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On the synthesis of 3-terminal RC networks*

by

K.M. Adams, B.Sc., M.Sc., D.C.Ae.

Summary

This report is concerned with the synthesis of 3-terminal RC networks according to three specified network functions. Some necessary conditions for physical realizability are derived from first principles by an algebraic method. The possibility of synthesis is shown to depend upon a certain inequality but the weakest conditions required in order to satisfy it are not yet known.

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Notation

v_α	the voltage rise from node o to node α
i_α	the current entering node α from sources external to the network
$i_{\alpha\beta}$	the current flowing from node α to node β in the branch joining nodes α and β
$V_\alpha, I_\alpha, I_{\alpha\beta}$	the Laplace transforms of v_α, i_α, $i_{\alpha\beta}$ respectively
t	time
λ	the complex frequency, independent variable of the Laplace transform
λ'	a linear or bilinear function of λ
$\underline{C}$	the network capacitance matrix
$\underline{G}$	the network conductance matrix
$P =$	$\underline{G} + \lambda\underline{C}$
$\underline{Z}$	the three pole (Strecker-Feldtkeller) impedance matrix
$\underline{A}, \underline{Q}$	R-matrices
$\underline{E}, \underline{M}, \underline{F}$	square matrices
$\underline{E}'$	the transpose of $\underline{E}$
$\det\underline{E}$	the determinant of $\underline{E}$
$\underline{E}^{(k)}$	the k^{th} compound of $\underline{E}$
$\mathrm{adj}^{(k)}\underline{E}$	the k^{th} adjugate compound of $\underline{E}$
$(\mathrm{adj}\underline{E})^{(k)}$	the k^{th} compound of the adjugate of $\underline{E}$
$\underline{E}_{ij}^{(k)}$	the element in the i^{th} row, j^{th} column of $\underline{E}^{(k)}$
$\dotplus$	the direct sum
$\underline{1}_k$	unit matrix of order k
$\underline{K}, \underline{K}_i, \underline{Y}$	transformation matrices
$U_k =$	$P_{11}^{(k)}$
$W_k =$	$P_{11}^{(k-1)}$

$\delta, \delta_i, \varpi, \varpi_i, R_{ij}, \varepsilon_i, \varepsilon, a, b, m$ constants

$f, f_i, g, H, H_i, \omega, \omega_i, r, r_i, S, T, h$ polynomials in λ

$(H)_{k-1}$, $(f)_{k-1}$ etc. the polynomials corresponding to H, f etc. in the succeeding cycle

λ_i	zeros of polynomials in λ or λ'
$\underline{\Omega}$	a diagonal matrix
$U \gg 0$	the coefficients of powers of λ in the polynomial U are all non-negative
$U//W$	the poles of $\frac{U}{W}$ separate the zeros of $\frac{U}{W}$
$f_i \overset{c}{=} r_i (\text{mod.} U_k)$	f_i is congruent with r_i, modulo U_k.

On the synthesis of 3-terminal RC networks

Introduction

Rapid developments in the fields of communications and servomechanisms have in recent years emphasised the need for a study of networks containing no transformers. Servomechanism engineers have been specially interested in transfer function synthesis, and since their requirements have generally been for low frequency characteristics, resistance capacitance networks have received particular attention. In 1952 Fialkow and Gerst (Ref. 3) gave a general solution of the problem of realizing a given transfer function by means of either a three terminal or two terminal-pair resistance capacitance network; and two years later (Ref. 4) succeeded in extending their method to give a realization by means of networks composed of resistances, capacitances, and (self) inductances.

On the other hand, for communications purposes it is often desired to realize a network not only for a specified transfer function, but also for specified input and output impedances (or admittances). Furthermore, three terminal networks are of particular engineering interest in that the input and output have a common terminal which it is often convenient to earth.

The present work is concerned with the synthesis of such networks. It is assumed that the input, output and transfer impedances are given, and the problem then is to obtain a realization by means of a network containing two kinds of elements only. Some necessary conditions for physical realizability have been derived from first principles by an algebraic method. These conditions are themselves generally well known, but the method of derivation was adopted because it suggested the approach to synthesis. The possibility of synthesis is shown to depend upon a certain inequality, but it is not yet known what are the weakest conditions required in order to satisfy it. By way of illustration two numerical examples are worked out; it is shown that for each of them the method is capable of yielding an infinity of solutions.

1. Analytical Preliminaries

Consider a network containing two kinds of elements and having n+1 nodes. Three of these nodes are to be regarded as input, output, and common earth respectively. For the basic case we shall take the two kinds of elements to be resistance and capacitance; other combinations may be obtained by a simple transformation of the basic case, as will be shown in Sect. 4. Let the nodes be numbered so that the input is node n, output n-1, and earth (reference) node 0.

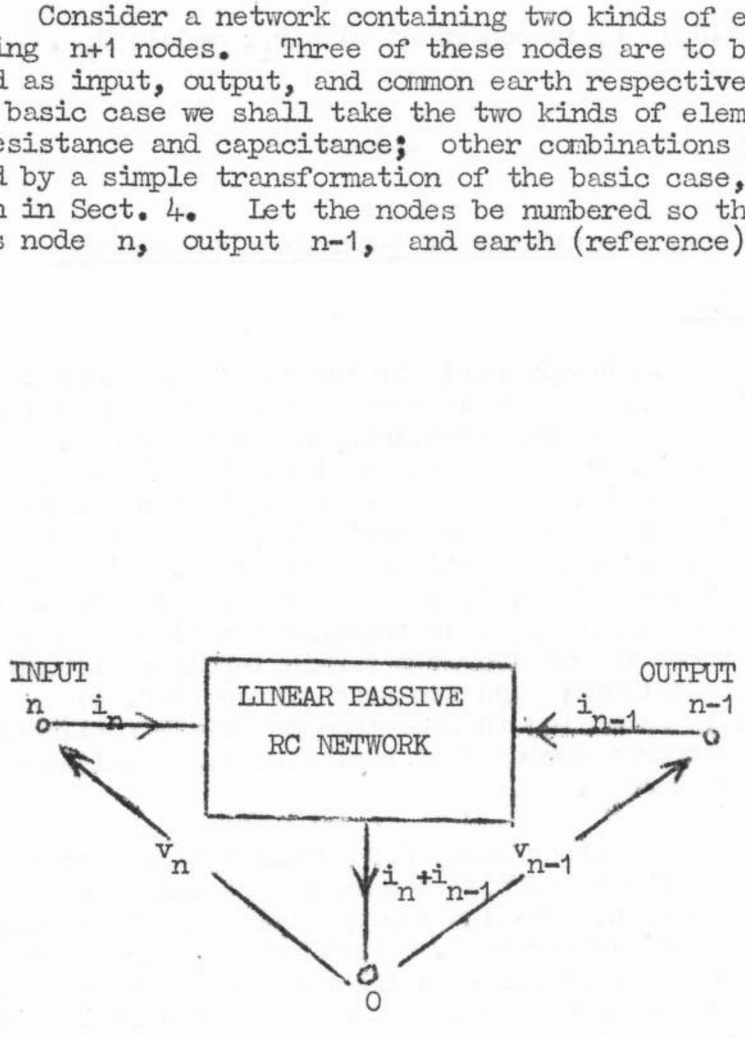

Fig. 1.1

Take the positive direction of voltage and current as shown in Fig. 1.1. Denote the branch joining nodes α and β by $(\alpha\beta)$. Then the equation for the current in $(\alpha\beta)$ is

$$C_{\alpha\beta} \frac{d}{dt} (v_\alpha - v_\beta) + G_{\alpha\beta} (v_\alpha - v_\beta) = i_{\alpha\beta}^{*} \quad \text{.....(1.1)}$$

* Summation on repeated suffices is not implied.

where

$C_{\alpha\beta}$ is the capacitance in $(\alpha\beta)$

$G_{\alpha\beta}$ is the conductance in $(\alpha\beta)$

$i_{\alpha\beta}$, a function of time, is the current in $(\alpha\beta)$, the positive direction being from node α to node β

v_α a function of time, is the voltage rise from node 0 to node α.

$$\ldots\ldots\ldots\ldots(1.2)$$

Now take the 'λ-multiplied' Laplace transform of (1.1) defined by

$$V(\lambda) = \lambda \int_0^\infty v(t)\, e^{-\lambda t}\, dt \qquad \ldots\ldots\ldots\ldots(1.3)$$

and neglecting the initial conditions, obtain

$$(\lambda C_{\alpha\beta} + G_{\alpha\beta})\,(V_\alpha - V_\beta) = I_{\alpha\beta} \qquad \ldots\ldots\ldots\ldots(1.4)$$

(The more usual s or p symbol for the Laplace transform **variable** has here been replaced by λ to agree with the established notation for complex frequency and for λ-matrices).

When Kirchhoff's current law, in the form

$$\sum_{\substack{\beta=0 \\ \beta\neq\alpha}}^{n} I_{\alpha\beta} = I_\alpha \qquad \alpha = 1, \ldots\ldots, n\,, \qquad \ldots\ldots\ldots\ldots(1.5)$$

where I_α is the current entering node α from sources external to the network, is applied to (1.4), there results:

$$\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ I_{n-1} \\ I_n \end{bmatrix} = \underline{P} \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ V_{n-1} \\ V_n \end{bmatrix} \qquad \ldots\ldots\ldots\ldots(1.6)$$

where $P_{\alpha\beta}$ (the element in row α, column β of $\underline{P}$) $= P_{\beta\alpha}$

$$= -(\lambda C_{\alpha\beta} + G_{\alpha\beta}) \quad \alpha \neq \beta$$

$$P_{\alpha\alpha} = \sum_{\substack{\beta=0 \\ \beta\neq\alpha}}^{n} (\lambda C_{\alpha\beta} + G_{\alpha\beta}) \qquad \ldots\ldots\ldots\ldots(1.7)$$

From lemma 1, it follows that $\underline{P}$ is positive definite for all $\lambda > 0$, if and only if the linear graph * of the network consists of only one separable part. In what follows we shall suppose this condition is satisfied so that the Strecker-Feldtkeller impedance matrix always exists. The treatment of the degenerate case, in which $\underline{P}$ is singular, calls for some device such as the scattering matrix, which, however, will not be discussed here.

