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Stability of the Compressible Laminar Boundary
Layer with an External Pressure Gradient

- by -

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S U M M A R Y

The small perturbation theory of the stability of the laminar boundary layer, as first considered in detail by Tollmien and Schlichting for incompressible flow and applied to compressible flow by Lees and Lin, is extended in this paper to include compressible flows with a pressure gradient in the main stream.

The analysis shows that if normal modes of perturbations of the boundary layer are considered, the approximate solutions for the perturbation flow in terms of the steady velocity and temperature distributions of the boundary layer, as developed by Lees and Lin for compressible flow without a pressure gradient, hold to the same order of approximation in the presence of a pressure gradient. The solutions are valid for large values of the Reynolds number R and the parameter αR , where α is the wave number of the disturbance considered, with the additional restriction that α should lie between the extreme limits, R and R^{-1} . The Reynolds number referred to here is based upon the boundary layer thickness. Moreover, it is found that for a given boundary layer profile, the differential equations governing the behaviour of the disturbance in an inviscid fluid are independent of the pressure gradient. Thus the general criteria established by Lees and Lin for the stability of inviscid flows can be taken over with little or no modification in proof.

The boundary layer velocity and temperature distributions for the compressible flow were obtained by transformation of the general 'similar velocity profile' incompressible solutions, in which the incompressible main stream velocity in the x_1^* direction is taken as $u_{i1}^* = \bar{c}(x_1^*)^m$, where \bar{c} and m are constants. In applying this method, the viscosity was taken to be proportional to the absolute temperature and the Prandtl number was taken to be unity.

/Calculation ...

Calculation of the neutral stability characteristics gave the following general results:

- (i) For any given free stream Mach number, an increasing negative pressure gradient ($m > 0$) increases the stability of the laminar layer, i.e. the value of the stability limit (defined as the Reynolds number below which there is laminar stability) increases, and the range of unstable disturbance frequencies and velocities becomes smaller.
- (ii) With a positive pressure gradient the stability decreases as the Mach number increases.
- (iii) For a sufficiently large negative pressure gradient there is a range of Mach number, varying from about 1.3 to 2.4, for which the boundary layer is completely stable for all Reynolds numbers; however, the stability limit always decreases at large enough values of the Mach number.

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S Y M B O L S

In general asterisks will denote mean values of dimensional quantities, while unstarred symbols will denote the corresponding non-dimensional quantities. A prime will denote differentiation with respect to distance normal to the solid surface, unless stated otherwise.

- A suffix 0 will refer to conditions at the solid surface.
- A suffix 1 will refer to local values in the main stream, except for y_1^* , which will denote the boundary layer thickness.
- A suffix 10 will refer to adiabatic stagnation in the main stream.
- A suffix i will refer to the incompressible variable obtained from the corresponding compressible variable using the transformation (11), given on page 12.

| <u>Dimensional Quantity</u> | <u>Dimensionless Quantity</u> | <u>Dimensional Measure</u> |
|--|--|----------------------------|
| Co-ordinate measured along surface | | |
| x^* | x | l |
| Coordinate measured perpendicular to surface | | |
| y^* | y | l |
| Time | | |
| t^* | t | l/u_1^* |
| Velocity components in the x and y directions respectively | | |
| u^* | $w(x,y)+f(x,y)e^{i\alpha(x-ct)}$ | u_1^* |
| v^* | $v(x,y)+a\phi(x,y)e^{i\alpha(x-ct)}$ | u_1^* |
| Density | | |
| ρ^* | $\rho(x,y)+r(x,y)e^{i\alpha(x-ct)}$ | ρ_1^* |
| Pressure | | |
| p^* | $p(x,y)+\bar{w}(x,y)e^{i\alpha(x-ct)}$ | p_1^* |
| Temperature | | |
| T^* | $T(x,y)+\theta(x,y)e^{i\alpha(x-ct)}$ | T_1^* |
| Viscosity | | |
| μ^* | $\mu(x,y)+\bar{m}(x,y)e^{i\alpha(x-ct)}$ | μ_1^* |
| Kinematic viscosity | | |
| $\nu^* = \mu^*/\rho^*$ | ν | ν_1^* |
| Conductivity | | |
| k^* | $k(x,y)+\mathcal{K}(x,y)e^{i\alpha(x-ct)}$ | $C_p \mu_1^*$ |

| <u>Dimensional Quantity</u> | <u>Dimensionless Quantity</u> | <u>Dimensional Measure</u> |
|---------------------------------|--|----------------------------|
| Wave number of disturbance | α | $1/l$ |
| Phase velocity of disturbance | c | u_1^* |
| Specific heats | $\gamma = C_p/C_v$ | |
| C_v, C_p | | |
| $R = \frac{u_1^* l}{v_1^*}$ | Reynolds number. | |
| $a = \sqrt{\gamma \bar{R} T^*}$ | local velocity of sound, \bar{R} = gas constant. | |
| $M_1 = u_1^*/a_1$ | local Mach number at edge of boundary layer. | |
| $\sigma = C_p u^*/k^*$ | Prandtl number. | |
| δ | boundary layer thickness. | |
| δ^* | boundary layer displacement thickness. | |
| ψ | boundary layer momentum thickness. | |
| m | external pressure gradient parameter, defined by $u_{11}^* = \bar{c}(x_1^*)^m$. | |
| ω | index in the viscosity-temperature relationship $\mu^* = (T^*)^\omega$. | |
| β | $= \frac{2m}{m+1}$. | |

The characteristic measure l will in general be taken as the boundary layer thickness δ . When l is taken as ψ or δ^* the non-dimensional quantities will be written with a suffix ψ or δ^* respectively.

1. Introduction

The mechanism of the transition from laminar to turbulent flow is one of the oldest fundamental problems of hydrodynamics that remains unsolved. The perturbation method is an attempt, which is only partially successful, to explain transition by finding the conditions under which the viscous equations of motion admit of non-uniform unsteady solutions in the form of either damped, neutral or amplified disturbances, the normal modes of which are usually considered separately. Amplified, self-excited disturbances, when they exist, mean that the boundary layer is in a state of unstable dynamic equilibrium, and that transition will occur when the disturbance amplitudes become sufficiently large.

The restriction of the method to small disturbances (introduced to make the mathematics tractable) reduces the value of the results, since there is left a large gap in our knowledge of the behaviour of the flow between the beginning of the amplification of a small disturbance and the onset of full scale turbulence.* In consequence, it is found that turbulence develops much later than is given by the condition that a small disturbance should be just amplified. In spite of this, once verified, the theory can be of great interest in giving an insight into the origin of transition and in providing at least a sufficient condition for the stability of laminar flows.

This method of dealing with laminar stability was first applied to boundary layer flow by Tietjens (reference 1), followed by other German investigators, notably Tollmien and Schlichting (references 2 and 3). However, the theory was held in doubt by many, chiefly because of the simplified and approximate mathematical approach that had to be made to deal with the problem, and the success of G.I. Taylor's theory of transition (reference 4). Taylor ascribed the instability of the laminar boundary layer to the presence of finite, non-selfexcited, disturbances, which in most cases would arise from the external turbulence of the main flow. A more satisfactory approach to the small perturbation theory has now been given by Lin (reference 5) for incompressible flow, and verification of the asymptotic expansions used by him has been given by Wasow (reference 6). Moreover, full experimental confirmation of the theory has been made by Schubauer and Skramstad (reference 7), and there is now no doubt that in the absence of such sources of finite disturbances as surface imperfections and external turbulence, a major cause of transition to turbulence in boundary layers derives from their inherent instability under certain circumstances. Presumably, when disturbances present are large enough, whether self-excited or imposed by external turbulence or surface condition, transition ensues. From this point of view, Taylor's approach and the approach in the small perturbation theory can be reconciled, although a complete linking of the two approaches is still to be made.

* German investigators (references 8 and 9) have calculated the amplification of the disturbances for incompressible flow and have attempted to correlate the rate of amplification with the occurrence of transition, in spite of the restriction of the theory to small disturbances. Two estimates thus made for the amplification factor required to give transition on a flat plate differ by over 100 percent.

The extension of the theory to parallel compressible flows was made by Lees and Lin (references 10 and 11), and in the present paper it is found that in all essentials the theory developed by them is applicable to flows with an external pressure gradient, and that the stability again depends on the local velocity and temperature profiles of the boundary layer only. In both cases there is a Reynolds number, based upon the boundary layer thickness, above which the laminar flow is inherently unstable, and self-excited disturbances appear. Below this so-called minimum critical Reynolds number, which depends on the Mach number and pressure gradient in the main stream, all disturbances are damped out, and the boundary layer is completely stable.

The physical reason for the behaviour of the disturbances can be seen in a study of the interchange of energy between the disturbance and the mean flow. If a normal mode of the disturbance, of non-dimensional (real) phase velocity c_r is considered, it is found that across a critical section of the boundary layer, where the mean flow velocity u^* equals the disturbance phase velocity, there is a phase shift in the x component of the disturbance velocity f . This results in an apparent stress, the value of which is

$$\tau_{c_r}^* = \rho_1^* (u_1^*)^2 \frac{\alpha}{2} \pi \frac{T_{c_r}^2}{(w'_{c_r})^2} \left[\frac{d}{dy} \left(\frac{w'}{T} \right) \right]_{w=c_r}$$

where

w is the non-dimensional mean velocity parallel to the surface in the x-direction,

$$w' = \partial w / \partial y,$$

T is the non-dimensional temperature,

ρ is the density,

α is the wave number of the disturbance,

subscripts 1 and c_r refer to values at the wall and at the critical layer.

According as $\tau_{c_r}^*$ is greater or less than zero, energy is absorbed or lost by the disturbance, whilst if $\tau_{c_r}^* = 0$, there is no transfer of energy. A further apparent shear stress is introduced by the action of viscosity near the wall; this, however, is always positive, and through its action energy is gained by the disturbance from the mean flow. Finally, the disturbance loses energy throughout the boundary layer by the action of viscous dissipation.

If a disturbance is to be damped, the shear stress $\tau_{c_r}^*$ must be sufficiently large negatively to ensure that the energy absorbed by the mean flow at the inner critical layer at $w = c$, together with the energy dissipated by the viscous forces is greater than the energy added to the disturbance through the destabilising action of viscosity near the wall.

/Generally ...

Generally speaking, therefore, if the quantity

$$- \left[\frac{d}{dy} \left(\frac{w'}{T} \right) \right]_{w=c_r}$$

which is a function of the shape of the velocity and temperature profiles of the boundary layer, is large enough, the boundary layer will be stable. The principal effect of pressure gradient can thus be seen in the way that it changes the shape of the boundary layer velocity distribution.

However, for main stream Mach numbers greater than unity, an additional factor enters into the problem. The disturbances for which

$$c_r > 1 - \frac{1}{M_1},$$

where $M_1 = u_1^*/a_1^*$ (the local Mach number of the main stream), are termed subsonic disturbances (the phase velocity of the disturbance relative to an observer moving with the velocity of the free stream is less than the local velocity of sound), and for some values of R such disturbances can be made to satisfy the differential equations and boundary conditions for the disturbance. However, the amplified supersonic disturbances for which

$$c_r < 1 - \frac{1}{M_1}$$

do not exist, for it is found that they cannot satisfy the boundary conditions at the 'edge' of the boundary layer. Thus, for a given boundary layer profile, if the only values of c required to make

$$\left[\frac{d}{dy} \left(\frac{w'}{T} \right) \right]_{w=c_r}$$

sufficiently large to enable the disturbance of this phase velocity to gain energy, are such that

$$c_r < 1 - \frac{1}{M_1},$$

there can be no solution of the disturbance equations for amplified disturbances, and the boundary layer is completely stable for all Reynolds numbers.

For an insulated surface and flow with no pressure gradient, this condition is in fact never obtained. But for some Mach numbers and with a negative (favourable) pressure gradient or by withdrawing heat away from the boundary layer, it is possible to obtain complete laminar stability, and it is this eventuality which is of particular practical interest. The latter case has been calculated by Lees (reference 11) and the former is dealt with in this paper.

An attempt to make the present paper completely self-contained in all mathematical details would make it prohibitively large, and, in any case, would only be repeating much of Lees' and Lin's work. The author has, therefore tried to make the main section of the paper intelligible without reference to Lees' and Lin's reports, keeping the mathematics to a minimum, whilst the mathematical analysis that is original has been included in appendices, but it will be found that the latter can only be studied to advantage in detail if reference is made to Lees and Lin.

2. Outline of the Mathematical Theory of Stability

2.1 The perturbation equations

Except in special cases, the only method of finding the conditions under which the viscous equations of motion admit of a solution in the form of a perturbation is by linearisation of the equations. This can be done by considering the harmonic components of a small disturbance. Thus, we take a typical normal mode in the form

$$q(x, y) \exp i\alpha(x - ct) \dots\dots\dots (1)$$

where $c = c_r + ic_i$, c_r and c_i real

α is real and positive

q may be complex.

It has been shown by Squire (reference 12) that for parallel incompressible flow three-dimensional disturbances are more stable than two-dimensional disturbances; the restriction of the mode (1) to two dimensions is thus justified for incompressible flow. This result has not been proved for compressible flow, but physical considerations lead us to expect that there cannot be any fundamental difference in the behaviour of disturbances in this respect between the compressible and incompressible case.

Such a two-dimensional stability theory can be applied formally to approximately parallel flows or flows over slightly curved surfaces, where the velocity in the boundary layer normal to the surface is small. The validity of such applications can be assessed in the light of work done by Görtler (reference 13), who has studied the three-dimensional disturbance problem for incompressible flow over curved surfaces. Görtler found that boundary layer profile had relatively little effect on three-dimensional disturbances, but that a convex surface was stabilising and a concave surface devastatingly destabilising. In general, two-dimensional disturbances determine the stability of flow over flat or convex surfaces, with a positive or negative pressure gradient, but for concave surfaces instability is first brought about by three-dimensional disturbances. In particular, when considering the stability of the laminar boundary layer on normal aerofoils the two-dimensional theory is usually applicable.

Returning now to equation (1), we see that the disturbance is damped if $c_i < 0$, is amplified if $c_i > 0$, and is neutral if $c_i = 0$. The neutral disturbance forms the boundary between stability and instability of the laminar layer, and the occurrence of the latter will eventually lead to transition. It is assumed that transition occurs when the amplification of the disturbance reaches a certain level, this corresponding to a larger Reynolds number than for the neutral disturbance, as the analysis shows.

To obtain the perturbation equations, the disturbance quantities in the form (1) are introduced into the full viscous equations of motion, which are then linearised by neglecting squares and products of the disturbances, and applying the normal boundary

/layer ...

layer approximations to the mean flow quantities, for which the boundary layer equations are also assumed to hold. Because of the boundary layer approximations the theory is essentially applicable for large Reynolds numbers only, and the large R condition is also made use of in obtaining the solutions as series expansions.

The details of this procedure are given in Appendix A1, where it is found that the additional restriction that

$$R^{-1} \ll O(\alpha) \ll R,$$

where R is the Reynolds number based on the boundary layer thickness, is needed to obtain the perturbation equations. We can write this alternatively, though less precisely, as

$$\alpha = O(1) \\ \text{or } \alpha^* = O(1/\delta),$$

i.e. the wave length, $2\pi/\alpha^*$, of the disturbance is of the order of magnitude of the boundary layer thickness. This is the assumption made by Pretsch¹⁵ in his treatment of the incompressible case of flow with a pressure gradient. From the various numerical calculations that have been made, it appears that this assumption is correct.

We find that in the limit $R \rightarrow \infty$, the differential equations for the disturbances reduce to those with no pressure gradient. In particular, the equation for ϕ ($\alpha \phi e^{i(x-ct)}$ is the non-dimensional disturbance velocity normal to the surface),

$$\frac{d}{dy} \left\{ \frac{(w-c) \phi' - w' \phi}{T - M_1^2 (w-c)^2} \right\} = \frac{\alpha^2 (w-c)}{T} \phi, \dots\dots\dots (2)$$

is the same as the basic equation used by Lees and Lin (reference 10) in their treatment of the inviscid case of stability without a pressure gradient.

The equations of motion, of continuity, of energy and of state yield five linear differential equations for five disturbance quantities f, ϕ, w, r, θ , which are equivalent to six homogeneous equations in six independent variables, with six linearly independent solutions. The six variables chosen are

$$f, f', \phi, \frac{w}{M_1}, \theta, \theta', \dots\dots\dots (3)$$

Three methods of solution of these equations are considered: convergent series solution in powers of $(\alpha R)^{-1/3}$, and asymptotic series solutions in powers of $(\alpha R)^{-1}$ and in powers of $(\alpha R)^{-1/2}$. In each case the initial approximation as a function of the mean flow quantities is found to be independent of x, and is thus not directly dependent on the pressure gradient. The influence of the pressure gradient is brought about entirely through its effect on the mean flow, i.e. by its effect on the local steady boundary layer velocity and temperature distributions.

The boundary conditions to be satisfied by the disturbances at the 'edge' of the boundary layer and at the surface form six homogeneous equations in the variables (3).

/Since ...

Since these must also satisfy six homogeneous differential equations, there must be a restriction on the values of the parameters occurring in the equations (viz. α , c , R and M_1) in the form of a secular or characteristic equation. The derivation of the secular equation is the same with or without a pressure gradient and need not be entered into here (for details see Appendix A3 and reference 10). For a given boundary layer profile and chosen values of M_1 and c_i (the amount of damping or amplification), the secular equation gives a relationship between α and R , and α and c_r . There are, however, important exceptions to this when the main stream flow is supersonic. In the first instance, for the neutral supersonic disturbance, i.e. for

$$c_i = 0, \quad c_r < 1 - 1/M_1,$$

the boundary condition for ϕ at the edge of the boundary layer is automatically satisfied by both solutions of the differential equation (2) for ϕ , see Appendix A4. Hence there is no characteristic equation to be satisfied and a disturbance of any wave length and velocity can exist for all Reynolds numbers. However, for an amplified supersonic disturbance, $c_i > 0$, the secular equation exists, but has no solution, i.e. there is no amplified solution to the disturbance equation which can satisfy the boundary conditions, and the boundary layer must then be stable to small disturbances of all wave lengths and velocities.

2.2 The inviscid case

The differential equation for ϕ in the case $R \rightarrow \infty$ is

$$\frac{d}{dy} \left\{ \frac{(w-c) \phi' - w' \phi}{T - M_1^2 (w-c)^2} \right\} = \alpha^2 \frac{(w-c)}{T} \phi, \quad \dots \dots (4)$$

the same as given by Lees and Lin for flow without a pressure gradient. Two particular integrals are

$$\left. \begin{aligned} \phi_1(y; \alpha^2, c, M_1^2) &= (w-c) \sum_0^{\infty} \alpha^{2n} h_{2n}(y; c, M_1^2) \\ \phi_2(y; \alpha^2, c, M_1^2) &= (w-c) \sum_0^{\infty} \alpha^{2n} k_{2n+1}(y; c, M_1^2) \end{aligned} \right\} \dots \dots (5)$$

where

$$\left. \begin{aligned} h_{2n} &= \int_{y_0}^y \left(\frac{T}{(w-c)^2} - M_1^2 \right) dy \int_{y_0}^y \frac{(w-c)^2}{T} h_{2n-2} dy \\ h_0 &= 1 \\ k_{2n+1} &= \int_{y_0}^y \left(\frac{T}{(w-c)^2} - M_1^2 \right) dy \int_{y_0}^y \frac{(w-c)^2}{T} k_{2n-1} dy \\ k_1 &= \int_{y_0}^y \left(\frac{T}{(w-c)^2} - M_1^2 \right) dy \end{aligned} \right\} \dots \dots (6)$$

As shown by Lin (reference 5) the path of integration in (6) must lie below the singular point $y = y_c$ (where $w=c$) in the complex y -plane.

A study of the characteristics of the solution of (4) has been made by Lees and Lin (reference 10), and the only modification of their analysis required for the case of flow with a pressure gradient is a minor one that occurs in the proof of the disturbance energy relations, and involves the mean flow equations of motion. A summary of the more important results which they obtained that are relevant to the present discussion are given here.

There is a transfer of energy between the main flow and the disturbance motion at the critical layer where $w = c$. This occurs through a change in the relative phase of the x- and y-components of the disturbance motion at $y = c$ and the appearance of a Reynolds stress there. Since, for the neutral disturbance, this shear stress is proportional to

$$\left[\frac{d}{dy} \left(\frac{w'}{T} \right) \right]_{w=c}, \quad (c = c_r),$$

it is found that if

- | | | | |
|-------|--|--|-------------|
| (i) | $\left[\frac{d}{dy} \left(\frac{w'}{T} \right) \right]_{w=c} > 0,$ | energy passes from the mean flow to the disturbance, | } . . . (7) |
| (ii) | $\left[\frac{d}{dy} \left(\frac{w'}{T} \right) \right]_{w=c} = 0,$ | there is no exchange of energy between the mean and disturbance motions, | |
| (iii) | $\left[\frac{d}{dy} \left(\frac{w'}{T} \right) \right]_{w=c} < 0,$ | the disturbance loses energy to the mean flow. | |

These results can be correlated with the three cases of amplified, neutral and damped disturbances:

Amplified disturbances $c_i > 0$.

The subsonic disturbances take the form of outgoing waves exponentially attenuated for large y . A sufficient condition for their existence is that

$$\frac{d}{dy} \left(\frac{w'}{T} \right) = 0 \quad \text{for some } w > 1 - 1/M_1.$$

Neutral disturbances $c_i = 0$.

Subsonic disturbances take the form of waves travelling parallel to the x-axis, exponentially attenuated in y for large y . A necessary and sufficient condition for their existence is that

$$\frac{d}{dy} \left(\frac{w'}{T} \right) = 0 \quad \text{for some } w > 1 - 1/M_1 = c_s, \text{ say,}$$

and then the phase velocity of the disturbance is c_s .

In the case of supersonic disturbances both incoming and outgoing disturbances exist and are not attenuated. The characteristic equation does not exist and we may say that the characteristic values are continuous. Note that in this event any of the three cases (7) can hold, since the ingoing and outgoing disturbances can be of different amplitudes.

/Damped . . .

Damped disturbances $c_i < 0$.

The disturbances travel inward and are attenuated with y as $y \rightarrow \infty$.

These results for the inviscid case show that the stability depends essentially on the variation of the quantity

$$\frac{d}{dy} \left(\frac{w'}{T} \right) = \frac{1}{T} \frac{d^2 w}{dy^2} - \frac{1}{T^2} \frac{dw}{dy} \frac{dT}{dy}$$

across the boundary layer. For flow past an insulated surface we can apply the transformation given in section 3, so that

$$\frac{d}{dy} \left(\frac{w'}{T} \right) = K \left(\frac{1}{T^3} \frac{d^2 w}{d\zeta^2} + \frac{2(\gamma-1)M_1^2}{T^4} w \left(\frac{dw}{d\zeta} \right)^2 \right)$$

where $\zeta = \sqrt{\frac{m+1}{2}} \sqrt{\frac{u_{i1}^*}{v_{10}^* x_i^*}} y_i^*$, and $K > 0$.

For a main stream flow given by

$$u_{i1}^* = \bar{c} (x_i^*)^m$$

we have the following results (Appendix B1),

$$\frac{dw}{d\zeta} > 0; \quad \frac{dw}{d\zeta} \rightarrow 0 \text{ as } \zeta \rightarrow \infty;$$

$$\frac{d^2 w}{d\zeta^2} = - \frac{dw}{d\zeta} \int_0^\zeta w d\zeta + \beta (w^2 - 1);$$

where $\beta = 2m/(m+1)$.

We can thus draw the following conclusions about the behaviour of $\frac{d}{dy} \left(\frac{w'}{T} \right)$:-

If $\beta < 0$ (positive pressure gradient)

$\frac{d}{dy} \left(\frac{w'}{T} \right)$ is always positive at $\zeta = 0$, and negative if ζ is large enough*. Hence it is zero for some $w = c_s$, and provided $c_s < 1 - 1/M_1$, the inviscid boundary layer will be unstable for all values of M_1 .

If $\beta > 0$ (negative pressure gradient)

$\frac{d}{dy} \left(\frac{w'}{T} \right)$ is negative at $\zeta = 0$, and will remain negative for all ζ if M_1 is small enough or β large enough, but it will become positive for some ζ if M_1 is made large enough. Hence for small M_1 and large β the boundary layer will be stable, but if M_1 is chosen large enough it can always be made unstable.

The intermediate case with $\beta = 0$ always has the degenerate neutral solution at $w = c_s = 0$, which gives $\alpha = c = 0$; apart from this, the general behaviour is as for $\beta > 0$.

* Since $\frac{d^2 w}{d\zeta^2} \rightarrow 0$, for $\zeta \rightarrow \infty$, as $-\frac{dw}{d\zeta}$, but the positive Mach number term $\rightarrow 0$ as $\left(\frac{dw}{d\zeta} \right)^2$. /The ...

The foregoing conclusions apply only for the subsonic disturbances; if, however, $c_s < 1 - 1/M_1$, we have a supersonic disturbance and only neutral and damped disturbances can exist, so that the boundary layer is stable. This is dealt with more fully in the Appendix. Numerical calculations are required to determine the values of M_1 and β for which this occurs, and such calculations have been carried out in the present paper (see section 3). The effect of a non-insulated surface is also important in this connection, and the values of the ratio of the temperature at the wall to the main stream temperature, T_0^*/T_1^* , which give stability have been calculated by Lees (reference 11) for zero pressure gradient.

2.3 The viscous solution

It has already been mentioned that the initial approximations for the solution of the stability problem for a viscous boundary layer with a pressure gradient depends only on the local velocity and temperature distributions of the boundary layer flow. The general conclusions and expressions obtained by Lees (reference 11) are therefore applicable to the case being considered in this paper of flow with a pressure gradient.

In particular, Lees' considerations of the energy balance in the boundary layer, which have been summarized in section 1, hold good, and the characteristic equation for the neutral disturbance obtained by him can be used in our case. For proof of the existence of neutral or amplified disturbances 'adjacent' to the inviscid neutral disturbances (the Heisenberg criterion) and other mathematical details the reader is referred to his paper.

The secular equation for the neutral subsonic disturbance, as obtained by Lees, is

$$E(\alpha, c_r, M_1^2) = F(z), \quad c_r > 1 - 1/M_1, \quad \dots\dots\dots (8)$$

$$z = \left(\frac{\alpha R w_c'}{\nu_c} \right)^{1/3} / y_c$$

where ν is the non-dimensional kinematic viscosity. Here $F(z)$ is the Tietjens function (equation (24), Appendix A), and E is a function of the inviscid solutions only, equation (5). E and F are complex, and equation (8) is equivalent to two real equations. The form of the equation when expressed as a function of the boundary layer temperature and velocity distributions, and suitable for numerical solution, is given in section 3 of Appendix A.

