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The Unsteady Aerodynamic Forces on Deforming, Low Aspect Ratio Wings and Slender Wing-Body Combinations Oscillating Harmonically in a Compressible Flow

-by-

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SUMMARY

A method is presented whereby the 'Slender Body Theory' can be applied to the determination of the unsteady aerodynamic forces acting on slender wings and wing-body combinations experiencing harmonic deformations in a compressible flow. The analysis holds for subsonic and supersonic speeds, subject to restrictions which are stated and discussed.

A simplification of the method is also introduced which is applicable to many practical cases and calculations are performed on this basis which lead to numerical results for:

1. 'Equivalent Constant Derivatives' for a deforming slender delta wing using modal functions which are polynomials of the spanwise parameter;

2. 'Rigid' Force Coefficients for a pitching and plunging, slender, wing-body combination.

These results are given as closed expressions and in tabular form and some of the results are also shown in graphical form.

Both the derivatives and the 'Rigid' force coefficients are defined in such a way as to agree with the usual British Sign Convention.

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LIST OF SYMBOLS

The use of a 'bar' over a symbol denotes that it is the amplitude of the harmonically oscillating quantity represented by the symbol itself.

Symbol	Description	Defined in Section
A, B	portmanteau symbols in eqn. to reference axis	Appendix I
$F(x)$	function of $\Pi(x)$	7
G_s	constants in $H(\delta)$	4
$H(\delta)$	torsional mode	4
L	unsteady lift on rigid wing or body	6
L_z, $L_{\dot{z}}$, L_α, $L_{\dot{\alpha}}$	'Rigid' force coefficients	6
L_n	constants in series expansion for $\bar{\phi}$	3
M	unsteady moment on rigid wing or body	6
M_z, $M_{\dot{z}}$, M_α, $M_{\dot{\alpha}}$	'Rigid' force coefficients	6
M_∞	freestream Mach number	2
$Ne_n^{(2)}$	modified Mathieu function of the third kind	3
O( )	'of order'	
P_n	constants in series expansion for $\bar{\phi}$	3
Q_r	Lagrangian generalised force	4
$R(x)$	local radius of body of revolution	7
R_o	Maximum ' ' ' ' '	7
S	planform area of wing	Appendix I
T	factor giving length of wing-body combination forebody as proportion of wing root chord	Appendix VI

U_{∞}	freestream velocity	2
V	factor giving total length of wing-body combination	Appendix VI
AR	wing aspect ratio	Appendix I
a	speed of sound in free stream	2
b	maximum wing span	Fig. 1
c, c_r, c_m	local, root and mean chords	Appendix I
$f(x,y)$	deformation function	3
g_r	constants in $h(\delta)$	4
$h(\delta)$	flexural mode	4
k	frequency - Mach No. parameter $= \frac{\nu s}{2a}$	3
$(l_z)_{rs}, (l_{\dot{z}})_{rs}$, $(l_{\alpha})_{rs}, (l_{\dot{\alpha}})_{rs}$	equivalent constant derivatives	Appendix II
l	distance of reference section from wing root	Fig. 2
l_B	length of body of revolution	Fig. 3
$(m_z)_{rs}, (m_{\dot{z}})_{rs}$, $(m_{\alpha})_{rs}, (m_{\dot{\alpha}})_{rs}$	equivalent constant derivatives	Appendix II
m, m_c, m_b	factors giving position of reference axis for wing, wing-body combination and body respectively	Appendix VI
p_n	constants in series expansion for $\bar{\phi}$	3
p, p_{∞}	local and freestream pressures	2
Δp	differential pressure	3
q_s	Lagrangian generalised coordinates	4
r	polar, radial, coordinate	7
r, s	indices and suffices in modal functions	4
$s(x)$	local wing semispan	Fig. 1

s_o	$b/2$	
se_n	Mathieu function (periodic)	3
t	time	
u,v,w	perturbation velocities	2
$\bar{w}$	with suffices - various upwash conditions	
x,y,z	right-handed Cartesian coordinates	Fig. 1
$x_o(y)$	equation of reference axis	4
z_o	amplitude at reference section in flexural mode	4
$\Gamma(x)$	local cross-sectional area of body	7
Λ	factor giving length of conical nose on body	Fig. 4
Π	$= - \bar{w}\Gamma$	7
α_o	amplitude at reference section in torsional mode	4
β_n	functions of λ	Appendix III
$\gamma_{rs}, b_{rs}, c_{rs}$	flutter force coefficients	4
δ	non-dimensional spanwise parameter	4
ε	dimensionless amplitude or thickness	2
ζ	elliptical coordinate; polar angle	3 and 7
η	elliptical coordinate	3
K	$= \frac{\nu}{a} = \frac{2k}{s}$	7
λ	delta planform factor	Fig. 2
ν	angular frequency	3
$\rho,\ \rho_\infty$	local and freestream densities	2
σ	$= \frac{R_o}{s_o}$ ratio of max. body radius to max. wing span	7

τ	position of reference section	
	$= \tau.b/2$	Appendix I
ϕ	perturbation velocity potential	2
$\omega, \omega_r, \omega_m$	local, root and mean frequency parameters	2
$\omega_c = \omega_r = \dfrac{\nu c_r}{U_\infty}$, etc.	Suffix c is used for wing-body combination in place of 'r'	Appendix V

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1. Introduction

In this paper a method is given whereby the aerodynamic forces can be calculated for slender, low aspect ratio wings deforming harmonically in a compressible flow.

The method is applied to a slender, cropped, delta wing and certain flutter modes are assumed which take the form of polynomials in the spanwise parameter. Freedom of the wing root is allowed for so that body freedoms can be included. In the latter part of the paper the aerodynamic forces on a pitching and plunging, slender, wing-body combination are evaluated.

The basis of the method is the 'Slender-Body Theory' which has been applied in connection with the (quasi-steady) stability derivatives for slender, wings and wing-body combinations (refs. 1,2,3,4,5,6).

The application to an oscillating and deforming wing has, very recently, been studied by Merbt and Landahl (ref. 8).

The solution of the 'cross-sectional' problem, for the wing, is analogous to that of a two-dimensional flat plate oscillating in a compressible flow and has been treated by Timman* (ref. 9) and Reissner*(ref. 10).

The use of the 'Slender-Body Theory' allows the

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* Only the regular part of their solution is required in this case.

analysis to apply at subsonic and supersonic speeds subject to certain restrictions on Aspect Ratio, Mach number, Frequency Parameter, Slenderness. Validity at subsonic speeds depends on an approximate satisfaction of the Kutta-Jowkowski condition.

The assumptions of linearised, thin aerofoil theory are used, the fluid is perfect, the flow irrotational and harmonic motions are considered throughout.

2. The Slender-Body Theory

The coordinate system used is shown in Figure 1 where right-handed rectangular axes are drawn from an origin, O, fixed in the wing, with the x-axis parallel to the main stream and the z-axis upward. It is assumed that the wing is a thin, flat plate oscillating about its position of zero incidence in the plane $z = 0$, but always lying in the immediate vicinity of the plane.

The perturbation velocity potential, ϕ, satisfies the equation

$$\nabla^2\phi = \frac{1}{a^2}\left(U_\infty \frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)^2 \phi \qquad \text{.........(2.1)}$$

The conditions holding at the surface of the wing are specified by a prescribed 'downwash' $w(x,y,0,t)$ and the stipulation that the relative normal velocity of the air and of the wing is zero.

Applying the assumptions of the 'Slender Body Theory' implies that the x-derivatives in equation (2.1) are neglected and the two-dimensional flow at any cross-section is then given by the wave equation,

$$\phi_{yy} + \phi_{zz} = \frac{1}{a^2} \cdot \phi_{tt} \qquad \text{.............(2.2)}$$

The approximation equation (2.2) is satisfied (ref. 13,5) if:

$$\left|1 - M_\infty^2\right| \left(\frac{ds}{dx}\right)^2 << 1 \qquad \text{.............(2.3)a}$$

where M_∞ is the freestream Mach number and s is the local semi-span, or for a triangular wing if

$$\left|1 - M_\infty^2\right| \mathcal{R}^2 << 16 \qquad \text{.............(2.3)b}$$

where $\mathcal{R}$ is the aspect ratio.

If the influence of the time-derivative term in (2.2) is so small that it can be neglected the equation reduces to Laplace's equation, as for steady flow, and the difference between the unsteady and steady flow cases manifests itself entirely in the linearised Bernoulli Pressure equation,

$$p - p_{\infty} = - \rho_{\infty} \left(U_{\infty} \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial t} \right) \quad \ldots\ldots\ldots\ldots(2.4)$$

This implies that the root frequency parameter must be small (see ref. 11 cases 2 and 5).

3. Solution of the Potential Equation

Assuming harmonic motion of angular frequency, ν, equation (2.2) becomes,

$$\bar{\phi}_{yy} + \bar{\phi}_{zz} + \frac{\nu^2}{a^2} \bar{\phi} = 0 \quad \ldots\ldots\ldots\ldots(3.1)$$

where

$$\bar{\phi}(x,y,z)\, e^{i\nu t} = \phi(x,y,z,t) \quad \ldots\ldots\ldots\ldots(3.2)$$

The potential ϕ is subject to the following boundary conditions:

(a) ϕ is bounded everywhere in the flow and at infinity all disturbances should disappear in the proper manner, thus,

i) $\phi, \phi_z, \phi_y \longrightarrow 0$ as $\sqrt{y^2 + z^2} \longrightarrow \infty$

ii) The solution at infinity should represent waves travelling <u>outwards</u> from the origin.

(b) At any point on the wing the prescribed normal velocity must be equal to the normal derivative of ϕ at that point.