Under the conditions of lemma 1, then, it is now valid to write

$$\begin{bmatrix} V_n \\ V_{n-1} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_n \\ I_{n-1} \end{bmatrix} \qquad \ldots\ldots\ldots\ldots(1.9)$$

where

$$Z_{11} = P_{11}^{(n-1)} / P_{11}^{(n)}$$

$$Z_{12} = Z_{21} = -P_{12}^{(n-1)} / P_{11}^{(n)}$$

$$Z_{22} = P_{22}^{(n-1)} / P_{11}^{(n)} \qquad \ldots\ldots\ldots\ldots(1.10)$$

where $P_{ij}^{(k)}$ is the element in row i, column j, of the k^{th} compound of the matrix $\underline{P}$ ** .

If the network is now regarded as a 'black-box' with three accessible terminals, then its properties are determined completely by the functions Z_{11}, Z_{12}, Z_{22}. The actual performance will of course depend on the values of current and voltage at time $t = 0$; but these will have no effect on the circuit parameters of a linear network. It will henceforth be assumed that all physical information has been reduced to

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* For a more detailed discussion see Ref. 5.

** A discussion of compound matrices may be found in Ref. 1 ch. 5.

a form expressible in terms of the Z functions; clearly the Z functions must be rational functions.

2. R-Matrices

The discussion in this section is purely algebraic, but is of such importance to the derivation of the synthesis algorithm that it has been included in the main body of the text and not relegated to an appendix.

The concept of R-matrix was introduced into network theory in 1937 by Burington (Ref. 2); it provides a method of formulating purely topological properties in algebraic terms.* The argument advanced here hinges on the key theorem of Sect. 2.2. Some of Fialkow and Gerst's results are obtained as corollaries of this theorem, in Sect. 2.4.

2.1. Definition

If $\underline{A}$ is matrix, with elements A_{ij}, such that

(i) $\underline{A}$ is square, of order n x n, say

(ii) $A_{ij} = A_{ji} \leqslant 0$
$\quad = - R_{ij} = - R_{ji}$, say $\qquad (i \neq j)$

(iii) $A_{ii} \geqslant 0$
$\quad = R_{ii}$, say

(iv) $\sum_{j=1}^{n} A_{ij} \geqslant 0 \qquad i = 1, ... n$
$\quad = \delta_i$, say $\qquad$ ....(2.1)

then $\underline{A}$ is said to be an R-matrix. This definition differs from that of Burington who instead employed the condition **

$$\sum_{j=1}^{n} |A_{ij}| \leqslant 2 A_{ii} \qquad ....(2.2)$$

Conditions (2.1) are implicit in the work of Fialkow and Gerst, and will be found more manageable than (2.2). It will now be

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* Ref. 5, Part (ii).

** This arose from his use of mesh instead of nodal equations.

clear from (1.7) that $\underline{C}$, $\underline{G}$, and $\underline{P}$, for $\lambda \geq 0$, are all R-matrices provided all resistors and capacitors are positive - a condition certainly satisfied by passive networks. Moreover, if $\underline{P}$ is an R-Matrix for all $\lambda \geq 0$, then $\underline{P}$ is the admittance matrix of a physically realizable network.

2.2. Theorem

If $\underline{A}$ is an R-matrix, and $A_{11} = R_{11} \neq 0$, then $\underline{Q} = \underline{K}_1' \, \underline{A} \, \underline{K}_1$ is an R-matrix where

$$\underline{K}_1 = \begin{bmatrix} 1 & R_{12}/R_{11} & R_{13}/R_{11} & \cdots & R_{1n}/R_{11} \\ 0 & 1 & 0 & \cdots & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & & \cdot\cdot \\ 0 & 0 & 0 & & 1 \end{bmatrix} \qquad \ldots .(2.3)$$

and $\underline{K}_1'$ denotes the transpose of $\underline{K}_1$

Proof

By simple multiplication, Q is seen to be

$$\begin{bmatrix} R_{11} & 0 & 0 & & 0 \\ 0 & (R_{22} - \frac{R_{12}\,R_{12}}{R_{11}}) & -(R_{23} + \frac{R_{12}\,R_{13}}{R_{11}}) & \cdots & -(R_{2n} + \frac{R_{12}\,R_{1n}}{R_{11}}) \\ 0 & -(R_{23} + \frac{R_{12}\,R_{13}}{R_{11}}) & (R_{33} - \frac{R_{13}\,R_{13}}{R_{11}}) & \cdots & -(R_{3n} + \frac{R_{13}\,R_{1n}}{R_{11}}) \\ \cdot\cdot & \cdots & \cdots & & \cdots \\ 0 & -(R_{2n} + \frac{R_{12}\,R_{1n}}{R_{11}}) & -(R_{3n} + \frac{R_{13}\,R_{1n}}{R_{11}}) & \cdots & (R_{nn} - \frac{R_{1n}\,R_{1n}}{R_{11}}) \end{bmatrix}$$

i.e.

$$
\underline{Q} = \begin{bmatrix}
A_{11} & 0 & 0 & \cdots & P \\
0 & A_{11}^{(2)}/A_{11} & A_{12}^{(2)}/A_{11} & \cdots & A_{1,n-1}^{(2)}/A_{11} \\
0 & A_{12}^{(2)}/A_{11} & A_{22}^{(2)}/A_{11} & \cdots & A_{2,n-1}^{(2)}/A_{11} \\
\cdot\cdot & \cdot\cdot & \cdot\cdot & & \cdot\cdot \\
0 & A_{1,n-1}^{(2)}/A_{11} & A_{2,n-1}^{(2)}/A_{11} & \cdots & A_{n-1,n-1}^{(2)}/A_{11}
\end{bmatrix}
$$

............(2.4)

then

(i) $\underline{Q}$ is square

(ii) $Q_{ij} = Q_{ji} = -\frac{1}{R_{11}} (R_{11} R_{ij} + R_{1i} R_{1j}) \leq 0$

$i \neq j,$

$i, j = 2, \ldots n,$

in virtue of (ii) and (iii) of (2.1)

$Q_{ij} = 0$, if either i or $j = 1$, $i \neq j$.

(iii) $Q_{ii} = (1/R_{11})(R_{11}R_{ii} - R_{1i}R_{1i}) \geq 0, \quad i > 1,$

Since from (2.1), (iv)

$R_{11} \geq R_{1i}$

$R_{ii} \geq R_{1i}$

$Q_{11} = R_{11} > 0.$

(iv) $\sum_{j=1}^{n} Q_{ij} = \sum_{j=2}^{n} (A_{ij} - R_{1i}R_{1j}/R_{11}), \; i, j \neq 1$

$= \delta_i + R_{1i} + (R_{1i}/R_{11})(\delta_1 - R_{11})$, from (2.1), (iv)

$= \delta_i + R_{1i}\,\delta_1/R_{11}$ ....(2.5)

≥ 0, in virtue of (2.1), (iv).

Hence $\underline{Q}$ is an R-matrix.

Corollary

Let $\underline{A}$ be positive definite, and let $\underline{K}_k$ be defined as*

$$\underline{1}_{k-1} \dotplus \begin{bmatrix} 1 & -A_{12}^{(k)}/A_{11}^{(k)} & -A_{13}^{(k)}/A_{11}^{(k)} & \cdots & -A_{1,n-k+1}^{(k)}/A_{11}^{(k)} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

then

$$\underline{K}'_{k-1}\ \underline{K}'_{k-2} \cdots \underline{K}'_1\ \underline{A}\ \underline{K}_1\ \underline{K}_2 \cdots \underline{K}_{k-1}$$

$$= \text{diag} \begin{bmatrix} A_{11} \\ A_{11}^{(2)}/A_{11} \\ A_{11}^{(3)}/A_{11}^{(2)} \\ \cdots \\ A_{11}^{(k-1)}/A_{11}^{(k-2)} \end{bmatrix} \dotplus 1/A_{11}^{(k-1)} \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} & \cdots & A_{1,n-k+1}^{(k)} \\ A_{12}^{(k)} & A_{22}^{(k)} & \cdots & A_{2,n-k+1}^{(k)} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1,n-k+1}^{(k)} & & \cdots & A_{n-k+1,n-k+1}^{(k)} \end{bmatrix}$$

is an R-matrix.

This follows immediately by induction from the proof of the main theorem, and in virtue of lemma 2.

The condition '$\underline{A}$ is positive definite' ensures that

$$A_{11}^{(k-1)} \neq 0 \qquad k = 2, \ldots, n.$$

2.3 Inverse $\underline{K}$ Transformations

Suppose now that the R-matrix $\underline{Q}$ is given. The question then arises, how may an R-matrix $\underline{A}$ be obtained from $\underline{Q}$ such that $A_{ij}^{(2)}/A_{11} = Q_{i+1,j+1}$ $i, j = 1, \ldots, n-1$? (2.5) provides the clue.

* The symbol $\dotplus$ denotes the direct sum. $\underline{1}_{k-1}$ is unit matrix of order k-1.

Let $\varpi_i = \sum_{j=2}^{n} Q_{ij}$

$$= \delta_i + R_{1i}\,\delta_1/R_{11} \qquad \ldots\ldots\ldots\ldots(2.5)$$

also $$\sum_{i=2}^{n} R_{1i} = R_{11} - \delta_1\ , \qquad \ldots\ldots\ldots\ldots(2.6)$$

from (2.1), (iv) and (ii).

