gamma}{\lambda}\,\delta\left(\frac{dC_3}{d\,x}\right)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(15)$$

contd.

The evaluation of the individual terms of Equation (14) is given in Appendix 1. The resulting form of the equation is:-

$$\delta U = 0 = \int_0^l \left\{F_{21}\delta C_2 + F_{22}\delta\left(\frac{dC_2}{d\,x}\right) + F_{31}\delta C_3 + F_{32}\delta\left(\frac{dC_3}{d\,x}\right)\right\} dx \quad (16)$$

where F_{21} is a function of C_2, $\frac{dC_3}{d\,x}$, and $\frac{dM_1}{d\,x}$

F_{22} ' ' ' ' $\frac{dC_2}{d\,x}$, C_3, M_1, $\frac{dM_1}{d\,x}$, L_1

F_{31} ' ' ' ' $\frac{dC_2}{d\,x}$, C_3, M_1, L_1

F_{32} ' ' ' ' C_2, $\frac{dC_2}{d\,x}$, $\frac{dC_3}{d\,x}$, $\frac{dM_1}{d\,x}$, L_1

Integrating Equation (16) by parts:-

$$\left.\begin{aligned}\delta U &= 0 \\ &= \Big[F_{22}\delta C_2 + F_{32}\delta C_3\Big]_0^l + \int_0^l \left[\left\{F_{21} - \frac{d(F_{22})}{dx}\right\}\delta C_2 + \left\{F_{31} - \frac{d(F_{32})}{dx}\right\}\delta C_3\right] dx \\ &\text{and tip condition gives: } - (C_2)_{x=l} = (C_3)_{x=l} = 0 \\ &\text{therefore } (\delta C_2)_{x=l} = (\delta C_3)_{x=l} = 0 \\ &\text{but otherwise arbitrary}\end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots(17)$$

4. Results

Rewriting Equation (17) to give equations for C_2 and C_3 :-

$$\left.\begin{aligned}
&K_{21}\frac{d^2C_2}{dx^2}+K_{22}C_2+K_{23}\frac{dC_3}{d\,x}=+K_{24}Z+K_{25}\frac{dZ}{dx}+K_{26}\frac{dL_1}{d\,x}\\
&K_{31}\frac{d^2C_3}{dx^2}+K_{33}C_3+K_{32}\frac{dC_2}{d\,x}=+K_{35}\frac{dZ}{dx}\\
&\text{and boundary conditions:-}\\
&K_{21}\left(\frac{dC_2}{d\,x}\right)_{x=0}+K'_{23}(C_3)_{x=o}=+K'_{24}(M_1)_{x=o}+K_{25}(Z)_{x=o}+K_{26}(L_1)_{x=o}\\
&K_{31}\left(\frac{dC_3}{d\,x}\right)_{x=o}+K'_{32}(C_2)_{x=o}=+K_{35}(Z)_{x=o}\\
&(C_2)_{x=l}=(C_3)_{x=l}=0
\end{aligned}\right\}\quad(18)$$

where

$$K_{21}=\frac{A_{33}c}{5}(15\beta^2+10\beta+3)+\frac{b\omega^2}{6\,G}\left(\frac{1}{t_w}+\frac{1}{t'_w}\right)-A_{13}\gamma c(3\beta-1)$$

$$K_{22}=-\frac{4A_{11}\omega}{c}$$

$$K_{23}=-\frac{8A_{13}}{5\lambda}\left(\frac{15}{2}\beta^2+6\beta+1\right)\qquad K_{32}=\frac{8A_{13}}{15\lambda}\left(\frac{15}{2}\beta^2+6\beta+1\right)$$

$$K'_{23}=-\frac{4A_{13}}{5\lambda}(6\beta+1)\qquad K'_{32}=\frac{4A_{13}}{15\lambda}(15\beta^2+6\beta+1)$$

$$K_{24}=-\frac{3A_{13}\omega}{4b\lambda c}\qquad K'_{24}=-\frac{A_{13}\omega}{4b\lambda c}$$

$$K_{25}=-\frac{\omega}{8Gc}\left(\frac{1}{t_w}-\frac{1}{t'_w}\right)$$

$$K_{26}=\frac{A_{33}\omega}{8bc}-\frac{\omega}{16Gc^2}\left(\frac{1}{t_w}+\frac{1}{t'_w}\right)-\frac{A_{13}\gamma\omega}{8bc\,\lambda}$$