Let the motion of the point, (x,y), on the wing be represented by,

$$z = f(x,y,t) = \bar{f}(x,y)\, e^{i\nu t} \quad \ldots\ldots\ldots\ldots(3.3)$$

$\bar{f}(x,y)$ will be referred to as the '<u>deformation function.</u>'

According to the usual assumptions of the linearised theory the vertical velocity of the point (x,y) is given by,

$$w = \frac{dz}{dt} = \frac{\partial z}{\partial t} + U_{\infty}\frac{\partial z}{\partial x}$$

$$= \left[i\nu\bar{f}(x,y) + U_{\infty}\bar{f}_x(x,y) \right] e^{i\nu t} ,$$

or,

$$\bar{w}(x,y) = i\nu\,\bar{f}(x,y) + U_{\infty}\bar{f}_x(x,y) \quad \text{............(3.4)}$$ *

The condition is satisfied over the projection of the wing in the plane $z = 0$ and takes the form,

$$\bar{w}(x,y) = \bar{\phi}_z(x,y,0) = i\nu\bar{f}(x,y) + U_{\infty}\bar{f}_x(x,y) \quad \text{..(3.5)}$$

(c) Outside the wing and outside the wake $\phi(x,y,z,t)$ must be continuous in planes, x = constant, and since it is antisymmetric in z must satisfy the condition,

$$\phi(x,y,0,t) = 0 .$$

By transforming equation (3.1) to the Elliptical Coordinates, (ζ,η), where,

$$\left.\begin{aligned} y &= s\cosh\eta\cos\zeta \\ z &= s\sinh\eta\sin\zeta \end{aligned}\right\} \quad \text{............(3.6)}$$

and $s(x)$ is the local semi span, Merbt and Landahl (Ref 8) have derived a solution in terms of Mathieu functions.

Using the notation of reference 18 the solution takes the form,

$$\bar{\phi}(x,\eta,\zeta) = \sum_{n=1}^{\infty} p_n \,\mathrm{Ne}_n^{(2)}(\eta,k)\,\mathrm{se}_n(\zeta,k)$$

with $\quad k = \dfrac{\nu s}{2a} \quad$ ....(3.7)

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* The use of a 'bar' over a symbol denotes that it is the amplitude of the harmonically oscillating quantity represented by the symbol itself.

where the coefficients, p_n, are to be determined from boundary condition (b), equation (3.5) which becomes in elliptical coordinates,

$$\bar{\phi}_\eta\,(x,0,\zeta) = s(x)\,\bar{w}\,(x,s\cos\zeta)\sin\zeta \quad \ldots\ldots(3.8)$$

Differentiating (3.7) with respect to η and putting $\eta = 0$ (on the wing) gives.-

$$\bar{\phi}_\eta(x,0,\zeta) = \sum_{n=1}^{\infty} p_n \left[\frac{\partial}{\partial\eta}\left(Ne_n^{(2)}(\eta,k)\right)\right]_{\eta=0} se_n(\zeta,k)$$

$$= \sum_{n=1}^{\infty} P_n\, se_n(\zeta,k) \quad \ldots\ldots\ldots\ldots(3.9)$$

writing,

$$P_n(k) = p_n \frac{\partial}{\partial\eta}\left(Ne_n^{(2)}(\eta,k)\right)_{\eta=0} \quad \ldots\ldots\ldots\ldots(3.10)$$

Now if $\bar{w}(x,\zeta)$ is bounded and is a continuous function of ζ, the series representing $\bar{\phi}_\eta(x,0,\zeta)$ in (3.9) will be uniformly convergent.

Multiplying (3.9) by $se_n(\zeta,k)$ and integrating over the range, 0 to π, the coefficients P_n may be determined in an analogous manner to Half-Range Fourier Coefficients since an orthogonality relation exists for the Mathieu function $se_n(\zeta,k)$ (see ref. 16).

The P_n are thus given by;

$$P_n(k) = \frac{2s}{\pi}\int_0^{\pi} \bar{w}(x,\zeta_1)\sin\zeta_1\, se_n(\zeta_1,k)\, d\zeta_1 \quad (3.11)$$

and finally the p_n from (3.10).

The solution, (3.7), is now completely determined and the pressure distribution on the wing will be given from (3.7) with $\eta = 0$.

As discussed in section 2 it is possible, under certain conditions, to suppress the time- dependent term in equation (3.1) and the solution (3.7) then reduces to.-

$$\bar{\phi} = \sum_{n=1}^{\infty} e^{-n\eta}\, L_n \sin n\zeta \quad \ldots\ldots\ldots\ldots(3.12)$$

where the L_n are given by,

$$-n\,L_n = P_n'(x) = \frac{2s}{\pi}\int_0^{\pi} \bar{w}.\sin\zeta_1.\,\sin n\zeta_1.\,d\zeta_1 \qquad \text{.............(3.13)}$$

On the wing,

$$_w\bar{\phi} = -\sum_{n=1}^{\infty} \frac{P_n'}{n}.\,\sin n\zeta \qquad \text{.............(3.14)}$$

The differential pressure across the wing plane is given as,

$$\overline{\Delta p} = 2\rho_\infty U_\infty \left({}_w\bar{\phi}_x + \frac{i\nu}{U_\infty}.\,{}_w\bar{\phi} \right)_{z=+0} \qquad \text{.......(3.15)}$$

4. Symmetric Flutter Characteristics Typical of a Slender Delta Wing

The simplest, pointed, low aspect ratio wing satisfying the assumptions of the 'Slender Body Theory' is the slender delta wing.

Accordingly the analysis as developed in section 3 is applied to the wing shown in Figure 2.*

To describe the possible flutter modes of the wing a reference axis, $x_o(y)$ is used (see Fig. 2) given by the equation,

$$x_o = mc_r + \frac{2c_r}{b}\,(1-\lambda)\,(1-m)\,|y| \qquad \text{.........(4.1)}$$

The applicability of an axis such as this to delta wings is discussed by Woodcock (ref. 17).

For any particular flutter motion it is then prescribed that sections parallel to the line of flight will twist about the reference axis according to some modal function, such sections remaining themselves undistorted, whilst the reference axis itself translates according to another modal function, each degree of freedom so involved being associated with a Lagrangian generalised coordinate, q_s.

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* Details of the wing are given in Appendix I .

The generalised coordinates are defined at reference sections given by

$$|y| = l \qquad \text{.............(4.2)}$$

and all motions are measured relative to the mean position of the wing (plane, $z = 0$).

A non-dimensional spanwise parameter, δ, is introduced such that

$$y = \delta l \qquad \text{.............(4.3)}$$

and $|\delta| = 1$ at the reference sections.

Each degree of freedom will lead to an equation of motion and a generalised force, Q_r, which can be expressed conveniently in terms of force coefficients as (omitting $e^{i\nu t}$),

$$* \quad \frac{Q_r}{\rho_\infty l^3 U_\infty^2} = \sum_s (-\gamma_{rs}\,\omega_m^2 + b_{rs}\,i\omega_m + c_{rs})\,.\,q_s \qquad \text{.............(4.4)}$$

on the assumptions of the linearised theory, where,

$s = r =$ number of degrees of freedom

and ω_m = mean frequency parameter

$= \dfrac{\nu c_m}{U_\infty}$. <u>Note</u>.- c_m is the <u>mean</u> chord of the <u>half</u>-wing. See Appendix I.

In what follows the suffices + and - will indicate whether a function applies only for $y > 0$ or $y < 0$ respectively.

4.1. <u>Uncoupled Modes</u>

Let there be one uncoupled mode in flexure and one uncoupled mode in torsion described by the modal functions, $h(\delta)$ and $H(\delta)$ respectively, so defined that,

$$|h(\pm 1)| = |H(\pm 1)| = 1\,. \qquad \text{.............(4.5)}$$

If $ze^{i\nu t}$ represents the translation of a point on the reference axis, measured from the mean position,

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* The generalised coordinates and forces are amplitude functions but the bar notation is not used in their case.

positive upwards, and $\alpha e^{i\nu t}$ the rotation, positive leading edge down, of the section through that point, then,

$$\left.\begin{aligned} z &= z_o h(\delta) \\ \text{and} \quad \alpha &= \alpha_o H(\delta) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4.6)$$

where z_o and α_o are the amplitudes at the reference sections, $y = \pm\, l$.

Now, if the generalised coordinates are chosen so that,

$$\left.\begin{aligned} q_1 &= \frac{z_o}{l} \\ \text{and} \quad q_2 &= \frac{c_m}{l}\,\alpha_o \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4.7)$$

then the deformation function $\bar{f}(x,y)$ of equation (3.3) takes the form,

$$\bar{f} = l h q_1 + (x-x_o)\,\frac{l}{c_m}\,H q_2 \quad \ldots\ldots\ldots\ldots(4.8)$$

It will be convenient, for the aerodynamic problem, to consider the functions, h and H, each to be polynomials in δ: thus, for symmetrical modes;

$$\left.\begin{aligned} h &= \sum_r g_r\,|\delta|^r \\ \text{and} \quad H &= \sum_s G_s\,|\delta|^s \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4.9)$$

$$r,s = 0,1,2,\ \ldots$$

Equation (4.8) becomes, for the half-wing for which $y > 0$,

$$\bar{f}_+ = l\left\{\sum_r g_r\delta^r\right\} q_1 + (x-x_{o+})\,\frac{l}{c_m}\left\{\sum_s G_s\delta^s\right\} q_2 \quad \ldots\ldots\ldots\ldots(4.10)$$

It is now required to find the two generalised forces Q_1 and Q_2.

Thus,

$$* \quad Q_1 \cdot \delta q_1 = \iint_{S_+} \overline{\Delta p} \cdot (\delta\bar{f}_+)_{q_1}\, dx\, dy \quad \ldots\ldots\ldots(4.11)$$

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* 'S_+' indicates that the area of integration is over the wing planform for which $0 < s(x) < y$ only.

where $\left[\overline{\Delta} p \, dx \, dy\right]$ is the total (incremental) aerodynamic force at the point (x,y) on the wing and is given by equation (3.15) when the velocity potential derived for the assumed deformation function is substituted.

From (4.10),

$$(\delta\bar{f}_+)_{q_1} = l \left\{ \sum_r g_r \delta^r \right\} \delta q_1$$

and the force, Q_1, is seen to be built up from a sum of integrals of the form,

$$\left[Q_1\right]_r = l^2 \iint_{S_+} \overline{\Delta} p. \, g_r \, \delta^r . \, dx \, d\delta \; ....(4.12)$$

In the same way the force, Q_2, is expressed as a sum of integrals of the form

$$\left[Q_2\right]_s = \frac{l^2}{c_m} \iint (x-x_{o+}) \, \overline{\Delta} p. \, G_s. \, \delta^s. \, dx \, d\delta \quad (4.13)$$

It will be clear that many of the integrals (4.12) and (4.13) will be identical, apart from constant factors.