The relation between α and the Reynolds number R_{cr} obtained from the neutral characteristic equation (8), forms a curve having two branches extending to $R = \infty$, meeting at some minimum Reynolds number, R_{crmin} , below which the boundary layer is stable. The interior of the curve corresponds to values of α and R for which disturbances are amplified. An approximate formula for R_{crmin} is

$$R_{crmin} = \frac{25 [T(c)]^{1+\omega} w_0'}{c^4 \sqrt{1-M_1^2} (1-c)^2} \dots\dots\dots (9)$$

and c is determined by

$$(1 - 2\lambda) v(c) = 0.58, \dots\dots\dots (9a)$$

where

$$v(c) = \frac{-\pi w'_0 c}{T_0} \left[\frac{T^2}{(w')^3} \frac{d}{dy} \left(\frac{w'}{T} \right) \right]_{w=c}$$

$$\lambda = \frac{w'_0(y_c)}{c} - 1, \text{ the suffix } 0 \text{ refers to surface values.}$$

If (8) has no solution for any $c > 1 - 1/M_1$, only damped or neutral supersonic disturbances can exist and the boundary layer is stable for all Reynolds numbers.

3. Numerical Methods and Results

Solution of the characteristic equation (8) for the neutral disturbance enables us to assess the effect of Mach number and pressure gradient on the boundary layer stability. For this we need to know the variation of velocity and temperature across the boundary layer quite accurately. The common approximate methods for calculating the boundary layer velocity, such as the Pohlhausen, do not usually give sufficient accuracy, as can be seen by comparing the results of Pretsch (references 15 and 16) for incompressible flow stability, who used the exact boundary layer profiles calculated by Hartree (reference 18), and those of Schlichting (reference 17), who used the Pohlhausen method. This state of affairs arises from the fact that the solution of the stability problem depends principally on the variation of $\frac{d^2 w}{dy^2}$ across the boundary layer, and not merely on the velocity w . In this paper, therefore, the exact similar flow solutions calculated by Hartree for the incompressible boundary layer with a main stream velocity

$$u_{i1}^* = \bar{c} (x_1^*)^m \dots\dots\dots (10)$$

have been used to obtain the compressible boundary layer profiles. For an insulated surface, with the Prandtl number σ taken as unity and the viscosity taken proportional to the absolute temperature ($\omega = 1$), this can be done by means of the transformations*

$$w = \frac{u_i}{u_{i1}} = \frac{u^*}{u_1^*};$$

$$y_i^* = \left(\frac{a_1}{a_{10}} \right)^{\frac{\gamma+1}{\gamma-1}} \int_0^{y^*} \frac{\rho}{\rho_1} dy^* = \left(\frac{a_1}{a_{10}} \right)^{\frac{\gamma+1}{\gamma-1}} \int_0^{y^*} \frac{1}{T} dy^*$$

$$x_i^* = \int_0^{x^*} \left(\frac{a_1}{a_{10}} \right)^{\frac{3\gamma-1}{\gamma-1}} dx^* \dots\dots\dots (11)$$

Application of these transformations to the Hartree solutions gives the compressible boundary layer

*These are the same as the equations given by Illingworth (reference 19), and very similar to those given by Stewartson (reference 20).

velocity distributions having a main stream velocity given by the transformation of equation (10) using (11), and this is

$$u_1^* = \bar{c}(x^*)^m \left\{ 1 - \frac{(10m+1)}{10(2m+1)} \left(\frac{\bar{c}(x^*)^m}{a_{10}} \right)^2 + \dots \right\} \dots \dots (12)$$

The boundary layer temperature distribution for an insulated surface, as obtained by Crocco, is

$$T^*/T_1^* = 1 + \frac{\gamma-1}{2} M_1^2 (1 - w^2).$$

Recently, Lees (reference 21) has made some calculations of the stability limit for flow over a circular arc aerofoil, using formula (9) for the minimum critical Reynolds number*. He employs the Dorodnitzn transformation (reference 22), and uses a modification of the Pohlhausen method to obtain the velocity in the boundary layer. In place of the index m which determines the pressure gradient he employs a modified Pohlhausen parameter

$$\Lambda \approx \frac{\delta^2}{v_1} \frac{du_1^*}{dx^*}.$$

The relationship between m and Λ is

$$\Lambda = m \left\{ \frac{K_{\delta^*} + \frac{\gamma-1}{2} M_1^2 (K_{\delta^*} + K_{\delta^*})}{I_2(\Lambda)} \right\}^2 \dots \dots (13)$$

where

$$I_2(\Lambda) = .30 + .4175 \frac{\gamma-1}{2} M_1^2 - (.0083 + .0094 \frac{\gamma-1}{2} M_1^2) \Lambda - .0001 \frac{\gamma-1}{2} M_1^2 \Lambda^2,$$

and K_{δ^*} and K_{δ^*} are defined by equation (15).

Using these formulae it was found that the few calculations Lees has made for $R_{cr.min.}$ agree approximately with the results of those made in the present paper.

A detailed solution of the neutral characteristic equation (8) using the transformation (11), has been made for a moderate, favourable, pressure gradient with $m = .429$ ($\beta = .6$), and Mach numbers up to 1.3. The results are given in Table I and illustrated in Figures 1 and 2. Details of the mathematical working involved is given in Appendix B.

In addition, approximate neutral stability curves for the same pressure gradient were drawn (Figure 3), using approximate formulae for the upper and lower branches of the curves (see Appendix B6) and formula (9) for $R_{cr.min.}$ This approximate method was also applied to flow with a positive pressure gradient, $m = -.0477$, $\beta = -.1$, and the results are shown in Figure 4. For comparison, the neutral curves obtained by Pretsch (reference 15) for $M_1 = 0$, using a different method of solution of the characteristic equation have been included. It should be mentioned that the

/approximate

*Weil (reference 23) has used the same method to calculate the stability over a biconvex aerofoil, but only for two Mach numbers, $M_1 = 1.5$, and $M_1 = 4.0$.

approximate method cannot give the shape of the upper part of the curve at the smaller Reynolds numbers, and these portions have been sketched in using the accurate calculations as a guide.

The minimum critical Reynolds number given by equation (9) was calculated from the Hartree incompressible boundary layer solutions using the transformation (11) for a full range of values of m , and Mach numbers up to 3. The resulting curves are shown in Figure 5, whilst the corresponding solutions of (9a) for c are plotted in Figure 8. The discrepancy between these values of $R_{cr.min.}$ and those obtained from the accurate calculations (Figure 2) is due to the extreme sensitivity of $R_{cr.min.}$ to the values of c when c approaches $1 - 1/M_1$. The small errors in the values of c obtained from equation (9a), resulting from the assumption that λ is small, can then cause large errors in $R_{cr.min.}$.

It was found that the most suitable non-dimensional length to employ was the boundary layer momentum thickness ψ . Since numerous stability calculations use the displacement thickness δ^* , we include here the formula relating ψ and δ^* ,

$$\frac{\delta^*}{\psi} = \frac{\frac{\gamma-1}{2} M_1^2 (K_\psi + K_{\delta^*}) + K_{\delta^*}}{K_\psi} = \frac{\gamma-1}{2} M_1^2 (1 + H) + H, \dots \dots (14)$$

$$H = \frac{K_{\delta^*}}{K_\psi} = \frac{\delta_i^*}{\psi_i}$$

where K_ψ and K_{δ^*} are defined by

$$\left. \begin{aligned} K_\psi &= \sqrt{\frac{u_{i1}^*}{v_{10}^* x_i^*}} \int_0^\infty \frac{u_i^*}{u_{i1}^*} \left(1 - \frac{u_i^*}{u_{i1}^*}\right) dy_i^* \\ K_{\delta^*} &= \sqrt{\frac{u_{i1}^*}{v_{10}^* x_i^*}} \int_0^\infty \left(1 - \frac{u_i^*}{u_{i1}^*}\right) dy_i^* \end{aligned} \right\} \dots \dots (15)$$

and their values are plotted against β in Figure 9, $\beta = 2m/(m+1)$.

These neutral stability calculations confirm the general conclusions drawn in section 2.2. Thus, for the inviscid case, $R = \infty$, with $\beta = 0$ and $\beta = -0.1$, the boundary layer is unstable, and the range of α which gives instability increases as the Mach number increases; but when $\beta > 0$, it is stable until the Mach number exceeds a certain value, which increases as β increases. Thus, for $\beta = 0$, this Mach number is 0 (see Lees' results, reference 11), whilst for $\beta = 0.6$, it is about 2.8.

For finite values of R the results show that the stability decreases as β is reduced and M_1 is increased, except for a restricted range of Mach numbers varying between the limits 1.3 and 2.4 and a favourable pressure gradient (β greater than about 0.6). In this region the values of

$$\left[\frac{d}{dy} \left(\frac{w'}{T} \right) \right]_{w=c}, \quad c > 1 - 1/M_1, \quad / \text{are } \dots$$

are never large enough to enable the disturbance to gain energy, there are no subsonic amplified disturbances and the boundary layer is stable for all Reynolds numbers. The critical values of β above which this happens are plotted against Mach number in Figure 6. A similar state of affairs can occur when heat is withdrawn from the boundary layer. This case has been calculated in Lees' paper for zero pressure gradient, and it is interesting to note that the shapes of the graphs of critical temperature ratio $T_{0, cr}$ versus M_1 (as plotted by Lees) and the critical values of β versus M_1 are very similar (Figures 6 and 7).

This effect is important because it leads to the possibility of obtaining complete laminar stability over entire flying surfaces at some supersonic speeds. In this respect, stabilisation of the boundary layer by heat withdrawal appears to offer greater possibilities than stabilisation through favourable pressure gradients. For example, calculations made by Lees show that, at a Mach number of 3 and at a height of 50,000 ft., the heat radiated from the surface under conditions of thermal equilibrium is sufficient to give complete stability of parallel laminar flow without a pressure gradient. In contrast, even for the highest favourable pressure gradients, the possible range of Mach numbers for which complete stability is attainable on an insulated surface, is only from 1.3 to 2.4.

This general conclusion is confirmed by some calculations recently made by Lees (reference 21) for the stability of flow over an insulated circular arc aerofoil, using equation (9) to obtain $R_{cr, min.}$. He found that the stabilising effect of a negative pressure gradient gave laminar stability only at the mid-section of the aerofoil, with a Mach number of 1.5; above and below this Mach number the flow was unstable.

4. Conclusions

The analysis carried out shows that the solution of the stability problem for the laminar boundary layer with an external pressure gradient depends only on the local temperature and velocity distributions in the boundary layer. Thus the general criteria and formulae obtained by Lees and Lin for compressible flow without a pressure gradient can be applied to flow with a pressure gradient. The only restrictions to this conclusion are that the Reynolds number should be large, and that only the initial approximations to the solutions of the stability equations should be used.

Both theory and numerical calculations show that an increasing negative (favourable) pressure gradient is always stabilising, and that increasing Mach number is always destabilising if large enough. The minimum critical Reynolds number below which the boundary is stable for disturbances of all velocities and wave lengths, decreases as the Mach number increases, except for $m > 0$ (favourable pressure gradient), and Mach numbers between 1.3 and 2.4. In this region the boundary layer will be completely stable for all Reynolds numbers if m is made large enough.

Application of these results to obtain a lower limit for the transition Reynolds number is possible, provided the free stream turbulence is low. Thus, transition cannot occur below a Reynolds number equal to $R_{cr.min.}$, as calculated from the theory; however, it will normally occur at a considerably higher Reynolds number. This is because transition is a large scale perturbation phenomenon, whilst the stability theory applies strictly to vanishingly small disturbances. The appearance of small amplified disturbances does not ensure transition. In spite of this it may be possible to calculate transition by postulating that it will occur if the degree of amplification of a small disturbance of the theory reaches a certain level. However, a full explanation of the transition to turbulence cannot be given without investigating the stability of finite disturbances of the boundary layer. At present the mathematical difficulties of this problem seem to be insurmountable.

Finally, it should be stated that there has been no attempt to verify the compressible small perturbation theory by experiment. Thus, although the incompressible theory cannot now be held in doubt, there is need for a check on the validity and accuracy of the compressible theory, especially in predictions that are not physically obvious, such as the non-occurrence of amplified supersonic disturbances.

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A P P E N D I X A

Mathematics of the Theory of Stability.

The modifications that are introduced into the mathematical theory of compressible boundary layer stability by the presence of an external pressure gradient are dealt with here in detail. The analysis follows that of Lees and Lin (references 10 and 11) very closely, and constant reference will be made to their work.

1. The differential equations for the disturbance

The complete viscous compressible flow equations in two dimensions* are:

the equations of motion,

$$\rho^* \left(\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} \right) = - \frac{\partial p^*}{\partial x^*} + \mu^* \left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{1}{3} \frac{\partial^2 v^*}{\partial x^* \partial y^*} + \frac{4}{3} \frac{\partial^2 u^*}{\partial x^{*2}} \right) + \frac{\partial \mu^*}{\partial x^*} \left(\frac{4}{3} \frac{\partial u^*}{\partial x^*} - \frac{2}{3} \frac{\partial v^*}{\partial y^*} \right) + \frac{\partial \mu^*}{\partial y^*} \left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*} \right) \dots \dots \dots (1)$$

$$\rho^* \left(\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} \right) = - \frac{\partial p^*}{\partial y^*} + \mu^* \left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{1}{3} \frac{\partial^2 u^*}{\partial x^* \partial y^*} + \frac{4}{3} \frac{\partial^2 v^*}{\partial y^{*2}} \right) + \frac{\partial \mu^*}{\partial y^*} \left(\frac{4}{3} \frac{\partial v^*}{\partial y^*} - \frac{2}{3} \frac{\partial u^*}{\partial x^*} \right) + \frac{\partial \mu^*}{\partial x^*} \left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*} \right), \dots \dots \dots (2)$$

the equation of continuity,

$$\frac{\partial \rho^*}{\partial t^*} + u^* \frac{\partial \rho^*}{\partial x^*} + v^* \frac{\partial \rho^*}{\partial y^*} + \rho^* \left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} \right) = 0, \dots \dots \dots (3)$$

the equation of energy,

$$J \rho^* c_v \left(\frac{\partial T^*}{\partial t^*} + u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} \right) + p^* \left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} \right) = J \left[k^* \left(\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}} \right) + \frac{\partial k^*}{\partial x^*} \frac{\partial T^*}{\partial x^*} + \frac{\partial k^*}{\partial y^*} \frac{\partial T^*}{\partial y^*} \right] + \mu^* \left\{ - \frac{2}{3} \left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} \right)^2 + 2 \left(\frac{\partial u^*}{\partial x^*} \right)^2 + 2 \left(\frac{\partial v^*}{\partial y^*} \right)^2 + \left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*} \right)^2 \right\}, \dots \dots \dots (4)$$

*We have not included gravitational effects here, as Lees and Lin have done in their discussion of the general equations.

and the equation of state,

$$p^{\#}/\rho^{\#} = R T^{\#} . \quad \dots\dots (5)$$

We now suppose that a perturbation is introduced into the boundary layer in the form of a small two-dimensional disturbance. We consider the disturbance to be resolved into its normal modes, one of which we take to be of the form

$$q(x, y) \exp i\alpha(x - ct), \quad \dots\dots (6)$$

corresponding to a dependent variable Q , in accordance with the scheme given in the list of symbols. In particular, it should be noticed that the non-dimensional perturbation velocity in the y -direction is taken as $\alpha \phi(x, y) e^{i\alpha(x-ct)}$.

We shall take our dimensional length measure to be the boundary layer thickness δ , so that

$$x = x^{\#}/\delta, \quad y = y^{\#}/\delta, \quad t = t^{\#}u_1^{\#}/\delta, \quad R = u_1^{\#} \delta/v_1^{\#},$$

and if we write

$$x' = x/R, \quad v' = Rv, \quad +$$

in place of x and v , then, according to boundary layer theory, all the non-dimensional mean quantities and their derivatives will be of order unity.

The equations to be satisfied by the disturbances (6) are found by substituting the quantities

$$Q(x, y) + q(x, y) e^{i\alpha(x-ct)}$$

into equations(1) to (5), neglecting terms of order higher than the first in the disturbances, and applying the boundary layer equations and approximations for steady flow to the mean quantities Q .

Following this procedure, we find for the first equation of motion expressed in non-dimensional form

$$\begin{aligned} & \alpha \rho \left\{ i(w-c)f + \frac{\partial w}{\partial y} \phi \right\} + \frac{1}{R} \left\{ r \left(w \frac{\partial w}{\partial x} + v' \frac{\partial w}{\partial y} \right) + \rho \left(f \frac{\partial w}{\partial x} + w \frac{\partial f}{\partial x} + v' \frac{\partial f}{\partial y} \right) \right\} \\ & = - \frac{i \alpha \bar{w}}{\gamma M_1^2} + \frac{1}{R} \left[- \frac{1}{\gamma M_1^2} \frac{\partial \bar{w}}{\partial x} + \mu \left\{ \frac{\partial^2 f}{\partial y^2} + \frac{1}{3} i \alpha^2 \frac{\partial \phi}{\partial y} - \frac{4}{3} \alpha^2 f \right\} \right. \\ & \quad \left. + \bar{m} \frac{\partial^2 w}{\partial y^2} + \frac{\partial \bar{m}}{\partial y} \frac{\partial w}{\partial y} + \frac{\partial \mu}{\partial y} \left(\frac{\partial f}{\partial y} + i \alpha^2 \phi \right) \right] \\ & + \frac{\alpha}{R^2} \left[\frac{4}{3} i \left(f \frac{\partial \mu}{\partial x} + \bar{m} \frac{\partial w}{\partial x} - 2\mu \frac{\partial f}{\partial x} \right) - \frac{2}{3} \left(\frac{\partial \phi}{\partial y} \frac{\partial \mu}{\partial x} + i \bar{m} \frac{\partial v'}{\partial y} \right) \right. \\ & \quad \left. + \frac{\partial \phi}{\partial x} \frac{\partial \mu}{\partial y} + \frac{1}{3} \mu \frac{\partial^2 \phi}{\partial x' \partial y} \right] \\ & + \frac{1}{R^3} \left[\frac{1}{3} \bar{m} \frac{\partial^2 v'}{\partial x' \partial y} + \frac{4}{3} \left(\mu \frac{\partial^2 f}{\partial x'^2} + \bar{m} \frac{\partial^2 w}{\partial x'^2} + \frac{\partial \bar{m}}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial \mu}{\partial x} \frac{\partial f}{\partial x} \right) \right. \\ & \quad \left. - \frac{2}{3} \frac{\partial \bar{m}}{\partial x} \frac{\partial v'}{\partial y} + \frac{\partial \bar{m}}{\partial y} \frac{\partial v'}{\partial x} \right] \quad \dots\dots (7) \end{aligned}$$

*The prime will be used only in this section to denote the new variables as defined by these equations. Elsewhere it will always denote differentiation with respect to the coordinate normal to the surface.

In general we require that the order of magnitude N of each disturbance q and its derivatives be much less than the order of magnitude of the corresponding mean quantity Q and its derivatives, which are themselves of order unity in terms of the non-dimensional quantities x' , y , etc. Thus

$$O(q) = N \ll O(Q) = 1.$$

Therefore, the maximum order of magnitude of the last two brackets on the right hand side of equation (7) are αR^{-2} and R^{-3} , whilst the remaining terms are of maximum order $\alpha^2 R^{-1}$, α , and R^{-1} . Thus, without going into detailed considerations of the magnitudes of each of the terms forming the equation, it can be seen that for large Reynolds numbers these last two brackets can be neglected provided

$$R^{-1} \ll \alpha \ll R,$$

or

$$1/d \ll \alpha^{\#} \ll d/\delta^2, \text{ where } d \text{ is a typical geometrical length.}$$

Less rigorously we may say that

$$\alpha^{\#} = O(1/\delta),$$

so that the wave length of the disturbance must be of the order of magnitude of the boundary layer thickness. It is just this condition that Pretsch introduced when considering the stability problem for the incompressible case with a pressure gradient (reference 15). Making the same assumption, we find for (7)

$$\begin{aligned} \alpha \rho \left\{ i(w-c)f + \frac{\partial w}{\partial y} \phi \right\} + \frac{1}{R} \left\{ r \left(w \frac{\partial w}{\partial x'} + v' \frac{\partial w}{\partial y} \right) + \rho \left(f \frac{\partial w}{\partial x'} + w \frac{\partial f}{\partial x'} + v' \frac{\partial f}{\partial y} \right) \right\} \\ = - \frac{i \alpha \bar{w}}{\gamma M_1^2} + \frac{1}{R} \left[- \frac{1}{\gamma M_1^2} \frac{\partial \bar{w}}{\partial x'} + \mu \left\{ \frac{\partial^2 f}{\partial y^2} + \frac{1}{3} i \alpha^2 \frac{\partial \phi}{\partial y} - \frac{4}{3} \alpha^2 f \right\} \right. \\ \left. + \bar{m} \frac{\partial^2 w}{\partial y^2} + \frac{\partial \bar{m}}{\partial y} \frac{\partial w}{\partial y} + \frac{\partial \mu}{\partial y} \left(\frac{\partial f}{\partial y} + i \alpha^2 \phi \right) \right]. \end{aligned} \quad \dots\dots (8)$$

If we proceed to the limit $R \rightarrow \infty$, equation (8) gives

$$\alpha \rho \left\{ i(w-c)f + \frac{\partial w}{\partial y} \phi \right\} = - \frac{i \alpha \bar{w}}{\gamma M_1^2}$$

and this is the same as the inviscid equation for compressible flow without a pressure gradient given on page 24 of reference 10.

The procedure for dealing with the remaining equations (2) to (5) is precisely similar. For the

/second ...

second equation of motion we get

$$\begin{aligned} & \alpha^2 \rho \left\{ i(w-c)\phi \right\} + \frac{\alpha \rho}{R} \left\{ \phi \frac{\partial v'}{\partial y} + v' \frac{\partial \phi}{\partial y} + w \frac{\partial \phi}{\partial x'} \right\} + \frac{1}{R^2} \left\{ \rho f \frac{\partial v'}{\partial x'} + r \left(w \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y} \right) \right\} \\ & = - \frac{1}{\gamma M_1^2} \frac{\partial \varpi}{\partial y} + \frac{\alpha}{R} \left[\frac{4}{3} \mu \frac{\partial^2 \phi}{\partial y^2} + \frac{\mu}{3} i \frac{\partial f}{\partial y} - \mu \alpha^2 \phi + i \bar{m} \frac{\partial w}{\partial y} + \frac{4}{3} \frac{\partial \phi}{\partial y} \frac{\partial \mu}{\partial y} - \frac{2}{3} i f \frac{\partial \mu}{\partial y} \right] \\ & + \frac{1}{R^2} \left[2i \alpha^2 \frac{\partial \phi}{\partial x'} + \frac{1}{3} \left(\mu \frac{\partial f}{\partial x' \partial y} + \bar{m} \frac{\partial^2 w}{\partial x' \partial y} \right) + \frac{4}{3} \left(\bar{m} \frac{\partial^2 v'}{\partial y^2} + \frac{\partial \bar{m}}{\partial y} \frac{\partial v'}{\partial y} \right) \right. \\ & \quad \left. - \frac{2}{3} \left(\frac{\partial \mu}{\partial y} \frac{\partial f}{\partial x'} + \frac{\partial \bar{m}}{\partial y} \frac{\partial w}{\partial x'} \right) + i \alpha^2 \phi \frac{\partial \mu}{\partial x'} + \frac{\partial \mu}{\partial x'} \frac{\partial f}{\partial y} + \frac{\partial \bar{m}}{\partial x'} \frac{\partial w}{\partial y} \right] \\ & + \frac{\alpha}{R^3} \left[\mu \frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial \mu}{\partial x'} \frac{\partial \phi}{\partial x'} + i \bar{m} \frac{\partial v'}{\partial x'} \right] + \frac{1}{R^4} \left[\bar{m} \frac{\partial^2 v'}{\partial x'^2} + \frac{\partial \bar{m}}{\partial x'} \frac{\partial v'}{\partial x'} \right]. \end{aligned} \dots\dots\dots (9)$$

Taking $R^{-1} \ll \alpha \ll R$, for large Reynolds numbers, this yields

$$\begin{aligned} & \alpha^2 \rho \left\{ i(w-c)\phi \right\} + \frac{\alpha \rho}{R} \left\{ \phi \frac{\partial v'}{\partial y} + v' \frac{\partial \phi}{\partial y} + w \frac{\partial \phi}{\partial x'} \right\} \\ & = - \frac{1}{\gamma M_1^2} \frac{\partial \varpi}{\partial y} + \frac{\mu \alpha}{R} \left\{ \frac{4}{3} \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{3} i \frac{\partial f}{\partial y} - \alpha^2 \phi \right\} + \frac{\alpha}{R} \left\{ i \bar{m} \frac{\partial w}{\partial y} + \frac{4}{3} \frac{\partial \phi}{\partial y} \frac{\partial \mu}{\partial y} - \frac{2}{3} i f \frac{\partial \mu}{\partial y} \right\} \end{aligned} \dots\dots\dots (10)$$

The equation of continuity (3) becomes

$$\begin{aligned} & \alpha R \left[i(w-c)r + \rho \left(\frac{\partial \phi}{\partial y} + i f \right) + \phi \frac{\partial \rho}{\partial y} \right] + f \frac{\partial \rho}{\partial x'} + \rho \frac{\partial f}{\partial x'} + w \frac{\partial r}{\partial x'} + v' \frac{\partial r}{\partial y} \\ & \quad + r \left(\frac{\partial w}{\partial x'} + \frac{\partial v'}{\partial y} \right) \\ & = 0 \end{aligned}$$

and for large R this is

$$i(w-c)r + \rho \left(\frac{\partial \phi}{\partial y} + i f \right) + \frac{\partial \rho}{\partial y} \phi = 0. \dots\dots\dots (11)$$

We shall take the Prandtl number $\sigma_p = C_p \mu^{\frac{1}{2}} / k^{\frac{1}{2}}$, to be constant, so that, with C_p constant, $k^{\frac{1}{2}} \ll \mu^{\frac{1}{2}}$. Under these circumstances the perturbation equation corresponding

