REPORT NO. 42

January, 1951

THE COLLEGE OF AERONAUTICS

CRANFIELD

THE EQUATIONS OF MOTION AND ENERGY AND THE VELOCITY PROFILE OF A TURBULENT BOUNDARY LAYER IN A COMPRESSIBLE FLUID

-by-

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MEP

1. Introduction and Summary

As far as the author is aware the derivation of the equations of motion and energy for a turbulent boundary layer in a compressible fluid have not as yet been given in detail in any publication. To meet a possible need in this connection this paper puts on record the analysis underlying the equations quoted by the author in Chapter X of the forthcoming Vol. III of Modern Developments in Fluid Dynamics.[1]

In the absence of further experimental data and of an adequate knowledge of the physics of turbulence, particularly in high speed flow, it is impossible to make use of these equations except in a few simple cases. One of these cases is considered here viz., mean parallel flow in the boundary layer on a flat plate with the Prandtl number equal to unity. The object was to investigate the effect of compressibility and heat transfer on the form of the velocity profile of the boundary layer. The crude method that serves for a theory in deriving the well known 'log' law in an incompressible fluid is adapted here to a compressible fluid, the underlying justification being that since the method gives something close to the right result in the first instance, it should indicate with reasonable accuracy any change that arises due to compressibility.

It is concluded that for Mach numbers of the order of 2.5 or less and for a wide range of heat transfer conditions the form of the velocity profile in the turbulent boundary layer will differ very little from that for an incompressible fluid and the same Reynolds number. This result is in agreement with existing experimental results. For higher Mach numbers, however, small differences will become apparent particularly for cases of considerable heat transfer from the surface.

/2. List ...

2 List of principle notation

v_α, V_α, v'_α	total, mean and fluctuating velocity components, in tensor notation
x_α	coordinates of reference in tensor notation
U , u'	mean and fluctuating velocity components parallel to x-axis (2 dimensional Cartesian axes)
V , v'	mean and fluctuating velocity components parallel to y-axis
u	U/U_1
M	Mach number
x	distance downstream from leading edge along surface
y	distance normal to surface
ρ	density
t	time
μ	coefficient of viscosity
$p_{\alpha\beta}$	stress tensor component
p	pressure (mean of principal stresses)
$e_{\alpha\beta}$	strain tensor component
c_p	coefficient of specific heat at constant pressure
J	mechanical equivalent of heat
k	coefficient of thermal conductivity
T	temperature
Φ	dissipation function
q	resultant velocity
T_H	total temperature $= T + \frac{q^2}{2 J c_p}$
σ	Prandtl number $= \mu c_p/k$
δ	boundary layer thickness.

Suffix o refers to the undisturbed stream, suffix 1 to the free stream just outside the boundary layer, and suffix w refers to conditions at the wall. Dashes are used to denote turbulent components, and a bar is used to denote an average that is taken over an interval of time large compared with the periods of the turbulent fluctuations. Other terms used are defined in the text.

/3. Equations ...

3. Equations of motion and continuity for a turbulent, compressible fluid

Using conventional tensor notation and neglecting body forces the equations of motion may be written

$$\left.\begin{aligned}
\rho \frac{\partial v_\alpha}{\partial t} &= \frac{\partial}{\partial x_\beta}\left(p_{\alpha\beta} - \rho v_\beta v_\alpha\right) - v_\alpha \frac{\partial \rho}{\partial t} \\
\text{where} \quad p_{\alpha\beta} &= -\left(p + \frac{2}{3}\mu\Delta\right)\delta_{\alpha\beta} + \mu e_{\alpha\beta}, \\
\Delta &\equiv \frac{\partial v_\alpha}{\partial x_\alpha}, \\
e_{\alpha\beta} &\equiv \frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} \\
\delta_{\alpha\beta} &= 0, \text{ if } \alpha \neq \beta \\
&= 1, \text{ if } \alpha = \beta
\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (1)$$

If now in equation (1) we substitute $V_\alpha + v'_\alpha$ for v_α, $\rho + \rho'$ for ρ, etc. where the dashed terms denote the fluctuating components and the undashed terms denote the mean components, and the mean of the equation is taken, we obtain

$$\begin{aligned}
\rho \frac{\partial V_\alpha}{\partial t} + \overline{\rho' \frac{\partial v'_\alpha}{\partial t}} = \frac{\partial}{\partial x_\beta}\Big(p_{\alpha\beta} + \overline{p'_{\alpha\beta}} - \rho V_\alpha V_\beta - \rho\, \overline{v'_\alpha v'_\beta} \\
- \overline{\rho' v'_\alpha}\, V_\beta - \overline{\rho' v'_\beta}\, V_\alpha - \overline{\rho' v'_\alpha v'_\beta}\Big) \\
- V_\alpha \frac{\partial \rho}{\partial t} - \overline{v'_\alpha \frac{\partial \rho'}{\partial t}},
\end{aligned} \qquad \ldots\ldots\ldots\ldots (2)$$