4.2. Coupled Modes

The deformation function, $\bar{f}$, now takes the form:

$$\bar{f}_+ = \sum_r \left\{ lh(\delta) + (x-x_{o+}) \frac{l}{c_m} H(\delta) \right\} . \, q_r \quad ..(4.14)$$

there now being r degrees of freedom.

The functions h and H are as defined before in equation (4.9) with $r = s$ so that equation (4.14) becomes,

$$\bar{f}_+ = \sum_r q_r \left\{ l \, g_r + (x-x_{o+}) \frac{l}{c_m} G_r \right\} \delta^r \quad ...(4.15)$$

and Q_r is given by :

$$Q_r . \, \delta q_r = \iint_{S_+} \overline{\Delta} p \, (\delta\bar{f}_+)_{q_r} \, dx \, dy .$$

Finally, from (4.15),

$$Q_r = l^2 \iint\limits_{S_+} \overline{\Delta p} \left(g_r + \frac{(x-x_{o+})}{c_m} \cdot G_r \right) \delta^r \cdot dx\, d\delta \qquad \text{............(4.16)}$$

This is a sum of integrals like (4.12) and (4.13).

5. Calculation of the Velocity Potential and the Generalised Forces for the Assumed Modes of Paragraph 4

The coefficients, P_n, in the series representation of $\bar{\phi}$ are given by (3.11) and since $\bar{f}(x,y)$ is expressed in polynomials of $|\delta|$, and hence of $|y|$, integrals of the following type are met with:

$$\int \cos^s \zeta_1 \sin \zeta_1 \; se_n(\zeta_1, k)\, d\zeta_1 \; . \qquad \text{............(5.1)}$$

Such integrals can be written as the sum of integrals of the form

$$\int \sin p\zeta_1 \cdot se_n(\zeta_1, k)\, d\zeta_1 \qquad \text{............(5.2)}$$

Using the Fourier Series expansion of $se_n(\zeta, k)$ integrals such as (5.2) become;

$$\sum_{r=1}^{\infty} B_r^{(n)}(k) \int \sin r\zeta_1 \cdot \sin p\zeta_1\, d\zeta_1 \qquad \text{........(5.3)}$$

When s is even (equation 5.1), the limits on (5.3) are 0 to π, quite straight forwardly, as indicated by equation (3.11) and only a finite number of terms is obtained for (5.1). When s is odd, owing to the assumption of symmetry, the limits on (5.3) reduce to 0 to $\pi/2$ (or $\pi/2$ to π) and an infinite series is obtained for (5.3), and hence for (5.1). However, only a few terms need be retained in practice.

The velocity potential on the wing, $\bar{\phi}$, is now fully determined and the corresponding loading is given by equation (3.15).

The generalised forces give rise to integrals like (omitting constants),

$$\iint\limits_{S_+} \left(\bar{\phi}_x + \frac{i\nu}{U_\infty} \bar{\phi}_w \right) \delta^r \, dx\, d\delta \qquad \text{............(5.4)}$$

$$\text{and} \quad \iint\limits_{S_+} x \left({}_{w}\bar{\varnothing}_x + \frac{i\nu}{U_\infty} {}_{w}\bar{\varnothing} \right) \delta^s \cdot dx\, d\delta \quad \ldots\ldots\ldots\ldots (5.5)$$

These integrations must, in general, be done by a graphical or numerical means, except when the time-dependent term in the potential equation (3.6) is suppressed and the Mathieu functions take on their degenerate forms.

The application of the analysis to antisymmetric flutter modes follows the same general lines as given for symmetric modes.

5.1. Equivalent Constant Flutter Derivatives

By analogy with the flutter derivatives of two-dimensional (strip) theory it is possible to define a set of 'equivalent constant derivatives'.

These derivatives are constant over the span of the wing and give the correct generalised forces when interpreted in the conventional sense.

The lift and moment on a strip of unit width are defined in terms of derivatives such as

$$l_z,\ l_{\dot{z}},\ l_\alpha,\ l_{\dot{\alpha}} \quad *$$

$$m_z,\ m_{\dot{z}},\ m_\alpha,\ m_{\dot{\alpha}} \qquad \text{(ref. 18)}$$

where the 'stiffness' derivatives include the 'inertia' derivatives, $l_{\ddot{z}}$, $m_{\ddot{z}}$, $l_{\ddot{\alpha}}$, $m_{\ddot{\alpha}}$.

Equivalent constant derivatives,

$$(l_z)_{rs},\ (l_{\dot{z}})_{rs},\qquad\text{etc.}$$

$$(m_z)_{rs},\ (m_{\dot{z}})_{rs},$$

are defined from the force coefficients of equation (4.4) in Appendix II.

As with the force coefficients the first suffix refers to the generalised force and the second to the mode.

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* See Appendix II for discussion of sign convention.

Apart from the analogy with 'two-dimensional' derivatives the concept of equivalent constant derivatives is useful in that it facilitates direct comparison of sets of derivatives derived for different modes. For example, direct derivatives in one freedom are made independent of the modes in other freedoms. (See Appendix III).

6. Representative Results

The preceding analysis has been applied to the case of a triangular wing (Fig. 2, $\lambda = 0$) using uncoupled modes.

Equivalent Constant Derivatives have been calculated for the flexural modes;

$$h(\delta) = |\delta|^r \quad ; \quad r = 0, 1, 2.$$

and torsional modes;

$$H(\delta) = |\delta|^s \quad ; \quad s = 0, 1.$$

Modes such as these have been taken in pairs, one in flexure, $|\delta|^{r_1}$ and one in torsion, $|\delta|^{s_1}$, giving six 'sets' of derivatives.

The accompanying table of numerical results (Table I) shows the order of the derivatives and their signs (for $m = \frac{1}{2}$) and a set of general expressions for the derivatives is given in Appendix III together with the results of a calculation on a cropped delta for $r_1 = s_1 = 0$ only.

The 'damping' derivatives $l_{\dot{a}}$, $m_{\dot{z}}$ and $m_{\dot{a}}$ are plotted against 'm' in Figs. 8 9 and 10

By taking $r_1 = s_1 = 0$ and $m = 1$ the derivatives are obtained for a rigid pitching and plunging wing referred to the trailing edge - use of the usual transformation formulae then refers the derivatives to any other axis.

This has been done for a triangular wing ($\lambda = 0$), a cropped delta ($\lambda = 1/7$) and a rectangular wing ($\lambda = 1$) for an axis at $0.500\ c_r$ and the results are presented in Table II.

In Fig. 11 the 'cross-damping' derivatives $m_{\dot{z}}$, $l_{\dot{a}}$ have been plotted against 'm' for these wings.

In this case of a pitching and plunging wing the generalised forces Q_1 and Q_2 have simple interpretations and do in fact represent the unsteady lift and moment amplitudes on the <u>complete</u> wing, i.e.,

$$\frac{L}{\rho_\infty U_\infty^2 S} = \frac{Q_1}{\rho_\infty U_\infty^2 l^3} \times \frac{\tau l^2}{c_r}, \quad (\text{+ve upwards})$$

and

$$\frac{M}{\rho_\infty U_\infty^2 S c_r} = \frac{Q_2}{\rho_\infty U_\infty^2 l^3} \times \frac{\tau l^2}{c_r^2}, \quad (\text{+ve nose downwards}) \qquad \text{............(6.1)}$$

The expressions for lift and moment will be in terms of the dimensionless amplitudes,

$$\left(\frac{z_o}{c_r}\right) \text{ and } a_o$$

and the relevant frequency parameter will be,

$$\omega_r = \frac{\nu c_r}{U_\infty} = \omega_m \left(\frac{2}{1+\lambda}\right).$$

It is convenient in this connection to define a set of force coefficients for rigid motions only since in later paragraphs unsteady lifts and moments on rigid bodies and wing-body combinations are considered.

Coefficients L_z, $L_{\dot{z}}$ ... etc. are defined by the expressions;

$$\frac{L}{\rho_\infty U_\infty^2 S} = -(L_z + i\omega_r L_{\dot{z}})\left(\frac{z_o}{c_r}\right) - (L_a + i\omega_r L_{\dot{a}})\, a_o$$

and

$$\frac{M}{\rho_\infty U_\infty^2 S} = (M_z + i\omega_r M_{\dot{z}})\left(\frac{z_o}{c_r}\right) + (M_a + i\omega_r M_{\dot{a}})\, a_o \qquad \text{............(6.2)}$$

and these will be referred to as 'Rigid' force coefficients.

As for the definitions giving the equivalent constant derivatives (Appendix II) these rigid force coefficients are signed to agree with the normal British flutter sign convention

TABLE I

$-l_z\,(AR)^{-1}$				$+l_\alpha\,(AR)^{-1}$			
s \ r	0	1	2	s \ r	0	1	2
0	.0296	.00768	.00374	0	.781	.996	1.18
1	.0296	.00768	.00374	1	1.00	.955	1.00
$m_z\,(AR)^{-1}$				$m_\alpha\,(AR)^{-1}$			
s \ r	0	1	2	s \ r	0	1	2
0	.00376	.00431	.00450	0	.0530	.0530	.0530
1	.00264	.00214	.00200	1	.0334	.0334	.0334
$+l_{\dot z}\,(AR)^{-1}$				$+l_{\dot\alpha}\,(AR)^{-1}$			
s \ r	0	1	2	s \ r	0	1	2
0	.786	.954	.981	0	.735	1.02	1.30
1	.786	.954	.981	1	.915	.905	.817
$m_{\dot z}\,(AR)^{-1}$				$-m_{\dot\alpha}\,(AR)^{-1}$			
s \ r	0	1	2	s \ r	0	1	2
0	.0542	-.0225	-.135	0	.114	.114	.114
1	.0900	.0379	-.0405	1	.195	.195	.195

Derivatives for $m = \frac{1}{2}$; $\omega_m = 0.2$

TABLE II

Equivalent Constant Derivatives for rigid wings plunging and pitching about an axis at $\frac{1}{2} c_r$

Equivalent Constant Derivatives	Triangular Wing ($\lambda=0$)	Cropped Delta ($\lambda=1/7$)	Rectangular Wing ($\lambda=1$)
l_z	$-\frac{\pi}{6}\omega_m^2$ (.524)	$-.585\,\omega_m^2$	$-\frac{\pi}{4}\omega_m^2$ (.785)
$l_{\dot{z}}$	$+\frac{\pi}{4}$ (.785)	$+.785$	$+\frac{\pi}{4}$ (.785)
l_α	$-\frac{\pi}{12}\omega_m^2 + \frac{\pi}{4}$ (.262) (.785)	$-.245\omega_m^2 + .785$	$+\frac{\pi}{4}$ (.785)
$l_{\dot{\alpha}}$	$+\frac{5\pi}{16}$ (.981)	$+1.07$	$+\frac{3\pi}{8}$ (1.18)
m_z	$\frac{\pi}{12}\omega_m^2$ (.262)	$.245\,\omega_m^2$	0
$m_{\dot{z}}$	$-\frac{\pi}{16}$ (.197)	$-.0345$	$\frac{\pi}{8}$ (.392)
m_α	$\frac{\pi}{20}\omega_m^2 - \frac{\pi}{16}$ (.157) (.197)	$.137\omega_m^2 - .0845$	$\frac{\pi}{48}\omega_m^2 + \frac{\pi}{8}$ (.0655)(.392)
$m_{\dot{\alpha}}$	$-\frac{\pi}{8}$ (.392)	$-.386$	$-\frac{\pi}{16}$ (.197)

NOTE: Figures in brackets are decimal equivalents of fractions of π.