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to the energy equation (4) is

$$\begin{aligned}
 & \alpha \rho \left(\theta \frac{\partial T}{\partial y} + i(w-c)\theta \right) + \frac{1}{R} \left[r \left(w \frac{\partial T}{\partial x} + v' \frac{\partial T}{\partial y} \right) + \rho \left(f \frac{\partial T}{\partial x} + w \frac{\partial \theta}{\partial x} + v' \frac{\partial \theta}{\partial y} \right) \right] \\
 & \quad + (\gamma-1) \alpha \rho T \left(\frac{\partial \theta}{\partial y} + i f \right) + \frac{\gamma-1}{R} \left[w \left(\frac{\partial w}{\partial x} + \frac{\partial v'}{\partial y} \right) + p \frac{\partial f}{\partial x} \right] \\
 & = \frac{1}{R} \left[\frac{\gamma}{\sigma} \left\{ \mu \left(\frac{\partial^2 \theta}{\partial y^2} - \alpha^2 \theta \right) + \frac{\partial \mu}{\partial y} \frac{\partial \theta}{\partial y} + \frac{\partial}{\partial y} \left(\bar{m} \frac{\partial T}{\partial y} \right) \right\} \right. \\
 & \quad \left. + \gamma(\gamma-1) M_1^2 \left\{ \bar{m} \left(\frac{\partial w}{\partial y} \right)^2 + 2\mu \frac{\partial w}{\partial y} \left(\frac{\partial f}{\partial y} + i \alpha^2 \theta \right) \right\} \right] \\
 & + \frac{\alpha}{R^2} \left[\frac{\gamma i}{\sigma} \left\{ 2\mu \frac{\partial \theta}{\partial x} + \theta \frac{\partial \mu}{\partial x} + \bar{m} \frac{\partial T}{\partial x} \right\} \right. \\
 & \quad \left. + \gamma(\gamma-1) M_1^2 \left\{ \frac{4}{3} \mu i f \left(2 \frac{\partial w}{\partial x} - \frac{\partial v'}{\partial y} \right) + \frac{4}{3} \mu \frac{\partial \theta}{\partial y} \left(2 \frac{\partial v'}{\partial y} - \frac{\partial w}{\partial x} \right) \right. \right. \\
 & \quad \left. \left. + 2\mu \frac{\partial \theta}{\partial x} \frac{\partial w}{\partial y} \right\} \right] \\
 & + \frac{1}{R^3} \left[\frac{\gamma}{\sigma} \left\{ \bar{m} \frac{\partial^2 T}{\partial x^2} + \mu \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial \bar{m}}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial \mu}{\partial x} \frac{\partial \theta}{\partial x} \right\} \right. \\
 & \quad \left. + \gamma(\gamma-1) M_1^2 \left\{ \frac{4}{3} \bar{m} \left(\frac{\partial w^2}{\partial x^2} - \frac{\partial w}{\partial x} \frac{\partial v'}{\partial y} + \frac{\partial v'^2}{\partial y^2} \right) + \frac{4}{3} \mu \frac{\partial f}{\partial x} \left(2 \frac{\partial w}{\partial x} - \frac{\partial v'}{\partial y} \right) \right. \right. \\
 & \quad \left. \left. + 2i \alpha^2 \mu \theta \frac{\partial v'}{\partial x} + 2 \frac{\partial v'}{\partial x} \left(\bar{m} \frac{\partial w}{\partial y} + \mu \frac{\partial f}{\partial y} \right) \right\} \right] \\
 & + \frac{1}{R^4} 2\gamma(\gamma-1) M_1^2 \alpha \mu \frac{\partial \theta}{\partial x} \frac{\partial v'}{\partial x} \\
 & + \frac{1}{R^5} \gamma(\gamma-1) M_1^2 \bar{m} \left(\frac{\partial v'}{\partial x} \right)^2 \dots \dots \dots (12)
 \end{aligned}$$

For large R, with $R^{-1} \ll \alpha \ll R$ again, this reduces to

$$\begin{aligned}
 & \alpha \rho \left(\theta \frac{\partial T}{\partial y} + i(w-c)\theta \right) + \frac{1}{R} \left[r \left(w \frac{\partial T}{\partial x} + v' \frac{\partial T}{\partial y} \right) + \rho \left(f \frac{\partial T}{\partial x} + w \frac{\partial \theta}{\partial x} + v' \frac{\partial \theta}{\partial y} \right) \right] \\
 & = -\alpha(\gamma-1) \rho T \left(\frac{\partial \theta}{\partial y} + i f \right) - \frac{(\gamma-1)}{R} \left[w \left(\frac{\partial w}{\partial x} + \frac{\partial v'}{\partial y} \right) + p \frac{\partial f}{\partial x} \right] \\
 & + \frac{\gamma}{\sigma R} \left[\mu \left(\frac{\partial^2 \theta}{\partial y^2} - \alpha^2 \theta \right) + \frac{\partial \mu}{\partial y} \frac{\partial \theta}{\partial y} + \frac{\partial}{\partial y} \left(\bar{m} \frac{\partial T}{\partial y} \right) \right] \\
 & + \frac{\gamma(\gamma-1)}{R} M_1^2 \left[\bar{m} \left(\frac{\partial w}{\partial y} \right)^2 + 2\mu \frac{\partial w}{\partial y} \left(\frac{\partial f}{\partial y} + i \alpha^2 \theta \right) \right] \dots \dots \dots (13)
 \end{aligned}$$

Finally, the equation of state gives

$$\frac{p}{\rho} = \frac{r}{\rho} + \frac{\theta}{T} \dots \dots \dots (14)$$

Equations (8), (10), (11), (13), and (14) are the linearised disturbance equations for flow with a pressure gradient. It is of particular interest to note that for infinite Reynolds number they reduce to the equations with no pressure gradient (cf. the equations given in reference 10);

/thus ...

thus the inviscid stability problem is independent of the external pressure gradient, except in as far as it affects the local boundary layer distributions of velocity and temperature. In particular, the differential equation for ϕ (equation (112) of reference (10)) in inviscid flow

$$\frac{d}{dy} \left\{ \frac{(w-c) \frac{d\phi}{dy} - \frac{dw}{dy} \phi}{T - M_1^2 (w-c)^2} \right\} = \alpha^2 \frac{(w-c)}{T} \phi, \quad \dots\dots\dots (15)$$

is the same with and without a pressure gradient.

The next step in the analysis is to introduce a new system of dependent variables defined as

$$\begin{aligned} z_1 &= f, & z_2 &= \frac{\partial f}{\partial y}, & z_3 &= \phi, \\ z_4 &= \frac{w}{M_1^2}, & z_5 &= \theta, & z_6 &= \frac{\partial \theta}{\partial y}. \end{aligned}$$

With the aid of these transformations it will be possible to reduce the perturbation differential equations to a system of six linear homogeneous first order differential equations in the variables z_1, \dots, z_6 . Thus we obtain immediately

$$\frac{\partial z_1}{\partial y} = z_2, \quad \frac{\partial z_5}{\partial y} = z_6 \quad \dots\dots\dots (16)$$

whilst equation (11) becomes

$$\frac{\partial z_3}{\partial y} = -i z_1 - \frac{1}{\rho} \frac{\partial \rho}{\partial y} z_3 - i(w-c) \left\{ \frac{M_1^2}{p} z_4 - \frac{1}{T} z_5 \right\}. \quad \dots\dots\dots (17)$$

Using equations (8) and (17), we can find the value of $\partial z_2 / \partial y$; equation (8) gives

$$\begin{aligned} \frac{\mu}{p} \left\{ \frac{\partial z_2}{\partial y} + \frac{1}{3} i \alpha^2 \frac{\partial z_3}{\partial y} \right\} &= \alpha \rho \left\{ i(w-c) z_1 + \frac{\partial w}{\partial y} z_3 \right\} + \frac{1}{\gamma} i \alpha z_4 \\ &+ \frac{1}{R} \left[\frac{1}{\gamma} \frac{\partial z_4}{\partial x'} + \rho \left(M_1^2 \frac{z_4}{p} - \frac{z_5}{T} \right) \left(w \frac{\partial w}{\partial x'} + v' \frac{\partial w}{\partial y} \right) \right. \\ &\quad \left. + \rho \left(w \frac{\partial z_1}{\partial x'} + v' z_2 + \frac{\partial w}{\partial x'} z_1 \right) \right. \\ &\quad \left. + \frac{4}{3} \mu \alpha^2 z_1 - \left\{ \bar{m} \frac{\partial^2 w}{\partial y^2} + \frac{\partial \bar{m}}{\partial y} \frac{\partial w}{\partial y} + \frac{\partial \mu}{\partial y} (z_2 + i \alpha^2 z_3) \right\} \right]. \end{aligned}$$

Taking μ^* to be a single valued function of the temperature, T^* , so that $\bar{m} = \theta \frac{d\mu}{dT}$, we can write the last equation as

$$\frac{\partial z_2}{\partial y} + \frac{1}{3} i \alpha^2 \frac{\partial z_3}{\partial y} = \frac{\alpha R}{\mu} \left[\rho \left\{ i(w-c) z_1 + \frac{\partial w}{\partial y} z_3 \right\} + \frac{1}{\gamma} i z_4 \right] + O(1), \quad \dots\dots\dots (18)$$

where $O(1)$ stands for a linear function of the perturbation quantities $z_1 \dots z_6$, $\partial z_1 / \partial x'$ and $\partial z_4 / \partial x'$, which is of order unity in R and is regular in M_1^2 , and is thus of maximum

order of magnitude α^2 or 1, assuming that $\frac{\partial Z}{\partial x} \leq O(Z)$.[‡]
 Eliminating $\partial Z_3/\partial y$ between (17) and (18) we find that

$$\frac{\partial Z_2}{\partial y} = \frac{\alpha R}{\mu} \left\{ \rho \left[i(w-c)Z_1 + \frac{\partial w}{\partial y} Z_3 \right] + \frac{1}{\gamma} iZ_4 \right\} + O(1) \dots\dots\dots (19)$$

The only difference between this equation and the corresponding equation with no pressure gradient (equation (69) reference 10⁺), lies in the terms forming the expression written as $O(1)$, and it is found that for all the methods of solution considered the latter expression can be neglected. This is dealt with in greater detail in the next section.

In an exactly similar manner it can be shown that the remaining equations for $\partial Z_4/\partial y$ and $\partial Z_6/\partial y$ are of the same form as found by Lees and Lin. For completeness they are included here.

$$\frac{\partial Z_4}{\partial y} = \gamma \frac{-R i \alpha^2 (w-c) \rho Z_3 + \alpha O(1)}{R + \frac{4}{3} i \alpha^2 \mu \frac{w-c}{p} \gamma M_1^2}, \dots\dots\dots (20)$$

$$\frac{\partial Z_6}{\partial y} = \frac{\sigma R}{\gamma \mu} \alpha \left\{ \gamma \rho \left[i(w-c)Z_5 + \frac{\partial T}{\partial y} Z_3 \right] - (\gamma-1) \left[\frac{\partial p}{\partial y} Z_3 + i(w-c)M_1^2 Z_4 \right] \right\} + O(1). \dots\dots\dots (21)$$

In these equations $O(1)$ again denotes a function linear in $Z_1, \dots, Z_6, \partial Z_1/\partial x', \partial Z_3/\partial x', \partial Z_4/\partial x',$ and $\partial Z_5/\partial x'$, which is of a maximum order of magnitude α^2 or 1.

2. Solution of the perturbation equations

There are three principal methods of solving the perturbation differential equations; in each case, as is shown by Lees and Lin, the terms denoted by $O(1)$ in the equations are negligible to the first order of approximation. The first order solutions are in fact the only ones that are considered. Through the dropping of these terms the dependence of the differential equations on x and v is lost, and the equations may be solved for any given distribution of velocity and temperature along the y -axis in exactly the same fashion as given by Lees and Lin for parallel flows. We shall summarize the methods developed by Lees and Lin here.

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[‡]Note that this type of relation does not hold for derivatives normal to the surface, due to the rapid variation of the importance of viscosity in the y -direction at the critical layer.

⁺A number of minor errors will be found in at least some copies of this report.

In the first method solutions are found as power series in the small parameter

$$\epsilon = (\alpha R)^{-1/3}$$

and in terms of the new dependent variable

$$\eta = (y - y_c)/\epsilon \quad \text{where } w(y_c) = c.$$

The initial approximation solutions for Z_1 and Z_3 are as follows -

$$X_{11} = \int H_{1/3}^{(1)} \left[\frac{2}{3} (i\zeta)^{3/2} \right] \zeta^{1/2} d\zeta,$$

$$X_{12} = \int H_{1/3}^{(2)} \left[\frac{2}{3} (i\zeta)^{3/2} \right] \zeta^{1/2} d\zeta,$$

$$X_{13} = 1,$$

$$X_{14} = X_{15} = X_{16} = 0,$$

$$X_{31} = -i\alpha \left(\frac{w_c'}{v_c} \right)^{1/3} \left\{ \zeta \int H_{1/3}^{(1)} \left[\frac{2}{3} (i\zeta)^{3/2} \right] \zeta^{1/2} d\zeta - \int H_{1/3}^{(1)} \left[\frac{2}{3} (i\zeta)^{3/2} \right] \zeta^{3/2} d\zeta \right\},$$

$$X_{32} = -i\alpha \left(\frac{w_c'}{v_c} \right)^{1/3} \left\{ \zeta \int H_{1/3}^{(2)} \left[\frac{2}{3} (i\zeta)^{3/2} \right] \zeta^{1/2} d\zeta - \int H_{1/3}^{(2)} \left[\frac{2}{3} (i\zeta)^{3/2} \right] \zeta^{3/2} d\zeta \right\},$$

$$X_{33} = -i\alpha \left(\frac{w_c'}{v_c} \right)^{-1/3} \zeta,$$

$$X_{34} = X_{35} = X_{36} = 0,$$

where $\zeta = \left(\frac{w_c'}{v_c} \right)^{1/3} \eta,$

and where the suffix c denotes values at the critical layer where $w = c$, and X_{11} and X_{31} are the initial approximations to Z_1 and Z_3 respectively, and $H_{1/3}^{(1)}$, $H_{1/3}^{(2)}$ are Hankel functions of the first and second kind of order $1/3$.

Another set of solutions is obtained by expanding the solutions in power series in $(\alpha R)^{-1}$. This gives a pair of asymptotic series, and the initial approximation depends on the inviscid equation (15) only. Since this does not depend directly on the pressure gradient, the same results expressed as functions of w and T , are obtained for the solutions with or without a pressure gradient.

The third set of solutions is obtained by putting

$$Z_i = f_i \exp \left\{ (\alpha R)^{\frac{1}{2}} \int g \, dy \right\}, \quad i = 1, \dots, 6$$

and expanding f_i as a power series in $(\alpha R)^{-\frac{1}{2}}$. The initial approximation in terms of w and T is again independent of the pressure gradient. The method gives four independent solutions, but the expansions are valid only in the regions given by

$$-\frac{7\pi}{6} < \arg(\zeta) < \frac{\pi}{6}, \quad -\frac{7\pi}{6} < \arg(\zeta \sigma c^{1/3}) < \frac{\pi}{6} \dots \dots \dots (22)$$

in the complex ζ -plane. These solutions are

$$\left. \begin{aligned} (Z_1, Z_3, Z_4, Z_5) &= (1, 0, 0, 0) \exp \left\{ \pm (\alpha R)^{\frac{1}{2}} \int_{y_c}^y \sqrt{\frac{1}{\nu}(w-c)} \, dy \right\}, \\ (Z_2, Z_6) &= (\alpha R)^{\frac{1}{2}} \left(\pm \sqrt{\frac{1}{\nu}(w-c)}, 0 \right) \exp \left\{ \pm (\alpha R)^{\frac{1}{2}} \int_{y_c}^y \sqrt{\frac{1}{\nu}(w-c)} \, dy \right\}, \\ (Z_1, Z_3, Z_4, Z_5) &= (0, 0, 0, 1) \exp \left\{ \pm (\alpha R)^{\frac{1}{2}} \int_{y_c}^y \sqrt{\frac{i\sigma}{\nu}(w-c)} \, dy \right\}, \\ (Z_2, Z_6) &= (\alpha R)^{\frac{1}{2}} \left(0, \pm \sqrt{\frac{i\sigma}{\nu}(w-c)} \right) \exp \left\{ \pm (\alpha R)^{\frac{1}{2}} \int_{y_c}^y \sqrt{\frac{i\sigma}{\nu}(w-c)} \, dy \right\}. \end{aligned} \right\} \dots \dots \dots (23)$$

3. The characteristic equation

In the preceding sections it has been shown that all the initial approximations for the solution of the disturbance equations, and also the exact differential equation for the inviscid case, depend only on the local velocity and temperature distributions in the boundary layer, and do not depend directly on the pressure gradient. We can thus say immediately that the bulk of the stability theory proved for compressible flow without a pressure gradient by Lees and Lin will hold for the case in which we are interested at present. In fact, the only proofs given by them which cannot be taken over exactly are those in which recourse is made to the boundary layer equations of the mean motion.

In particular, the boundary value problem and its solution can be quoted directly from their work. Thus in reference 10 it was shown that the boundary value problem leads to a characteristic equation in c , α , M_1 , and R . Expressed in terms of the perturbation velocity ϕ it is

$$E(\alpha, c, M_1^2) = F(z),$$

where $F(z)$ is the Tietjens function and has been calculated (reference 11), and

$$/F(z) \dots$$

$$F(z) = 1 + \int_{+\infty}^{-z} H_{1/3}^{(1)} \left[\frac{2}{3} (i\zeta)^{3/2} \right] \zeta^{3/2} d\zeta / z \int_{+\infty}^z H_{1/3}^{(1)} \left[\frac{2}{3} (i\zeta)^{3/2} \right] \zeta^{1/2} d\zeta, \quad \dots\dots\dots (24)$$

$$z = - \left(\frac{w'_c}{y_c} \right)^{1/3} \eta_0, \quad \eta_0 \text{ is the value of } \eta \text{ at } y = 0,$$

and E is a function depending on the inviscid solution only, namely

$$E(\alpha, c, M_1^2) = - \frac{1}{y_c} \left| \begin{array}{l} \phi_{11}, \phi'_{12} + \bar{\beta} \phi_{12} \\ \phi_{21}, \phi'_{22} + \bar{\beta} \phi_{22} \end{array} \right| \div \left| \begin{array}{l} \frac{T_0 \phi'_{11} + M_1^2 w'_c \phi_{11}}{T_0 - M_1^2 c^2}, \phi'_{12} + \bar{\beta} \phi_{12} \\ \frac{T_0 \phi'_{21} + M_1^2 w'_c \phi_{21}}{T_0 - M_1^2 c^2}, \phi'_{22} + \bar{\beta} \phi_{22} \end{array} \right| \quad \dots\dots\dots (25)$$

$$\bar{\beta} = \sqrt{1 - M_1^2 (1-c)^2}$$

where ϕ_{r1} and ϕ_{r2} are the values of the two solutions of the inviscid equation (15) at the wall and at infinity respectively (see equation (5) on page 8).

Equation (25) holds provided

$$1 - M_1^2 (1-c_r)^2 \neq 0,$$

corresponding to the neutral supersonic and sonic disturbances. The significance of this restriction will be dealt with in section 4.

The complex characteristic equation leads to two equations relating c_i, c_r, α, M_1 and R . It is usual to consider chosen values of c_i (representing the amount of damping or amplification of the disturbance) and M_1 , and to plot the relationship between α and R given by these equations. In particular, if c_i is chosen to be zero the neutral stability curves are obtained. In this case the calculation of the solutions of the characteristic equation is not too laborious and Lees (reference 11) gives the following modified form of (25) for computational purposes, when solved for α as a function of R and M_1 .

$$\alpha = \frac{\frac{w'_c}{T_0} (1 - \alpha^2 H_2) \left[\frac{\sqrt{1 - M_1^2 (1-c)^2}}{(1-c)^2} \left(1 - \sum_{n=1}^{\infty} \alpha^{2n} N_{2n} \right) - \sum_1^{\infty} \alpha^{2n+1} N_{2n+1} \right]}{(u-L) \left[1 - \sum_2^{\infty} \alpha^{2n} M_{2n} + \frac{\sqrt{1 - M_1^2 (1-c)^2}}{(1-c)^2} \left(\alpha H_1 - \sum_1^{\infty} \alpha^{2n+1} M_{2n-1} \right) \right]} \quad \dots\dots\dots (26)$$

where $L = \frac{w'_c}{T_0} R \left(K_1 + \frac{T_0}{w'_c} \right),$

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and the corresponding value of R is given by

$$\alpha R = \frac{v_c}{w_c' (1+\lambda)^3} \left(\frac{zw_0'}{c} \right)^3, \quad \lambda = \frac{w_0' (y_c)}{c} - 1.$$

Here u is determined from

$$u + iv = 1 + \frac{w_0' c}{T_0} \left(\frac{\phi_{22}' + \bar{\beta} \phi_{22}}{\phi_{12}' + \bar{\beta} \phi_{12}} \right),$$

and

$$\left. \begin{aligned} H_{2n}(c, M_1^2) &= h_{2n}(y_1; c, M_1^2), & H_0 &= 1 \\ K_{2n+1}(c, M_1^2) &= k_{2n+1}(y_1; c, M_1^2), \\ H_{2n-1}(c, M_1^2) &= \left[\frac{1 - M_1^2 (1-c)^2}{(1-c)^2} \right]^{-1} h_{2n}'(y_1; c, M_1^2), \\ K_{2n}(c, M_1^2) &= \left[\frac{1 - M_1^2 (1-c)^2}{(1-c)^2} \right]^{-1} k_{2n+1}'(y_1; c, M_1^2), & K_0 &= 1, \end{aligned} \right\} \dots\dots\dots (27)$$

where

$$h_{2n}(y; c, M_1^2) = \int_0^y \left[\frac{T}{(w-c)^2} - M_1^2 \right] dy \int_0^y \frac{(w-c)^2}{T} h_{2n-2}(y; c, M_1^2) dy$$

$$h_0 = 1. \quad n \geq 1$$

Here the integration must be carried out in the complex plane, and then, because of the condition (2), the path of integration must lie below the point $y = y_c$.

$$k_{2n+1}(y; c, M_1^2) = \int_0^y \left[\frac{T}{(w-c)^2} - M_1^2 \right] dy \int_0^y \frac{(w-c)^2}{T} k_{2n-1}(y; c, M_1^2) dy$$

$$n \geq 1$$

$$k_1 = \int_0^y \left[\frac{T}{(w-c)^2} - M_1^2 \right] dy$$

and

$$\left. \begin{aligned} N_2 &= H_2 \\ N_n &= K_1 H_{n-1} - K_n \\ M_n &= H_2 H_{n-2} - H_n \end{aligned} \right\} n \geq 3.$$

For inviscid flow the secular equation reduces to

$$\begin{vmatrix} \phi_{11} & \phi'_{12} + \bar{\beta} \phi_{12} \\ \phi_{21} & \phi'_{22} + \bar{\beta} \phi_{22} \end{vmatrix} = 0 \quad \dots\dots\dots (28)$$

and since $\phi_{21} = 0$, this is

$$\phi'_{22} + \bar{\beta} \phi_{22} = 0. \quad \dots\dots\dots (29)$$

4. The supersonic disturbance

The boundary conditions for the disturbance velocity are that $\phi = 0$ at $y = 0$, and ϕ is bounded as $y \rightarrow \infty$. For the latter case we need only consider the inviscid equation

$$\frac{d}{dy} \left\{ \frac{(w-c)\phi' - w'\phi}{T - M_1^2 (w-c)^2} \right\} = \alpha^2 \frac{(w-c)}{T} \phi, \quad \dots\dots\dots (30)$$

the solution of which becomes

$$\phi \sim e^{\pm \bar{\beta} y}, \quad \text{as } y \rightarrow \infty. \quad \dots\dots\dots (31)$$

Here $\bar{\beta} = \alpha \sqrt{\Omega} = \alpha \sqrt{(1-M_1^2 (1-c)^2)}$, $\alpha > 0$, and we define $\bar{\beta}$ uniquely by making a cut along the negative real axis.

The boundary condition imposed on (31) means that we must have $\phi \sim e^{-\bar{\beta} y}$. Then if we put

$$\Omega = |\Omega| (\cos \theta + i \sin \theta), \quad \pi > \theta > -\pi,$$

we get

$$\cos \frac{\theta}{2} = \sqrt{\frac{1}{2}(1 + \cos \theta)}$$

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1}{2}(1 - \cos \theta)}, \quad \begin{matrix} + \text{ for } \pi > \theta > 0 \\ - \text{ for } 0 > \theta > -\pi \end{matrix}$$

therefore,

$$\bar{\beta} = \frac{\alpha}{\sqrt{2}} \left\{ \sqrt{|\Omega| + \Omega_r} \pm i \sqrt{|\Omega| - \Omega_r} \right\} \begin{matrix} + \text{ for } \Omega_i > 0 \\ - \text{ for } \Omega_i < 0 \end{matrix} \quad \dots\dots\dots (32)$$

$$\Omega_r = \mathcal{R}(\Omega) = 1 - M_1^2 [(1-c_r)^2 - c_i^2]$$

$$\Omega_i = \mathcal{I}(\Omega) = 2 M_1^2 c_i (1-c_r).$$

/These ...

These results show that if

- (i) $c_i > 0$ (i.e. the disturbance is amplified), $\Omega_i > 0$, $\bar{\beta}_i > 0$, and we have disturbances in the form of y attenuated waves travelling outwards.
- (ii) $c_i < 0$ (damped disturbance), $\Omega_i < 0$, $\bar{\beta}_i < 0$, we have a damped disturbance travelling inwards.
- (iii) $c_i = 0$, and $\Omega_r > 0$ (a subsonic neutral disturbance), then $\bar{\beta}_i = 0$, and the disturbance is attenuated in the y -direction, and travels parallel to the surface.
- (iv) $c_i = 0$, and $\Omega_r < 0$ (a supersonic neutral disturbance), $\bar{\beta}_r = 0$, and both solution $e^{\pm \beta y}$ satisfy the boundary conditions at $y = \infty$, so that both inward and outward travelling waves, of constant amplitude in the y -direction, can exist.

In the last case (iv) the boundary condition at $y = \infty$ is automatically satisfied by both solutions of (30), and therefore the six boundary conditions to be satisfied by the solutions of the six differential equations (16) to (20) are reduced to five. There is, therefore, no characteristic equation (25) to be satisfied by α , c , M_1 , and R . It is then apparent that neutral disturbances of any wave length and frequency with

$$c_r < 1 - 1/M_1, \quad c_i = 0,$$

can exist for all Reynolds numbers.

In the calculation made by Lees (reference 11) it has been assumed that neutral and amplified supersonic disturbances have no significance in the stability problem. This can be confirmed by investigating the secular equation for the supersonic disturbances.