We are given ϖ_i and R_{11} ; we wish to determine R_{1i}, δ_i, δ_1, all positive, such that

$$Q_{ij} + (R_{1i}R_{1j})/R_{11} \leqslant 0,\ i \neq j \qquad \ldots\ldots\ldots\ldots(2.7)$$

$$i,\ j = 2,\ \ldots,n.$$

Summation on i in (2.5) yields, in virtue of (2.6),

$$-\sum_{i=2}^{n} \varpi_i = -\sum_{i=2}^{n} \delta_i + \delta_1^2/R_{11} - \delta_1$$

or putting

$$\sum_{i=2}^{n} \varpi_i = \varpi, \text{ and } \sum_{i=2}^{n} \delta_i = \delta \qquad \ldots\ldots\ldots\ldots(2.8)$$

obtain

$$\delta_1^2 - R_{11}\,\delta_1 + R_{11}\,(\varpi - \delta) = 0 \qquad \ldots\ldots\ldots\ldots(2.9)$$

In order to satisfy (2.7), suppose we now demand that R_{1i} is less than the minimum value of

$$(-R_{11}\,Q_{ij})^{\frac{1}{2}} = m\,R_{11}, \text{ say } i,\ j = 2,\ \ldots\ n,\ i \neq j \qquad \ldots.(2.10)$$

then $$\delta_1 \geq R_{11} - m(n-1)\,R_{11} \text{ from } (2.6) \qquad \ldots\ldots\ldots\ldots(2.11)$$

and $$\delta_1/R_{11} = 1/2\ \ 1 \pm \left[1 - 4(\varpi-\delta)/R_{11}\right]^{1/2} \text{ from } (2.9) \qquad \ldots\ldots\ldots\ldots(2.12)$$

(2. 1) and (2.12) now imply

$$1 - m\,(n-1) \leqslant 1/2\,\left\{1 \pm (1-4\,(\varpi-\delta)/R_{11})^{1/2}\right\}$$

i.e. either $1-4\,(\varpi-\delta)/R_{11} \geqslant (1 - 2\,m\,(n-1)\,)^2$, if $m(n-1) < 1/2$

or $\quad 1 - 4\,(\omega-\delta)/R_{11} \leq (1 - 2\,m\,(n-1)\,)^2$, if $m(n-1) > 1/2$

subject to the condition $1 - 4\,(\omega-\delta)/R_{11} \geqslant 0$ ....(2.13)

Hence a solution always exists; in fact as (2.13) shows an infinity of solutions can be obtained. δ can be suitably chosen, thus fixing δ_1, as a result of (2.12). Finally R_{1i} is chosen to satisfy (2.6) and (2.10).

This then completes the inverse $\underline{K}_1$ transformation. Inverse $\underline{K}_k$ transformations can be similarly treated.

2.4. <u>λ-R-Matrices</u>

Consider $\underline{P} = \lambda\underline{C} + \underline{G}$. As we have seen, $\underline{G}$ and $\underline{C}$ are R-matrices, so that $\underline{P}$ is an R-matrix for all $\lambda \geqslant 0$. We shall call $\underline{P}$ a λ-R-matrix. The $\underline{K}$ transformation theory of Sect. 2.2 will now apply if we put $\underline{A} = \underline{P}$ and qualify all statements with the remark, 'for all $\lambda \geqslant 0$'.

Moreover, since $-R_{11}Q_{ij}$, $i \neq j$, is formed from non-negative quantities by the operations of addition and multiplication only, it is clear that all the coefficients of powers of λ in $-P_{ij}^{(2)}$, $i = j$, are non-negative. Following Fialkow and Gerst we shall denote this by

$$- P_{ij}^{(2)} \gg 0. \; i \neq j. \qquad \qquad .... \; (2.13a)$$

Similarly

$$R_{11} \sum_{j=2}^{n} Q_{ij} = R_{11}\,(\delta_i + R_{1i}\,\delta_1/R_{11})$$

involves only positive quantities and the operations of addition and multiplication, so that

$$\sum_{j=1}^{n-1} P_{ij}^{(2)} \gg 0.$$

Therefore

$$P_{ii}^{(2)} = \sum_{j=1}^{n-1} P_{ij}^{(2)} + \sum_{\substack{j=1 \\ j\neq i}}^{n-1} (- P_{ij}^{(2)}\,) \gg 0. \qquad \text{from (2.13a)}$$

It now follows by induction that

$$\begin{aligned} P_{ii}^{(k)} &\gg 0 & & \\ & & & i, j = 1, \dots n-k+1 \\ P_{ij}^{(k)} &\ll 0 & i \neq j & \end{aligned} \qquad \text{.............(2.14)}$$

where the last statement means

$$- P_{ij}^{(k)} \gg 0 .$$

Much of the theory of the preceeding sections can now be carried over to λ-R-matrices, if we replace the sign $\geqslant$ by $\gg$. However, the 'δ-equations' of (2.5), cannot in general be solved by the methods of Sect. 2.3. It is in fact necessary to employ some additional algebraic device. In the next section the solution of the δ-equations will be discussed in terms of the algebra of congruences,* and the problem of synthesis reduced to finding polynominals to satisfy a certain inequality. Whether, however, this inequality condition can always be satisfied is not proven. All that can be said at present is that it can be satisfied in many cases.

3. Synthesis

The problem of synthesis may now be formally stated thus: given $\underline{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{bmatrix}$, find a λ-R-matrix $\underline{P}$ of order n (say) linear in λ such that

$$Z_{11} = P_{11}^{(n-1)}/P_{11}^{(n)} , \quad Z_{12} = - P_{12}^{(n-1)}/P_{11}^{(n)}, \quad Z_{22} = P_{22}^{(n-1)}/P_{11}^{(n)} \qquad \text{.............(3.1)}$$

Once $\underline{P}$ is known, the network can easily be constructed. Clearly $\underline{P}$ can be determined by applying the inverse

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* Ref. 6, Chapter 8, 10.

$\underline{K}_{n-1}, \underline{K}_{n-2}, \dots \underline{K}_1$ transformations in succession on the matrix

$$\begin{bmatrix} Z_{11} & -Z_{12} \\ -Z_{12} & Z_{22} \end{bmatrix}$$

provided such transformations are possible when the quantities involved are rational functions. It will be sufficient to consider the general case of an inverse $\underline{K}_k$ transformation, but before doing this we shall see what conditions Z_{11}, Z_{12}, Z_{22} must satisfy if they are to have a network representation.

3.1. The Conditions of Physical Realizability

First it is clear from lemma 4, note 2, and (2.14) that the poles and zeros of Z_{11}, Z_{22} are real, non-positive, distinct and that the zeros must separate the poles. Also from lemma 4, corollary, the degree of the numerator of Z_{11} is one less than or equal to the degree of the denominator, and similarly for Z_{22}.

Since from (2.14) $-P_{ij}^{(k)}$ is never negative for $\lambda \geqslant 0$, it is evident that Z_{12} has no zeros on the positive real axis. However, there is no restriction on the zeros being elsewhere in the complex plane, as can be seen by replacing J_j^2 in lemma 4 by $Y_{j1}^{(k-1)} Y_{j2}^{(k-1)} / Y_{11}^{(k)} Y_{11}^{(k)}$, which can be positive, or negative or zero. Observe that since $P_{12}^{(n-1)}$ and $P_{11}^{(n)}$ may have common factors, it is not necessary that (2.14) should apply (in practice it very often does not) to the numerator of Z_{12}. However, since the zeros of $P_{ii}^{(k)}$ are all real and non-positive, then this condition does apply to both numerator and denominator of Z_{11} and Z_{22}.

Since $\begin{bmatrix} Z_{11} & -Z_{12} \\ -Z_{12} & Z_{22} \end{bmatrix}$ must be an R-matrix, $\lambda > 0$,

then $$Z_{11} - Z_{12} = \frac{P_{11}^{(n-1)} + P_{12}^{(n-1)}}{P_{11}^{(n)}} \geqslant 0, \quad \lambda > 0, \quad \dots(3.2)$$

so that, for example, the coefficients of the highest and lowest powers of λ in Z_{11} must not be less than the corresponding coefficients in Z_{12} ; with of course a similar result for Z_{22}. This raises the interesting point that it is not sufficient for only the R-matrix property to hold; separation of the poles and zeros of the input and output impedance functions is required as well.

Next, since from Jacobi's Theorem

$$P_{11}^{(n-1)} \; P_{22}^{(n-1)} - P_{12}^{(n-1)} \; P_{12}^{(n-1)} = P_{11}^{(n)} \; P_{11}^{(n-2)} ,$$

we must have

$$Z_{11} \, Z_{22} - Z_{12} \, Z_{12} = P_{11}^{(n-2)} / P_{11}^{(n)} \qquad \ldots\ldots\ldots\ldots(3.3)$$

so that $$P_{11}^{(n-2)} / P_{11}^{(n-1)} = \frac{P_{11}^{(n-2)} / P_{11}^{(n)}}{P_{11}^{(n-1)} / P_{11}^{(n)}} = \frac{\det \underline{Z}}{Z_{11}} \qquad \ldots\ldots\ldots(3.4)$$

By lemma 4, the poles and zeros of this function are real, non-positive, distinct; the poles must separate the zeros; and the degree of the numerator is equal to or one less than the degree of the denominator. We can of course interchange the roles of Z_{11} and Z_{22} ; it is purely a matter of convenience as to which way the nodes should be numbered.

Because of the possibility of factors common to the numerator and denominator of a Z function, we cannot uniquely identify the various $P_{ij}^{(k)}$ polynominals with the corresponding polynominals derived from the Z functions. However, it will be most convenient to equate $P_{11}^{(n)}$ to the lowest common multiple, L, of the denominators of Z_{11}, $-Z_{12}$, Z_{22} and det $\underline{Z}$. We then put

$$P_{11}^{(n-1)} = L \, Z_{11}$$
$$P_{12}^{(n-1)} = L \, Z_{12}$$
$$P_{22}^{(n-1)} = L \, Z_{22}$$
$$P_{11}^{(n-2)} = L \det \underline{Z} \qquad \ldots\ldots\ldots\ldots(3.5)$$

From lemma 3, it follows that if $P_{ii}^{(k)}$ is divisible by the factor $(\lambda + \lambda_j)^s$, then, provided $P_{\alpha\beta}^{(k-1)}$ is a minor determinant of $P_{11}^{(k)}$, $P_{\alpha\beta}^{(k-1)}$ is divisible by $(\lambda + \lambda_j)^x$, where $x \geqslant s-1$. A corollary of the result is that Z_{12} must have simple poles only, as can also be seen from the proof of lemma 4.