7. The Pitching and Plunging Slender Body of Revolution

A set of flutter force coefficients for a pointed slender body of revolution can be calculated in an analogous manner to those of the wing by adopting polar coordinates instead of elliptical coordinates when solving the potential equation and in the specification of the boundary conditions.

Probably the only case of interest is the rigid pitching and plunging body and accordingly this case will be dealt with. The cartesian coordinate system for the body is the same as for the wing and is shown in Figure 3

In each cross-section, x = const., take polar coordinates;

$$\left.\begin{aligned} y &= r\cos\zeta \\ z &= r\sin\zeta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(7.1)$$

then the potential equation ($\cancel{3.6}$ 3.1) transforms to;

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial\bar{\phi}}{\partial r}\right) + \frac{1}{r^2}\cdot\frac{\partial^2\bar{\phi}}{\partial\zeta^2} + \frac{\nu^2}{a^2}\bar{\phi} = 0 \quad \ldots\ldots\ldots(7.2)$$

Consider the body movements to consist of a vertical translation (+ve upwards) and pitching about an axis,

$$x = x_o = m_B\, l_B \quad \ldots\ldots\ldots\ldots\ldots(7.3)$$

parallel to the y-axis (nose-down pitching +ve).

By analogy with equation (4.6) we define z_o to be the amplitude of the displacement of the point, $x = x_o$, on the body axis and a_o to be the amplitude of the inclination of the body axis to the Ox axis. Then the motions produce at a point, x, on the body axis the total upward displacement,

$$(z_o + (x-x_o)\,a_o)\,e^{i\nu t}\ .$$

Taking the body length, l_B, as a reference length the vertical velocity (equation 3.4) is,

$$\bar{w}(x) = i\nu\left\{l_B\left(\frac{z_o}{l_B}\right) + \left(x-x_o + \frac{U}{i\nu}\right)a_o\right\} \quad \ldots\ldots\ldots(7.4)$$

Writing $\Gamma(x)$ for the local cross-sectional area of the body, the potential near the body takes the form;

$$(\bar{\phi})_{r \to R} = - \bar{w}(x) \frac{\Gamma(x)}{\pi r} . \sin \zeta \quad \ldots\ldots\ldots\ldots(7.5)$$

The pressure at any point on the surface of the body is

$$\bar{p} = \rho_{\infty} U_{\infty} \left\{ \frac{\partial II}{\partial x} + \frac{i\nu}{U_{\infty}} .II \right\} \frac{\sin \zeta}{\pi r}$$

where

$$II(x) = - \bar{w}(x) \Gamma(x) \quad \ldots\ldots\ldots\ldots(7.6)$$

The unsteady lift and moment, L and M follow by integration of (7.6) along the body.

The 'rigid' force coefficients of equations (6.2) have been calculated for a cylindrical body, with a conical nose as shown in Fig. 4. These are based on the Aspect Ratio of the geometrically similar wing having its root-chord equal to the length of the body and

$$\frac{b}{2} = \frac{R_o}{\sigma} \quad \text{(see Figure 7)}$$

where R_o is the maximum (base) radius of the body.

The coefficients are given in Appendix IV.

By putting $\sigma = 1$ and $\Lambda = (1-\lambda)$ it will be seen that the expressions of Appendix IV are identical with those that would be given for the rigid cropped delta wing using the equivalent constant derivatives of Appendix III with $B = 0$ and equations (6.1).

8. The Slender Wing-Body Combination

8.1 The rigid pitching and plunging combination

A set of 'rigid' force coefficients will now be derived for the slender wing-body combination shown in Figure 8.

This problem will be dealt with rather differently from the wing and slender body cases in that the velocity potential will be found, not directly as a solution of Laplace's Equation, but from the two-dimensional potential for incompressible flow normal to a flat plate.

The required potential will not generally satisfy the two-dimensional wave equation (2.2) and hence the solution will be subject to similar restrictions as the wing solution for $k \to 0$.

Using the Joukowski Transformation (see Fig. 6) the velocity potential for the flow around the body configuration of Fig. 6 (see ref. 4) due to a motion w of any section in a fluid at rest can easily be found and conveniently expressed in two parts,

$$_B(\phi) = -w\sqrt{S^2\left(1+\frac{R^2}{S^2}\right)^2 - 4y^2} + w\sqrt{R^2-y^2}$$

and

$$_W(\phi) = -w\sqrt{S^2\left(1+\frac{R^4}{S^4}\right) - y^2\left(1+\frac{R^4}{y^4}\right)}$$

$$\ldots\ldots\ldots\ldots(8.1)$$

where $B(\phi)$ is the potential on the body $(r=R)$ and $w(\phi)$ is the potential on the wing $(\zeta = 0 \text{ or } \pi,\ y \equiv r)$.

It will be clear that the force coefficients for the wing-body combination of Fig. 5 can be considered to be the addition of two sets of force coefficients; viz.,

(i) the force coefficients for a triangular wing on a cylindrical body, downstream of the lateral plane through the wing leading edge and body junction.

(ii) the force coefficients for a pointed body upstream of the wing leading edge.

The coefficients (ii) have been calculated in section 7 (Appendix IV).

The coefficients (i) can be calculated using (8.1) with the axes and notation of Figure 7.

For the combination, the velocity, $\bar{w}$, will take the same form as for the body alone, i.e. equation (7.4), thus,

$$\bar{w}(x) = i\nu\left\{\left[c_r\left(\frac{z_o}{c_r}\right) - x_o\alpha_o + \frac{U}{i\nu}\alpha_o\right] + \alpha_o x\right\} \quad (8.2)$$

The loading distribution is given by the pressure equation (3.15) and lift and moment by:

$$L = \int_{\sigma c_r}^{c_r}\left\{\int_o^R (\bar{\Delta p})_B . dy + \int_R^S (\bar{\Delta p})_W . dy\right\} dx$$

and

$$M = \int_{\sigma c_r}^{c_r}\left\{\int_o^R (\bar{\Delta p})_B . dy + \int_R^S (\bar{\Delta p})_W . dy\right\}(x-x_o)\, dx$$

$$\ldots\ldots\ldots\ldots(8.3)$$

Complete expressions for lift and moment on the clyindrical body and triangular wing lead to the 'rigid' force coefficients which are given in Appendix V. These coefficients like those of the body are based on the triangular wing having root chord, c_r, and maximum semispan $s_o = b/2 = R_o/\sigma$.

The force coefficients for the whole wing-body combination of Figure 5 are given in Appendix VI and the variation of the 'damping' force coefficients $L_{\dot{z}}$, $L_{\dot{\alpha}}$, $M_{\dot{z}}$, $M_{\dot{\alpha}}$ with σ is shown graphically in Fig. 12.

In adding the appropriate coefficients of Appendices IV and V the definitions of Figure 5 and Appendix VI were used and again the triangular wing is used as a basis.

9. Discussion

The use of the 'Slender Body Theory' for unsteady flow problems leads to a solution for the aerodynamic forces which does not involve long computation and many geometrical and other parameters can be carried along in the analysis without having to be specified definitely at the outset.

The restrictions of the theory as discussed in section 2 seem to be somewhat severe but there is evidence to show (ref. 8) that for a rigid triangular wing of aspect ratio, 1, at a Mach number of 1.25 and for a frequency parameter, ω_r, up to 6, the theory appears to be quite valid. Furthermore, results for an aspect ratio of 0.5 show that when the time derivative terms are neglected the results differ from those given by the complete solution only if $\omega_r > 2$ for a Mach number range of 0 - 1.25 (rigid triangular wing).

Owing to the need to evaluate several terms of the Mathieu function series when deriving the full solution it is much longer than the simplified case (for small root frequency parameter) and it would always be worthwhile to question whether the full solution is really necessary in any specified case.

With the type of wing to which this analysis can be applied, it is very unlikely that the root frequency parameter will exceed about 0.5 so that in many cases the simplified approach would suffice.

The force coefficients γ,b,c of equation (4.4) are dependent on Mach number and frequency only through the parameter, k, in the general solution, consequently, in the simplified case which implies that $k \to 0$, the coeff-icients are independent of frequency and Mach number.

It has been found, both experimentally and theoretically that the variation of flutter force coefficients with frequency decreases as aspect ratio decreases so that this is not a surprising result from a theory which is correct for $Æ \to 0$.

The preceding remarks can be taken to apply equally well, in principle, to the wing-body combination.

It is interesting to note that the analysis used by Lawrence and Gerber (ref. 19) (subsonic) when taken to the limit $Æ \to 0$ gives results for a rigid wing which agree with those found here and by Garrick (ref. 14).

In this connection it is also interesting to study their results when plotted against aspect ratio. The slopes of the curves (force coefficients) at zero aspect ratio are correctly those given by 'Slender Body Theory' but, in general, the curves depart from their original tangents extremely rapidly. It might be suggested therefore that force coefficients derived using 'Slender Body Theory', if applied outside their range of reasonable validity, will give magnitudes which, in general, will be very different from the 'true' values. (See also refs. 20 and 21 for which $Æ = 3$).