Thus, for the inviscid case, the secular equation is

$$\phi'_{22} + \bar{\beta} \phi_{22} = 0. \quad \dots\dots (33)$$

The first order approximation in α for ϕ_2 gives (see equation (5) in the main text)

$$\left. \begin{aligned} \phi_{22} &= (1-c) K_1(c) \\ \phi'_{22} &= (1-c) \left\{ \frac{1 - M_1^2 (1-c)^2}{(1-c)^2} \right\} \\ K_1 &= \int_0^{y_1} \frac{T - M_1^2 (w-c)^2}{(w-c)^2} dy \end{aligned} \right\} \dots\dots (34)$$

and the path of integration lies below the point $y = y_c$ in the complex y -plane.

Taylor expansion about the point $y = y_c$ gives

$$K_1(c, M_1^2) = -\frac{T_0}{w_0' c} + \frac{T_c^2}{(w_c')^3} \left[\frac{d}{dy} \left(\frac{w'}{T} \right) \right]_{w=c} (\ln|c| - i\pi) + \dots$$

Thus neglecting the higher powers of c , (33) can be written

$$\alpha \sqrt{1 - M_1^2 (1-c)^2} \left\{ -\frac{T_0}{w_0' c} + \frac{T_c^2}{(w_c')^3} \left[\frac{d}{dy} \left(\frac{w'}{T} \right) \right]_{w=c} (\ln|c| - i\pi) \right\} + \frac{1 - M_1^2 (1-c)^2}{(1-c)^2} = 0. \dots\dots\dots (35)$$

To study an amplified disturbance we suppose that the phase velocity c has a small imaginary component so that $c = c_r + i c_i$, say. In this case if c_i is small enough it is clear that (35) is approximately

$$\alpha \left\{ -\frac{T_0}{w_0' c_r} + \frac{T_{c_r}^2}{(w_{c_r}')^3} \left[\frac{d}{dy} \left(\frac{w'}{T} \right) \right]_{w=c_r} (\ln(c_r) - i\pi) \right\} + \frac{\sqrt{1 - M_1^2 (1-c_r)^2}}{(1-c_r)^2} = 0. \dots\dots\dots (36)$$

For supersonic disturbances and Ω_1 small

$$\bar{\beta} = \alpha \sqrt{1 - M_1^2 (1-c)^2} \doteq \alpha \left\{ \frac{1}{2} \left| \frac{\Omega_i}{\Omega_r} \right| \sqrt{|\Omega_r|} \pm i \sqrt{\Omega_r} \right\} \doteq \pm \alpha i \sqrt{M_1^2 (1-c_r)^2 - 1}, \quad \begin{array}{l} + \text{ for } c_i > 0 \\ - \text{ for } c_i < 0. \end{array}$$

Thus considering the supersonic case and taking the real and imaginary parts of (36)

$$\ln(c_r) \frac{T_{c_r}^2}{(w_{c_r}')^3} \left[\frac{d}{dy} \left(\frac{w'}{T} \right) \right]_{w=c_r} = \frac{T_0}{w_0' c_r},$$

$$- \pi \alpha \frac{T_{c_r}^3}{(w_{c_r}')^3} \left[\frac{d}{dy} \left(\frac{w'}{T} \right) \right]_{w=c_r} \pm \sqrt{M_1^2 (1-c_r)^2 - 1} = 0, \quad \begin{array}{l} + \text{ for } c_i > 0 \\ - \text{ for } c_i < 0. \end{array}$$

/Therefore ...

Therefore

$$-\frac{\pi \alpha}{\ln(c_r)} \frac{T_0}{w_0' c_r} + \sqrt{M_1^2 (1-c_r)^2 - 1} = 0, \begin{matrix} + \text{ for } c_i > 0 \\ - \text{ for } c_i < 0. \end{matrix}$$

Since $\alpha > 0$, $\ln(c_r) < 0$, and excluding the sonic case $c_r = 1 - 1/M_1$, this has no solution for $c_i > 0$. Thus, at least for small values of α and c , amplified supersonic disturbances cannot exist, for they cannot be made to satisfy the boundary conditions as $y \rightarrow \infty$.

It is to be expected that the same result will hold for the viscous case, since the breakdown in the boundary conditions occurs in the main stream, where viscosity can be neglected.

The mathematical analysis of the supersonic disturbance thus predicts that neutral disturbances with continuous characteristic values and damped disturbances with discrete characteristic values should exist, but amplified, self-excited, disturbances cannot exist. The physical significance of these results is not clear. The abrupt change from continuous to discrete characteristic values does not seem to be physically reasonable, and since only small non-amplified supersonic disturbances are mathematically possible their existence could not be established experimentally. We note that the former difficulty can be removed by altering the boundary conditions from

$$\phi \text{ is bounded as } y \rightarrow \infty, \text{ to } \phi \rightarrow 0, \text{ as } y \rightarrow \infty.$$

Since the whole mechanism of laminar instability is considered to arise from small perturbations within the boundary layer, this revised boundary condition seems plausible. If it is used the secular equation will hold for the neutral supersonic disturbance, and it may be considered as the limiting case of the damped supersonic disturbance.

A P P E N D I X B

Calculations of the Neutral Stability
Characteristics

1. The transformation of compressible boundary layer flow
into a corresponding incompressible flow

When the Prandtl number σ and the index ω in the viscosity temperature relationship $\mu^* \propto T^{*\omega}$ are taken to be unity^{*}, and when the surface is insulated, the compressible boundary layer equations can be transformed into the incompressible equations in the new variables u_i^* , y_i^* , x_i^* , defined by

$$w_i = w$$

$$\frac{u_{i1}^*}{a_{10}} = \frac{u_1^*}{a_1} = M_1 \dots \dots \dots (37)$$

$$x_i^* = \int_0^{x^*} \left(\frac{a_1}{a_{10}} \right)^{\frac{3\gamma-1}{\gamma-1}} dx^* \dots \dots \dots (38)$$

$$y_i^* = \left(\frac{a_1}{a_{10}} \right) \int_0^{y^*} \left(\frac{\rho}{\rho_{10}} \right) dy^*$$

$$= \left(\frac{a_1}{a_{10}} \right)^{\frac{\gamma+1}{\gamma-1}} \int_0^{y^*} \left(\frac{a_1}{a} \right)^2 dy^*$$

$$= \left(\frac{a_1}{a_{10}} \right)^{\frac{\gamma+1}{\gamma-1}} \int_0^{y^*} \frac{dy^*}{T} \dots \dots \dots (39)$$

Note that in these equations

$$w = u^*/u_1^*, \quad w_i = u_i^*/u_{i1}^*,$$

suffix 1 refers to values at the edge of the boundary layer,

suffix 10 refers to stagnation in the main stream.

With an insulated surface the boundary layer temperature distribution is dependent only on the local main stream Mach number and the velocity ratio $w = w_i$, and is given by the Crocco formula

$$T^* = \frac{T^*}{T_1^*} = 1 + \frac{\gamma-1}{2} M_1^2 (1-w^2). \dots \dots \dots (40)$$

/Proof ...

^{*} ω is about 0.8 for air under normal conditions.

Proof of the transformation formulae (37) - (39) have been given by Stewartson (reference 20) and Illingworth (reference 19).

For finding the steady compressible boundary layer velocity distribution we shall use the exact solutions of the incompressible equations for the main stream velocity distribution

$$u_{i1}^* = \bar{c} (x_i^*)^m, \quad \bar{c} \text{ constant.} \quad \dots\dots\dots (41)$$

The differential equation that arises for this case is

$$\frac{d^3 F}{d\zeta^3} + F \frac{d^2 F}{d\zeta^2} = \beta \left[\left(\frac{dF}{d\zeta} \right)^2 - 1 \right] \quad \dots\dots\dots (42)$$

$$\beta = 2m/(m+1)$$

$$F = (2-\beta)^{-\frac{1}{2}} f, \quad f'(\eta_i) = u_{i1}^*/u_{i1}^* = w,$$

$$\zeta = (2-\beta)^{-\frac{1}{2}} \eta_i, \quad \eta_i = (u_{i1}^*/v_{10}^* x_i^*)^{\frac{1}{2}} y_i^*.$$

This equation has been solved by Hartree (reference 18), and his results are the basis for the calculations made in this paper.

In solving the stability problem we shall have need to use the derivatives of w with respect to y. These can be obtained from (39) using the following results obtained from (42)

$$\left| (2-\beta) \frac{dw}{d\eta_i} \right|_0^{\eta_i} = -w \int_0^{\eta_i} w d\eta_i + \int_0^{\eta_i} w^2 d\eta_i + \beta \int_0^{\eta_i} (w^2 - 1) d\eta_i,$$

$$(2-\beta) \frac{d^2 w}{d\eta_i^2} = -\frac{dw}{d\eta_i} \int_0^{\eta_i} w d\eta_i + \beta (w^2 - 1),$$

$$(2-\beta) \frac{d^3 w}{d\eta_i^3} = -\frac{d^2 w}{d\eta_i^2} \int_0^{\eta_i} w d\eta_i + w \frac{dw}{d\eta_i} (2\beta - 1),$$

$$(2-\beta) \frac{d^4 w}{d\eta_i^4} = -\frac{d^3 w}{d\eta_i^3} \int_0^{\eta_i} w d\eta_i + 2w \frac{d^2 w}{d\eta_i^2} (\beta - 1) + \left(\frac{dw}{d\eta_i} \right)^2 (2\beta - 1).$$

Derivatives of w up to the fourth are required in solving the secular equation, but for the determination of $R_{cr. min.}$ only the first and second derivatives are needed. These have been calculated for a number of values of β by Pretsch (reference 15), and can be used conveniently in obtaining the approximate values of these derivatives. His results are reproduced in Figures 10 and 11.

The Hartree solutions of (42) expressing w as a function of η_i , can be transformed into the compressible solution in the variable

$$y = y^*/y_1^* = y^*/\delta^*, \quad \text{say.}$$

Then, using (39),

$$y = \frac{\int_0^{\eta_i} T d\eta_i}{\int_0^{\eta_{i1}} T d\eta_i} = \frac{\int_0^{\eta_i} T d\eta_i}{b}, \quad \text{say,} \quad \dots\dots\dots (43)$$

where $b = \int_0^{\eta_{i1}} T d\eta_i$.

For the 'edge' of the boundary layer we take η_{i1} such that

$$\frac{u_1^*(\eta_{i1})}{u_{i1}^*} = .9995.$$

The compressible boundary layer distribution $w(y)$ thus obtained corresponds to the transformation of the incompressible main flow velocity distribution $u_{i1}^* = \bar{c}(x_i^*)^m$, given by equations (37) and (38). We find in fact that

$$u_1^{*2} = \frac{\bar{c}^2 a_{10}^2 x_i^{*2m}}{a_{10}^2 + \frac{\gamma-1}{2} \bar{c}^2 x_i^{*2m}},$$

$$a_1^2 = \frac{a_{10}^4}{a_{10}^2 + \frac{\gamma-1}{2} \bar{c}^2 x_i^{*2m}},$$

$$\frac{dx_i^*}{dx_i^*} = \left[1 + \frac{\gamma-1}{2} \frac{\bar{c}^2}{(a_{10})^2} x_i^{*2m} \right]^{\frac{3\gamma-1}{2(\gamma-1)}}.$$

In general there is no exact solution of these equations, but we can obtain the solution as a power series in $\bar{c}(x_i^*)^m/a_{10}$. Thus, if $\gamma = 1.4$,

$$u_1^* = \bar{c} x_i^{*m} \left[1 - \frac{(10m+1)}{10(2m+1)} \left(\frac{\bar{c} x_i^{*m}}{a_{10}} \right)^2 + \dots \right] \quad \dots\dots\dots (44)$$

Normally we require the compressible velocity u^* as a function of y^*/δ^* or y^*/θ^* , where δ^* is the boundary layer displacement thickness and θ^* is the momentum thickness. Now

$$/ \frac{y^*}{y_1^*} \dots$$

$$\begin{aligned} \frac{y^*}{y_1^*} &= \frac{\int_0^\infty \left\{ \frac{u_i^* \rho^*}{u_{i1}^* \rho_1^*} \left(1 - \frac{u_i^*}{u_1^*} \right) \right\} dy^*}{y_1^*} \\ &= \frac{\int_0^\infty \frac{u_i^*}{u_{i1}^*} \left(1 - \frac{u_i^*}{u_1^*} \right) d\eta_i}{\int_0^{\eta_{i1}} T d\eta_i} \\ &= K_{\vartheta} / b, \quad \text{where } K_{\vartheta} = \int_0^\infty w(1-w) d\eta_i \quad \dots\dots\dots (45) \\ &\quad \text{and } b = \int_0^{\eta_{i1}} T d\eta_i, \end{aligned}$$

and therefore

$$\frac{y^*}{y} = y \frac{b}{K_{\vartheta}} \quad \dots\dots\dots (46)$$

Similarly

$$\begin{aligned} \frac{\delta^*}{y_1^*} &= \frac{\int_0^\infty (1-w) dy^*}{y_1^*} \\ &= \frac{\int_0^\infty (T-w) d\eta_i}{\int_0^{\eta_{i1}} T d\eta_i} \\ &= \frac{\int_0^\infty \left(1-w + \frac{\gamma-1}{2} M_1^2 [w(1-w) + (1-w)] \right) d\eta_i}{b} \\ &= \frac{K_{\delta^*} + \frac{\gamma-1}{2} M_1^2 (K_{\vartheta} + K_{\delta^*})}{b} \quad \dots\dots\dots (47) \end{aligned}$$

where $K_{\delta^*} = \int_0^\infty (1-w) d\eta_i$.

Therefore

$$\frac{y^*}{\delta^*} = y \frac{b}{\left(K_{\delta^*} + \frac{\gamma-1}{2} M_1^2 [K_{\vartheta} + K_{\delta^*}] \right)} \quad \dots\dots\dots (48)$$

From these results it follows that

$$\left. \begin{aligned} a_{\vartheta} &= a \frac{K_{\vartheta}}{b} \\ R_{\vartheta} &= R \frac{K_{\vartheta}}{b} \end{aligned} \right\} \quad \dots\dots\dots (49)$$

remembering here that $a = a^* \delta$, $R = \frac{u_1^* \delta}{v_1^*}$.

Pretsch has also calculated the variation of K_{β} and K_{δ} with β for the Hartree solutions, and these are shown in graphical form in Figure 9.

2. Numerical solution of the neutral stability equation

The method of solution of the neutral secular equation (26) has been given by Lees (reference 11), and his method is quoted here. Once the coefficients of α in (26) have been obtained (see Appendix B3) from the velocity and temperature distributions of the boundary layer, the procedure for solving (26) is as follows.

From the Tietjens function $F(z)$, the function

$$\bar{\Phi}(z) = 1/(1 - F(z))$$

is plotted (Figure 12).

For a given Mach number and a chosen value of $c > 1 - 1/M_1$, the value of v (see equation (26)) is calculated from

$$v = - \frac{\pi w'_0 c}{T_0} \frac{T_c}{(w'_c)^2} \left(\frac{w''_c}{w'_c} - \frac{T'_c}{T_c} \right) = \frac{w'_0 c}{T_0} \Im K_1, \dots\dots\dots (50)$$

where $\Im K_1$ is the imaginary part of K_1 .

$\lambda = \frac{w'_0 \cdot y_c}{c} - 1$, is usually small, and the first order approximations for $\bar{\Phi}_r$ and $\bar{\Phi}_i$, the real and imaginary parts of $\bar{\Phi}$, are

$$\begin{aligned} \bar{\Phi}_i^{(0)}(z^0) &= v \\ \bar{\Phi}_r^{(0)}(z^0) &= u^0. \end{aligned} \dots\dots\dots (51)$$

Using the values of v obtained from (50) z^0 and u^0 are found from Figure 12, and these give the value of

$$\alpha R = \frac{v_c}{w'_c (1+\lambda)^3} \left(\frac{z w'_0}{c} \right)^3.$$

The first order approximations for u and z can be improved by using

$$\bar{\Phi}_i^{(n+1)}(z^{(n+1)}) = \frac{(1+\lambda)v}{(1+\lambda u^{(n)})^2 + \lambda^2 v^2},$$

$$u^{(n+1)} = \bar{\Phi}_r^{(n+1)}(z^{(n+1)}) \left[\frac{(1+\lambda u^{(n)})^2 + \lambda^2 v^2}{(1+\lambda)(1+\lambda u^{(n)})} \right] - \frac{\lambda v^2}{1+\lambda u^{(n)}},$$

using always the value of v given by (50).

/Finally ...

Finally the value of α is given by (26), which can be solved by an iteration process; the whole procedure is repeated for a number of values of $c > 1-1/M_1$ to obtain the complete neutral stability curve of α versus R .

3. Evaluation of the integrals appearing in the characteristic equation

In the method given in the last section for solving the neutral characteristic equation, the values of the coefficients, H, K, M, N , of α were required. The evaluation of these coefficients, which are multiple integrals of the boundary layer velocity distribution, constitutes the bulk of the work needed in numerical solution of the neutral stability problem. In the present case it was found advantageous to alter the method of evaluating these integrals in some details from that given by Lees (ibid).

Following Lees in equation (26) for α , we neglect powers of α higher than the second. The coefficients that have to be evaluated are, therefore

$$\begin{aligned}
 H_1 &= \int_0^{y_1} \frac{(w-c)^2}{T} dy \\
 N_2 = H_2 &= \int_0^{y_1} \frac{T-M_1^2 (w-c)^2}{(w-c)^2} dy \int_0^y \frac{(w-c)^2}{T} dy \\
 K_1 &= \int_0^{y_1} \frac{T-M_1^2 (w-c)^2}{(w-c)^2} dy \\
 N_3 &= \int_0^{y_1} \frac{T-M_1^2 (w-c)^2}{(w-c)^2} dy \int_0^y \frac{(w-c)^2}{T} dy \int_0^y \frac{T-M_1^2 (w-c)^2}{(w-c)^2} dy \\
 M_3 &= \int_0^{y_1} \frac{(w-c)^2}{T} dy \int_0^y \frac{T-M_1^2 (w-c)^2}{(w-c)^2} dy \int_0^y \frac{(w-c)^2}{T} dy.
 \end{aligned} \tag{52}$$

The general method of evaluation is to split each of the integrals into two portions, from $y = 0$ to $y = y_j$ say, and from $y = y_j$ to $y = y_1$, so that the first region always contains the singularity of the integral at $w = c$. This is then calculated by expansion in Taylor series about the point $y = y_c$ and integrating term by term (bearing in mind that the path of integration always lies below the point $y = y_c$ in the complex y -plane. The second part of the integral can be found by a straightforward method of numerical integration, such as the Simpson parabolic rule. In general, we

shall denote the two integrals thus formed by the additional suffices 1 and 2. Thus, for example

$$K_{11} = \int_0^{y_j} \frac{T}{(w-c)^2} dy ,$$

$$K_{12} = \int_{y_j}^{y_1} \frac{T}{(w-c)^2} dy .$$

Then

$$K_1 = K_{11} + K_{12} - M_1^2 y_1 .$$

3.1. Evaluation of $K_1(c)$

We put

$$\frac{T}{(w-c)^2} = \frac{T}{(w-c)^2 (y-y_c)^2 \psi^2}$$

where

$$\psi(y) = 1 + \frac{w_c''}{2w_c'} (y-y_c) + \frac{w_c'''}{3!w_c'} (y-y_c)^2 + \dots$$

and expand T/ψ^2 in a Taylor series

$$\frac{T}{\psi^2} = \left(\frac{T}{\psi^2}\right)_c + \left(\frac{T}{\psi^2}\right)'_c (y-y_c) + \left(\frac{T}{\psi^2}\right)''_c \frac{(y-y_c)^2}{2!} + \dots$$

where the suffix c denotes values at $y = y_c$.

(a) We therefore get

$$K_{11}(c) = \frac{1}{w_c'^2} \int_0^{y_j} \frac{dy}{(y-y_c)^2} \left[\left(\frac{T}{\psi^2}\right)_c + \left(\frac{T}{\psi^2}\right)'_c (y-y_c) + \left(\frac{T}{\psi^2}\right)''_c \frac{(y-y_c)^2}{2!} + \dots \right]$$

$$y_j > y_c .$$

The real and imaginary parts of K_{11} are

$$\begin{aligned} \Re K_{11} &= \frac{1}{w_c'^2} \left[-\left(\frac{T}{\psi^2}\right)_c \frac{1}{y-y_c} + \left(\frac{T}{\psi^2}\right)'_c \ln|y-y_c| + \left(\frac{T}{\psi^2}\right)''_c \frac{(y-y_c)}{2!} \right. \\ &\quad \left. + \left(\frac{T}{\psi^2}\right)'''_c \frac{(y-y_c)^2}{2 \cdot 3!} + \dots \right]_0^{y_j} \end{aligned}$$

$$= \left| A(y) \right|_0^{y_j}, \text{ say,}$$

$$= A(y_j) - A(0),$$

/and ...

and

$$\oint K_{11} = \frac{\pi}{(w'_c)^2} \left(\frac{T}{\gamma^2} \right)'_c$$

This integral has been shown by Lees to be the only effective imaginary contribution of the coefficients $K_1, H_1, \text{etc.}$

The method of evaluating the derivatives of $\frac{T}{\gamma^2}$ is given in section 3.6 of this appendix.

(b) To calculate K_{12} we use the transformation (43) given on page 34, to express it as an integral of the incompressible boundary layer velocity distribution; thus

$$\begin{aligned} K_{12} &= \frac{1}{b} \left\{ \int_{\eta_{ij}}^{\eta_{i1}} \frac{T^2}{w^2} d\eta_i + 2c \int_{\eta_{ij}}^{\eta_{i1}} \frac{T^2}{w^3} d\eta_i + 3c^2 \int_{\eta_{ij}}^{\eta_{i1}} \frac{T^2}{w^4} d\eta_i + \dots \right\} \\ &= \frac{1}{b} \left\{ \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^2} d\eta_i + 2h \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 1 \right) d\eta_i + h^2 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i \right. \\ &\quad + 2c \left[\int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^3} d\eta_i + 2h \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^3} - w \right) d\eta_i + h^2 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^3} - \frac{2}{w} + w \right) d\eta_i \right] \\ &\quad + 3c^2 \left[\int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^4} d\eta_i + \dots \right] \\ &\quad \left. + \dots \right\} \end{aligned}$$

where $h = \frac{\gamma-1}{2} M_1^2$

and $b = \int_0^{\eta_{i1}} T d\eta_i$

The number of terms of this expression that need to be taken varies with M_1, c and the pressure gradient parameter β . For the calculations made in the present paper it was found necessary to go as far as the tenth power of c in some instances.

The integrals forming the terms of this expansion can be calculated directly from Hartree's incompressible boundary layer solutions (reference 18), using Simpson's rule.

3.2. Evaluation of $H_1(c)$

This integral can be transformed immediately into one in the compressible variables:

$$H_1(c) = \frac{1}{b} \left\{ \int_0^{\eta_{i1}} w^2 d\eta_1 - 2c \int_0^{\eta_{i1}} w d\eta_1 + c^2 \eta_{i1} \right\}$$

$$= \left| B(y) \right|_0^{y_1}, \text{ say.}$$

3.3. Evaluation of $H_2(c) = N_2(c)$

We can split the integral up into the following terms (see Lees):

$$H_{2_{11}}(c) = \int_0^{y_j} \frac{T}{(w-c)^2} dy \int_0^y \frac{(w-c)^2}{T} dy$$

$$H_{2_2}(c) = \int_0^{y_1} \int_0^y \frac{(w-c)^2}{T} dy dy$$

$$P(c) = \int_{y_j}^{y_1} \frac{T}{(w-c)^2} dy \int_y^{y_1} \frac{(w-c)^2}{T} dy.$$

Then

$$\mathcal{R} H_2(c) = \mathcal{R} H_{2_{11}}(c) - M_1^2 H_{2_2}(c) + K_{12}(c) H_1(c) - P(c)$$

$$\mathcal{F} H_2(c) \doteq 0.$$

(a) We find that

$$H_{2_{11}} = \int_0^{y_j} \frac{T}{(w-c)^2} dy \left\{ \int_{y_c}^y \frac{(w-c)^2}{T} dy + \left| B(y) \right|_0^{y_c} \right\}$$

where $B(y)$ has already been calculated in finding H_1 .