$$K_{31}=A_{33}c\left(\frac{\omega^2}{3\lambda^2}-\frac{2\omega}{5\lambda}+\frac{1}{7}\right)+\frac{4A_{13}\beta^2\gamma c}{\lambda^2}$$

$$K_{33}=-\frac{4A_{11}}{5\lambda c}(6\beta+1)$$

$$K_{35}=\frac{A_{33}(6\beta+1)}{30b\lambda^2}+\frac{A_{11}\gamma^2}{2b\lambda^2}$$

$$\beta = \frac{EA\ A_{11}}{c} \qquad \gamma = \frac{EA\ A_{13}}{c}$$

$$\lambda = (1+\beta) \qquad \omega = (1+3\beta)$$

Equation (18) enables the values of C_2 and C_3 to be found and hence completes the solution.

5. Comments

Validity of Approximation

In order to derive Equation (18) from the initial assumption of Equation (1) it has been found necessary to introduce the further assumption of Equation (11) that $\gamma^2 \frac{d^2B}{dx^2}$ is zero. It will be seen that this is correct when γ is zero, that is when the box is unswept or when there are no spar booms. The values of the K_{ij} for these two special cases are given in Appendix 2.

For $\gamma^2 \frac{d^2B}{dx^2}$ to be negligible when there is sweep and relatively large boom area, $\frac{d^2B}{dx^2}$ must be small compared to B as γ itself will not necessarily be small. The degree to which this is sufficiently true is not immediately apparent. Accordingly two cases (ref. 3) have been analysed where γ is relatively large.

The first of these two cases is a 45° swept box having a boom area such that $\frac{A}{ct} = 0.34$. Considering the case of loading by a normal shear force applied at the tip, with $L_1 = 0$, and assuming $C_3 = 0$, it was found from the theory that:-

$$\frac{\gamma^2 c^2}{2} \cdot \frac{d^2B}{dx^2} = 0.04\ \lambda B, \text{ at the root } (x=0).$$

In this case $\gamma = 0.75$.

A 60° swept wing was the subject of the second example, and here, with $A/ct = 0.266$ it was found, for similar loading that:-

$$\frac{\gamma^2 c^2}{2} \cdot \frac{d^2B}{dx^2} = 0.006\ \lambda B, \text{ at the root.}$$

where $\gamma = 0.60$.

These examples thus indicate that Equation (11) also holds for relatively large γ, and in fact good agreement with experimentally derived results was obtained in these cases (ref. 3).

Unswept Case

Appendix 2 §2 shows that in the unswept case, all coupling terms between C_2 and C_3 of Equation (18) are zero. The equation for C_2 is thus explicit and gives torsion constraint plus the effect of unequal webs. The equation for C_3 gives shear lag effect.

Thus the method is of value in considering the effect of the distribution of end load carrying capacity in unswept boxes.

REFERENCES

1.	Hemp, W.S.	On the application of oblique coordinates to the problems of plane elasticity and swept wing structures. R. and M. 2754 (Jan. 1950).
2.	Hemp, W.S.	Constraint effects on a swept box. Unpublished (May 1953).
3.	Griffin, K.H., and Howe, D.	Constraint effects in box beams. Lecture to British Association, 1955. (To be published in Engineering).

APPENDIX 1

Evaluation of the individual components of Equation (14)

$$\int_{-c}^{c} T_1 \delta T_1 \, dy = \frac{8c}{3} C_2 \delta C_2 - \gamma c \frac{(3\beta-1)}{6b\lambda^2}\left(M_1 + \frac{\gamma L_1}{2}\right)\delta\left(\frac{dC_2}{d\,x}\right) + \frac{2\gamma^2 c^3}{9\lambda^2}(3\beta-1)^2 \frac{dC_2}{d\,x}\,\delta\left(\frac{dC_2}{d\,x}\right)$$

$$+ \frac{2\gamma c^2}{3\lambda}(3\beta-1)\left(\frac{\omega}{\lambda}-1\right)C_3\delta\left(\frac{dC_2}{d\,x}\right) + \frac{1}{2b\lambda}\left(M_1+\frac{\gamma L_1}{2}\right)\left(1-\frac{\omega}{\lambda}\right)\delta C_3 + \frac{2\gamma c^2}{3\lambda}(3\beta-1)\left(\frac{\omega}{\lambda}-1\right)\frac{dC_2}{dx}\cdot\delta C_3$$

$$+ 2c\left(\frac{9}{5} - \frac{2\omega}{\lambda} + \frac{\omega^2}{\lambda^2}\right)C_3\delta C_3$$