Fig. 10 shows that for an axis at the trailing edge of the wing, no matter which torsional modes are chosen, the direct damping derivative $m_{\dot\alpha}$ is zero indicating that an undamped pitching oscillation would be possible - for all axis positions $0 \leq m \leq 1$ the derivative gives positive damping.

When $\omega_r \to 0$ the 'rigid' force coefficients give the values of lift and moment for the steady case as found by Jones, Spreiter and others (refs. 7 and 4.)

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APPENDIX I

The Cropped Delta -

Definitions and Geometrical Properties

See Figure 2 : -

Mean chord, $c_m = \frac{1}{2} c_r (1 + \lambda)$ (1)

Area, $S = b\, c_m$ (2)

Aspect Ratio, $AR = \frac{b^2}{S} = \frac{b}{c_m}$ (3)

Local semispan, $s = \frac{b}{2c_r(1-\lambda)} . x$ for,

$0 \leqslant x \leqslant (1-\lambda)c_r$ (4)

Reference section position, $l = \tau . \frac{b}{2}$ (5)

Spanwise parameter, $\delta = \frac{y}{l}$ (6)

Local chord, $c = \frac{2c_m}{(1+\lambda)} . (1 - \tau(1-\lambda)\delta)$ (7)

Ratio, $\frac{l}{c_m} = \frac{AR . \tau}{2}$ (8)

Reference Axis, $x_o = mc_r + \frac{2c_r}{b} (1-\lambda)(1-m) |y|$

$\equiv A + B |\delta|$ (9)

thus, $\frac{A}{c_m} = \frac{2m}{1+\lambda}$

and $\frac{bB}{lc_m} = 4 \frac{(1-\lambda)}{(1+\lambda)} (1-m)$ (10)

Frequency parameter, (Mean) $\omega_m = \frac{\nu c_m}{U_\infty}$ (11)

APPENDIX II

Equivalent Constant Derivatives - Definitions

The following equivalent constant derivatives are appropriate to a system with the usual British sign convention i.e. z-axis downward, lift positive upwards, moment and angle of attack positive nose-up.

(a) Uncoupled Modes

i. One mode in flexure, one mode in torsion

$$(-\gamma_{11}\omega_m^2 + c_{11}) = l_z \int_0^{1/\tau} h^2 \,.\, d\delta$$

$$(-\gamma_{12}\omega_m^2 + c_{12}) = l_a \int_0^{1/\tau} \frac{c}{c_m} h.H \, d\delta$$

$$(-\gamma_{21}\omega_m^2 + c_{21}) = (-m_z) \int_0^{1/\tau} \frac{c}{c_m} H.h \, d\delta$$

$$(-\gamma_{22}\omega_m^2 + c_{22}) = (-m_a) \int_0^{1/\tau} \left(\frac{c}{c_m}\right)^2 H^2 \,.\, d\delta$$

$$b_{11}\,\omega_m = \omega_m .\, l_{\dot z} \int_0^{1/\tau} \frac{c}{c_m} \,.\, h^2 \, d\delta$$

$$b_{12}\,\omega_m = \omega_m \, l_{\dot a} \int_0^{1/\tau} \left(\frac{c}{c_m}\right)^2 h.H. \, d\delta$$

$$b_{21}\,\omega_m = \omega_m \, (-m_{\dot z}) \int_0^{1/\tau} \left(\frac{c}{c_m}\right)^2 H.h \, d\delta$$

$$b_{22}\,\omega_m = \omega_m \, (-m_{\dot a}) \int_0^{1/\tau} \left(\frac{c}{c_m}\right)^3 H^2 \, d\delta \,.$$

ii. r modes in flexure, s modes in torsion

Derivatives such as $(l_z)_{rr^1}$ and $(m_z)_{sr^1}$

will be defined in the same manner as in (i) by integrals such as,

$$\int_0^{1/\tau} h_r . h_{r'} \, d\delta \quad \text{and} \quad \int_0^{1/\tau} \frac{c}{c_m} H_s . h_{r'} . d\delta \quad \text{respectively.}$$

(b) <u>Coupled Modes</u>

$$(-\gamma_{rs}\,\omega_m^2 + c_{rs})_{l_z} = (l_z)_{rs} \int_0^{1/\tau} h_r . h_s \, d\delta$$

$$(-\gamma_{rs}\,\omega_m^2 + c_{rs})_{l_a} = (l_a)_{rs} \int_0^{1/\tau} \frac{c}{c_m} h_r H_s \, d\delta$$

etc.

$$(b_{rs}\,\omega_m)_{l_{\dot{a}}} = \omega_m (l_{\dot{a}})_{rs} \int_0^{1/\tau} \left(\frac{c}{c_m}\right)^2 h_r H_s \, d\delta$$

etc. - by analogy with (a)i.

A term such as $(-\gamma_{rs}\,\omega_m^2 + c_{rs})_{l_z}$ is obtained from the real part of the term in the total expression for Q_r which involves both h_r and h_s ; and $(b_{rs}\,\omega_m)_{l_{\dot{z}}}$ is obtained from the imaginary part of the same term: other terms are obtained in a similar manner.

APPENDIX III

Results of a calculation using uncoupled modes

(a) Equivalent Constant Derivatives for Triangular Wing (λ=0)

Flexural mode $h(\delta) = |\delta|^r$, $r = 0,1,2$

Torsional mode $H(\delta) = |\delta|^s$, $s = 0,1.$

$r=0, s=0$

$$l_z = -\frac{\pi}{6}\,\omega_m^2\, AR$$

$$l_{\dot{z}} = +\frac{\pi}{4} \cdot AR$$

$$l_a = -\left\{\frac{\pi}{6} \cdot (.863 - 1.36m)\,\omega_m^2 - \frac{\pi}{4}\right\} AR$$

$$l_{\dot{a}} = +\frac{3\pi}{16}\,(1.82 - 1.15m)\, AR$$

$$m_z = \frac{\pi}{2}\,(.288 - .455m)\,\omega_m^2 \cdot AR$$

$$m_{\dot{z}} = -\frac{3\pi}{4}\,(.121 - .288m)\, AR$$

$$m_a = \frac{3\pi}{4}\left\{2(.0710 - .200m + .158m^2)\,\omega_m^2 - \frac{1}{2}\,(.243 - .576m)\right\} AR$$

$$m_{\dot{a}} = \frac{\pi}{2}\,(-.353 + .703m - .354m^2)\, AR$$

$r = 1, s = 0$

$$l_z = -\frac{3}{5\pi}\,\omega_m^2\, AR$$

$$l_{\dot{z}} = +\frac{3}{\pi} \cdot AR$$

$$l_a = +3\left\{(-.139 + .205m)\,\omega_m^2 + .333\right\} AR$$

$$l_{\dot{\alpha}} = +\,3\,(.516 - .349m)\ A\!R$$

$$m_z = 3\,(.139 - .205m)\,\omega_m^2 \cdot A\!R$$

$$m_{\dot{z}} = -\,3\,(.182 - .349m)\ A\!R$$

$$m_\alpha = \frac{3\pi}{4}\left\{(.142 - .400m + .317m^2)\,\omega_m^2 + (-\,.121 + .288m)\right\} A\!R$$

$$m_{\dot{\alpha}} = \frac{\pi}{2}\,(-\,.353 + .703m - .354m^2)\ A\!R$$

<u>r = 2, s = 0</u>

$$l_z = -\,\frac{5\pi}{168} \cdot \omega_m^2\ A\!R$$

$$l_{\dot{z}} = +\,\frac{5\pi}{16} \cdot A\!R$$

$$l_\alpha = -\,6\pi\left\{\frac{1}{24}\,(.490 - .690m)\,\omega_m^2 - .0625\right\} A\!R$$

$$l_{\dot{\alpha}} = +\,\frac{15\pi}{16}\,(.690 - .490m)\ A\!R$$

$$m_z = \frac{3\pi}{2}\,(.082 - .115m)\,\omega_m^2\ A\!R$$

$$m_{\dot{z}} = -\,\frac{15\pi}{4}\,(.0730 - .123m)\ A\!R$$

$$m_\alpha = \frac{3\pi}{4}\left\{2(.0711 - .200m + .158m^2)\,\omega_m^2 - (.121 - .288m)\right\} A\!R$$

$$m_{\dot{\alpha}} = -\,\pi\,(.178 - .354m + .176m^2)\ A\!R$$

<u>r = 0, s = 1.</u>

$$l_z = -\,\frac{\pi}{6}\,\omega_m^2\ A\!R$$

$$l_{\dot{z}} = +\,\frac{\pi}{4} \cdot A\!R$$

$$l_a = -3\left\{\tfrac{1}{2}(.220 - .353m)\,\omega_m^2 - .333\right\} \mathcal{R}$$

$$l_{\dot{a}} = +3\,(.442 - .274m)\,\mathcal{R}$$

$$m_z = 3\,(.110 - .176m)\,\omega_m^2 \,.\, \mathcal{R}$$

$$m_{\dot{z}} = -3\,(.107 - .274m)\,\mathcal{R}$$

$$m_a = \frac{15}{2}\left\{8\,(.00359 - .0100m + .00800m^2)\,\omega_m^2 - \frac{4}{\pi}\,(.0430 - .0930m)\right\} \mathcal{R}$$

$$m_{\dot{a}} = 30\,(-.0250 + .0492m - .0243m^2)\,\mathcal{R}$$

------

r = 1, s = 1.

$$l_z = -\frac{3}{5\pi}\,.\,\omega_m^2\,\mathcal{R}$$

$$l_{\dot{z}} = +\frac{3}{\pi}\,.\,\mathcal{R}$$

$$l_a = -\left\{4\,(.0592 - .0911m)\,\omega_m^2 - \frac{3}{\pi}\right\} \mathcal{R}$$

$$l_{\dot{a}} = +\frac{5}{2}\,(0.540 - 0.355m)\,\mathcal{R}$$

$$m_z = 6\,(.0395 - .0612m)\,\omega_m^2 \,.\, \mathcal{R}$$

$$m_{\dot{z}} = -\frac{15}{2}\,(.0540 - .118m)\,\mathcal{R}$$

$$m_a = \frac{15}{2}\left\{(.0287 - .0800m + .0640m^2)\,\omega_m^2 - \frac{4}{\pi}\,(.0430 - .0930m)\right\} \mathcal{R}$$

$$m_{\dot{a}} = 30\,(-.0250 + .0492m - .0243m^2)\,\mathcal{R}$$

-----

r = 2, s = 1.