After expansion of the integrand in powers of $(y-y_c)$ and neglecting powers of y_c and (y_j-y_c) greater than the fifth, we finally get

$$H_{2_{11}}(c) = \dots$$

$$\begin{aligned}
 H_{2_{11}}(c) &= \left[\frac{1}{3} \frac{(y-y_c)^2}{2} + \frac{1}{12} \left(\frac{\gamma^2}{T} \right)_c \left(\frac{T}{\gamma^2} \right)'_c \frac{(y-y_c)^3}{3} + \left\{ \frac{1}{15} \left(\frac{\gamma^2}{T} \right)_c \left(\frac{T}{\gamma^2} \right)''_c \right. \right. \\
 &\quad - \left. \frac{1}{20} \left(\frac{\gamma^2}{T} \right)_c^2 \left(\frac{T}{\gamma^2} \right)'_c{}^2 \right\} \frac{(y-y_c)^4}{4} + \left\{ \frac{1}{36} \left(\frac{\gamma^2}{T} \right)_c \left(\frac{T}{\gamma^2} \right)'''_c \right. \\
 &\quad - \left. \frac{7}{120} \left(\frac{\gamma^2}{T} \right)_c^2 \left(\frac{T}{\gamma^2} \right)''_c \left(\frac{T}{\gamma^2} \right)'_c + \frac{1}{30} \left(\frac{\gamma^2}{T} \right)_c^3 \left(\frac{T}{\gamma^2} \right)'_c{}^3 \right\} \frac{(y-y_c)^5}{5} + \dots \Bigg|_0^{y_j} \\
 &\quad + K_{11} B(y_c) \\
 &= \left[C(y) \right]_0^{y_j} + K_{11} B(y_c), \text{ say.}
 \end{aligned}$$

(b) With the same procedure as for finding H_1 , we get

$$\begin{aligned}
 H_{2_2}(c) &= \frac{1}{b^2} \left\{ \int_0^{\eta_{i1}} d\eta_i \int_0^{\eta_i} w^2 d\eta_i + h \int_0^{\eta_{i1}} (1-w^2) d\eta_i \int_0^{\eta_i} w^2 d\eta_i \right. \\
 &\quad - \left. 2c \left[\int_0^{\eta_{i1}} d\eta_i \int_0^{\eta_i} w d\eta_i + h \int_0^{\eta_{i1}} (1-w^2) d\eta_i \int_0^{\eta_i} w d\eta_i \right] \right. \\
 &\quad \left. + c^2 \left[\frac{1}{2} \eta_{i1}^2 + h \int_0^{\eta_{i1}} (1-w^2) \eta_i d\eta_i \right] \right\}.
 \end{aligned}$$

(c) There is no singularity in $P(c)$, and we can therefore transform it into the incompressible form in the same manner as we did for H_{2_2} . The final formula obtained is

$$\begin{aligned}
 P(c) &= \frac{1}{b^2} \left\{ \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^2} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i + 2h \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 1 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \right. \\
 &\quad \left. + h^2 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \right. \\
 &\quad + 2c \left[\int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^3} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i - \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^2} d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \right. \\
 &\quad \left. + 2h \left\{ \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i - \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 1 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \right\} \right. \\
 &\quad \left. + h^2 \left\{ \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^3} - 2 \frac{1}{w} + w \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i - \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \right\} \right]
 \end{aligned}$$

$$\begin{aligned}
 & + c^2 \left[3 \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^4} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i - 4 \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^3} d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^2} (\eta_{i1} - \eta_i) d\eta_i \right. \\
 & + 2h \left\{ 3 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^4} - \frac{1}{w^2} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i - 4 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \right. \\
 & \quad \left. + \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 1 \right) (\eta_{i1} - \eta_i) d\eta_i \right\} \\
 & + h^2 \left\{ 3 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^4} - \frac{2}{w^2} + 1 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i - 4 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^3} - \frac{2}{w} + w \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \right. \\
 & \quad \left. + \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 2 + w^2 \right) (\eta_{i1} - \eta_i) d\eta_i \right\} \\
 & + c^3 \left[4 \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^5} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i - 6 \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^4} d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i + 2 \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^3} (\eta_{i1} - \eta_i) d\eta_i \right. \\
 & + 2h \left\{ 4 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^5} - \frac{1}{w^3} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i - 6 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^4} - \frac{1}{w^2} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \right. \\
 & \quad \left. + 2 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^3} - \frac{1}{w} \right) (\eta_{i1} - \eta_i) d\eta_i \right\} \\
 & + h^2 \left\{ 4 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^5} - \frac{2}{w^3} + \frac{1}{w} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i - 6 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^4} - \frac{2}{w^2} + 1 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \right. \\
 & \quad \left. + 2 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^3} - \frac{2}{w} + w \right) (\eta_{i1} - \eta_i) d\eta_i \right\} \\
 & + \dots \dots \dots \left. \right\} .
 \end{aligned}$$

3.4. Evaluation of M_3

It is found that

$$\mathcal{R} M_3(c) = \mathcal{R} M_{31_1}(c) + H_1(c) \left[K_{12}(c) H_1(c) - 2P(c) \right] + Q(c) - M_1^2 M_{32}(c)$$

$$\mathcal{F} M_3(c) \doteq 0,$$

/where ...

where

$$M_{31_1}(c) = \int_0^{y_j} \frac{(w-c)^2}{T} dy \int_0^{y_j} \frac{T}{(w-c)^2} dy \int_0^y \frac{(w-c)^2}{T} dy$$

$$Q(c) = \int_{y_j}^{y_1} \frac{(w-c)^2}{T} dy \int_{y_j}^y \frac{T}{(w-c)^2} dy \int_y^{y_1} \frac{(w-c)^2}{T} dy$$

$$M_{32}(c) = \int_0^{y_1} \frac{(w-c)^2}{T} dy \int_y^{y_1} \int_0^y \frac{(w-c)^2}{T} dy dy.$$

(a)

$$\begin{aligned} M_{31_1}(c) &= B(y_j) \left\{ C(y_j) + A(y_j) B(y_c) \right\} \\ &- (w'_c)^2 \left| \frac{1}{6} \left(\frac{v^2}{T} \right) \frac{(y-y_c)^5}{5} - \frac{5}{36} \left(\frac{v^2}{T} \right)_c \left(\frac{T}{v^2} \right)'_c \frac{(y-y_c)^6}{6} \right|_0^{y_j} \\ &- B(y_c) \left| -\frac{(y-y_c)^2}{2} + \left(\frac{v^2}{T} \right)_c \left(\frac{T}{v^2} \right)'_c \left\{ \ln |y-y_c| + \frac{2}{3} \right\} \frac{(y-y_c)^3}{3} \right. \\ &\quad \left. + \left\{ \left(\frac{v^2}{T} \right)_c \left(\frac{T}{v^2} \right)''_c - \left(\frac{v^2}{T} \right)_c^2 \left(\frac{T}{v^2} \right)'_c \left(\ln |y-y_c| + \frac{3}{4} \right) \right\} \frac{(y-y_c)^4}{4} \right. \\ &\quad \left. + \left\{ \frac{1}{4} \left(\frac{v^2}{T} \right)_c \left(\frac{T}{v^2} \right)'''_c - \left(\frac{v^2}{T} \right)_c \left(\frac{T}{v^2} \right)''_c \left(\frac{T}{v^2} \right)'_c \left(\frac{1}{2} \ln |y-y_c| + \frac{7}{5} \right) \right. \right. \\ &\quad \left. \left. + \left(\frac{v^2}{T} \right)_c^3 \left(\frac{T}{v^2} \right)'_c \left(\ln |y-y_c| + \frac{4}{5} \right) \right\} \frac{(y-y_c)^5}{5} \right|_0^{y_j}. \end{aligned}$$

We put the coefficient of $B(y_c)$ in this equation equal to $|D(y)|_0^{y_j}$.

(b)

$$\begin{aligned} Q &= \frac{1}{b^3} \left\{ \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^2} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \right. \\ &\quad + 2h \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 1 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i + h^2 \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i \\ &\quad \left. + \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \right\} \end{aligned}$$

$$+2c \left[\int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^3} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i - \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^2} d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \right. \\ \left. - \int_{\eta_{ij}}^{\eta_{i1}} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^2} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \right]$$

$$+2h \left\{ \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i - \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 1 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \right. \\ \left. - \int_{\eta_{ij}}^{\eta_{i1}} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 1 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \right\}$$

$$+h^2 \left\{ \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - \frac{2}{w} + w \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \right.$$

$$- \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i$$

$$\left. - \int_{\eta_{ij}}^{\eta_{i1}} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \right\}$$

$$+c^2 \left[3 \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^4} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i - 4 \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^3} d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \right.$$

$$+ \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w} (\eta_{i1} - \eta_i) d\eta_i - 4 \int_{\eta_{ij}}^{\eta_{i1}} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^3} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i$$

$$+ 4 \int_{\eta_{ij}}^{\eta_{i1}} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^2} d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^2} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i$$

/+ 2h ...

$$\begin{aligned}
 & +2h \left\{ 3 \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^4} - \frac{1}{w^2} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \right. \\
 & -4 \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 1 \right) (\eta_{i1} - \eta_i) d\eta_i \\
 & -4 \int_{\eta_{ij}}^{\eta_{i1}} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i + 4 \int_{\eta_{ij}}^{\eta_{i1}} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 1 \right) \int_{\eta_i}^{\eta_{i1}} w d\eta_i \\
 & \left. + \int_{\eta_{ij}}^{\eta_{i1}} d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 1 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \right\} \\
 & +h^2 \left\{ 3 \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^4} - \frac{2}{w^2} + 1 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \right. \\
 & -4 \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^3} - \frac{2}{w} + w \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \\
 & + \int_{\eta_{ij}}^{\eta_{i1}} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 2 + w^2 \right) (\eta_{i1} - \eta_i) d\eta_i \\
 & -4 \int_{\eta_{ij}}^{\eta_{i1}} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^3} - \frac{2}{w} + w \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \\
 & +4 \int_{\eta_{ij}}^{\eta_{i1}} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \\
 & \left. + \int_{\eta_{ij}}^{\eta_{i1}} d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \right\} \\
 & + \dots \dots \dots \left. \right\} \cdot
 \end{aligned}$$

/(c) ...

(c)

$$\begin{aligned}
 M_{32}(c) = & \frac{1}{b^3} \left\{ \int_0^{\eta_{i1}} w^2 d\eta_i \int_{\eta_i}^{\eta_{i1}} d\eta_i \int_0^{\eta_i} w^2 d\eta_i + h \int_0^{\eta_{i1}} w^2 d\eta_i \int_{\eta_i}^{\eta_{i1}} (1-w^2) d\eta_i \int_0^{\eta_i} w^2 d\eta_i \right. \\
 & - 2c \left[\int_0^{\eta_{i1}} w^2 d\eta_i \int_{\eta_i}^{\eta_{i1}} d\eta_i \int_0^{\eta_i} w d\eta_i + \int_0^{\eta_{i1}} w d\eta_i \int_{\eta_i}^{\eta_{i1}} d\eta_i \int_0^{\eta_i} w^2 d\eta_i \right. \\
 & \left. \left. + m \left[\int_0^{\eta_{i1}} w^2 d\eta_i \int_{\eta_i}^{\eta_{i1}} (1-w^2) d\eta_i \int_0^{\eta_i} w d\eta_i + \int_0^{\eta_{i1}} w d\eta_i \int_{\eta_i}^{\eta_{i1}} (1-w^2) d\eta_i \right. \right. \right. \\
 & \left. \left. \left. \int_0^{\eta_i} w^2 d\eta_i \right] \right] \right. \\
 & + c^2 \left[\int_0^{\eta_{i1}} w^2 d\eta_i \int_{\eta_i}^{\eta_{i1}} \eta_i d\eta_i + 4 \int_0^{\eta_{i1}} w d\eta_i \int_{\eta_i}^{\eta_{i1}} d\eta_i \int_0^{\eta_i} w d\eta_i \right. \\
 & \left. + \int_0^{\eta_{i1}} d\eta_i \int_{\eta_i}^{\eta_{i1}} d\eta_i \int_0^{\eta_i} w^2 d\eta_i \right. \\
 & \left. + h \left\{ \int_0^{\eta_{i1}} w^2 d\eta_i \int_{\eta_i}^{\eta_{i1}} (1-w^2) d\eta_i \int_0^{\eta_i} d\eta_i + 4 \int_0^{\eta_{i1}} w d\eta_i \int_{\eta_i}^{\eta_{i1}} (1-w^2) d\eta_i \int_0^{\eta_i} w d\eta_i \right. \right. \\
 & \left. \left. + \int_0^{\eta_{i1}} d\eta_i \int_{\eta_i}^{\eta_{i1}} (1-w^2) d\eta_i \int_0^{\eta_i} w^2 d\eta_i \right\} \right] \\
 & - 2c^3 \left[\int_0^{\eta_{i1}} w d\eta_i \int_{\eta_i}^{\eta_{i1}} \eta_i d\eta_i + \int_0^{\eta_{i1}} d\eta_i \int_{\eta_i}^{\eta_{i1}} d\eta_i \int_0^{\eta_i} w d\eta_i \right. \\
 & \left. + h \left\{ \int_0^{\eta_{i1}} w d\eta_i \int_{\eta_i}^{\eta_{i1}} (1-w^2) d\eta_i \int_0^{\eta_i} d\eta_i + \int_0^{\eta_{i1}} d\eta_i \int_{\eta_i}^{\eta_{i1}} (1-w^2) d\eta_i \int_0^{\eta_i} w d\eta_i \right\} \right] \\
 & + c^4 \left[\int_0^{\eta_{i1}} d\eta_i \int_{\eta_i}^{\eta_{i1}} \eta_i d\eta_i + h \int_0^{\eta_{i1}} d\eta_i \int_{\eta_i}^{\eta_{i1}} (1-w^2) d\eta_i \int_0^{\eta_i} d\eta_i \right] \left. \right\}.
 \end{aligned}$$

3.5. Evaluation of N_3

The expanded form of N_3 is

$$N_3(c) = N_{31_1}(c) + K_{12}(c) \left[2H_{2_{11}}(c) + H_1(c)K_{12}(c) - P(c) \right] - N_{31_3}(c) \\ - 2M_1^2 \left\{ (y_1 - y_c)H_{2_{11}}(c) - J_{2_{11}}(c) + K_{12}(c) \left[H_1(c) - X(c) \right] \right. \\ \left. - [P - P_1] \right\} - M_1^4 N_{34}$$

where

$$N_{31_1}(c) = \int_0^{y_j} \frac{T}{(w-c)^2} dy \int_0^y \frac{(w-c)^2}{T} dy \int_y^{y_j} \frac{T}{(w-c)^2} dy$$

$$N_{31_3}(c) = \int_{y_j}^{y_1} \frac{T}{(w-c)^2} dy \int_{y_j}^y \frac{(w-c)^2}{T} dy \int_{y_j}^y \frac{T}{(w-c)^2} dy$$

$$J_{2_{11}}(c) = \int_0^{y_j} \frac{T}{(w-c)^2} dy \int_0^y \frac{(w-c)^2}{T} (y - y_c) dy$$

$$X(c) = \int_0^{y_1} \frac{(w-c)^2}{T} y dy$$

$$P_1(c) = \int_{y_j}^{y_1} \frac{T}{(w-c)^2} dy \int_y^{y_1} \frac{(w-c)^2}{T} dy$$

$$N_{34}(c) = \int_0^{y_1} dy \int_0^y \frac{(w-c)^2}{T} (y_1 - y) dy.$$

We find that

(a)

$$N_{31_1}(c) = H_{2_{11}} A(y_j) + K_{11} D(0) - \frac{1}{(w'_c)^2} \left| -\frac{1}{2} \left(\frac{T}{\gamma^2} \right)_c (y - y_c) \right. \\ \left. + \left(\frac{T}{\gamma^2} \right)'_c \left\{ \frac{1}{3} \ln. |y - y_c| - \frac{4}{9} \right\} \frac{(y - y_c)^2}{2} + \left(\frac{\gamma^2}{T} \right)_c \left(\frac{T}{\gamma^2} \right)'_c \left\{ \frac{1}{12} \ln. |y - y_c| + \frac{1}{144} \right\} \right. \\ \left. \times \frac{(y - y_c)^3}{3} \right|_0^{y_j}.$$

/(b) ...

$$\begin{aligned}
 (b) \quad N_{313} &= \frac{1}{b^3} \left\{ \int_{\eta_{ij}}^{\eta_{i1}} \frac{a_1}{w^3} d\eta_i + 2h \left\{ \int_{\eta_{ij}}^{\eta_{i1}} \frac{b_1}{w^3} d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} a_1 \left(\frac{1}{w^2} - 1 \right) d\eta_i \right\} \right. \\
 &+ h^2 \left\{ \int_{\eta_{ij}}^{\eta_{i1}} d_1 \frac{1}{w^2} d\eta_i + 4 \int_{\eta_{ij}}^{\eta_{i1}} b_1 \left(\frac{1}{w^2} - 1 \right) d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} a_1 \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i \right\} \\
 &+ 2c \left[\int_{\eta_{ij}}^{\eta_{i1}} a_2 \frac{1}{w^2} d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} a_1 \frac{1}{w^3} d\eta_i + 2h \left\{ \int_{\eta_{ij}}^{\eta_{i1}} b_2 \frac{1}{w^2} d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} a_2 \left(\frac{1}{w^2} - 1 \right) d\eta_i \right. \right. \\
 &+ \left. \int_{\eta_{ij}}^{\eta_{i1}} b_1 \frac{1}{w^3} d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} a_1 \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i \right\} + h^2 \left\{ \int_{\eta_{ij}}^{\eta_{i1}} d_2 \frac{1}{w^2} d\eta_i \right. \\
 &+ 4 \int_{\eta_{ij}}^{\eta_{i1}} b_2 \left(\frac{1}{w^2} - 1 \right) d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} a_2 \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} d_1 \frac{1}{w^3} d\eta_i \\
 &+ \left. \left. 4 \int_{\eta_{ij}}^{\eta_{i1}} b_1 \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} a_1 \left(\frac{1}{w^3} - \frac{2}{w} + w \right) d\eta_i \right\} \right] \\
 &+ c^2 \left[\int_{\eta_{ij}}^{\eta_{i1}} a_3 \frac{1}{w^2} d\eta_i + 4 \int_{\eta_{ij}}^{\eta_{i1}} a_2 \frac{1}{w^3} d\eta_i + 3 \int_{\eta_{ij}}^{\eta_{i1}} a_1 \frac{1}{w^4} d\eta_i + 2h \left\{ \int_{\eta_{ij}}^{\eta_{i1}} b_3 \frac{1}{w^2} d\eta_i \right. \right. \\
 &+ \int_{\eta_{ij}}^{\eta_{i1}} a_3 \left(\frac{1}{w^2} - 1 \right) d\eta_i + 4 \int_{\eta_{ij}}^{\eta_{i1}} b_2 \frac{1}{w^3} d\eta_i + 4 \int_{\eta_{ij}}^{\eta_{i1}} a_2 \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i \\
 &+ \left. 3 \int_{\eta_{ij}}^{\eta_{i1}} b_1 \frac{1}{w^4} d\eta_i + 3 \int_{\eta_{ij}}^{\eta_{i1}} a_1 \left(\frac{1}{w^4} - \frac{1}{w^2} \right) d\eta_i \right\} + h^2 \left\{ \int_{\eta_{ij}}^{\eta_{i1}} d_3 \frac{1}{w^2} d\eta_i \right. \\
 &+ 4 \int_{\eta_{ij}}^{\eta_{i1}} b_3 \left(\frac{1}{w^2} - 1 \right) d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} a_3 \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i + 4 \int_{\eta_{ij}}^{\eta_{i1}} d_2 \frac{1}{w^3} d\eta_i \\
 &+ 16 \int_{\eta_{ij}}^{\eta_{i1}} b_2 \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i + 4 \int_{\eta_{ij}}^{\eta_{i1}} a_2 \left(\frac{1}{w} - \frac{2}{w} + w \right) d\eta_i + 3 \int_{\eta_{ij}}^{\eta_{i1}} d_1 \frac{1}{w^4} d\eta_i \\
 &+ \left. \left. 12 \int_{\eta_{ij}}^{\eta_{i1}} b_1 \left(\frac{1}{w^4} - \frac{1}{w^2} \right) d\eta_i + 3 \int_{\eta_{ij}}^{\eta_{i1}} a_1 \left(\frac{1}{w^4} - \frac{2}{w^2} + 1 \right) d\eta_i \right\} \right]
 \end{aligned}$$

$$\begin{aligned}
 & + c^3 \left[\int_{\eta_{ij}}^{\eta_{i1}} a_4 \frac{1}{w^2} d\eta_i + 2 \int_{\eta_{ij}}^{\eta_{i1}} a_3 \frac{1}{w^3} d\eta_i + 6 \int_{\eta_{ij}}^{\eta_{i1}} a_2 \frac{1}{w^4} d\eta_i + 4 \int_{\eta_{ij}}^{\eta_{i1}} a_1 \frac{1}{w^5} d\eta_i \right. \\
 & + 2h \left\{ \int_{\eta_{ij}}^{\eta_{i1}} b_4 \frac{1}{w^2} d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} a_4 \left(\frac{1}{w^2} - 1 \right) d\eta_i + 2 \int_{\eta_{ij}}^{\eta_{i1}} b_3 \frac{1}{w^3} d\eta_i \right. \\
 & + 2 \int_{\eta_{ij}}^{\eta_{i1}} a_3 \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i + 6 \int_{\eta_{ij}}^{\eta_{i1}} b_2 \frac{1}{w^4} d\eta_i + 6 \int_{\eta_{ij}}^{\eta_{i1}} a_2 \left(\frac{1}{w^4} - \frac{1}{w^2} \right) d\eta_i \\
 & + 4 \int_{\eta_{ij}}^{\eta_{i1}} b_1 \frac{1}{w^5} d\eta_i + 4 \int_{\eta_{ij}}^{\eta_{i1}} a_1 \left(\frac{1}{w^5} - \frac{1}{w^3} \right) d\eta_i \left. \right\} + h^2 \left\{ \int_{\eta_{ij}}^{\eta_{i1}} d_4 \frac{1}{w^2} d\eta_i \right. \\
 & + 4 \int_{\eta_{ij}}^{\eta_{i1}} b_4 \left(\frac{1}{w^2} - 1 \right) d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} a_4 \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i + 2 \int_{\eta_{ij}}^{\eta_{i1}} d_3 \frac{1}{w^3} d\eta_i \\
 & + 8 \int_{\eta_{ij}}^{\eta_{i1}} b_3 \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i + 2 \int_{\eta_{ij}}^{\eta_{i1}} a_3 \left(\frac{1}{w^3} - \frac{2}{w} + w \right) d\eta_i + 6 \int_{\eta_{ij}}^{\eta_{i1}} d_2 \frac{1}{w^4} d\eta_i \\
 & + 24 \int_{\eta_{ij}}^{\eta_{i1}} b_2 \left(\frac{1}{w^4} - \frac{1}{w^2} \right) d\eta_i + 6 \int_{\eta_{ij}}^{\eta_{i1}} a_2 \left(\frac{1}{w^4} - \frac{2}{w^2} + 1 \right) d\eta_i + 4 \int_{\eta_{ij}}^{\eta_{i1}} d_1 \frac{1}{w^5} d\eta_i \\
 & \left. + 16 \int_{\eta_{ij}}^{\eta_{i1}} b_1 \left(\frac{1}{w^5} - \frac{1}{w^3} \right) d\eta_i + 4 \int_{\eta_{ij}}^{\eta_{i1}} a_1 \left(\frac{1}{w^5} - \frac{2}{w^3} + \frac{1}{w} \right) d\eta_i \right\} \\
 & + \dots \left. \right\},
 \end{aligned}$$

where

$$a_1 = \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^2} d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^2} d\eta_i$$

$$a_2 = \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^2} d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^3} d\eta_i - \int_{\eta_{ij}}^{\eta_i} \frac{1}{w} d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^2} d\eta_i$$

$$a_3 = 3 \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^2} d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^4} d\eta_i - 4 \int_{\eta_{ij}}^{\eta_i} \frac{1}{w} d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^2} d\eta_i + \int_{\eta_{ij}}^{\eta_i} d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^2} d\eta_i$$

/a₄ = ...

$$a_4 = 4 \int_{\eta_{ij}}^{\eta_i} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^5} d\eta_i - 6 \int_{\eta_{ij}}^{\eta_i} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^4} d\eta_i + 2 \int_{\eta_{ij}}^{\eta_i} d\eta_i \int_{\eta_{ij}}^{\eta_i} \frac{1}{w^2} d\eta_i$$

$$b_1 = \int_{\eta_{ij}}^{\eta_i} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 1 \right) d\eta_i$$

$$b_2 = \int_{\eta_{ij}}^{\eta_i} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i - \int_{\eta_{ij}}^{\eta_i} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 1 \right) d\eta_i$$

$$b_3 = 3 \int_{\eta_{ij}}^{\eta_i} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^4} - \frac{1}{w^2} \right) d\eta_i - 4 \int_{\eta_{ij}}^{\eta_i} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i + \int_{\eta_{ij}}^{\eta_i} d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 1 \right) d\eta_i$$

$$b_4 = 4 \int_{\eta_{ij}}^{\eta_i} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^5} - \frac{1}{w^3} \right) d\eta_i - 6 \int_{\eta_{ij}}^{\eta_i} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^4} - \frac{1}{w^2} \right) d\eta_i + 2 \int_{\eta_{ij}}^{\eta_i} d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i$$

$$d_1 = \int_{\eta_{ij}}^{\eta_i} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i$$

$$d_2 = \int_{\eta_{ij}}^{\eta_i} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^3} - \frac{2}{w} + w \right) d\eta_i - \int_{\eta_{ij}}^{\eta_i} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i$$

$$d_3 = 3 \int_{\eta_{ij}}^{\eta_i} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^4} - \frac{2}{w^2} + 1 \right) d\eta_i - 4 \int_{\eta_{ij}}^{\eta_i} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^3} - \frac{2}{w} + w \right) d\eta_i + \int_{\eta_{ij}}^{\eta_i} d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i$$

/a₄ = ...

$$d_4 = 4 \int_{\eta_{ij}}^{\eta_i} w^2 d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^5} - \frac{2}{w^3} + \frac{1}{w} \right) d\eta_i - 6 \int_{\eta_{ij}}^{\eta_i} w d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^4} - \frac{2}{w^2} + 1 \right) d\eta_i$$

$$+ 2 \int_{\eta_{ij}}^{\eta_i} d\eta_i \int_{\eta_{ij}}^{\eta_i} \left(\frac{1}{w^3} - \frac{2}{w} + w \right) d\eta_i.$$

(c)

$$J_{2,11}(c) = w_c'^2 K_{11} \left| \left(\frac{r^2}{T} \right)_c \frac{(y-y_c)^4}{4} - \left(\frac{r^2}{T} \right)_c \left(\frac{T}{r^2} \right)'_c \frac{(y-y_c)^5}{5} \right|_0^{y_c}$$

$$+ \left| \frac{1}{12} (y-y_c)^3 + \frac{1}{20} \left(\frac{r^2}{T} \right)_c \left(\frac{T}{r^2} \right)'_c \frac{(y-y_c)^4}{4} \right|_0^{y_c}$$

$$+ \left\{ \frac{1}{24} \left(\frac{r^2}{T} \right)_c \left(\frac{T}{r^2} \right)''_c - \frac{1}{30} \left(\frac{r^2}{T} \right)_c \left(\frac{T}{r^2} \right)'_c \right\} \frac{(y-y_c)^5}{5} \Big|_0^{y_j}$$

(d)

$$X(c) = \frac{1}{b^2} \left\{ \int_0^{\eta_{i1}} w^2 \eta_i d\eta_i + h \int_0^{\eta_{i1}} w^2 d\eta_i \int_0^{\eta_i} (1-w^2) d\eta_i \right.$$

$$- 2c \left[\int_0^{\eta_{i1}} w \eta_i d\eta_i + h \int_0^{\eta_{i1}} w d\eta_i \int_0^{\eta_i} (1-w^2) d\eta_i \right]$$

$$\left. + c^2 \left[\frac{\eta_{i1}^2}{2} + h \int_0^{\eta_{i1}} d\eta_i \int_0^{\eta_i} (1-w^2) d\eta_i \right] \right\}$$