To summarize we have the following:

Theorem

The necessary conditions that

$$\underline{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{bmatrix}$$

should be the Strecker-Feldtkeller impedance matrix of a 3-terminal RC network are:

(i) The functions Z_{11}, Z_{12}, Z_{22}, det $\underline{Z}$, $\dfrac{\det \underline{Z}}{Z_{11}}$ *, $\dfrac{\det \underline{Z}}{Z_{11}+Z_{22}-2Z_{12}}$ have the properties.-

(a) The poles are simple, real and non-positive

(b) The degree of the numerator is not greater than that of the denominator.

---

* Observe that in the special case when $P_{11}^{(n-1)}$, $P_{12}^{(n-1)}$, $P_{11}^{(n)}$ are co-prime, we have from

$$P_{11}^{(n-1)} P_{22}^{(n-1)} - P_{12}^{(n-1)} P_{12}^{(n-1)} = P_{11}^{(n)} P_{11}^{(n-2)}$$

that if $P_{11}^{(n-1)} // P_{11}^{(n)}$, then $P_{11}^{(n-1)} // P_{11}^{(n-2)}$ and $P_{22}^{(n-1)} // P_{11}^{(n)}$.

Also, provided $P_{11}^{(n-1)}$, $P_{22}^{(n-1)}$, $- P_{12}^{(n-1)}$ are not equal simultaneously, for some value of λ,

$$(P_{11}^{(n-1)} + P_{22}^{(n-1)} + 2\, P_{12}^{(n-1)}) // P_{11}^{(n-2)} .$$

However, in general if common zeros occur, it does not follow from $P_{11}^{(n-1)} // P_{11}^{(n)}$ alone that the other functions quoted possess this separation property.

(ii) The functions Z_{11}, Z_{22}, $\frac{\det \underline{Z}}{Z_{11}}$, $\frac{\det \underline{Z}}{Z_{11}+Z_{22}-2Z_{12}}$ have the properties.-

(a) The zeros are simple, real and non-positive

(b) The poles separate the zeros.

(iii)
$$Z_{11} \geqslant 0$$
$$Z_{12} \geqslant 0 \qquad \text{all} \quad \lambda \geqslant 0$$
$$Z_{22} \geqslant 0$$

(iv)
$$\left.\begin{aligned} Z_{11} - Z_{12} \geqslant 0 \\ Z_{22} - Z_{12} \geqslant 0 \end{aligned}\right\} \quad \text{all } \lambda \geqslant 0. \qquad \ldots\ldots\ldots\ldots(3.6)$$

Observe that (i)b and (ii)b ensure that the difference between the degrees of denominator and numerator is not greater than unity.

3.2. Solutions of the δ-equations

Suppose that we are given $P_{ij}^{(k+1)}/P_{11}^{(k)}$, $i,j = 1, \ldots, n-k$. We wish to determine $P_{ij}^{(k)}/P_{11}^{(k-1)}$, $i, j = 1, \ldots, n-k+1$. Let

$$P_{11}^{(k)} = U_k$$
$$P_{11}^{(k-1)} = W_k$$
$$P_{1i}^{(k)} = -f_{i-1}, \quad \sum_{j=1}^{n-k} f_j = f$$
$$\sum_{j=1}^{n-k+1} P_{ij}^{(k)} = \omega_{i-1}, \quad \sum_{j=1}^{n-k} \omega_j = \omega$$
$$\sum_{j=1}^{n-k} P_{ij}^{(k+1)} = H_i, \quad \sum_{j=1}^{n-k} H_i = H$$
$$\sum_{j=1}^{n-k+1} P_{ij}^{(k)} = g \qquad \ldots\ldots\ldots\ldots(3.7)$$

then the δ-equations of (2.5), when interpreted for the case of the inverse $\underline{K}_k$ transformation, become

$$\frac{\omega_i}{W_k} + \frac{g\, f_i}{U_k\, W_k} = \frac{H_i}{U_k} \qquad \text{.............(3.8)}$$

$$f + g = U_k \qquad \text{.............(3.9)}$$

hence $$f \overset{c}{\equiv} - g \ (\text{mod. } U_k) \ . \qquad \text{.............(3.10)}$$

Now $$\omega_i + \frac{g\, f_i}{U_k} = \frac{H_i\, W_k}{U_k} \qquad \text{.............(3.11)}$$

so that
$$g\, f_i \overset{c}{\equiv} H_i\, W_k \ (\text{mod. } U_k) \qquad \text{.............(3.12)}$$

and $$g\, f \overset{c}{\equiv} H\, W_k \ (\text{mod. } U_k) \qquad \text{.............(3.13)}$$

Also
$$\frac{U_k\, \omega_i}{W_k} + \frac{g\, f_i}{W_k} = H_i \qquad \text{.............(3.14)}$$

so that $$U_k\, \omega_i \overset{c}{\equiv} - g\, f_i \ (\text{mod. } W_k) \qquad \text{.............(3.15)}$$

and $$U_k\, \omega \overset{c}{\equiv} - g\, f \ (\text{mod. } W_k) \qquad \text{.............(3.16)}$$

Now let $f_i = \varepsilon_i\, U_k + r_i$, where ε_i is a constant and r_i is of lower degree than U_k, so that $g = (1-\varepsilon)\, U_k - r$.

where $$\sum_{j=1}^{n-k} \varepsilon_j = \varepsilon$$

and $$\sum_{j=1}^{n-k} r_j = r \qquad \text{.............(3.17)}$$

We now proceed to express the R-matrix condition of (2.1) in terms of ε_i and ε. Thus

$$U_k \geq f_i \geq 0 \ , \quad \lambda \geq 0 \qquad \text{.............(3.18)}$$

so that $1 - r_i/U_k \geq \varepsilon_i \quad - r_i/U_k$, $\lambda \geq 0$ ....(3.19)

For this to be possible, we must have

$$(r_i/U_k)_{max} - (r_i/U_k)_{min} \leq 1, \quad \lambda \geq 0 \qquad \text{............(3.20)}$$

Similarly $U_k \geq g \geq 0$ implies ....(3.21)

$$1 - r/U_k \geq \varepsilon \geq - r/U_k \;, \quad \lambda \geq 0 \qquad \text{............(3.22)}$$

and $$(r/U_k)_{max} - (r/U_k)_{min} \leq 1, \quad \lambda \geq 0 \qquad \text{............(3.23)}$$

Next, from (3.11) and (3.17), the condition $\omega_i \geq 0$ implies

$$W_k \; H_i \geq g \; f_i$$

i.e. $$W_k \; H_i \geq \varepsilon_i \; U_k \; g + r_i \; g \qquad \text{............(3.24)}$$

so that $$\varepsilon_i \leq - r_i/U_k + \frac{H_i \; W_k}{U_k \left[(1-\varepsilon) \; U_k - r\right]}, \quad \lambda \geq 0 \qquad \text{....(3.25)}$$

Similarly $\omega \geq 0$ implies

$$W_k \; H \geq g \; f \qquad \text{............(3.26)}$$

i.e. $$W_k \; H \geq \left[(1-\varepsilon) \; U_k - r\right] \; (\varepsilon U_k + r), \quad \lambda \geq 0 \qquad \text{.....(3.27)}$$

i.e. $$\varepsilon^2 U_k^2 - \varepsilon U_k (U_k - 2r) + W_k \; H + r^2 - U_k \; r \geq 0, \qquad \text{.....(3.28)}$$

whence <u>either</u>

$$(U_k - 2r)^2 \leq 4(W_k \; H + r^2 - U_k r), \; \lambda \geq 0$$

i.e. $$(U_k)^2 \leq 4 \; W_k \; H, \quad \lambda \geq 0 \qquad \text{............(3.29)}$$

<u>or</u> $$\varepsilon \gtrsim 1/2 - r/U_k + (1/4 - W_k H/U_k^2)^{1/2}, \lambda \geqslant 0 \qquad \ldots\ldots\ldots\ldots(3.30)$$

<u>or</u> $$\varepsilon \lesssim 1/2 - r/U_k - (1/4 - W_k H/U_k^2)^{1/2}, \lambda \geqslant 0 \qquad \ldots\ldots\ldots\ldots(3.31)$$

In order that (3.30) and (3.31) be consistent with (3.22) we must have

$$\left[(1/4 - W_k\ H/U_k^2)^{1/2}\right]_{max} + (r/U_k)_{max} - (r/U_k)_{min} \leqslant \tfrac{1}{2} \qquad \ldots\ldots\ldots\ldots(3.32)$$

and for (3.25) to be consistent with (3.19) we must have

$$\left[\frac{H_i\ W_k}{U_k\left[(1-\varepsilon)\ U_k - r\right]}\right]_{min} \gtrsim (r_i/U_k)_{max} - (r_i/U_k)_{min} \qquad \ldots\ldots\ldots\ldots(3.33)$$

We now proceed to examine the congruence conditions. From (3.13)

$$\begin{aligned} H\ W_k &\overset{c}{\equiv} g\ f\ (\text{mod. } U_k) \\ &\overset{c}{\equiv} - f^2\ (\text{mod. } U_k) \quad \text{by (3.10)} \\ &\overset{c}{\equiv} - r^2\ (\text{mod. } U_k) \quad \text{by (3.17) and (3.7)} \end{aligned}$$

then if H and U_k are co-prime, by Euclid's Algorithm there exist polynominals S, T such that

$$H\ T + U_k S = 1$$

i.e. $$H\ T \overset{c}{\equiv} 1\ (\text{mod. } U_k) \qquad \ldots\ldots\ldots\ldots(3.34)$$

i.e., H has an inverse with respect to modulo U_k. Hence

$$W_k \overset{c}{\equiv} - r^2\ T\ (\text{mod. } U_k) \qquad \ldots\ldots\ldots\ldots(3.35)$$

It then follows from lemma 5 that the zeros of U_k separate those of W_k provided r and U_k are co-prime. Henceforth we shall denote this by $U_k//W_k$.* If, however, U_k and H are not co-prime, we have, from lemma 5, that it is possible to choose r so that the poles of U_k/W_k separate the zeros, i.e. admitting the possibility of U_k and W_k having common

---

* The notation is borrowed from Geometry.

zeros. This relationship will also be denoted by $U_k // W_k$.
Next, it is required that the remaining parameters f_i be expressed in terms of r. From (3.13) and (3.12) we have

$$g\, f_i \overset{c}{=} H_i\, W_k \ (\text{mod. } U_k)$$

$$g\, f \overset{c}{=} H\ W_k \ (\text{mod. } U_k)$$

hence $$H\, g\, f_i \overset{c}{=} H_i\, H\, W_k \overset{c}{=} H_i\, g\, f \ (\text{mod. } U_k)$$

and so $$H\, f_i \overset{c}{=} H_i\, f \ (\text{mod. } U_k) \quad \text{.............(3.36)}$$

Strictly speaking this condition is not the most general relationship, it being necessary only if U_k and g are co-prime. However, it is sufficient for our purposes. Hence

$$H\, r_i \overset{c}{=} H_i\, r \ (\text{mod. } U_k) \quad \text{.............(3.37)}$$

which enables r_i to be determined.