$$l_z = -\frac{5\pi}{168}\ \omega_m^2 \cdot Æ$$

$$l_{\dot z} = -\frac{5\pi}{16} \cdot Æ$$

$$l_a = -\left\{\frac{5}{2}\ (.0792 - .117m)\ \omega_m^2 - 1.00\right\}\ Æ$$

$$l_{\dot a} = +\frac{15}{2}\ (.206 - .194m)\ Æ$$

$$m_z = -40\ (-.00495 + .00733m)\ \omega_m^2\ Æ$$

$$m_{\dot z} = 30\ (-.0359 + .0691m)\ Æ$$

$$m_a = \frac{15}{2}\left\{(.0287 - .0800m + .0640m^2)\ \omega_m^2 - \frac{4}{\pi}\ (.0430 - .0930m)\right\}\ Æ$$

$$m_{\dot a} = 30\ (-.0250 + .0492m - .0243m^2)\ Æ$$

(b) Equivalent Constant Derivatives for Cropped Delta (λ, general)

r = 0, s = 0 only

$$l_z = -\pi\,\beta_1\,\omega_m^2\ Æ$$

$$l_{\dot z} = +\frac{\pi}{4}\ Æ$$

$$l_a = -\left\{\pi(\beta_3 - \frac{A}{c_m}\beta_1 - \frac{bB}{\pi l c_m}\beta_2)\ \omega_m^2 - \frac{\pi}{4}\right\}\ Æ$$

$$l_{\dot a} = +\frac{3\pi}{16}\ \frac{(1+\lambda)^2}{1+\lambda+\lambda^2}\ \left(\frac{2}{1+\lambda} + 4\beta_1 - \frac{A}{c_m} - \frac{2\ b\ B}{3\pi l c_m}\right)\ Æ$$

$$m_z = \pi\ (\beta_3 - \frac{A}{c_m}\beta_1 - \frac{b\ B}{\pi\ l\ c_m}\ \beta_2)\ \omega_m^2 \cdot Æ$$

$$m_{\dot{z}} = \frac{3\pi}{4} \frac{(1+\lambda)^2}{1+\lambda+\lambda^2} \left(-\beta_4 + \frac{A}{4c_m} + \frac{b\,B}{6\pi l c_m} \right) \text{Æ}$$

$$\begin{aligned} m_{\alpha} = \frac{3\pi}{4} \cdot \frac{(1+\lambda)^2}{1+\lambda+\lambda^2} \Bigg\{ \Bigg[& \left(2\beta_8 - \frac{A}{c_m}\beta_3 - \frac{2\,b\,B}{\pi\,l\,c_m}\beta_9 \right) \\ & - \frac{A}{c_m}\left(\beta_3 - \frac{A}{c_m}\beta_1 - \frac{b\,B}{\pi l c_m} \cdot \beta_2 \right) \\ & - \frac{b\,B}{\pi l c_m}\left(\beta_6 - \frac{A}{c_m}\beta_2 - \frac{b\,B}{\pi l c_m} \cdot \beta_{10} \right) \Bigg] \omega_m^2 \\ & + \left(-\beta_4 + \frac{A}{4c_m} + \frac{b\,B}{6\pi l c_m} \cdot \right) \Bigg\} \text{Æ} \end{aligned}$$

$$\begin{aligned} m_{\dot{\alpha}} = \frac{\pi(1+\lambda)^3}{2(1+\lambda+\lambda^2+\lambda^3)} \Bigg\{ & -\left(\beta_7 + \beta_3 - \frac{A}{c_m}\beta_4 - \frac{b\,B}{\pi l c_m} \cdot \beta_5 \right) \\ & + \frac{A}{4c_m}\left(\frac{2}{1+\lambda} + 4\beta_1 - \frac{A}{c_m} - \frac{2\,b\,B}{3\pi l c_m} \right) \\ & + \frac{b\,B}{6\pi l c_m} \cdot \left(\frac{2}{1+\lambda} + 6\beta_2 - \frac{A}{c_m} - \frac{3\,b\,B}{4\pi l c_m} \right) \Bigg\} \end{aligned}$$

where;

$$\beta_1 = \frac{1 + 2\lambda}{6(1+\lambda)}$$

$$\beta_2 = \frac{1 + 3\lambda}{12(1+\lambda)}$$

$$\beta_3 = \frac{1 + 2\lambda - \lambda^2}{4(1+\lambda)^2}$$

$$\beta_4 = \frac{1 - \lambda}{3(1+\lambda)} = \frac{4}{3} \cdot \beta_5$$

$$\beta_6 = \frac{1 + 8\lambda - 4\lambda^2}{15(1+\lambda)^2}$$

$$\beta_7 = \frac{3 - 2\lambda + \lambda^2}{4(1 + \lambda)^2}$$

$$\beta_8 = \frac{(1 + 2\lambda - 2\lambda^2 + 2/3\,\lambda^3)}{5(1+\lambda)^3}$$

$$\beta_9 = \frac{(1 + 3\lambda - 3/2\,\lambda^2)}{15(1+\lambda)^2}$$

$$\beta_{10} = \frac{1 + 4\lambda}{20(1+\lambda)}$$

Expressions for the above derivatives when $\lambda = 1/7$ are:

$$l_z = -\,.585\,\omega_m^2 \cdot AR$$

$$l_{\dot z} = +\frac{\pi}{4}\ AR$$

$$l_a = -\left\{\pi(.141 - .226m)\,\omega_m^2 - \frac{\pi}{4}\right\} AR$$

$$l_{\dot a} = +\,.211\pi(1.86 - 1.12m)\ AR$$

$$m_z = \pi(.141 - .226m)\,\omega_m^2\ AR$$

$$m_{\dot z} = -\,.845\pi\,(.0944 - .281m)\ AR$$

$$m_a = .845\pi\left\{(.159 - .384m + .284m^2)\,\omega_m^2 - (.0944 - .281m)\right\} AR$$

$$m_{\dot a} = .645\pi\,(-\,.325 + .650m - .325m^2)\ AR$$

APPENDIX IV

'Rigid' Force Coefficients for a Slender Body of Revolution with conical nose

(see Figure 4)

Definitions:

Length of body $= l_B$

Length of conical nose $= \Lambda \, l_B$

Reference axis at distance $m_B \, l_B$ from nose

Frequency Parameter $\omega_B = \nu l_B / U_\infty$.

The force coefficients are based on the wing having a geometrically similar planform to the body, thus;

$$R_o = \frac{b}{2} \cdot \sigma \quad \text{and} \quad l_B = c_r \ ,$$

so that $\left(\dfrac{z_o}{l_B}\right) \equiv \left(\dfrac{z_o}{c_r}\right)$ and $\omega_B = \dfrac{\nu c_r}{U_\infty}$.

$$L_z = -\frac{\pi}{4}\sigma^2 \left(1 - \frac{2\Lambda}{3}\right) \omega_B^2 \cdot A\!\!R$$

$$L_{\dot z} = +\frac{\pi}{4}\sigma^2 \cdot A\!\!R$$

$$L_\alpha = -\frac{\pi}{4}\sigma^2 \left\{ \left(\frac{2\Lambda m_B}{3} + \frac{1}{2} - m_B - \frac{\Lambda^2}{4}\right) \omega_B^2 - 1 \right\} A\!\!R$$

$$L_{\dot\alpha} = +\frac{\pi}{4}\sigma^2 \left(2 - m_B - \frac{2\Lambda}{3}\right) A\!\!R$$

$$M_z = \frac{\pi}{4}\sigma^2 \left(\frac{2\Lambda m_B}{3} + \frac{1}{2} - m_B - \frac{\Lambda^2}{4}\right) \omega_B^2 \cdot A\!\!R$$

$$M_{\dot z} = -\frac{\pi}{4}\sigma^2 \left(\frac{2\Lambda}{3} - m_B\right) A\!\!R$$

$$M_\alpha = \frac{\pi}{4}\sigma^2 \left\{ \left(\frac{1}{3} - \frac{2\Lambda^3}{15} - m_B + m_B^2 + \frac{m_B\Lambda^2}{2} - \frac{2\Lambda m_B^2}{3}\right) \omega_B^2 - \left(\frac{2\Lambda}{3} - m_B\right) \right\} A\!\!R$$

$$M_{\dot\alpha} = -\frac{\pi}{4}\sigma^2 \, (1 - m_B)^2 \, A\!\!R$$

APPENDIX V

'Rigid' Force Coefficients for Triangular Wing on Cylindrical Body

(see Figure 7)

$$\omega_c \equiv \omega_r \cdot = \frac{\nu c_r}{U}$$

$$L_z = -\frac{\pi}{4}\left(\frac{1}{3} - \sigma^2 - \sigma^4 + \frac{5}{3}\sigma^3\right)\omega_c^2 \;\mathit{AR}$$

$$L_{\dot z} = +\frac{\pi}{4}\left(1 - \sigma^2\right)^2 \mathit{AR}$$

$$L_\alpha = -\frac{\pi}{4}\left\{\left[\frac{1}{4}\left(1 - 2\sigma^2 + \sigma^4 - 4\sigma^4 \ln\sigma\right) - m_c\left(\frac{1}{3} - \sigma^2 + \frac{5}{3}\sigma^3 - \sigma^4\right)\right]\omega_c^2 - \left(1 - \sigma^2\right)^2\right\}\mathit{AR}$$

$$L_{\dot\alpha} = +\frac{\pi}{4}\left\{\frac{2}{3}\left(2 - 3\sigma^2 + \sigma^3\right) - m_c\left(1 - \sigma^2\right)^2\right\}\mathit{AR}$$

$$M_z = \frac{\pi}{4}\left\{\frac{1}{4}\left(1 - 2\sigma^2 + \sigma^4 - 4\sigma^4 \ln\sigma\right) - m_c\left(\frac{1}{3} - \sigma^2 + \frac{5}{3}\sigma^3 - \sigma^4\right)\right\}\omega_c^2 \cdot \mathit{AR}$$

$$M_{\dot z} = -\frac{\pi}{2}\left\{\frac{1}{3}\left(1 - 4\sigma^3 + 3\sigma^4\right) - \frac{m_c}{2}\left(1 - \sigma^2\right)^2\right\}\mathit{AR}$$

$$M_\alpha = \frac{\pi}{4}\left\{\left[\left(\frac{1}{5} - \frac{\sigma^3}{3} + \sigma^4 - \frac{13}{15}\sigma^5\right) - \frac{m_c}{2}\left(1 - 2\sigma^2 + \sigma^4 - 4\sigma^4 \ln\sigma\right) + m_c^2\left(\frac{1}{3} - \sigma^2 + \frac{5}{3}\sigma^3 - \sigma^4\right)\right]\omega_c^2 - 2\left[\frac{1}{3}\left(1 - 4\sigma^3 + 3\sigma^4\right) - \frac{m_c}{2}\left(1-\sigma^2\right)^2\right]\right\}\mathit{AR}$$

$$M_{\dot\alpha} = -\frac{\pi}{2}\left\{\frac{1}{2}\left(1 - \sigma^2\right) - m_c\left(1 - \sigma^2 - \sigma^3 + \sigma^4\right) + \frac{m_c^2}{2}\left(1 - \sigma^2\right)^2\right\}\mathit{AR}$$