(e)

$$P_1(c) = \frac{1}{b^3} \left\{ \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^2} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 \eta_i d\eta_i + h \left\{ 2 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 1 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 \eta_i d\eta_i \right. \right.$$

$$\left. + \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^2} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \int_0^{\eta_i} (1-w^2) d\eta_i \right\}$$

$$+ h^2 \left\{ 2 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 1 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \int_0^{\eta_i} (1-w^2) d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i \right.$$

$$\left. \int_{\eta_i}^{\eta_{i1}} w^2 \eta_i d\eta_i \right\}$$

$$\begin{aligned}
 & + 2c \left[- \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^2} d\eta_i \int_{\eta_i}^{\eta_{i1}} w \eta_i d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^3} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 \eta_i d\eta_i \right. \\
 & + h \left\{ - 2 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 1 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w \eta_i d\eta_i - \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^2} d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \int_0^{\eta_i} (1-w^2) d\eta_i \right. \\
 & \left. + \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^3} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \int_0^{\eta_i} (1-w^2) d\eta_i - 2 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 \eta_i d\eta_i \right\} \\
 & + h^2 \left\{ - 2 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 1 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \int_0^{\eta_i} (1-w^2) d\eta_i - \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 2 + w^2 \right) d\eta_i \right. \\
 & \left. \int_{\eta_i}^{\eta_{i1}} w \eta_i d\eta_i \right. \\
 & \left. + 2 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \int_0^{\eta_i} (1-w^2) d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^3} - \frac{2}{w} + w \right) d\eta_i \right. \\
 & \left. \int_{\eta_i}^{\eta_{i1}} w^2 \eta_i d\eta_i \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + c^2 \left[\int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^2} d\eta_i \int_{\eta_i}^{\eta_{i1}} \eta_i d\eta_i - \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^3} d\eta_i \int_{\eta_i}^{\eta_{i1}} w \eta_i d\eta_i + 3 \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^4} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 \eta_i d\eta_i \right. \\
 & + h \left\{ \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^2} d\eta_i \int_{\eta_i}^{\eta_{i1}} d\eta_i \int_0^{\eta_i} (1-w^2) d\eta_i + 2 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 1 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} \eta_i d\eta_i \right. \\
 & - 4 \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^3} d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \int_0^{\eta_i} (1-w^2) d\eta_i - 8 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w \eta_i d\eta_i \\
 & \left. + 3 \int_{\eta_{ij}}^{\eta_{i1}} \frac{1}{w^4} d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \int_0^{\eta_i} (1-w^2) d\eta_i + 6 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^4} - \frac{1}{w^2} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 \eta_i d\eta_i \right\}
 \end{aligned}$$

/+ h^2 ...

$$\begin{aligned}
 & +h^2 \left\{ 2 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^2} - 1 \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} d\eta_i \int_0^{\eta_i} (1-w^2) d\eta_i + \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^3} - \frac{2}{w} + w \right) \int_{\eta_i}^{\eta_{i1}} \eta_i d\eta_i \right. \\
 & - 8 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^3} - \frac{1}{w} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w d\eta_i \int_0^{\eta_i} (1-w^2) d\eta_i - 4 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^3} - \frac{2}{w} + w \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w \eta_i d\eta_i \\
 & + 6 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^4} - \frac{1}{w^2} \right) d\eta_i \int_{\eta_i}^{\eta_{i1}} w^2 d\eta_i \int_0^{\eta_i} (1-w^2) d\eta_i + 3 \int_{\eta_{ij}}^{\eta_{i1}} \left(\frac{1}{w^4} - \frac{2}{w^2} + 1 \right) d\eta_i \\
 & \left. \int_{\eta_i}^{\eta_{i1}} w^2 \eta_i d\eta_i \right\} \\
 & + \dots \dots \dots \} .
 \end{aligned}$$

(f)

$$\begin{aligned}
 N_{34}(c) &= \frac{1}{b^2} \left\{ \int_0^{\eta_{i1}} d\eta_i \int_0^{\eta_i} w^2 (\eta_{i1} - \eta_i) d\eta_i + h \int_0^{\eta_{i1}} d\eta_i \int_0^{\eta_i} w^2 d\eta_i \int_{\eta_i}^{\eta_{i1}} (1-w^2) d\eta_i \right. \\
 & - 2c \left[\int_0^{\eta_{i1}} d\eta_i \int_0^{\eta_i} w (\eta_{i1} - \eta_i) d\eta_i + h \int_0^{\eta_{i1}} d\eta_i \int_0^{\eta_i} w d\eta_i \int_{\eta_i}^{\eta_{i1}} (1-w^2) d\eta_i \right] \\
 & \left. + c^2 \left[\int_0^{\eta_{i1}} \int_0^{\eta_i} (\eta_{i1} - \eta_i) d\eta_i d\eta_i + h \int_0^{\eta_{i1}} d\eta_i \int_0^{\eta_i} d\eta_i \int_{\eta_i}^{\eta_{i1}} (1-w^2) d\eta_i \right] \right\} .
 \end{aligned}$$

3.6. Evaluation of $(T/\gamma^2)_c$ and its derivatives

To calculate $\left(\frac{T}{\gamma^2}\right)_c^{(n)}$ we use a rather different

approach from that given by Lees, and express these derivatives directly in terms of the incompressible boundary layer velocity distribution and its derivatives, using the transformation (43).

Thus we find firstly that

$$\begin{aligned}
 \left(\frac{T}{\gamma^2}\right)_c &= T_c \\
 \left(\frac{T}{\gamma^2}\right)'_c &= T'_c - T_c \left(\frac{w''_c}{w'_c}\right)
 \end{aligned}$$

$/(T/\gamma^2) \dots$

$$\left(\frac{T}{\gamma^2}\right)''_c = T''_c - 2T'_c \left(\frac{w''_c}{w'_c}\right) + \frac{3}{2} T_c \left(\frac{w''_c}{w'_c}\right)^2 - \frac{2}{3} T_c \left(\frac{w''''_c}{w'_c}\right).$$

$$\begin{aligned} \left(\frac{T}{\gamma^2}\right)''''_c &= T''''_c - 3T''_c \left(\frac{w''_c}{w'_c}\right) + \frac{9}{2} T'_c \left(\frac{w''_c}{w'_c}\right)^2 - 2T'_c \left(\frac{w''''_c}{w'_c}\right) \\ &\quad + 3T_c \frac{w''''_c w'_c}{w'^2_c} - 3 T_c \left(\frac{w''_c}{w'_c}\right)^3 - \frac{1}{2} T_c \left(\frac{w^{iv}_c}{w'_c}\right). \end{aligned}$$

Whilst, using (43) we get

$$w = w_i$$

$$w' = w'_i \frac{b}{T}$$

$$w'' = \frac{b^2}{T^2} (w''_i - w'_i \frac{T'_i}{T})$$

$$w''' = \frac{b^3}{T^3} (w'''_i - \frac{3 T'_i}{T} w''_i + \frac{3 T'^2_i}{T^2} w'_i - \frac{T''_i}{T} w'_i)$$

$$\begin{aligned} w^{iv} &= \frac{b^4}{T^4} (w^{iv}_i - \frac{6 T'_i}{T} w'''_i + \frac{15 T'^2_i}{T^2} w''_i - \frac{4 T''_i}{T} w'_i \\ &\quad - \frac{15 T'^3_i}{T^3} w'_i + \frac{10 T''_i T'_i}{T^2} w'_i - \frac{T'''_i}{T} w'_i), \end{aligned}$$

where

$$w_i^{(n)} = w^{(n)}(\eta_i) = \frac{d^n w}{d \eta_i^n},$$

and

$$T_i^{(n)} = \frac{d^n T}{d \eta_i^n}.$$

So there results

$$\left(\frac{T}{\gamma^2}\right)_c = T_c$$

$$\left(\frac{T}{\gamma^2}\right)'_c = \frac{b}{T_c} \left\{ 2T'_i - T \left(\frac{w''_i}{w'_i}\right) \right\}_c$$

$$\begin{aligned} \left(\frac{T}{\gamma^2}\right)''_c &= \frac{b^2}{T_c^2} \left\{ \frac{5}{3} T''_i + \frac{1}{2} \frac{T'^2_i}{T} - 3 T'_i \left(\frac{w''_i}{w'_i}\right) + \frac{3}{2} T \left(\frac{w''_i}{w'_i}\right)^2 \right. \\ &\quad \left. - \frac{2}{3} T \left(\frac{w''''_i}{w'_i}\right) \right\}_c \end{aligned}$$

/ (T/\gamma^2 ...

$$\begin{aligned} \left(\frac{T}{\gamma^2}\right)'''_c &= \frac{b^3}{T_c^3} \left\{ \frac{3}{2} T_i''' - \frac{T_i'' T_i'}{T} - 4 T_i'' \left(\frac{w_i''}{w_i'}\right) + \frac{3}{2} \frac{T_i'^2}{T} \left(\frac{w_i''}{w_i'}\right) \right. \\ &\quad + \frac{9}{2} T_i' \left(\frac{w_i''}{w_i'}\right)^2 - 3 T \left(\frac{w_i''}{w_i'}\right)^3 - 2 T' \left(\frac{w_i''}{w_i'}\right) \\ &\quad \left. + 3 T \left(\frac{w_i''}{w_i'}\right) \left(\frac{w_i'''}{w_i'}\right) - \frac{1}{2} T \left(\frac{w_i^{IV}}{w_i'}\right) \right\}_c. \end{aligned}$$

Finally, using the fact that

$$T = 1 + h(1 - w^2), \quad h = \frac{\gamma-1}{2} M_1^2,$$

$$T_i' = -2hw w_i',$$

$$T_i'' = -2h(w_i'^2 + w w_i''),$$

$$T_i''' = -2h(3 w_i' w_i'' + w w_i'''),$$

we obtain

$$\left(\frac{T}{\gamma^2}\right)_c = 1 + h(1 - w_c^2)$$

$$\left(\frac{T}{\gamma^2}\right)'_c = \frac{b}{T} \left\{ -\frac{w_i''}{w_i'} - h(1-w^2) \left(\frac{w_i''}{w_i'}\right) + 4w w_i' \right\}_c$$

$$\begin{aligned} \left(\frac{T}{\gamma^2}\right)''_c &= \frac{b^2}{T^2} \left\{ -\frac{2}{3} \frac{w_i'''}{w_i'} + \frac{3}{2} \left(\frac{w_i''}{w_i'}\right)^2 + h \left(-\frac{10}{3} w_i'^2 + \frac{8}{3} w w_i'' \right) \right. \\ &\quad + h \left(-\frac{2}{3} \left(\frac{w_i'''}{w_i'}\right) (1-w^2) + \frac{3}{2} \left(\frac{w_i''}{w_i'}\right)^2 (1-w^2) \right) \\ &\quad \left. + \frac{2h^2 w^2 w_i'^2}{1+h(1-w^2)} \right\}_c \end{aligned}$$

$$\begin{aligned} \left(\frac{T}{\gamma^2}\right)'''_c &= \frac{b^3}{T^3} \left\{ -\frac{1}{2} \left(\frac{w_i^{IV}}{w_i'}\right) + 3 \left(\frac{w_i'''}{w_i'}\right) \left(\frac{w_i''}{w_i'}\right) - 3 \left(\frac{w_i''}{w_i'}\right)^3 \right. \\ &\quad + h \left[-\frac{1}{2} (1-w^2) \left(\frac{w_i^{IV}}{w_i'}\right) + 3(1-w^2) \left(\frac{w_i'''}{w_i'}\right) \left(\frac{w_i''}{w_i'}\right) \right. \\ &\quad \left. - 3(1-w^2) \left(\frac{w_i''}{w_i'}\right)^3 - w w_i'' \left(\frac{w_i'''}{w_i'}\right) + w w_i''' - w_i' w_i'' \right] \\ &\quad \left. + \frac{2h^2}{1+h(1-w^2)} \left[2 w_i'^2 + 5 w w_i'' \right] w w_i' \right\}_c. \end{aligned}$$

4. Calculation of the Minimum Critical Reynolds Number

The following estimate of $R_{cr.min.}$ is given by Lees (reference 11),

$$R_{cr.min} \approx \frac{25 [T(c)]^{1+\omega} w'_0}{c^4 \sqrt{1 - M_1^2} (1-c)^2} \dots\dots\dots (53)$$

where c is determined by

$$v(c) (1 - 2\lambda(c)) = 0.580$$

$$v(c) = - \frac{\pi w'_0 c}{T_0} \left[\frac{T^2}{(w')^3} \frac{d}{dy} \left(\frac{w'}{T} \right) \right]_{w=c}$$

$$\lambda(c) = \frac{w'_0 y_c}{c} - 1.$$

We can express the derivatives occurring in these equations in terms of the incompressible derivatives using (43); thus

$$w' = w'_i \frac{b}{T}$$

$$T' = T'_i \frac{b}{T}$$

so that

$$\frac{d}{dy} \left(\frac{w'}{T} \right) = b^2 \left(\frac{w''_i}{T^3} + \frac{2(\gamma-1) M_1^2 w w''_i}{T^4} \right)$$

then

$$v(c) = - \pi \frac{w'_0 c}{T_0^2} \frac{T_c^2}{w_{1c}} \left[\frac{w''_i}{w'^2_i} + \frac{2(\gamma-1) M_1^2 w}{T} \right]_{w=c} c$$

5. Approximate Calculation of the Upper and Lower Branches of the Neutral Stability Curve.

Expressions for the shape of the neutral stability curve for large Reynolds numbers have been obtained by Lees. These have been used in the present paper with an important modification of one of his results. From the neutral stability equation (26) Lees approximates α by

$$\alpha \approx \frac{w'_0 c}{T_0} \frac{1}{u} \sqrt{1 - M_1^2} (1-c)^2 \dots\dots\dots (54)$$

for large Reynolds numbers and small values of α and c .

/However ...

However, when c becomes appreciable the factor $(1-c)^2$ in (26) can no longer be neglected, and we must write for α

$$\alpha \approx \frac{w'_0 c}{T_0} \frac{1}{u} \frac{\sqrt{1 - M_1^2 (1-c)^2}}{(1-c)^2} \dots\dots\dots (55)$$

For many of the stability curves this modification is of minor importance, but when α remains finite as $R \rightarrow \infty$ (the unstable inviscid case), the effect of neglecting $(1-c)^2$ can become overwhelming. Thus for $\beta = -0.1$, at $R = \infty$, the formula (54) predicts that for the upper branch α decreases as the Mach number increases, whilst according to (55) it increases. Confirmation of equation (55) is given by calculations made by Pretsch (reference 15) for the incompressible case with $\beta = -0.1$, with which it agrees, as can be seen in Figure 4. In contrast, the values of α for the upper branch calculated from (54) are half the Pretsch values.

We shall give here the formulae used to calculate the branches of the neutral stability curves in terms of the incompressible derivatives of w , including the modifications introduced by using (55) in place of (54). All the equations hold for $\sigma = 1$ only.

5.1. Subsonic main stream velocity

Lower branch

$$R \approx b^5 \frac{(w'_{i0})^5 (1 - M_1^2)^{3/2}}{T_0^6 \alpha^4} \dots\dots\dots (56)$$

Upper branch

(a) If $\frac{d}{dy} \left(\frac{w'}{T} \right)$ vanishes for $w > 0$,

$$R \approx b^8 \frac{(w'_{i0})^8}{2\pi^2 T_c^8} \cdot \frac{1}{\left[\left\{ \frac{d^2}{dy^2} \left(\frac{w'}{T} \right) \right\}_0 \right]^2} \frac{1}{\alpha c^5} \frac{1}{(c-c_s)^2} \dots\dots\dots (57)$$

$$\alpha \approx b \frac{w'_{i0} c}{T_0} \frac{\sqrt{1 - M_1^2 (1-c)^2}}{(1-c)^2} \dots\dots\dots (58)$$

where c_s is the value of c for which $\left[\frac{d}{dy} \left(\frac{w'}{T} \right) \right]_c = 0$.

The appropriate formula for $\frac{d^2}{dy^2} \left(\frac{w'}{T}\right)$ in terms of the incompressible variables is

$$\frac{d^2}{dy^2} \left(\frac{w'}{T}\right) = b^3 \left[\frac{w_i'''}{T^4} + \frac{6 h w w_i' w_i''}{T^5} + \frac{4 h w_i'^3}{T^5} + \frac{8 h w w_i' w_i''}{T^5} + \frac{32 h^2 w^2 w_i'^2}{T^6} \right],$$

where $h = \frac{\gamma-1}{2} M_1^2$.

(b) If $\frac{d}{dy} \left(\frac{w'}{T}\right)$ does not vanish for some $w > 0$,

$$R \approx b'' \frac{(w_{i0}')^{11}}{2\pi^2 T_0^{16}} \frac{(1-M_1^2)^{5/2}}{\left[\left\{\frac{d}{dy} \left(\frac{w'}{T}\right)\right\}_0\right]^2} \frac{1}{\alpha^6} \dots (59)$$

Here

$$\frac{d}{dy} \left(\frac{w'}{T}\right) = b^2 \left(\frac{w_i''}{T^3} + \frac{4 h w w_i'^2}{T^4} \right),$$

6.2. Supersonic main stream velocity

(a) If $\frac{d}{dy} \left(\frac{w'}{T}\right) \neq 0$ for $w \geq 1 - 1/M_1$

$$R \approx b^2 \frac{(w_{i0}')^2 z_1^3}{(1 - 1/M_1)^3 \alpha} \dots (60)$$

The two values of u_1 and z_1 corresponding to the upper and lower branches are given by the solution of the equations

$$\left. \begin{aligned} \bar{\Phi}_i(z_1) &= v(c) = v(1 - 1/M_1) \\ &= \frac{-\pi w_{i0}' (1 - 1/M_1)}{T_0^2} \left[\frac{T^5}{(w_i')^3} \frac{d}{dy} \left(\frac{w'}{T}\right) \right]_{w=c=1-1/M_1} \end{aligned} \right\}$$

$$u_1 = \bar{\Phi}_R(z_1) \dots (61)$$

where $\bar{\Phi}(z)$ is the modified form of the Tietjens function, and is plotted in Figure 12.

(b) If $\frac{d}{dy} \left(\frac{w'}{T}\right) = 0$ for some $w > 1 - 1/M_1$, the lower branch is given by the single solution of (61) for u_1 and z_1 , and equation (61). The upper branch is given by equations (57), (58).

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T A B L E I

Reynolds Number, disturbance wave number and phase velocity for neutral stability.

Insulated surface, $\beta = 0.6$.

$M_1 = 0$

| c | α | R | α_{-3} | R_{-3} |
|--------|----------|----------------------|---------------|---------------------|
| .04905 | 0.1011 | 1.938×10^7 | .00808 | 1.549×10^6 |
| .04905 | 0.2469 | 6.372×10^7 | .01974 | 5.094×10^6 |
| .0966 | 0.2270 | 1.250×10^6 | .01815 | 0.999×10^5 |
| .0966 | 0.5316 | 3.054×10^6 | .04250 | 2.442×10^5 |
| .1426 | 0.3805 | 2.623×10^5 | .03042 | 2.097×10^4 |
| .1426 | 0.8306 | 4.1055×10^5 | .06641 | 3.282×10^4 |
| .1872 | 0.6028 | 8.873×10^4 | .04819 | 7.093×10^3 |
| .1872 | 1.0710 | 1.196×10^5 | .08563 | 9.562×10^3 |
| .1959 | 0.6366 | 7.742×10^4 | .05089 | 6.190×10^3 |
| .1959 | 1.1074 | 9.549×10^4 | .08854 | 7.634×10^3 |

$M_1 = 0.5$

| c | α | R | α_{-3} | R_{-3} |
|--------|----------|----------------------|---------------|---------------------|
| .04905 | .0800 | 2.503×10^7 | .006313 | 1.975×10^6 |
| .04905 | .1945 | 8.933×10^7 | .01535 | 7.049×10^6 |
| .0966 | .1826 | 1.591×10^6 | .01441 | 1.255×10^5 |
| .0966 | .4066 | 3.944×10^6 | .03208 | 3.112×10^5 |
| .1426 | .3111 | 3.366×10^5 | .02455 | 2.656×10^4 |
| .1426 | .6592 | 5.926×10^5 | .05202 | 4.676×10^4 |
| .1872 | .4847 | 1.105×10^5 | .03825 | 8.719×10^3 |
| .1872 | .8536 | 1.571×10^5 | .06736 | 1.240×10^4 |
| .2046 | .5818 | 7.889×10^4 | .04591 | 6.225×10^3 |
| .2046 | .8954 | 1.0314×10^5 | .07066 | 8.139×10^3 |

TABLE I (contd.)

$M_1 = 0.7$

| c | α | R | $\alpha_{\frac{g}{g}}$ | $R_{\frac{g}{g}}$ |
|--------|----------|----------------------|------------------------|---------------------|
| .04905 | .0633 | 3.171×10^7 | .00493 | 2.471×10^6 |
| .04905 | .1457 | 1.298×10^8 | .01135 | 1.012×10^7 |
| .0966 | .1491 | 1.982×10^6 | .01162 | 1.545×10^5 |
| .0966 | .3344 | 5.085×10^6 | .02606 | 3.963×10^5 |
| .1426 | .2478 | 4.171×10^5 | .01931 | 3.251×10^4 |
| .1426 | .5460 | 7.447×10^5 | .04255 | 5.804×10^4 |
| .1872 | .4000 | 1.365×10^5 | .03118 | 1.064×10^4 |
| .1872 | .6895 | 2.069×10^5 | .05374 | 1.613×10^4 |
| .2132 | .5478 | 8.753×10^4 | .04270 | 6.808×10^3 |
| .2132 | .7463 | 1.0948×10^5 | .05817 | 8.533×10^3 |

$M_1 = 0.9$

| c | α | R | $\alpha_{\frac{g}{g}}$ | $R_{\frac{g}{g}}$ |
|--------|----------|----------------------|------------------------|---------------------|
| .04905 | .03931 | 5.399×10^7 | .00301 | 4.140×10^6 |
| .04905 | .08999 | 1.935×10^8 | .00690 | 1.484×10^7 |
| .0966 | .0960 | 3.173×10^6 | .00736 | 2.433×10^5 |
| .0966 | .2187 | 7.886×10^6 | .01677 | 6.048×10^5 |
| .1426 | .1728 | 6.134×10^5 | .01325 | 4.704×10^4 |
| .1426 | .3523 | 1.2006×10^6 | .02702 | 9.207×10^4 |
| .1872 | .2614 | 2.080×10^5 | .02005 | 1.595×10^4 |
| .1872 | .4694 | 3.138×10^5 | .03600 | 2.406×10^4 |
| .2132 | .3479 | 1.166×10^5 | .02668 | 8.942×10^3 |
| .2132 | .5562 | 1.651×10^5 | .04265 | 1.266×10^4 |

Table I (contd.)

$M_1 = 1.1$

| c | α | R | α_{ψ} | R_{ψ} |
|-------|----------|----------------------|-----------------|---------------------|
| .0966 | .01614 | 1.9135×10^7 | .00121 | 1.437×10^6 |
| .0966 | .03623 | 5.220×10^7 | .00272 | 3.919×10^6 |
| .1426 | .0773 | 1.412×10^6 | .00580 | 1.060×10^5 |
| .1426 | .1599 | 2.798×10^6 | .01200 | 2.101×10^5 |
| .1872 | .1636 | 3.353×10^5 | .01228 | 2.517×10^4 |
| .1872 | .2951 | 5.3705×10^5 | .02216 | 4.032×10^4 |
| .2218 | .2612 | 1.464×10^5 | .01961 | 1.099×10^4 |
| .2218 | .4014 | 1.951×10^5 | .03014 | 1.465×10^4 |

$M_1 = 1.3$

| c | α | R | α_{ψ} | R_{ψ} |
|-------|----------|---------------------|-----------------|---------------------|
| .2333 | .00350 | 1.009×10^7 | .000257 | 7.411×10^5 |
| .2333 | .00485 | 1.443×10^7 | .000356 | 1.060×10^6 |

T A B L E II

Auxiliary functions used in the calculation of the neutral stability characteristics.

Insulated surface, $\beta = 0.6$.

| M_1 | c | λ | v | L | H_1 | $H_2=N_2$ | M_3 | N_3 |
|-------|--------|-----------|-------|--------|-------|-----------|--------|--------|
| 0 | .04905 | .0214 | .1023 | .2086 | .6604 | .2903 | .1243 | .1902 |
| | .0966 | .0377 | .2261 | .3929 | .5895 | .2842 | .1085 | .2041 |
| | .1426 | .0543 | .3600 | .6357 | .5252 | .2782 | .0945 | .2186 |
| | .1872 | .0709 | .5219 | .8879 | .4675 | .2732 | .0822 | .2366 |
| | .1959 | .0745 | .5571 | .9429 | .4560 | .2718 | .0798 | .2405 |
| 0.5 | .04905 | .0181 | .1006 | .1532 | .6518 | .2232 | .0936 | .1227 |
| | .0996 | .0340 | .2160 | .2813 | .5818 | .2243 | .0839 | .1378 |
| | .1426 | .0501 | .3697 | .4276 | .5184 | .2248 | .0748 | .1565 |
| | .1872 | .0663 | .5031 | .6037 | .4575 | .2212 | .0645 | .1676 |
| | .2046 | .0731 | .5687 | .6631 | .4395 | .2170 | .05945 | .1645 |
| 0.7 | .04905 | .0165 | .1000 | .0702 | .6438 | .1638 | .0680 | .0771 |
| | .0966 | .0323 | .2132 | .1685 | .5747 | .1718 | .0621 | .0892 |
| | .1426 | .0470 | .3424 | .2590 | .5121 | .1758 | .0569 | .1059 |
| | .1872 | .0629 | .4859 | .3778 | .4519 | .1807 | .0507 | .1235 |
| | .2132 | .0729 | .5879 | .4625 | .4239 | .1900 | .0493 | .1383 |
| 0.9 | .04905 | .0152 | .0993 | .0269 | .6335 | .0778 | .0268 | .0219 |
| | .0966 | .0306 | .2098 | .0332 | .5654 | .1036 | .0351 | .0346 |
| | .1426 | .0466 | .3338 | .0569 | .5038 | .1171 | .0369 | .0459 |
| | .1872 | .0622 | .4770 | .0551 | .4446 | .1248 | .0336 | .0636 |
| | .2132 | .0718 | .5543 | .0815 | .4171 | .1349 | .0332 | .0712 |
| 1.1 | .0966 | .0305 | .1805 | -.0889 | .5536 | .0235 | .0049 | -.0136 |
| | .1426 | .0458 | .3247 | -.1747 | .4933 | .0469 | .0116 | -.0046 |
| | .1872 | .0613 | .4555 | -.0488 | .4353 | .0629 | .0143 | .0053 |
| | .2218 | .0743 | .5695 | .01186 | .3986 | .0805 | .0177 | .0129 |
| 1.3 | .2333 | .0775 | .5799 | -.4979 | .3706 | .0183 | .0000 | -.0193 |

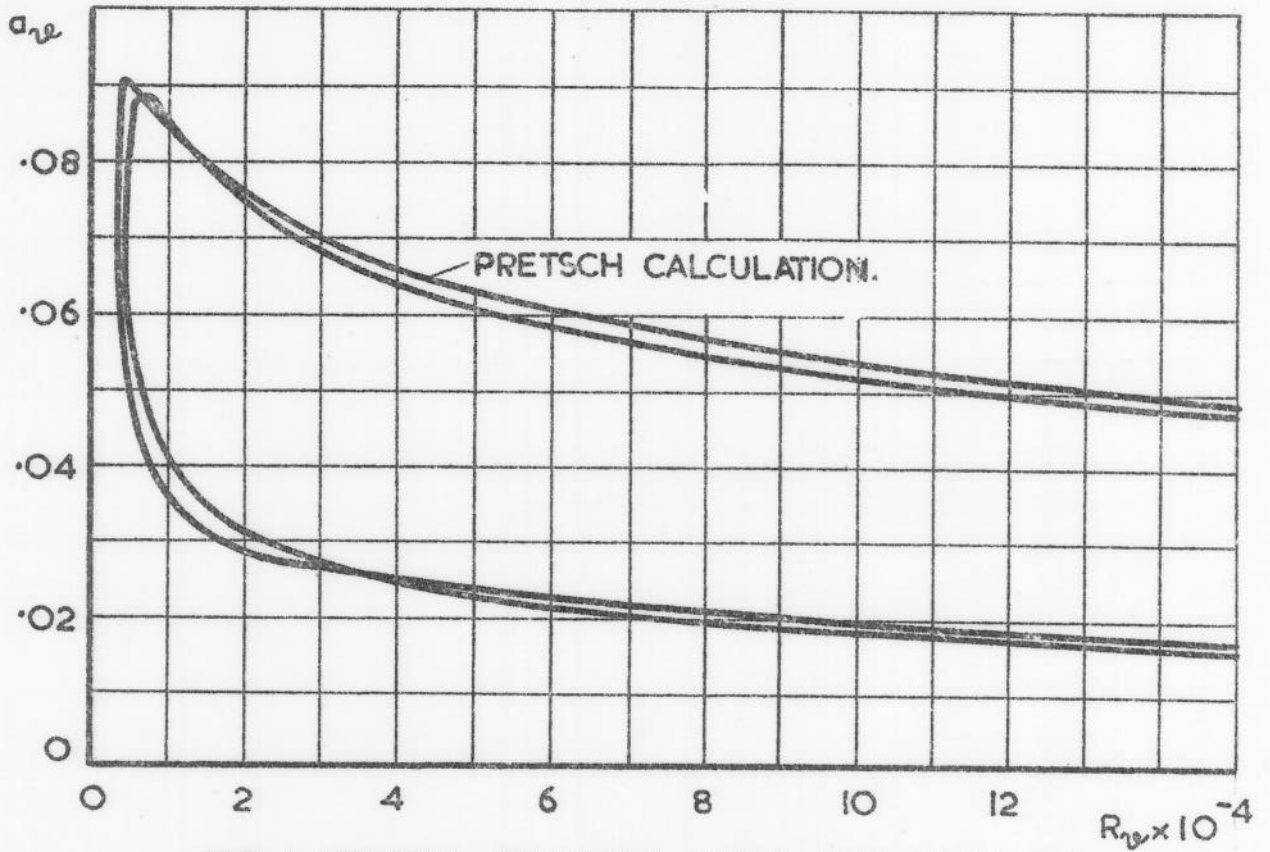


FIG. 1. NEUTRAL STABILITY CURVE FOR INSULATED SURFACE WITH $\beta = 0.6$.