$$\int_{-c}^{c} T_1 \delta S \, dy = \frac{c\omega}{6b\lambda}\left(M_1+\frac{\gamma L_1}{2}\right)\delta\left(\frac{dC_2}{d\,x}\right) - \frac{2\gamma c^3}{9\lambda}(9\beta^2-1)\frac{dC_2}{d\,x}\,\delta\left(\frac{dC_2}{d\,x}\right)$$

$$+ 2c^2\left(\frac{1}{15}+\beta-\frac{\omega^2}{3\lambda}\right)C_3\delta\left(\frac{dC_2}{d\,x}\right) + 4c^2\left(\frac{\omega}{3\lambda}-\frac{1}{5}\right)C_2\delta\left(\frac{dC_3}{d\,x}\right)$$

$$\int_{-c}^{c} S\delta T_1 \, dy = -\frac{Zc}{3b\lambda}\delta C_2 + 4c^2\left(\frac{\omega}{3\lambda}-\frac{1}{5}\right)\frac{dC_3}{d\,x}\delta C_2 + \frac{\gamma c L_1(3\beta-1)}{12\,b\lambda}\,\delta\left(\frac{dC_2}{d\,x}\right)$$

$$-\frac{2\gamma c^3}{9\lambda}(9\beta^2-1)\frac{dC_2}{d\,x}\,\delta\left(\frac{dC_2}{d\,x}\right) + \frac{L_1}{4b}\left(\frac{\omega}{\lambda}-1\right)\delta C_3 + 2c^2\left(\frac{1}{15}+\beta-\frac{\omega^2}{3\lambda}\right)\frac{dC_2}{d\,x}\delta C_3$$

$$\int_{-c}^{c} S\delta S \, dy = -\frac{L_1 c\omega}{12\,b}\,\delta\left(\frac{dC_2}{d\,x}\right) + 2c^3\left(\beta^2+\frac{2\beta}{3}+\frac{1}{5}\right)\frac{dC_2}{d\,x}\,\delta\left(\frac{dC_2}{d\,x}\right)$$

$$-\frac{Zc^2}{2b\lambda}\left(\frac{\omega}{3\lambda}-\frac{1}{5}\right)\delta\left(\frac{dC_3}{d\,x}\right) + 2c^3\left(\frac{\omega^2}{3\lambda^2}-\frac{2\omega}{5\lambda}+\frac{1}{7}\right)\frac{dC_3}{d\,x}\,\delta\left(\frac{dC_3}{d\,x}\right).$$

$$\int_{0}^{l}(T_W\delta T_W + T'_W\delta T'_W)\,dx = \frac{1}{2}\int_{0}^{l}(B\delta B + D\delta D)\,dx\,.$$

APPENDIX 2

Special Cases

1. Boom area zero.

i.e. $\omega = \lambda = 1$

$\beta = \gamma = 0$

$$K_{21} = \frac{3A_{33}c}{5} + \frac{b}{6G}\left(\frac{1}{t_w} + \frac{1}{t'_w}\right) \qquad K_{22} = -\frac{4A_{11}}{c}$$

$$K_{23} = -\frac{8A_{13}}{5} \qquad K_{32} = \frac{2A_{13}}{3}$$

$$K'_{23} = -\frac{4A_{13}}{5} \qquad K'_{32} = \frac{A_{13}}{3}$$

$$K_{24} = -\frac{3A_{13}}{4bc} \qquad K'_{24} = -\frac{A_{13}}{4bc}$$

$$K_{25} = -\frac{1}{8Gc}\left(\frac{1}{t_w} - \frac{1}{t'_w}\right) \qquad K_{26} = \frac{A_{33}}{8bc} - \frac{1}{16Gc^2}\left(\frac{1}{t_w} + \frac{1}{t'_w}\right)$$

$$K_{31} = \frac{2A_{33}c}{21} \qquad K_{33} = -\frac{A_{11}}{c}$$

$$K_{35} = \frac{A_{33}}{24b}$$

2. Zero Sweep

i.e. $\gamma = A_{13} = 0$

$$K_{21} = \frac{A_{33}c}{5}(15\beta^2+10\beta+3) + \frac{b\omega^2}{6\ G}\left(\frac{1}{t_w} + \frac{1}{t'_w}\right) \qquad K_{22} = -\frac{4A_{11}\omega}{c}$$

$$K_{23} = K_{32} = K'_{23} = K'_{32} = K_{24} = K'_{24} = 0$$

$$K_{25} = -\frac{\omega}{8Gc}\left(\frac{1}{t_w} - \frac{1}{t'_w}\right) \qquad K_{26} = \frac{A_{33}\omega}{8bc} - \frac{\omega}{16Gc^2}\left(\frac{1}{t_w} + \frac{1}{t'_w}\right)$$