Finally, $P_{ij}^{(k)}$ is determined from the relation

$$W_k\, P_{ij}^{(k+1)} + f_i\, f_j = U_k\, P_{i+1,j+1}^{(k)} \quad \text{.............(3.38)}$$

That this is possible, i.e. that the relation

$$W_k\, P_{ij}^{(k+1)} \overset{c}{=} -\, f_i\, f_i \ (\text{mod. } U_k)$$
$$\overset{c}{=} -\, r_i\, r_j \ (\text{mod. } U_k)$$

is valid, is proved in lemma 6.

Clearly from (3.38) we must have also the inequality condition

$$f_i\, f_j + W_k\, P_{ij}^{(k+1)} \leqslant 0, \quad \lambda \geqslant 0, \ i \neq j \text{.............(3.39)}$$

Equations (3.7) to (3.39) state the condition which any solution of the δ-equations must satisfy. The congruence relations enable us to calculate any of the desired parameters in terms of r. But there are still the inequality conditions to meet. With the restriction that $r, r_i, \varepsilon, \varepsilon_i$ etc. must satisfy the inequalities and that r may have certain restrictions on the location of its zeros these quantities may be chosen arbitrarily.

The success of this method as a general synthesis procedure depends on finding a method of choosing r and ε subject to the inequality conditions. At the present time it is not known whether this is always possible i.e. whether the conditions of the theorem of Sect. 3.1 are also sufficient. Certainly it appears possible in a large number of numerical cases and especially so if H and U_k have a common zero. The inequality conditions may be more simply expressed as:-

$$W_k H \geq g f, \quad \lambda \geq 0 \quad \text{i.e.} \quad \omega \geq 0$$

$$W_k H_i \geq g f_i, \lambda \geq 0 \quad \text{i.e.} \quad \omega_i \geq 0$$

$$f_i f_j \leq - W_k P_{ij}^{(k+1)}. \quad \lambda \geq 0 \quad \text{i.e.} \quad P_{i+1,j+1}^{(k)} \leq 0$$

$$i \neq j.$$

The central problem would appear to be to find f such that $\frac{W_k H}{f_i f_j}$ can be made arbitrarily large, for $\lambda \geq 0$, subject to $0 \leq f_i \leq U_k$. It is possible to guarantee the first inequality by suitably positioning the zeros of r, but more than this is not yet known.

3.3. Examples

(i) Synthesise

$$\underline{Z} = \frac{1}{35\lambda^3+148\lambda^2+200\lambda+86} \begin{bmatrix} 16\lambda^2+47\lambda+33 & 7\lambda^2+18\lambda+12 \\ 7\lambda^2+18\lambda+12 & 14\lambda^2+34\lambda+20 \end{bmatrix}$$

There are no common factors so that we have

$$P_{11}^{(n-1)} = 16\lambda^2 + 47\lambda + 33 \quad P_{12}^{(n-1)} = -(7\lambda^2 + 18\lambda + 12)$$

$$P_{12}^{(n-1)} = 14\lambda^2 + 34\lambda + 20 \quad P_{11}^{(n)} = 35\lambda^3 + 148\lambda^2 + 200\lambda + 82$$

$$P_{11}^{(n-2)} = \frac{P_{11}^{(n-1)}\,P_{22}^{(n-1)} - \left(P_{12}^{(n-1)}\right)^2}{P_{11}^{(n)}} = 5\lambda + 6 + \text{(zero remainder)}$$

$$P_{11}^{(n-1)} // P_{11}^{(n-2)}, \quad P_{12}^{(n-1)} // P_{11}^{(n-2)}$$

$$P_{11}^{(n-1)} + P_{12}^{(n-1)} \geq 0, \quad P_{22}^{(n-1)} + P_{12}^{(n-1)} \geq 0, \quad \lambda \geq 0.$$

Hence the necessary conditions are satisfied.

$$U_1 = 5\lambda + 6 \qquad H_1 = 9\lambda^2 + 29\lambda + 21$$

$$H_2 = 7\lambda^2 + 16\lambda + 8$$

$$H = 16\lambda^2 + 45\lambda + 29.$$

Since the functions $P_{11}^{(n-1)}$, $P_{12}^{(n-1)}$, $P_{22}^{(n-1)}$ are of degree 2, only a single inverse $\underline{K}_1$ transformation is required; we may take $W_1 = 1$.

By direct computation from Euclid's Algorithm we find $T = -0.510$. But $W_1 H \cong -r^2 \pmod{U_1}$. Hence

$$r^2 = 1.96 \quad \text{and so} \quad r = 1.4$$

$$r_1 H \cong r H_1 \pmod{U_1} \quad \text{Hence} \quad r_1 = 0.6$$

$$r_2 H \cong r H_2 \pmod{U_1} \quad \text{Hence} \quad r_2 = 0.8$$

observe that $W_1 H > 1/4\, U_1^2$

We now choose

$$\varepsilon_1 = 0.1, \quad \varepsilon_2 = 0.3, \quad \text{so that} \quad \varepsilon = 0.4$$

Then $f_1 = 0.5\lambda + 1.2, \quad f_2 = 1.5\lambda + 2.6$

$$f_1 f_2 = 0.75\lambda^2 + 3.1\lambda + 3.12 < 0.51\,(7\lambda^2 + 18\lambda + 12)$$

Thus all the inequality conditions are satisfied and the network is physically realizable.

From the relations

$$P_{11}\,P_{22} = P_{11}^{(2)} + f_1^2$$

$$P_{11}\,P_{23} = P_{12}^{(2)} + f_1 f_2$$

$$P_{11}\,P_{33} = P_{22}^{(2)} + f_2^2$$

$$P_{12} = -\,f_1$$

$$P_{13} = -\,f_2$$

we obtain

$$\underline{P} = \begin{bmatrix} 6 & -1.2 & -2.6 \\ -1.2 & 5.74 & -1.48 \\ -2.6 & -1.48 & 4.46 \end{bmatrix} + \lambda \begin{bmatrix} 5 & -0.5 & -1.5 \\ -0.5 & 3.25 & -1.25 \\ -1.5 & -1.25 & 3.25 \end{bmatrix}$$

and the final network is as shown if Fig. 3.1(a).

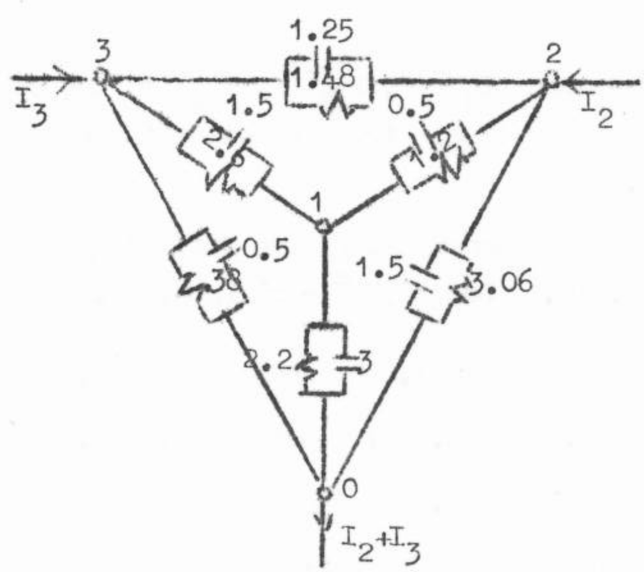

Fig. 3.1(a)

A second solution with 2 capacitors fewer than before may be obtained by putting $\varepsilon_1 = \varepsilon_2 = \varepsilon = 0.$ Then

$$f_1 = 0.6, \quad f_2 = 0.8, \quad f_1 f_2 = 0.48,$$

and

$$\underline{P} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3.2 & -1.4 \\ 0 & -1.4 & 2.8 \end{bmatrix} \lambda + \begin{bmatrix} 6 & -0.6 & -0.8 \\ -0.6 & 5.56 & -1.92 \\ -0.8 & -1.92 & 3.44 \end{bmatrix}$$

and the network is as shown in Fig. 3.1(b).

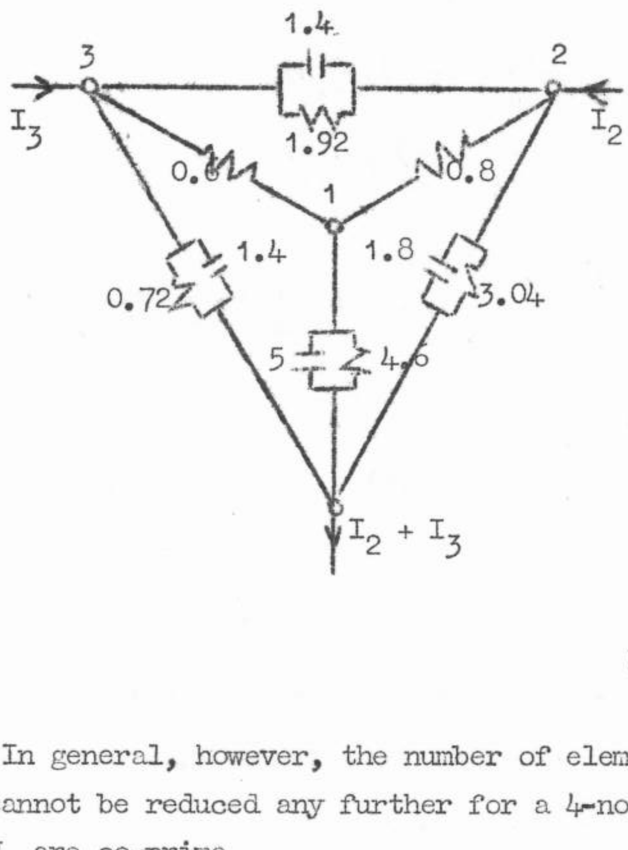

Fig. 3.1(b)

In general, however, the number of elements required cannot be reduced any further for a 4-node system if U_k and H are co-prime.