APPENDIX VI

'Rigid' Force Coefficients for a Slender Wing-Body Combination

(see Figure 5)

Definitions.-

$$\text{Total length of combination} = Vc_r \tag{1}$$

$$\text{Ratio of body length to wing root chord} = \frac{l_B}{c_r} = T \tag{2}$$

But $Vc_r = (l_B + c_r(1-\sigma))$, see figure 5,

so that from (2),

$$T = V - (1 - \sigma) \; . \tag{3}$$

Also ,
$$\left(\frac{z_o}{c_r}\right)_B = \left(\frac{z_o}{l_B}\right) = \frac{1}{T}\left(\frac{z_o}{c_r}\right) \tag{4}$$

and

$$\omega_B = T \cdot \omega_c \tag{5}$$

$$\omega_c \equiv \omega_r = \frac{\nu c_r}{U_\infty}$$

It is essential that the position of the reference axis should be unique when measured from the apex of the wing $(m_c\ c_r)$ and from the nose of the body $(m_B \cdot l_B)$. This requires that (see figure 5) :

$$m_B\, l_B = l_B - \sigma c_r + m_c c_r \; .$$

i.e.
$$m_B = 1 - \frac{(\sigma - m_c)}{T} \tag{6}$$

or
$$m_B = \frac{V - 1 + m_c}{V - 1 + \sigma} \tag{7}$$

$$\text{Length of conical nose} = \Lambda\, l_B = \Lambda\, Tc_r \tag{8}$$

This gives nose-length as a constant proportion of body length ahead of wing root. If a nose length which is a constant proportion of total combination length is stipulated then Λ must be replaced by an expression of the form;

$$\Lambda' = \Lambda\left\{\frac{V - 1 + \sigma}{V}\right\} = \text{const.} \tag{9}$$

Note: In the above definitions: $T \neq 0$
$V \neq (1-\sigma)$ (10)

'Rigid' Force Coefficients, based on the triangular wing, for the wing-body combination of figure 5 are then:

$$L_z = -\frac{\pi}{4}\left\{\frac{1}{3} + \sigma^2\left[\left(1 - \frac{2\Lambda}{3}\right)T - 1\right] + \sigma^3\,\frac{5}{3} - \sigma^4\right\}\omega_c^2 \cdot A\!R$$

$$L_{\dot z} = +\frac{\pi}{4}\left(1 - \sigma^2 + \sigma^4\right) A\!R$$

$$\begin{aligned} L_a = -\frac{\pi}{4}\Bigg[\Bigg\{&\left(\frac{1}{4} - \frac{m_c}{3}\right) \\ &+ \sigma^2\left[(m_c - \tfrac{1}{2}) - Tm_c\left(1 - \frac{2\Lambda}{3}\right) - T^2\left(\tfrac{1}{2} - \frac{2\Lambda}{3} + \frac{\Lambda^2}{4}\right)\right] \\ &+ \sigma^3\left[T\left(1 - \frac{2\Lambda}{3}\right) + \frac{5}{3}\, m_c\right] \\ &+ \sigma^4\left[\frac{1}{4} - \ln\sigma + m_c\right]\Bigg\}\omega_c^2 \\ &- \left(1 - \sigma^2 + \sigma^4\right)\Bigg] A\!R \end{aligned}$$

$$\begin{aligned} L_{\dot a} = +\frac{\pi}{4}\Bigg\{&\left(\frac{4}{3} - m_c\right) + \sigma^2\left[\left(1 - \frac{2\Lambda}{3}\right)T - (2 - m_c)\right] \\ &+ \sigma^3 \cdot \frac{5}{3} - \sigma^4 \cdot m_c\Bigg\} A\!R \end{aligned}$$

$$\begin{aligned} M_z = \frac{\pi}{4}\Bigg\{&\left(\frac{1}{4} - \frac{m_c}{3}\right) + \sigma^2\left[(m_c - \tfrac{1}{2}) - Tm_c\left(1 - \frac{2\Lambda}{3}\right) - T^2\left(\frac{1}{2} - \frac{2\Lambda}{3} + \frac{\Lambda^2}{4}\right)\right] \\ &+ \sigma^3\left[T\left(1 - \frac{2\Lambda}{3}\right) + \frac{5}{3}\, m_c\right] \\ &+ \sigma^4\left[\frac{1}{4} - \ln\sigma + m_c\right]\Bigg\}\omega_c^2 \cdot A\!R \end{aligned}$$

$$\begin{aligned} M_{\dot z} = -\frac{\pi}{4}\Bigg\{&\left(\frac{2}{3} - m_c\right) + \sigma^2\left[m_c + T\left(\frac{2\Lambda}{3} - 1\right)\right] \\ &- \frac{5}{3}\,\sigma^3 + \sigma^4\,(2 - m_c)\Bigg\} A\!R \end{aligned}$$

$$M_{a} = \frac{\pi}{4}\left[\left\{\left(\frac{1}{5} - \frac{m_c}{2} + \frac{m_c^2}{3}\right) + \sigma^2\left[m_c(1-m_c) + Tm_c^2\left(1 - \frac{2\Delta}{3}\right)\right.\right.\right.$$

$$- T^2 m_c\left(3 - \frac{4\Delta}{3} + \frac{\Delta^2}{2}\right)$$

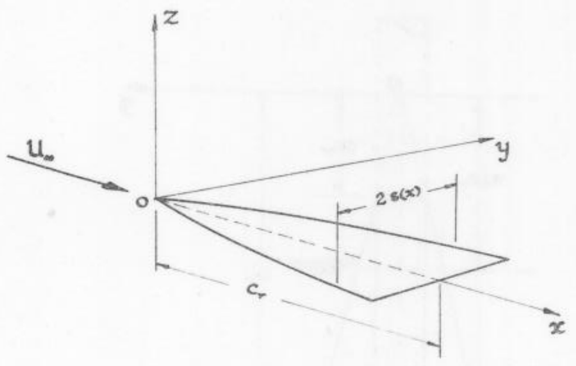
$$\left. + T^3\left(\frac{1}{3} - \frac{2\Delta}{3} + \frac{\Delta^2}{2} - \frac{2\Delta^3}{15}\right)\right]$$

$$+ \sigma^3\left[\frac{5m_c^2}{3} - \frac{1}{3} - 2m_c T\left(1 - \frac{2\Delta}{3}\right) - T^2\left(1 + \frac{2\Delta}{3} - \frac{\Delta^2}{2}\right)\right]$$

$$\left. + \sigma^4\left[1 - \frac{m_c}{2} - m_c^2 + T\left(1 - \frac{2\Delta}{3}\right)\right] - \frac{13}{15}\sigma^5\right\}\omega_c^2$$

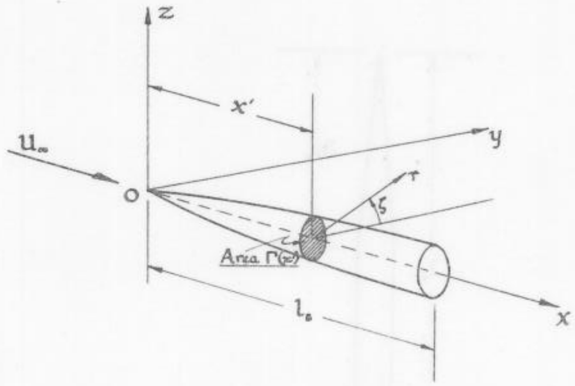
$$\left. + M_{\dot{z}}\right] AR$$

$$M_{\dot{a}} = -\frac{\pi}{4}\left\{(1 - m_c)^2 - \sigma^2\left[1 + (1 - m_c)^2\right] + \sigma^4\left[1 + (1 - m_c)^2\right]\right\} AR$$



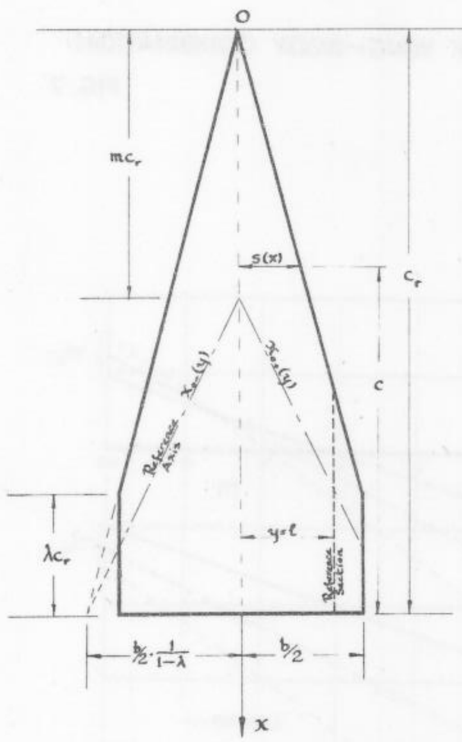
RECTANGULAR CO-ORDINATE SYSTEM

FIG. 1.