$$K_{31} = A_{33}c\left(\frac{\omega^2}{3\lambda^2} - \frac{2\omega}{5\lambda} + \frac{1}{7}\right) \qquad K_{33} = -\frac{4A_{11}}{5\lambda c}(6\beta+1)$$

$$K_{35} = \frac{A_{33}(6\beta+1)}{30b\lambda^2}$$

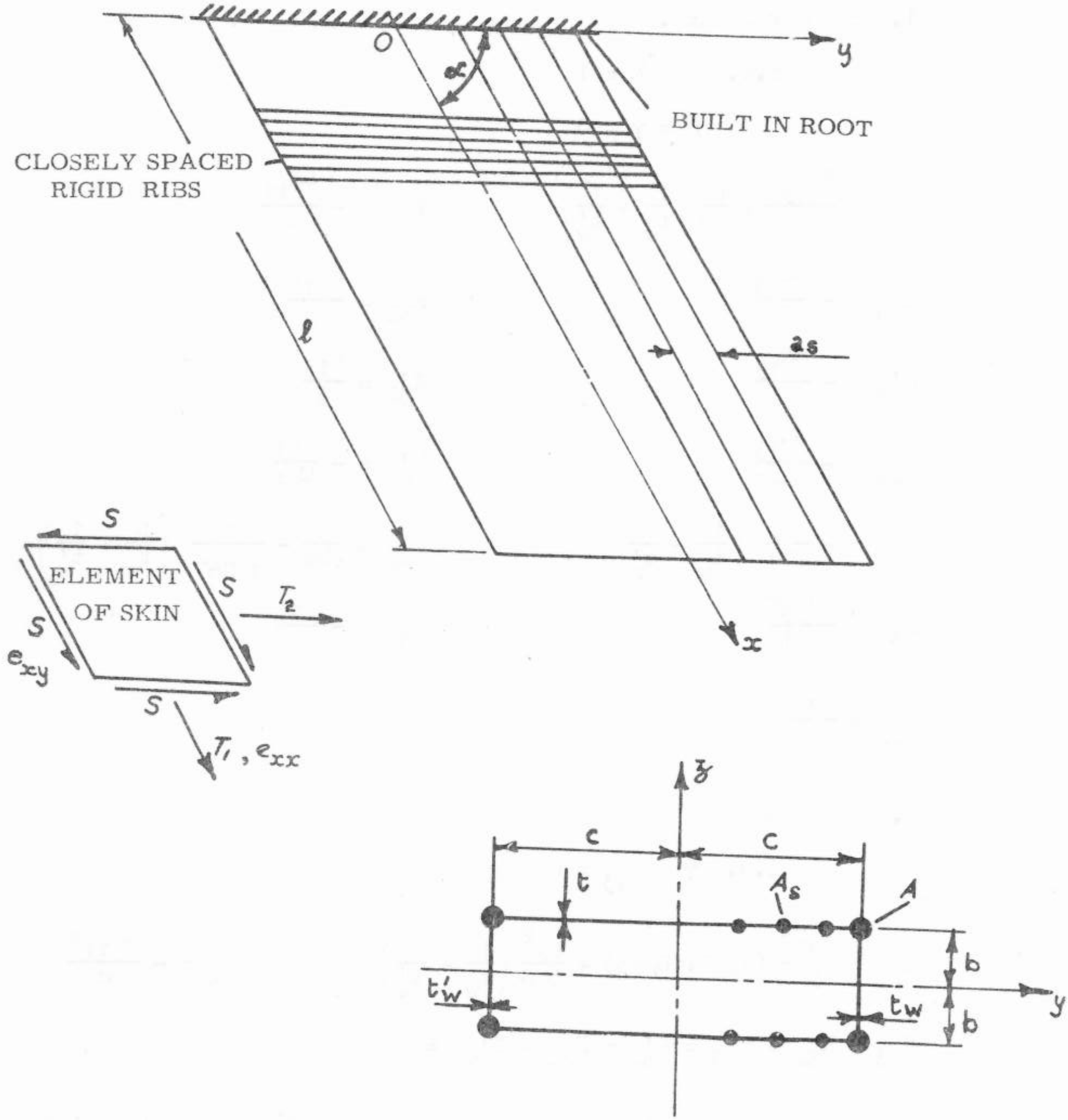

FIGURE 1. SWEPT BOX