(a) $M_1 = 0$.

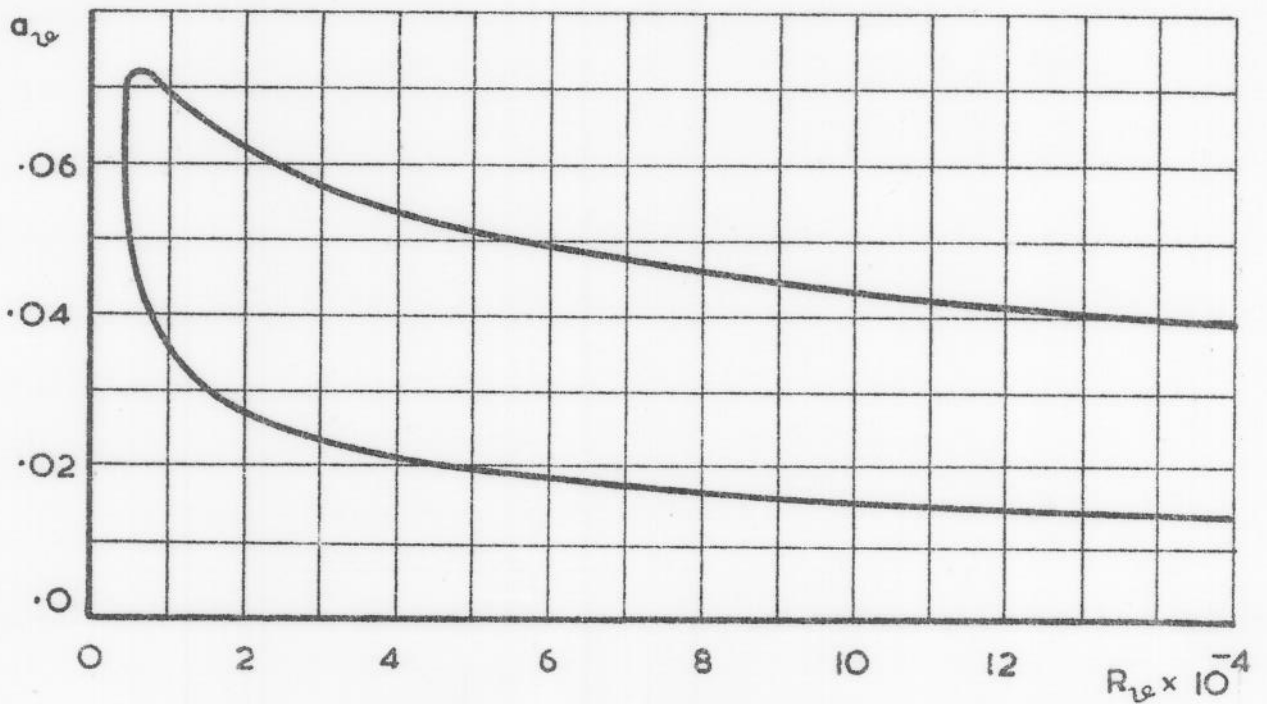


FIGURE I. CONTINUED.

(b) $M_1 = 0.5$.

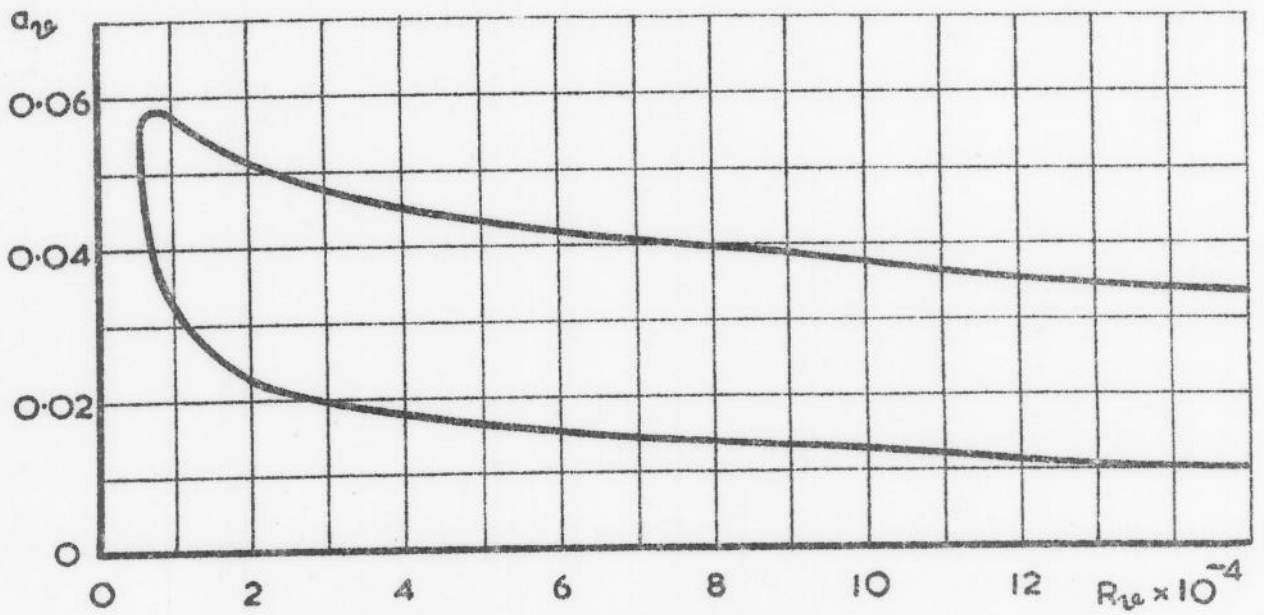


FIGURE 1. CONTINUED.

(c) $M_1 = 0.7$.

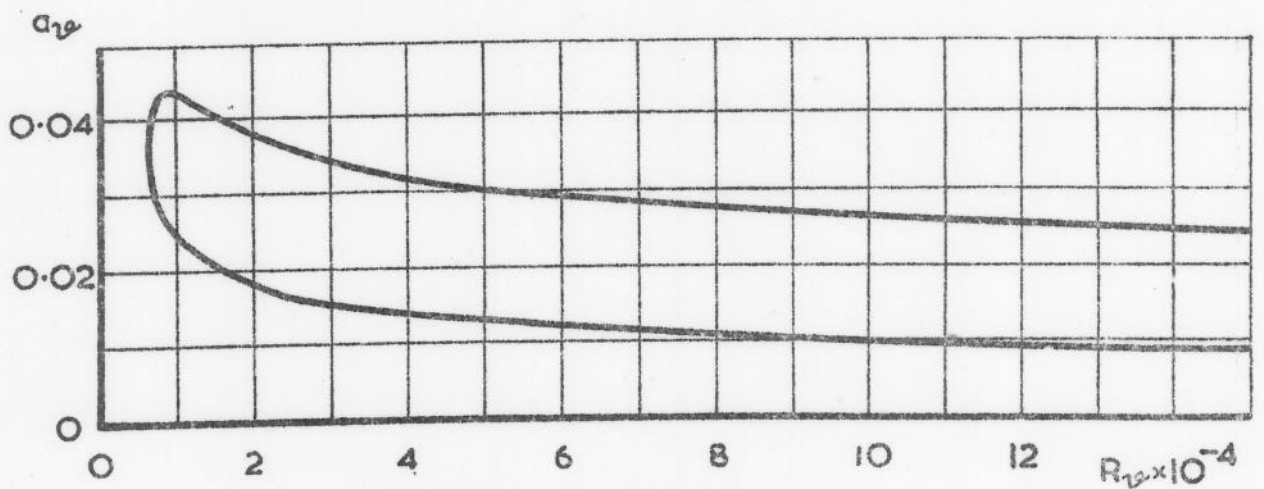


FIGURE 1. CONTINUED.

(d) $M_1 = 0.9$.

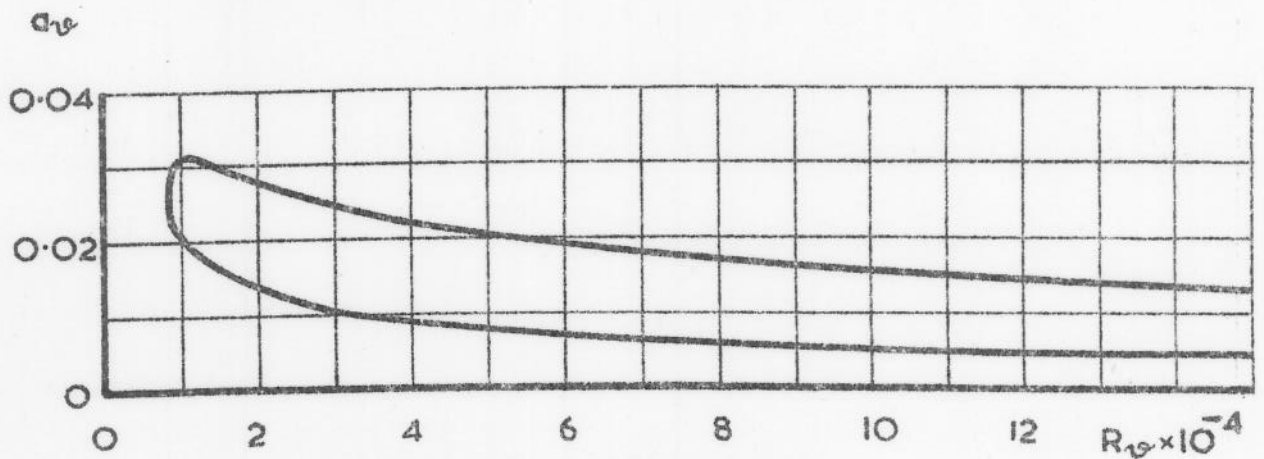


FIGURE 1. CONTINUED.

(e) $M_1 = 1.1$.

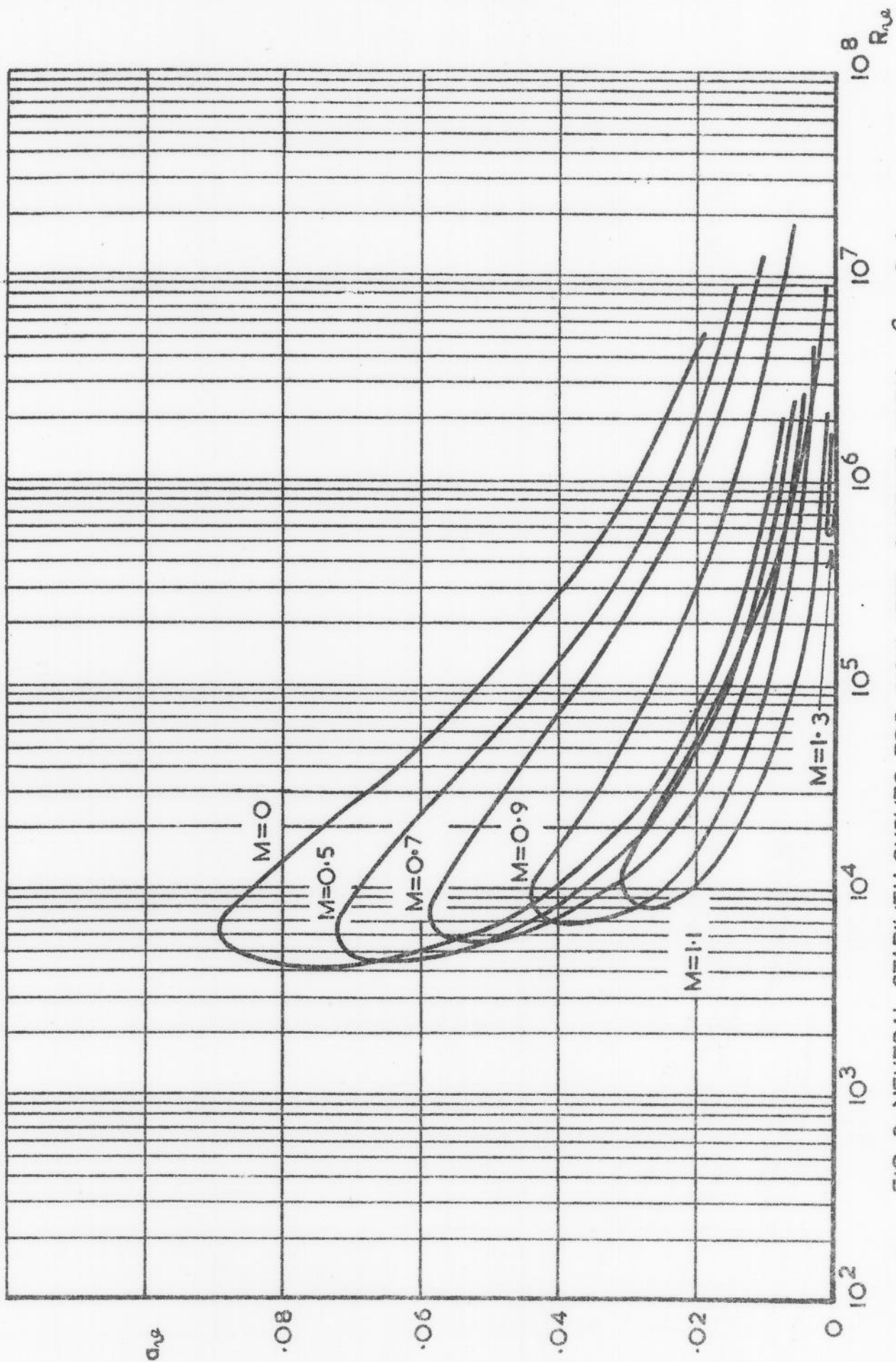


FIG. 2. NEUTRAL STABILITY CURVES FOR INSULATED SURFACE WITH $\beta = 0.6$.

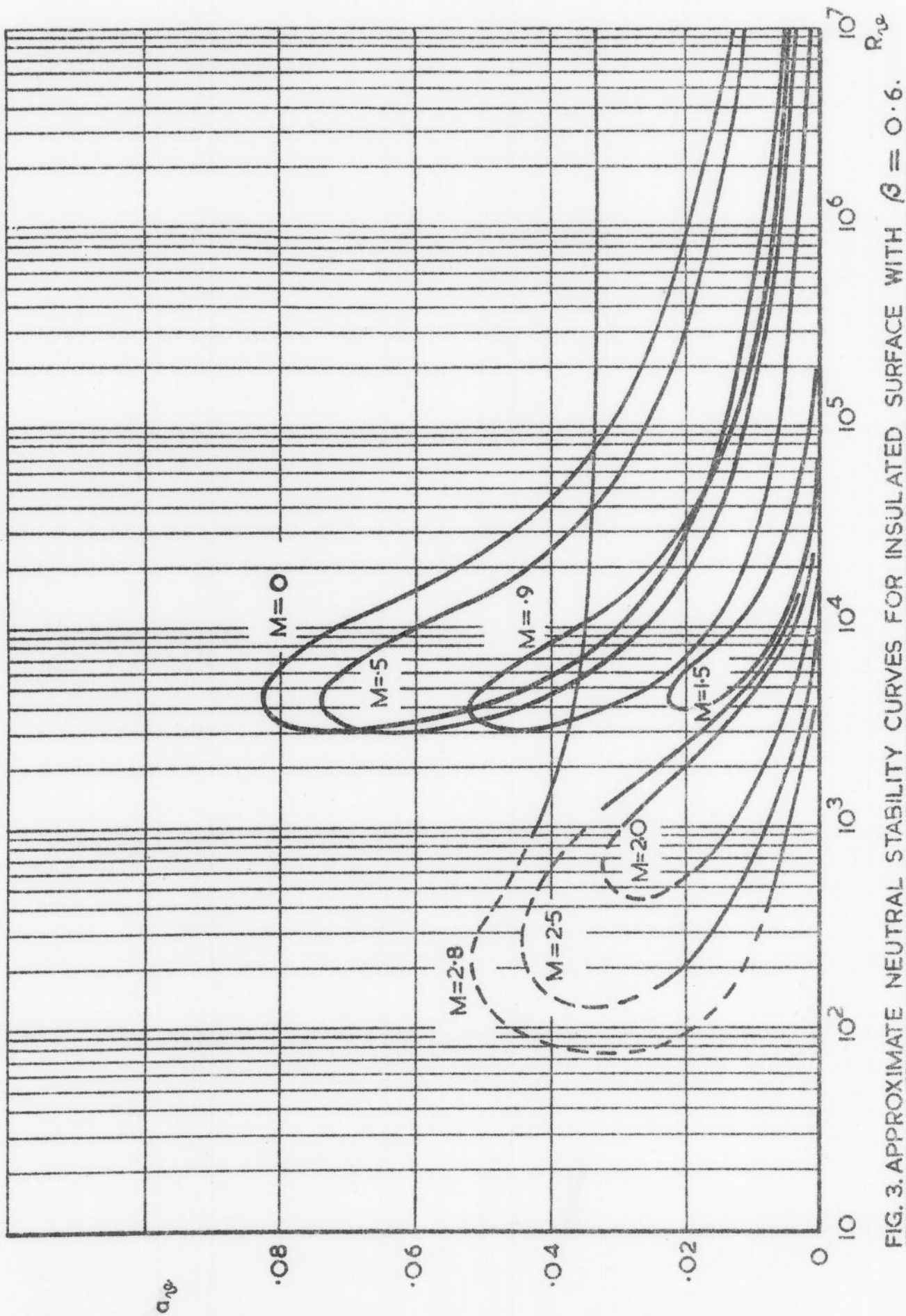


FIG. 3. APPROXIMATE NEUTRAL STABILITY CURVES FOR INSULATED SURFACE WITH $\beta = 0.6$.

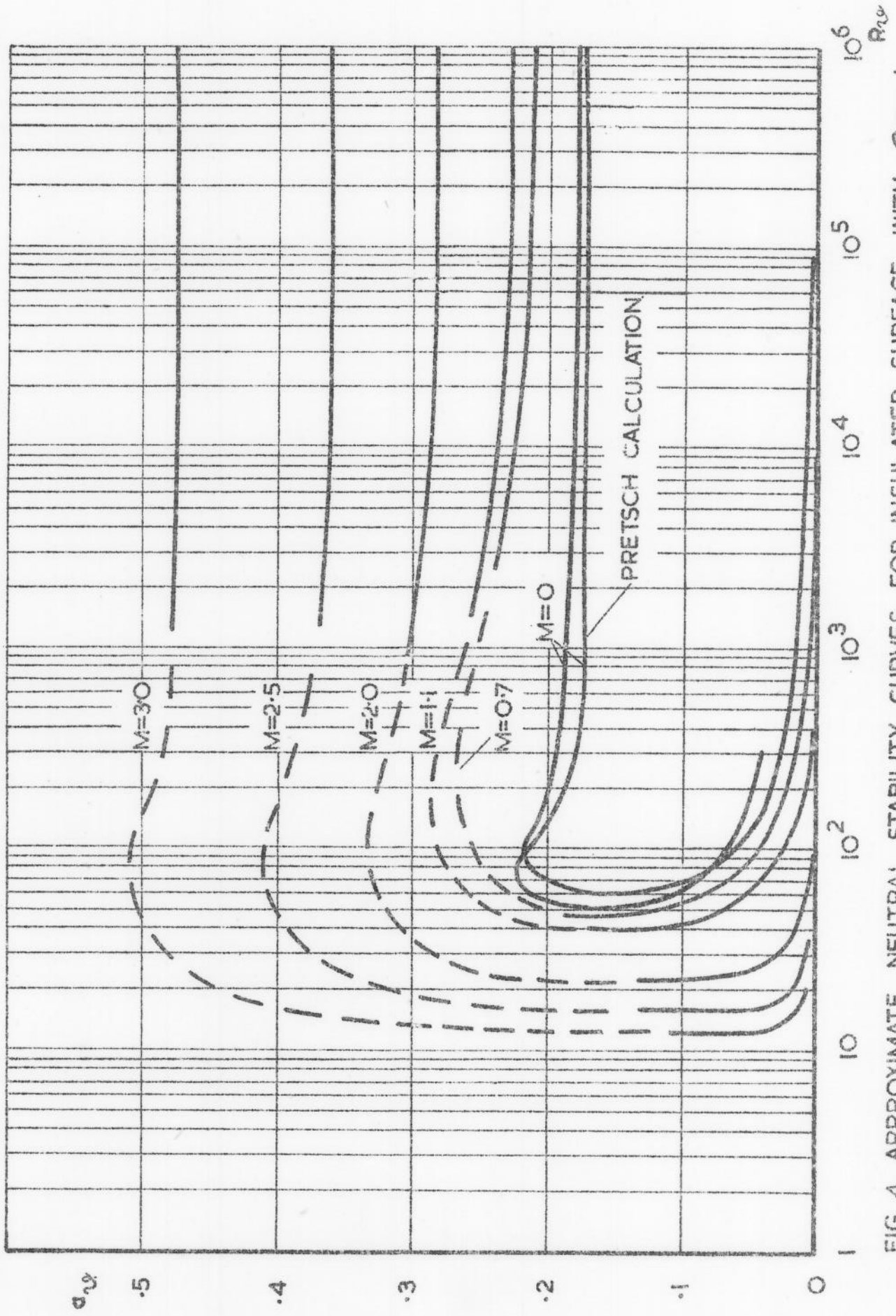


FIG. 4. APPROXIMATE NEUTRAL STABILITY CURVES FOR INSULATED SURFACE WITH $\beta = -1$.

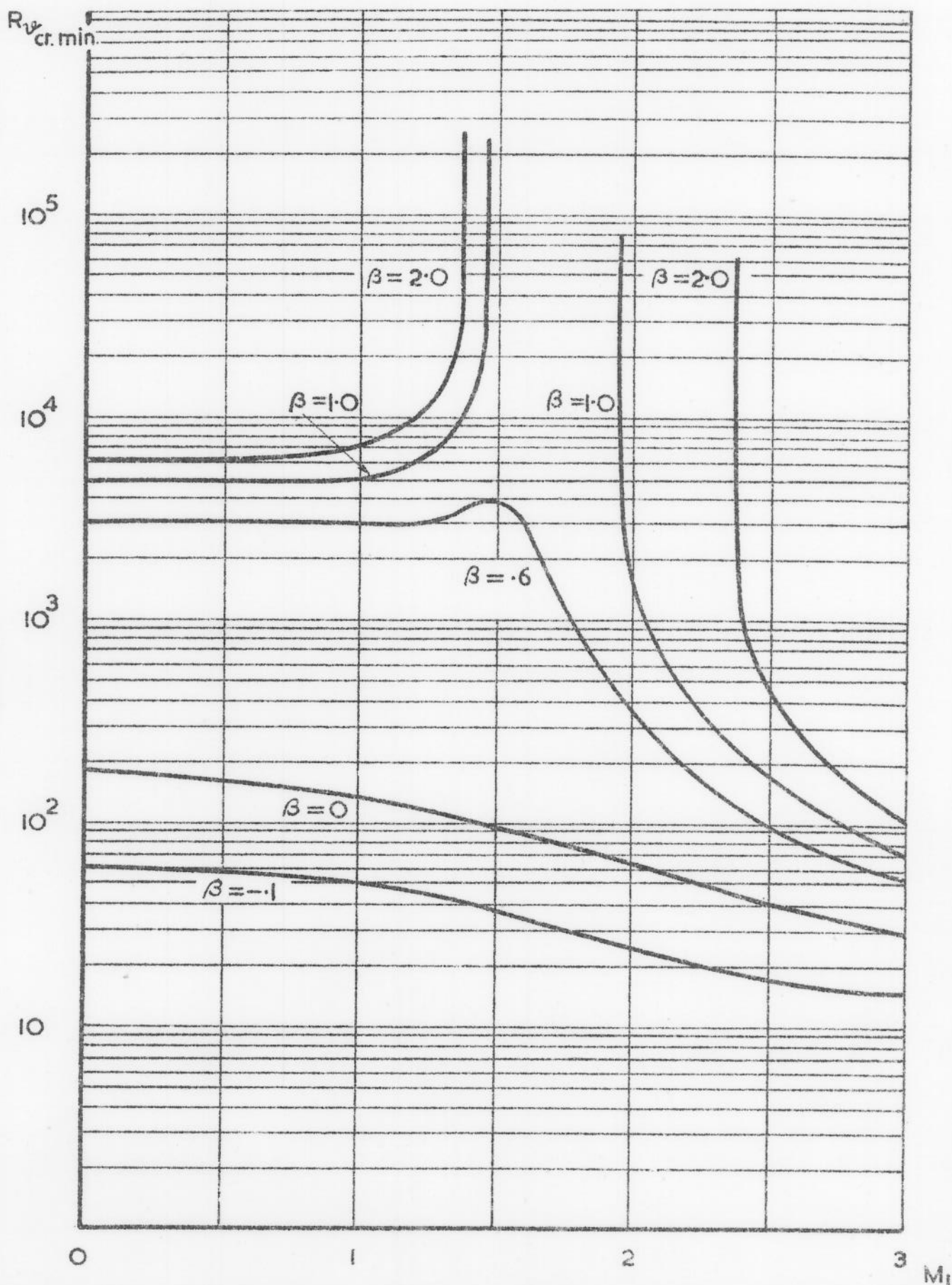


FIG. 5. MAXIMUM REYNOLDS NUMBER FOR STABILITY AGAINST MACH NUMBER.