(ii) Synthesise

$$Z_{11} = \frac{2}{\lambda+1}, \quad Z_{12} = \frac{1}{\lambda+1}, \quad Z_{22} = \frac{\lambda^2 + 4\lambda + 2}{(\lambda+1)(2\lambda+1)}.$$

We find $$\frac{Z_{11}\,Z_{22} - Z_{12}^2}{Z_{11}} = \frac{2\lambda^2 + 6\lambda + 3}{2(\lambda+1)(2\lambda+1)}.$$

Hence we take

$$P_{11}^{(3)} = 2(\lambda+1)\ (2\lambda+1)$$

$$P_{12}^{(3)} = -\ (\lambda+1)\ (2\lambda+1)$$

$$P_{22}^{(3)} = (\lambda+1)\ (\lambda^2+4\lambda + 2)$$

$$P_{11}^{(4)} = (\lambda+1)^2\ (2\lambda+1)$$

$$P_{11}^{(2)} = 2\lambda^2 + 6\lambda + 3.$$

It is easily verified that

$$P_{11}^{(3)}//P_{11}^{(4)},\quad P_{22}^{(3)}//P_{11}^{(4)}$$

$$P_{11}^{(3)}//P_{11}^{(2)},\quad P_{22}^{(3)}//P_{11}^{(2)}$$

$$P_{11}^{(3)} + P_{12}^{(3)} > 0$$

$$P_{12}^{(3)} + P_{22}^{(3)} > 0$$

, so that the necessary conditions are satisfied.

First cycle:

$$H_1 = (\lambda+1)\ (2\lambda+1)$$

$$H_2 = (\lambda+1)^3 \qquad\qquad U_2 = 2\lambda^2 + 6\lambda + 3$$

$$H = (\lambda+1)\ (\lambda^2 + 4\lambda + 2).$$

Choose $r = 1$. Then from $W_2H + r^2 \cong 0 \ (\mathrm{mod.}\ U_2)$

we find $W_2 = 12\lambda + 28.$

Also from $r_iH \cong rH_i \ (\mathrm{mod.}\ U_2)$

we find $r_1 = 2,\ r_2 = -1.$

We choose $f_1 = 2,\ f_2 = 1/2\ U_2 + r_2 = \lambda^2 + 3\lambda + 0.5$

From the relation $W_2\ P_{12}^{(3)} + f_1f_2 \cong 0 \ (\mathrm{mod.}\ U_2)$,

we find $P_{23}^{(2)} = -\ (12\lambda + 9)$

and similarly

$$P_{22}^{(2)} = 24\lambda + 20$$

$$P_{11}^{(2)} = 2\lambda^2 + 6\lambda + 3$$

$$P_{33}^{(2)} = 6.5\lambda^2 + 27.5\lambda + 18.75$$

so that

$$\underline{K}_1' \, \underline{P} \, \underline{K}_1 = \frac{1}{12\lambda+28} \begin{bmatrix} 2\lambda^2+6\lambda+3 & -2 & -(\lambda^2+3\lambda+0.5) \\ -2 & 24\lambda + 20 & -(12\lambda + 9) \\ -(\lambda^2+3\lambda+0.5) & -(12\lambda+9) & 6.5\lambda^2 + 27.5\lambda + 18.75 \end{bmatrix}$$

Second Cycle

$$H_1 = \lambda^2 + 3\lambda + 0.5$$

$$H_2 = 12\lambda + 9 \qquad\qquad U_1 = 12\lambda + 28$$

$$H_3 = 5.5\lambda^2 + 12.5\lambda + 9.25$$

$$H = 6.5\lambda^2 + 27.5\lambda + 18.75.$$

Again we choose $r = 1$, but find that in order to meet the inequality conditions, r needs to be larger. The choice is amended to $r = 14$. Then

$$W_1 = 19.6$$

$$r_1 = 1.48$$

$$r_2 = 26.6$$

$$r_3 = -14.0$$

We choose $f_1 = r_1$, $f_2 = r_2$, $f_3 = r_3 + 0.50\, U_1$

$= 1.48 \quad = 26.5 \quad = 6.0\lambda$.

In a similar manner to the first cycle we find

$$W_1\, P_{23} = 0 \qquad\qquad W_1\, P_{22} = 3.26\lambda + 2.17$$

$$W_1\, P_{24} = -\,(1.63\lambda + 0.35)$$

$$W_1\, P_{33} = 39.2$$

$$W_1\, P_{34} \quad - 6.31$$

$$W_1 \quad = 19.6 \qquad\qquad W_1\, P_{44} = 13.62\lambda + 20.4$$

Hence

$$\underline{P} = \begin{bmatrix} 0.61\lambda+1.43 & -0.076 & -1.33 & -0.306\lambda \\ -0.076 & 0.167\lambda+0.111 & 0 & -(0.083\lambda+0.018) \\ -1.33 & 0 & 2 & -0.322 \\ -0.306\lambda & -(0.083\lambda+0.018) & -0.322 & (0.695\lambda+0.67) \end{bmatrix}$$

The final network is as shown in Fig. 3.2. However, this

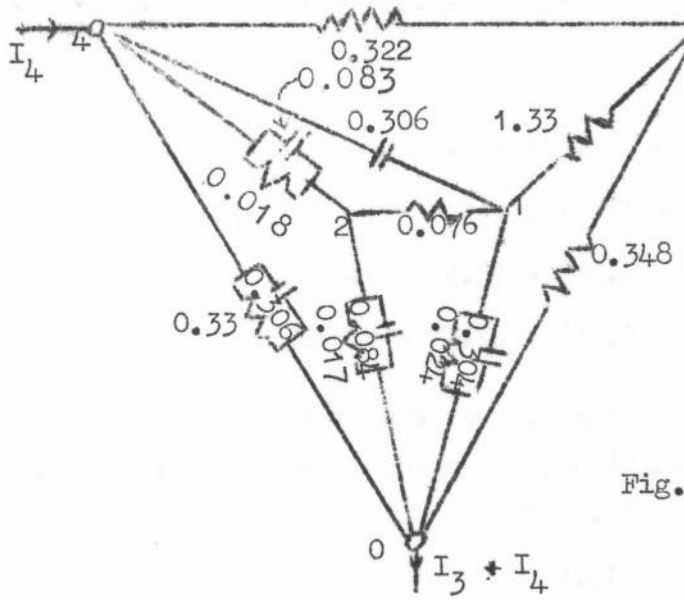

Fig. 3.2

network is by no means the simplest possible realization of the given functions.

4. Lambda-Transformations

So far we have considered only resistance-capacitance networks. However, the results of the previous sections can be taken over directly to other combinations of network elements provided the admittance matrix of the network can be expressed as a linear combination of two independent matrices. (This is essential for the validity of lemma 4). Possible combinations are tabulated below.

Network matrix	Physical Interpretation	λ-Transformation
$\lambda\underline{C} + \frac{1}{\lambda+a}\underline{\Gamma}$	Capacitance and inductance network: every inductor has a resistor in series with it and of value proportional to that of the inductor.	$\lambda' = \lambda(\lambda+a)$
$(\lambda+a)\,\underline{C} + \frac{1}{\lambda}\underline{\Gamma}$	capacitance and inductance network: every capacitor is shunted by a resistor of value inversely proportional to the capacitance.	$\lambda' = \frac{1}{\lambda(\lambda+a)}$
$\frac{\lambda}{1+a\lambda}\underline{C} + \frac{1}{\lambda}\underline{\Gamma}$	capacitance and inductance network: every capacitor has a resistor in series with it of value inversely proportional to the capacitance.	$\lambda' = \frac{1+\lambda a}{\lambda^2}$
$\lambda\underline{C} + (1+\frac{a}{\lambda})\underline{G}$	capacitance inductance network: every inductor shunted by a resistor of value proportional to the inductance.	$\lambda' = \frac{\lambda^2}{\lambda+a}$
$\underline{G} + \frac{\lambda}{1+a\lambda^2}\underline{C}$	resistance capacitance network: every inductor in series with a capacitor of value inversely proportional to the inductance.	$\lambda' = \frac{\lambda}{1+a\lambda^2}$
$\lambda\,\underline{C} + \frac{1}{\lambda}\underline{\Gamma}$	capacitance inductance network	$\lambda' = \lambda^2$
$\underline{G} + \frac{1}{\lambda}\underline{\Gamma}$	inductance resistance network	$\lambda' = \frac{1}{\lambda}$

Any linear combinations of the above combinations, provided the same two matrices appear in each sub-combination chosen, can also be treated by the preceeding method.

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(See also W.L. Ferrar ibid pp. 186-192).

Appendix

Lemma 1

If, in the notation of Sect. 2, $\underline{A}$ is an R-matrix such that, for each $i = 1, \dots n$, there exists at least one sequence

$$R_{ix}, R_{xy}, R_{y3}, \dots\dots\dots R_{st}, \delta_t ,$$

no member of which is zero,* then $\underline{A}$ is positive definite.

$$\dots\dots\dots\dots (A1.1)$$

Proof

(A1.1) and (2.1) ensure that

$$R_{ii} > 0 \qquad \dots\dots\dots\dots (A1.2)$$

Further,

$$- Q_{ij} = R_{ij} + R_{1i} R_{1j}/R_{11} > 0, \text{ if } R_{ij} \neq 0, i \neq j.$$

and

$$\sum_{j=2}^{n} Q_{ij} = \delta_i + R_{1i} \delta_1/R_{11} > 0, \text{ if } \delta_i \neq 0.$$

i.e. provided no suffix in (A1.1) is unity, $\underline{Q}$ also satisfies condition (A1.1).