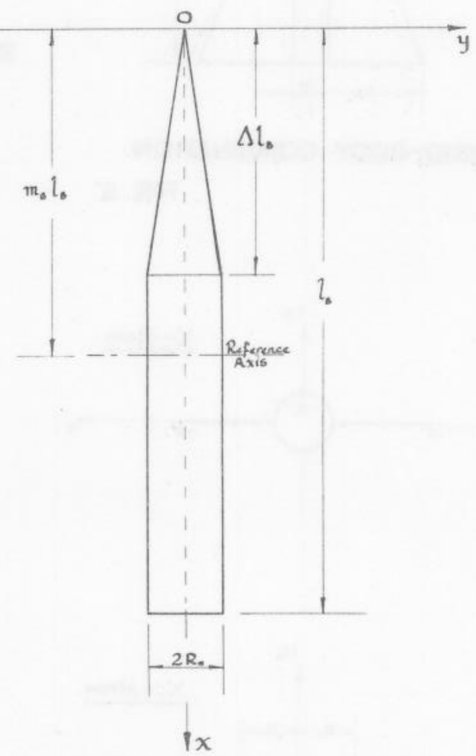
CO-ORDINATE SYSTEM FOR SLENDER BODY

FIG. 3.



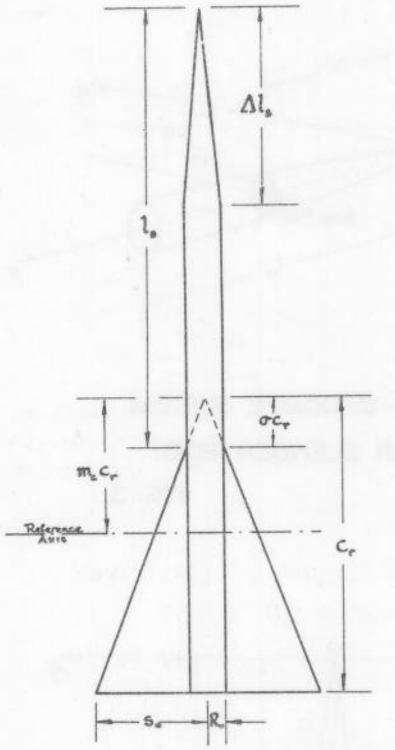
SLENDER DELTA WING

FIG. 2.



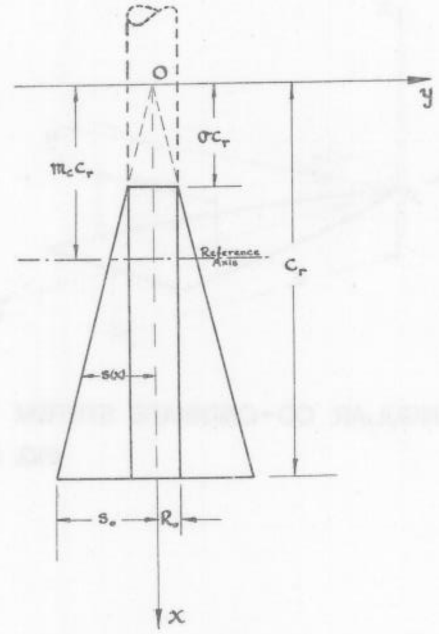
SLENDER BODY OF REVOLUTION

FIG. 4.



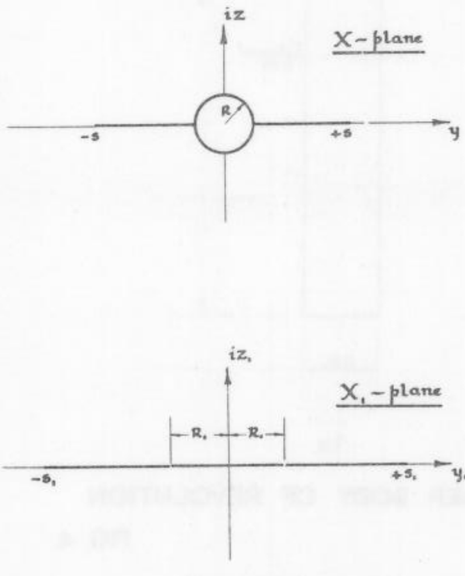
WING-BODY COMBINATION

FIG. 5.



REAR PART OF WING-BODY COMBINATION

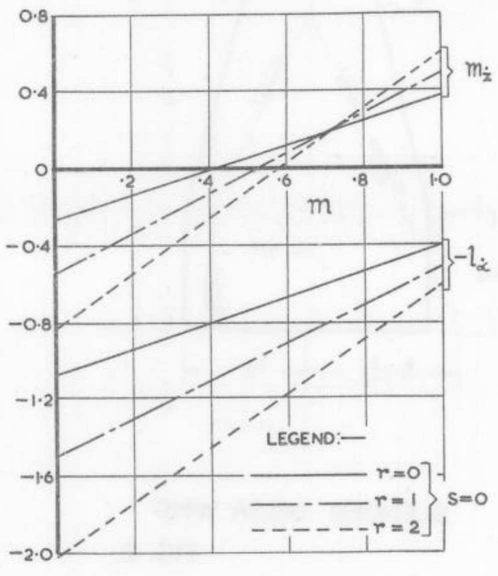
FIG. 7.



THE TRANSFORMATION

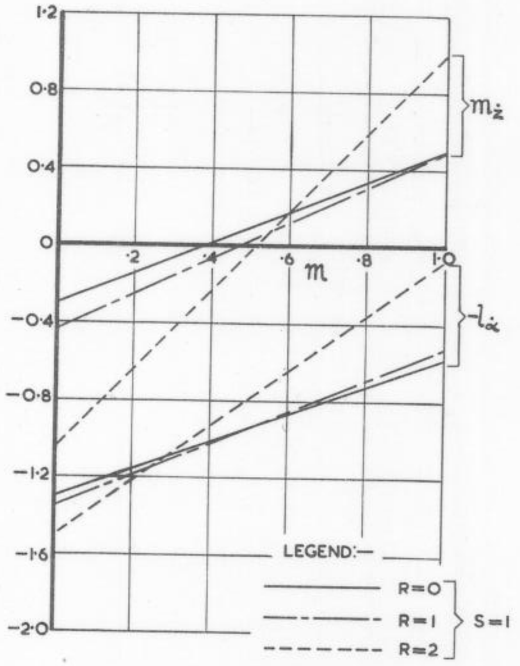
$$X_1 = X + \frac{R^2}{X}$$

FIG. 6.



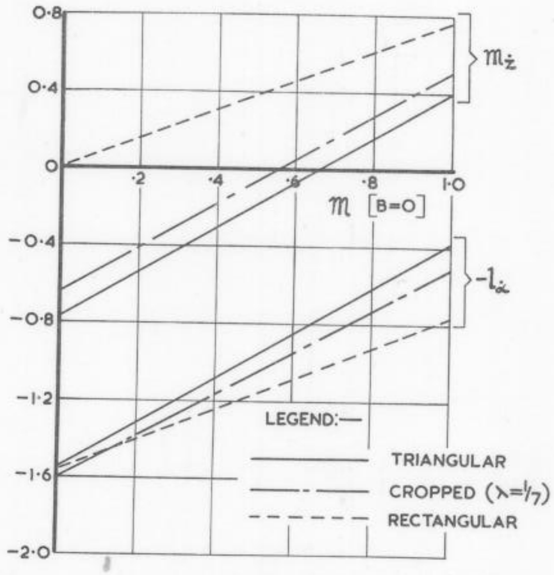
'CROSS' DAMPING DERIVATIVES FOR DEFORMING WING [λ=0]

FIG. 8.



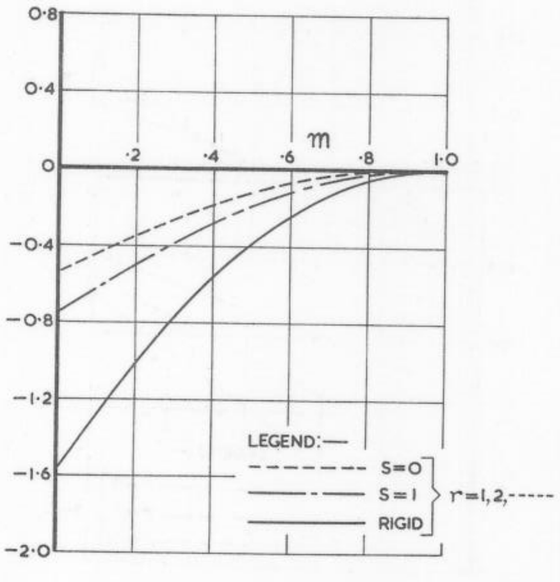
'CROSS' DAMPING DERIVATIVES
FOR DEFORMING WING [λ=O]

FIG. 9.



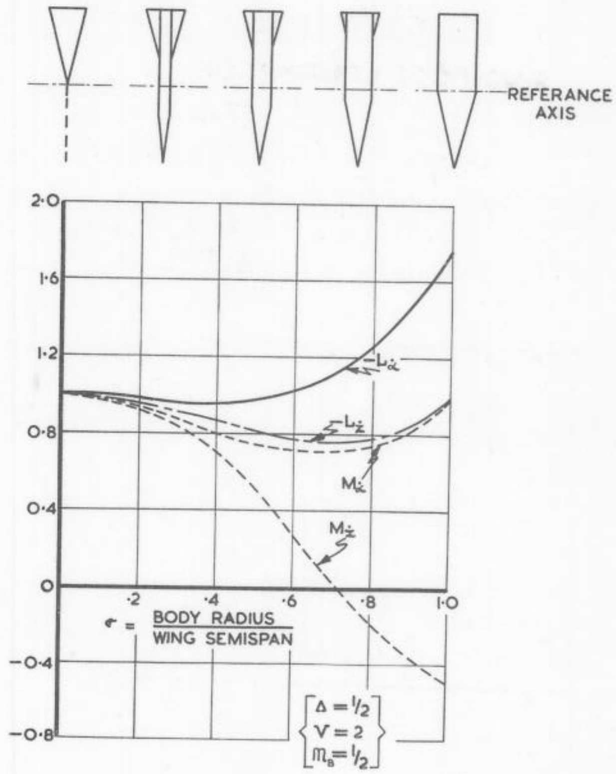
'CROSS' DAMPING DERIVATIVES
FOR RIGID WINGS

FIG. 11.



DAMPING DERIVATIVE $m_{\dot{\alpha}}$
FOR TRIANGULAR WING

FIG. 10.



RELATIVE VALUES OF
DAMPING FORCE COEFFICIENTS
FOR WING-BODY COMBINATION

FIG. 12.