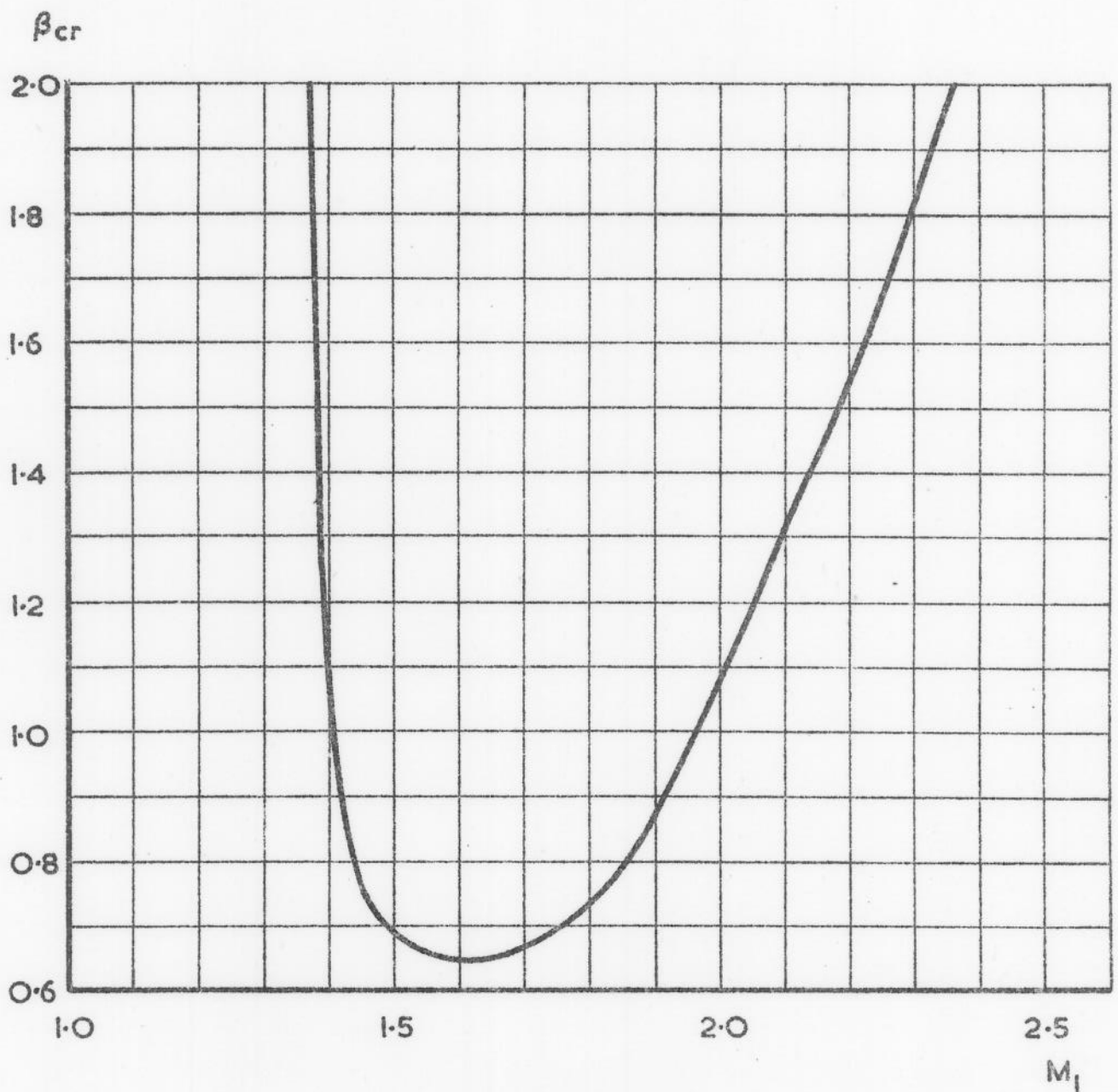


FIG. 6. THE CRITICAL PRESSURE GRADIENT COEFFICIENT β_{cr}
FOR COMPLETE STABILITY FOR ALL REYNOLD'S NUMBERS
WITH AN INSULATED SURFACE.

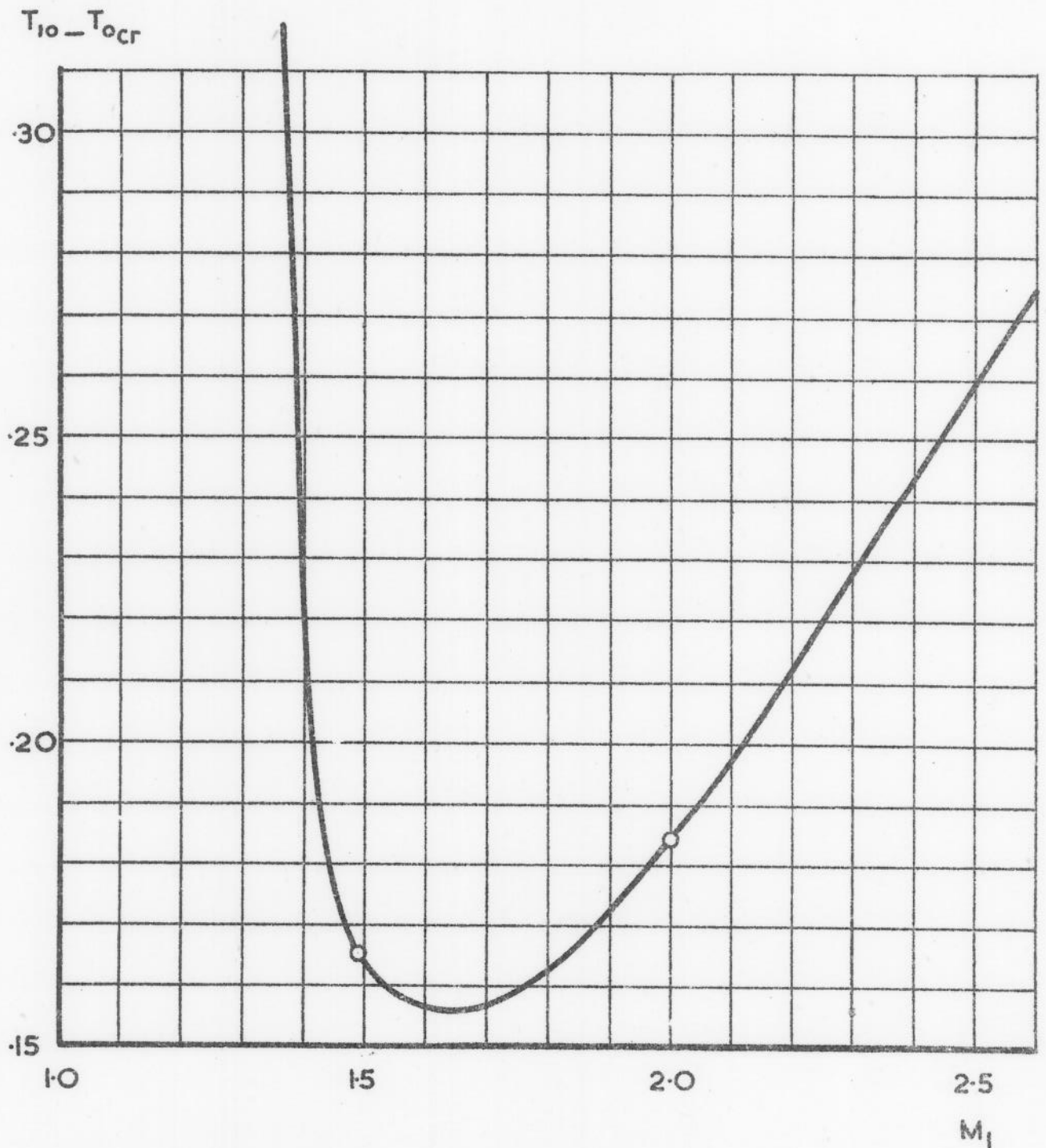


FIG. 7. THE CRITICAL TEMPERATURE RATIO T_{ocr} FOR COMPLETE STABILITY OF THE LAMINAR BOUNDARY LAYER WITHOUT A PRESSURE GRADIENT. (AFTER LEES: REFERENCE 1)).

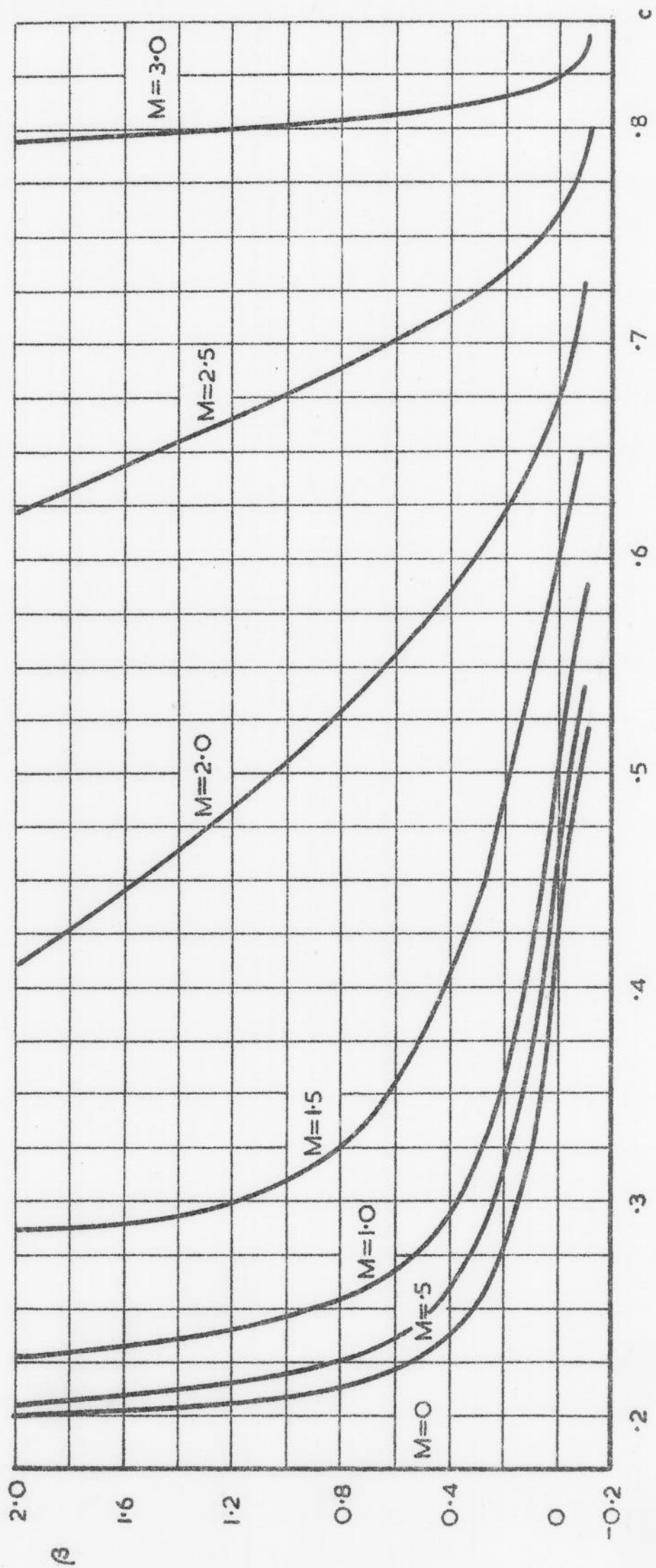


FIG. 8. PHASE VELOCITY c CORRESPONDING TO $R_{10} c.r. min.$ AGAINST PRESSURE GRADIENT.

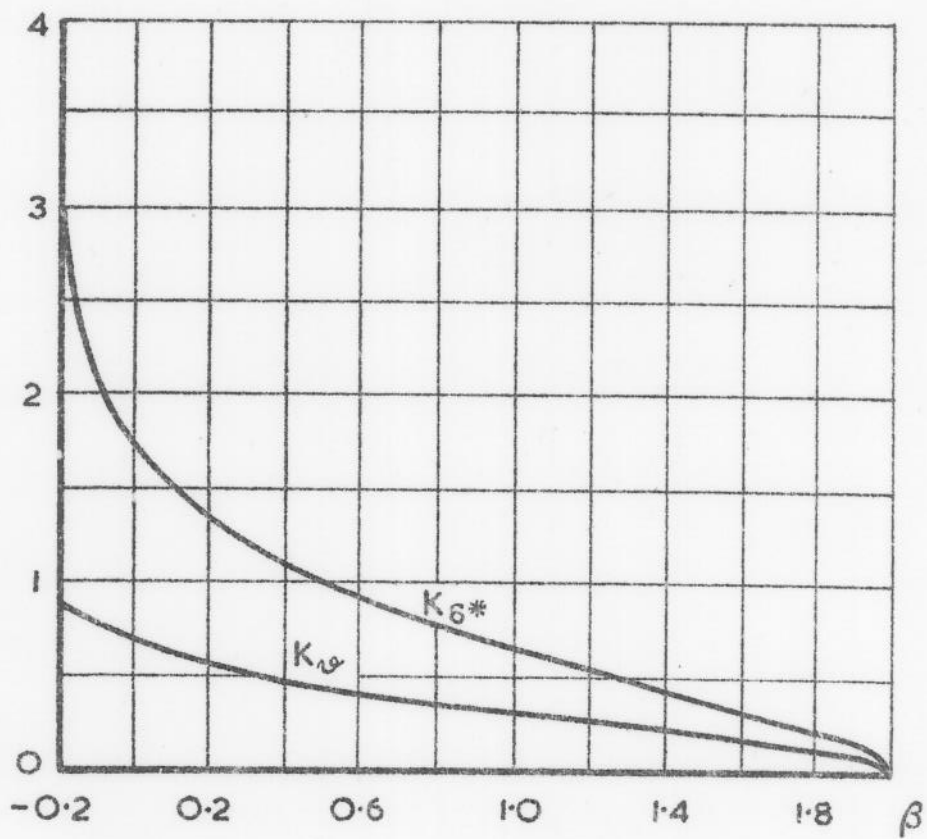


FIG. 9. VARIATION OF THE PARAMETERS $K_{1\psi}$ AND K_{δ^*} WITH PRESSURE GRADIENT. (TAKEN FROM REFERENCE 15.)

DERIVATIVES OF THE HARTREE VELOCITY PROFILES FOR THE
INCOMPRESSIBLE BOUNDARY LAYER. (TAKEN FROM REFERENCE 15).

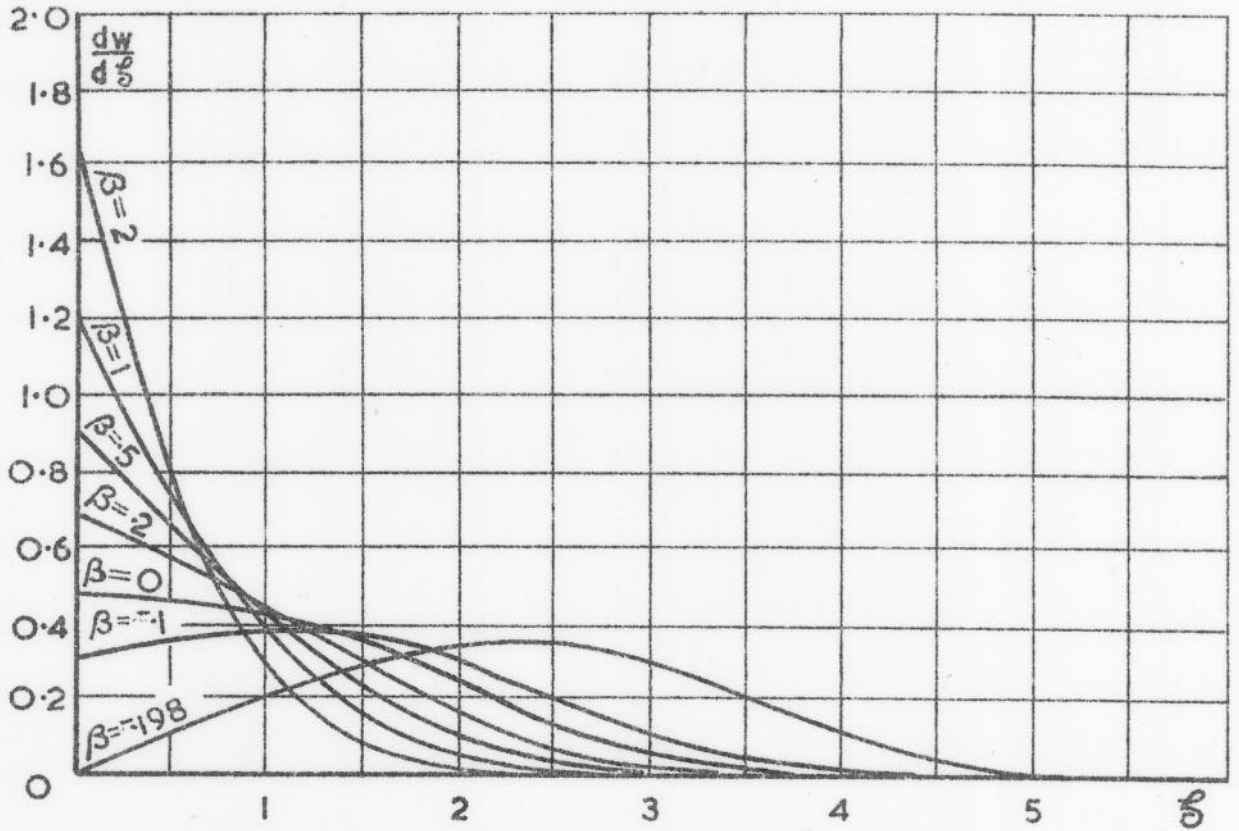


FIGURE 10.

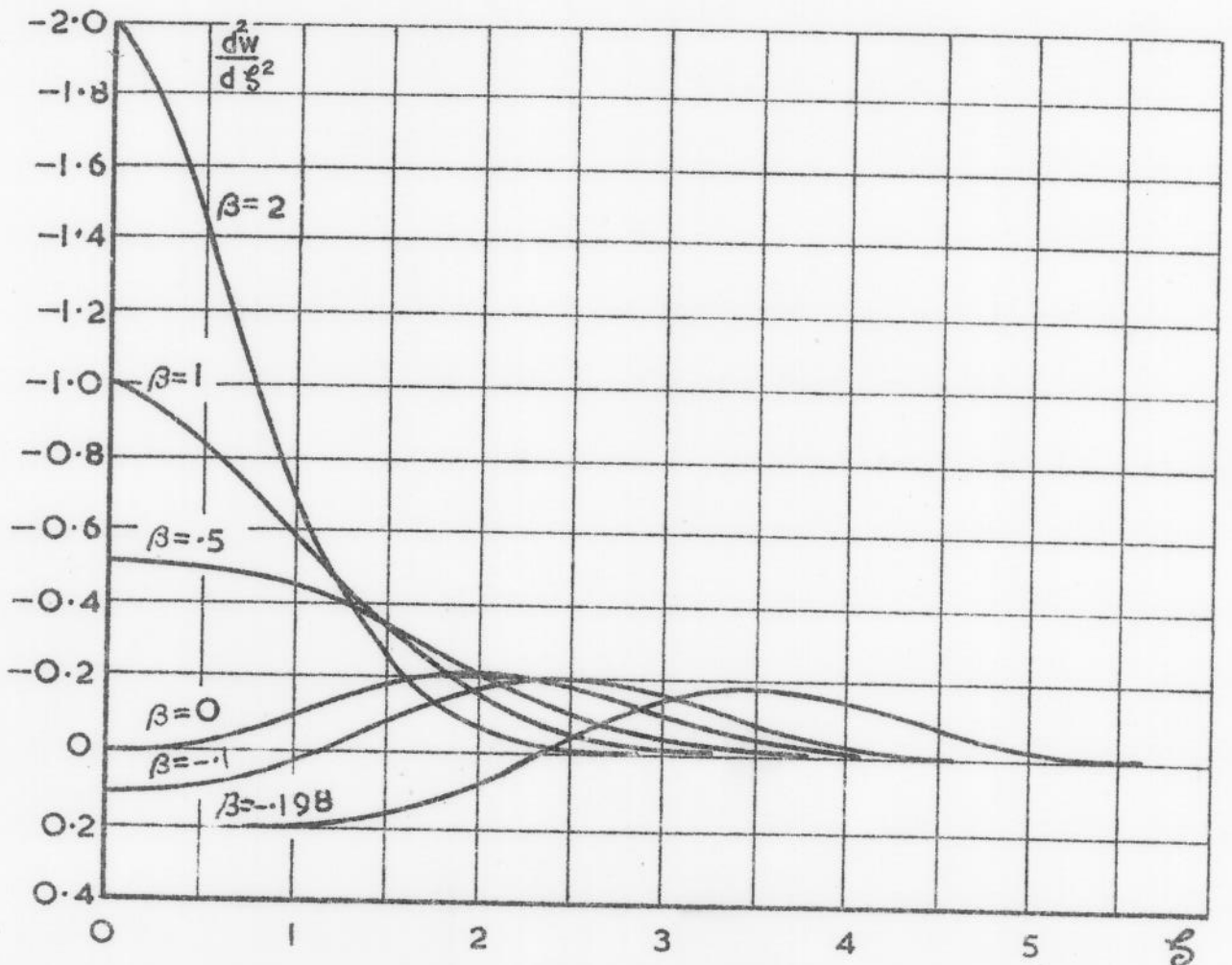


FIGURE 11.

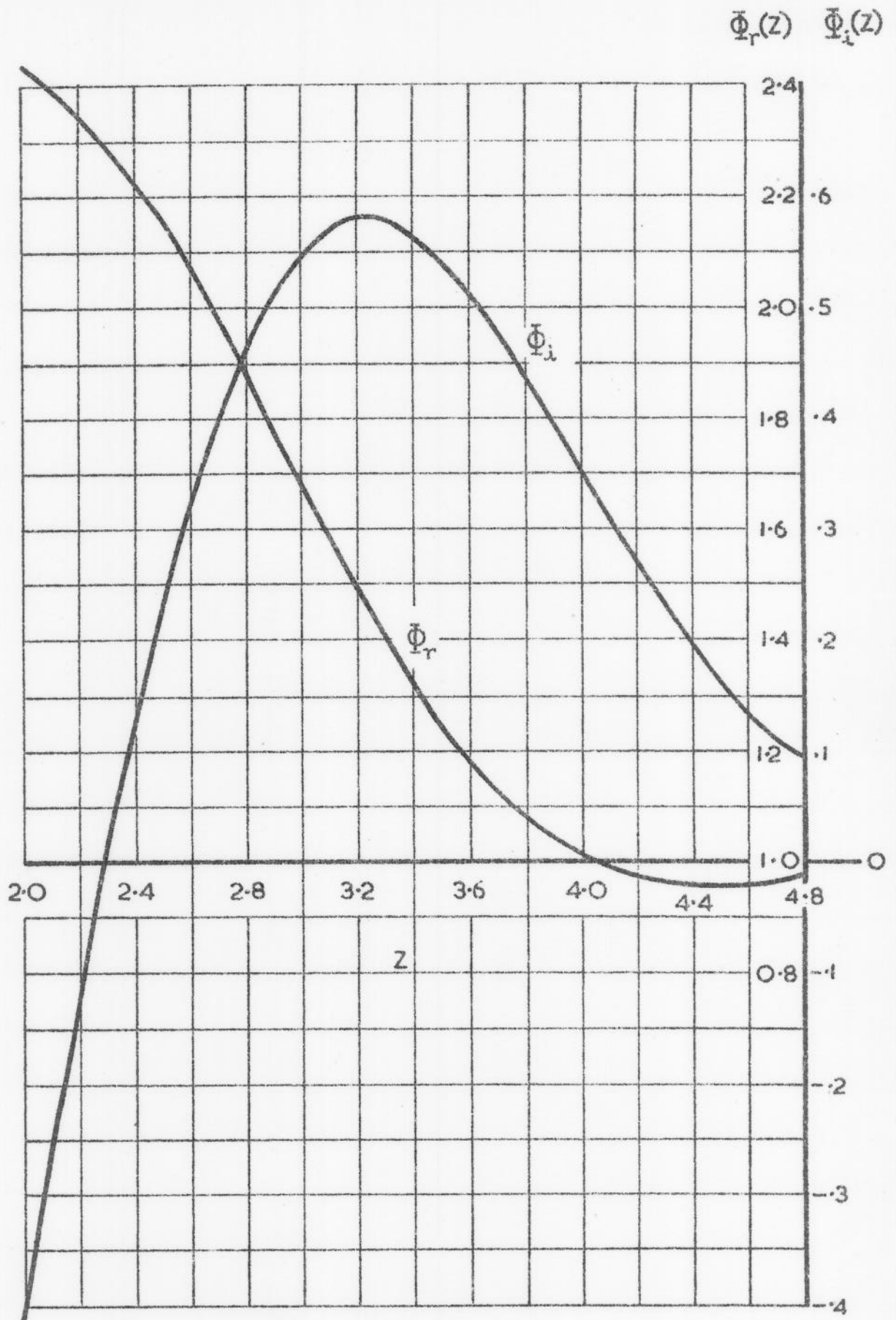


FIG. 12. THE REAL AND IMAGINARY PART OF THE
MODIFIED TIETJENS FUNCTION $\bar{\Phi}(z)$
 (TAKEN FROM REFERENCE 11.)