If, however, R_{s1} is a member of the sequence, then the next member is

either δ_1, in which case $\sum_{j=2}^{n} Q_{sj} > 0$

or R_{1t}, in which case $- Q_{st} > 0$, $s \neq t$.

Hence in all cases $\underline{Q}$ satisfies (A1.1). Therefore, by (A1.2),

$$Q_{ii} > 0 \qquad i = 2, \dots, n.$$

$$Q_{11} = R_{11} > 0.$$

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* If $\underline{A}$ is the conductance matrix of a network, this condition is necessary and sufficient that there be a conducting path between the earth node and every other node, i.e. the linear graph consists of only one separable part.

It then follows by induction that

$$\underline{K}_r' \underline{K}_{r-1}' \cdots\cdots \underline{K}_1' \underline{A} \underline{K}_1 K_2 \cdots\cdots \underline{K}_r$$

is also a matrix satisfying (A1.1) and (A1.2), so that in particular,

$$A_{11}^{(k)} > 0 \quad k = 1, \cdots\cdots n.$$

Hence $\underline{A}$ is positive definite.

Note 1

If $\underline{P}$ is the admittance matrix of a network whose linear graph consists of only one separable part, then $\underline{P}$ clearly satisfies (A1.1) for all $\lambda > 0$, so that $\underline{P}$ is positive definite, $\lambda > 0$.

Note 2

This result is normally obtained from physical considerations depending on the conservation of energy. However, this algebraic derivation is instructive in that it reveals another property of R-matrices (viz. the invariance of (A1.1) under $\underline{K}$ transformations) which is closely connected with the topology of the network.

Lemma 2

If $\underline{E}$ is a square matrix of order n such that $E_{11}^{(k)} E_{11}^{(k-1)} \neq 0$, then

$$\frac{\left(\underline{E}^{(k)}/E_{11}^{(k-1)}\right)_{ij}^{(2)}}{E_{11}^{(k)}/E_{11}^{(k-1)}} = \frac{E_{ij}^{(k+1)}}{E_{11}^{(k)}}, \quad i, j = 1, \cdots\cdots n-k.$$

(A similar result holds for rectangular matrices with suitable restrictions on i, j, k.)

Proof

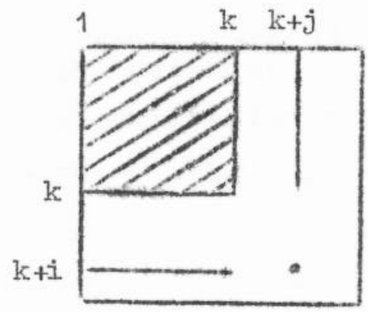
Let $\underline{M}$ be a submatrix of $\underline{E}$ of order $(k+1) \times (k+1)$, formed by adjoining to the leading $k \times k$ sub-matrix of $\underline{E}$

the elements of :

column (k+j), rows 1, ... k, row (k+i)

row (k+i), column 1,.... k, column (k+j)

and such that the lexical order of row and column suffices appearing in $\underline{E}$ is preserved. Then



$$\left(\underline{E}^{(k)}/E_{11}^{(k-1)}\right)_{ij}^{(2)} = \left(\underline{M}^{(k)}/M_{11}^{(k-1)}\right)_{11}^{(2)}$$

$$= \frac{M_{11}^{(k)}\, M_{22}^{(k)} - M_{12}^{(k)}\, M_{21}^{(k)}}{M_{11}^{(k-1)}\, M_{11}^{(k-1}} ,$$

which, on putting $\underline{N} = \text{adj.}\ \underline{M}$,

$$= \frac{N_{k+1,k+1}\, N_{kk} - N_{k+1,k}\, N_{k,k+1}}{M_{11}^{(k-1)}\, M_{11}^{(k-1)}}$$

$$= \frac{M_{11}^{(k-1)}\ \det \underline{M}}{M_{11}^{(k-1)} M_{11}^{(k-1)}} \quad \text{by Jacobi's Theorem*}$$

* i.e. $(\text{adj}\ \underline{M})^{(s)} = (\det \underline{M})^{(s-1)}\ \text{adj}^{(s)}\underline{M}$. (Ref. 1, ch. 5).

$$= E_{ij}^{(k)} / E_{11}^{(k-1)}. \text{ Hence } \frac{(\underline{E}^{(k)}/E_{11}^{(k-1)})_{ij}^2}{E_{11}^{(k)}/E_{11}^{(k-1)}} = \frac{E_{ij}^{(k+1)}}{E_{11}^{(k)}} ;i,j=1,..n-k$$

Lemma 3

If $\underline{E}$ is a real symmetric matrix of order n, then λ_i is a zero of multiplicity k of $\det(\underline{E} - \lambda \underline{1}_n)$ if and only if the rank of $(\underline{E} - \lambda_i \underline{1}_n)$ is n-k.

This result is well-known; it is proved in Ref. 8.

Corollary 1

If $\underline{E} + \lambda\underline{F}$ is a real symmetric matrix of order n, such that either (i) $\underline{F}$ is positive definite
or (ii) $\underline{E} + a\underline{F}$, where a is real and a scalar, is positive definite

then λ_i is a zero of multiplicity k of $\det(\underline{E} + \lambda\underline{F})$ if and only if the rank of $(\underline{E} + \lambda_i\underline{F})$ is n-k.

This is easily seen to be a re-statement of the lemma 3 if the transformations

(i) $\lambda = -\lambda'$, $\underline{X}'\,\underline{F}\,\underline{X} = \underline{1}_n$

or (ii) $\lambda = (a - \frac{1}{\lambda'})$, $\underline{X}'\,(a\underline{F} + \underline{E})\,\underline{X} = \underline{1}_n$ be made.

Corollary 2

If $(\underline{E} + \lambda\underline{F})_{11}^{(k)}$ has a zero λ_i of multiplicity s, then λ_i is also a zero of $(\underline{E} + \lambda\underline{F})_{ij}^{(k-1)}$ of multiplicity at least s-1

$$i,j = 1,2$$

This follows immediately from the fact that $(\underline{E} + \lambda_i\underline{F})_{ij}^{(k-1)}$ is a minor determinant of $(\underline{E} + \lambda_i\underline{F})_{11}^{(k)}$ which can thus have rank not greater than k-s.

Lemma 4

If $\underline{E} + \lambda\underline{F}$ is a real symmetric matrix of order n, such that either

(i) $\underline{F}$ is positive definite

or (ii) $\underline{F}$ is positive semi-definite and $\underline{E} + a\underline{F}$ is positive definite, where a is a real scalar, then (a) the poles and zeros of $\frac{(\underline{E} + \lambda\underline{F})^{(k-1)}_{ii}}{(\underline{E} + \lambda\underline{F})^{(k)}_{11}}$ are all real and simple; $i = 1,2.$

(b) the poles separate the zeros.

Proof

It is evident that the leading k x k sub-matrix of $\underline{E} + \lambda\underline{F}$ must also satisfy (i) and (ii).

The reality of the zeros of $(\underline{E} + \lambda\underline{F})^{(k)}_{11}$ is well-known; a proof is given in Ref. 1.*

To establish the separation property put,

in case (i), $\lambda' = \lambda$,
in case (ii), $\lambda' = \lambda + a$

and transform the matrices $\underline{E}$, $\underline{F}$ or $(\underline{E} + a\underline{F})$, $\underline{F}$ simultaneously to diagonal matrices. It is then valid to set the leading k x k sub-matrix of $\underline{E} + \lambda\underline{F}$ equal to

either (i) $\underline{Y}' (\underline{\Omega} + \lambda' \underline{1}_k) \underline{Y}$
or (ii) $\underline{Y}' (\underline{\Omega}\lambda' + \underline{1}_k) \underline{Y}$

where $\underline{\Omega}$ is diag $(\lambda_1, \lambda_2, \ldots\ldots \lambda_k)$ and $\underline{Y}$ is non-singular. Hence, by the Binet-Cauchy Theorem **,

$$\frac{(\underline{E} + \lambda\underline{F})^{(k-1)}_{ii}}{(\underline{E} + \lambda\underline{F})^{(k)}_{11}} = \text{either (i)} \quad \frac{J_1^2}{\lambda'+\lambda_1} + \frac{J_2^2}{\lambda'+\lambda_2} + \ldots + \frac{J_k^2}{\lambda'+\lambda_k}$$

$$\text{or} \quad \text{(ii)} \quad \frac{J_1^2}{\lambda'\lambda_1+1} + \frac{J_2^2}{\lambda'\lambda_2+1} + \ldots + \frac{J_k^2}{\lambda'\lambda_k+1}$$

where $J_j = Y^{(k-1)}_{ji} / Y^{(k)}_{11}$ is real.

It follows immediately that the poles are simple. Now arrange the λ_j in order so that

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* Chap. 3, Ex. 9.

** i.e. $(M\,N)^{(k)} = M^{(k)}\,N^{(k)}$ (Ref. 1, Chap. 5).

in case (i) $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_k$,

in case (ii) $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_k \geq 0$.

The last requirement is possible as a result of $\underline{F}$ being positive semi-definite. Then

$$(\underline{E} + \lambda\underline{F})_{11}^{(k-1)}$$

≥ 0 if $\lambda' = -\lambda_1$, case (i); $\lambda' = -1/\lambda_1$, case (ii).

≤ 0 if $\lambda' = -\lambda_2$, case (i); $\lambda' = -1/\lambda_2$, case (ii).

≥ 0 if $\lambda' = -\lambda_3$, case (i); $\lambda' = -1/\lambda_3$, case (ii),

etc. ...

where the equality sign occurs only if $J_i = 0$; i.e. numerator and denominator then have a common factor which can be cancelled out.

It follows there is at least one zero between consecutive poles. Since the degree of the numerator cannot exceed that of the denominator, it follows that the poles separate the zeros. Since the poles are simple, then the zeros must also be simple.

Note that in the event of some of the λ_i being coincident, the corresponding J_i^2 coefficient is replaced by a sum of squares.

Corollary 1

In case (i) the degree of the numerator is one less than that of the denominator. In case (ii), this also holds unless for a particular $J_j \neq 0$, $\lambda_j = 0$, in which case the degrees are equal.

Note 1

This result is usually attributed to Routh (Ref. 7) although it was effectively obtained by Salmon* some twenty years earlier and was probably also known to Sylvester.

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* Lessons in Higher Algebra.

However, Routh's treatment of the case of multiple zeros of $(\underline{E} + \lambda\underline{F})_{11}^{(k)}$ can hardly be regarded as satisfactory. The Binet-Cauchy Theorem provides a method which treats all cases equally easily. In conjunction with lemma 3 it is then possible to give an adequate account of the degeneracies that can occur in a network, non-existant impedance matrices excepted.

Note 2

If $\underline{P} = (\underline{G} + \lambda\underline{C})$ is the admittance matrix of a network and $\det \underline{P} \neq 0$, $\lambda > 0$, then $\underline{P}$ satisfies the conditions of the lemma. Here 'a' is any positive number. $\underline{G}$ and $\underline{C}$ are both positive semi-definite, as can easily be shown by adapting the proof of lemma 1 to the case of linear graphs consisting of several separable parts.

Corollary 2

If $(\underline{E} + \lambda\underline{F})_{ii}^{(k-1)}$ has a zero, λ_i, of multiplicity s, then λ_i is also a zero $(\underline{E} + \lambda\underline{F})_{11}^{(k)}$ of multiplicity at least $s-1$. This follows directly from the separation property and the partial fraction expansion.

Corollary 3

The matrix formed from $(\underline{E} + \lambda\underline{F})$ by adding the k^{th} row to the $(k-1)^{th}$ row and the k^{th} column to the $(k-1)^{th}$ column also satisfies the condition of the lemma. For, this transformation is non-singular and so does not affect the positive definite character. The leading element of the k^{th} compound of this matrix is easily seen to be

$$(\underline{E} + \lambda\underline{F})_{11}^{(k-1)} + (\underline{E} + \lambda\underline{F})_{22}^{(k-1)} + 2(\underline{E} + \lambda\underline{F})_{12}^{(k-1)}$$

by a Cauchy expansion* according to the $(k-1)^{th}$ row and column. Hence the poles of the function

* Ref. 1. Ch. 4.

$$\frac{(\underline{E} + \lambda\underline{F})_{11}^{(k-1)} + (\underline{E} + \lambda\underline{F})_{22}^{(k-1)} + 2\ (\underline{E} + \lambda\underline{F})_{12}^{(k-1)}}{(\underline{E} + \lambda\underline{F})_{11}^{(k-2)}}$$

separate the zeros.

Lemma 5

In the notation of Sect. 3.2, r can be so chosen that the zeros of U_k/W_k separate the poles, for $1 \leqslant k \leqslant n-1$.

Proof

The proof proceeds by induction. Suppose first that the poles of U_k/H separate the zeros. We shall denote this by $U_k//H$. Then

$$W_k\ H + r^2 \cong 0\ (\text{mod. } U_k).$$

Case 1

If U_k and H are co-prime, then at zeros of U_k, W_k must take the opposite sign to H, if we choose r co-prime to U_k. It then follows that the zeros of W_k are real and that $W_k//U_k$. It also follows that the leading coefficient of W_k must be positive since the degree of W_k is two less than that of H and since H has a positive leading coefficient. Then $W_k \gg 0$.

Case 2

If U_k and H have a common factor h, so that

$$U_k = h\ (U_k)o\ ,$$
$$H = h\ H_o$$

then we take

$$r = h\ r_o.$$

hence

$$W_k H_o + h\ r_o^2 \cong 0\ (\text{mod. } U_{k_o}).$$

Let T_o be the inverse of H_o with respect to modulo $(U_k)_o$

$$\text{i.e.}\quad H_o T_o \overset{c}{=} 1 \quad (\text{mod. } (U_k)_o).$$

Then

$$W_k \overset{c}{=} - h\, T_o\, r_o^2 \quad (\text{mod. } (U_k)_o)$$

and the general solution of the congruence is

$$(W_k)_1 + \chi\,(U_k)_o$$

where $(W_k)_1$ is a particular solution and χ is a polynominal of degree one less than the difference between the degrees of U_k and $(U_k)_o$.

We can ensure separation of the poles and zeros of W_k if we take the particular solution

$$W_k = h\,(W_k)_o.$$

The proof then proceeds as for case 1. Next, from the necessary realizability conditions we have that

$$(P_{11}^{(n-1)} + P_{22}^{(n-1)} + 2\,P_{12}^{(n-1)}) \;//\; P_{11}^{(n-2)}$$

so that the first H which arises satisfies the separation condition.

The proof will be completed if we can show that $H//U_k$ implies

$$U_{k-1} // (H)_{k-1}$$

i.e. $\quad W_k // (g+\omega)$

From (3.16)

$$U_k\omega \overset{c}{=} - g\, f \;(\text{mod. } W_k)$$
$$\overset{c}{=} - U_k g + g^2 \;(\text{mod. } W_k)$$

i.e. $U_k\,(g+\omega) \overset{c}{=} g^2 \;(\text{mod. } W_k).$

Then if U_k, W_k, g (and hence r) are co-prime $(\omega + g)$ takes the opposite sign to U_k at zeros of W_k.

In the case of common factors the proof proceeds as in case 2 for H. i.e. g may then be suitably chosen to endure separation. Hence, in both cases,

$$(g + \omega) \; // \; W_k$$

and $\omega + g >> 0$.

Lemma 6

In the notation of Sect. 3.2 it is always possible to calculate

$$P^{(k)}_{i+1,j+1} = \frac{W_k \, P^{(k+1)}_{ij} + f_i \, f_j}{U_k} .$$

Proof

The proof is by induction. Suppose the result is true for k. Then

$$U_k \, P^{(k)}_{i+1,j+1} = W_k \, P^{(k+1)}_{ij} + f_i \, f_j \quad \ldots\ldots\ldots\ldots (a)$$

i.e. $U_k \, P^{(k)}_{i+1,j+1} \cong f_i \, f_j \; (\text{mod. } W_k)$

From (3.15)

$$U_k \, \omega_i \cong - g \, f_i \quad (\text{mod. } W_k) \quad \ldots\ldots\ldots\ldots (b)$$

and $U_k \, (\omega + g) \cong g^2 \quad (\text{mod. } W_k) \quad \ldots\ldots\ldots\ldots (c)$

so that (a) becomes

$$\begin{aligned} U_k \, g^2 \, P^{(k)}_{i+1,j+1} &\cong g^2 \, f_i \, f_j \quad (\text{mod. } W_k) \\ &\cong U_k^2 \, \omega_i \, \omega_j \quad (\text{mod. } W_k), \text{ from (b)} \end{aligned}$$

Hence from (c)

$$(\omega + g) \, P^{(k)}_{i+1,j+1} \cong \omega_i \, \omega_j \quad (\text{mod. } W_k) \quad \ldots\ldots (d)$$

From (3.12) $W_k H_i \overset{c}{\equiv} g f_i \pmod{U_k}$

and from (3.10), (3.13)

$$W_k H \overset{c}{\equiv} - g^2 \pmod{U_k}$$

$$\overset{c}{\equiv} - f^2 \pmod{U_k}$$

From (3.36) $H_i f \overset{c}{\equiv} H f_i \pmod{U_k}$ ....(e)

Similarly $W_{k-1} (H_i)_{k-1} \overset{c}{\equiv} (g)_{k-1} (f_i)_{k-1} \pmod{W_k}$

$$W_{k-1} (H)_{k-1} \overset{c}{\equiv} - (g)^2_{k-1} \pmod{W_k}$$

$$\overset{c}{\equiv} - (f)^2_{k-1} \pmod{W_k}$$

$$(H_i)_{k-1}(f)_{k-1} \overset{c}{\equiv} (H)_{k-1} (f_i)_{k-1} \pmod{W_k}$$ ....(f)

But

$$(H)_{k-1} = \omega + g \quad \text{and} \quad \omega_i = (H_{i+1})_{k-1}$$

Now (d) becomes

$$(\omega + g) P^{(k)}_{i+1,j+1} \overset{c}{\equiv} (H_{i+1})_{k-1} (H_{j+1})_{k-1} \pmod{W_k}$$

or $$(H)_{k-1} P^{(k)}_{ij} \overset{c}{\equiv} (H_i)_{k-1} (H_j)_{k-1} \pmod{W_k}$$

Then if $(H)_{k-1}$ and W_k are co-prime, let $(H)_{k-1} (T)_{k-1} \overset{c}{\equiv} 1 \pmod{W_k}$ and so

$$P^{(k)}_{ij} \overset{c}{\equiv} (H_i)_{k-1} (H_j)_{k-1} T_{k-1} \pmod{W_k}$$ ...(g)

Therefore

$$W_{k-1} P^{(k)}_{ij} \overset{c}{\equiv} - (f)^2_{k-1} (H_i)_{k-1} (H_j)_{k-1} T^2_{k-1} \pmod{W_k},$$

from (f),

$$\overset{c}{\equiv} - (f_i)_{k-1} (f_j)_{k-1} \pmod{W_k}.$$

but this is simply a re-statement of (a) for $k-1$.

If $\omega + g$ and W_k contain a common factor, the argument can similarly be adapted as in lemma 5. The proof by induction will be completed if we can show the lemma to hold for $k = n-2$. That is, we require to show from (g) that

$$(P_{11}^{(n-1)} + P_{22}^{(n-1)} + 2\,P_{12}^{(n-1)})\,P_{12}^{(n-1)} \cong (P_{11}^{(n-1)} + P_{12}^{(n-1)})$$

$$(P_{22}^{(n-1)} + P_{12}^{(n-1)}) \pmod{P_{11}^{(n-2)}}$$

i.e. $$P_{11}^{(n-1)}\,P_{22}^{(n-1)} - (P_{12}^{(n-1)})^2 \cong 0 \pmod{P_{11}^{(n-2)}}.$$

but this is guaranteed by the realizability conditions. Hence the lemma is true for all k in $1 \leq k \leq n-2$.