Here $\overline{p'_{\alpha\beta}}$ represents the terms in $p_{\alpha\beta}$ arising from the fluctuations not obviously zero when the mean is taken, viz.,

$$\overline{p'_{\alpha\beta}} = -\frac{2}{3}\delta_{\alpha\beta}\,\overline{\mu'\Delta'} + \overline{\mu' e'_{\alpha\beta}},$$

$$\Delta' = \frac{\partial v'_\alpha}{\partial x_\alpha}, \quad e'_{\alpha\beta} = \frac{\partial v'_\alpha}{\partial x_\beta} + \frac{\partial v'_\beta}{\partial x_\alpha}.$$

The equation of continuity similarly yields

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_\alpha}(\rho V_\alpha) + \frac{\partial}{\partial x_\alpha}\overline{(\rho' v'_\alpha)} = 0. \qquad \ldots\ldots\ldots\ldots (3)$$

It will be seen that the fluctuations introduce effective stresses into the equations of motion insofar as they produce changes in the mean rates of transport of components of momentum across a surface. These stresses not only include those met with in incompressible flow theory,

/viz. ...

viz. $\rho \overline{v'_\alpha v'_\beta}$ (usually called Reynolds stresses) but also include terms involving the mean of products of ρ' and the velocity fluctuation components. These latter terms are probably small compared with the former at low Mach numbers, but in the absence of experimental data to guide us they cannot in general be ignored. Further it will be noted that the viscous stress terms now include terms involving the fluctuations in viscosity. However, in cases where the eddy stresses due to the velocity and density fluctuations are large compared with the viscous stresses, as in a turbulent boundary layer except very near the wall, it may be assumed that the fluctuating viscosity terms can be neglected with those involving the mean viscosity.

4. Boundary layer equations of motion (2 dimensions)

In two dimensions, equation (2) becomes

$$\rho \frac{\partial U}{\partial t} + \overline{\rho' \frac{\partial u'}{\partial t}} = - \frac{\partial p}{\partial x} - \frac{\partial}{\partial x}\left[\frac{2}{3}\mu\Delta - 2\mu\frac{\partial U}{\partial x}\right]$$
$$- \frac{\partial}{\partial x}\left[\rho U^2 + \rho \overline{u'^2} + 2\overline{\rho' u'}U\right] - \frac{\partial}{\partial x}\left[\frac{2}{3}\overline{\mu'\Delta'} - \overline{2\mu'\frac{\partial u'}{\partial x}}\right]$$
$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right] - \frac{\partial}{\partial y}\left[\rho UV + \rho\overline{u'v'} + \overline{\rho' u'}V + \overline{\rho' v'}U + \overline{\rho' u' v'}\right]$$
$$+ \frac{\partial}{\partial y}\overline{\left[\mu'\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)\right]} - U\frac{\partial \rho}{\partial t} - \overline{u'\frac{\partial \rho'}{\partial t}}, \qquad \ldots\ldots\ldots\ldots (4)$$

and

$$\rho \frac{\partial V}{\partial t} + \overline{\rho' \frac{\partial v'}{\partial t}} = - \frac{\partial p}{\partial y} - \frac{\partial}{\partial y}\left[\frac{2}{3}\mu\Delta - 2\mu\frac{\partial V}{\partial y}\right] - \frac{\partial}{\partial y}\left[\rho V^2 + \rho\overline{v'^2} + \overline{\rho' v'}V\right]$$
$$- \frac{\partial}{\partial y}\left[\frac{2}{3}\overline{\mu'\Delta'} - \overline{2\mu'\frac{\partial v'}{\partial y}}\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right]$$
$$- \frac{\partial}{\partial x}\left[\rho UV + \rho\overline{u'v'} + \overline{\rho' u'}V + \overline{\rho' v'}U + \overline{\rho' u' v'}\right] + \frac{\partial}{\partial x}\overline{\left[\mu'\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)\right]}$$
$$- V\frac{\partial \rho}{\partial t} - \overline{v'\frac{\partial \rho'}{\partial t}}. \qquad \ldots\ldots\ldots\ldots (5)$$

The equation of continuity becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho U) + \frac{\partial}{\partial y}(\rho V) + \frac{\partial}{\partial x}(\overline{\rho' u'}) + \frac{\partial}{\partial y}(\overline{\rho' v'}) = 0. \qquad \ldots\ldots\ldots\ldots (6)$$

If we now make the usual boundary layer assumptions,

/viz. ...

viz. the boundary layer thickness (δ) is small compared with x, and $\frac{V}{U} = O\left(\frac{\delta}{x}\right)$, and the rates of change of the mean velocity components and their derivatives in the direction parallel to the surface are small compared with the corresponding rates of change normal to the surface then equation (4) reduces to

$$\rho \frac{DU}{Dt} + \overline{\rho' \frac{\partial u'}{\partial t}} = - \frac{\partial p}{\partial x} - U \frac{\partial}{\partial x} (\rho\, U) - U \frac{\partial}{\partial y} (\rho\, V) + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial U}{\partial y} \right) \right]$$

$$- \frac{\partial}{\partial x} \left[\rho\, \overline{u'^2} + 2\overline{\rho' u'}\, U \right] - \frac{\partial}{\partial y} \left[\overline{\rho u'v'} + \overline{\rho' v'} U + \overline{\rho' u'v'} \right]$$

$$- \frac{\partial}{\partial x} \left[\frac{2}{3} \overline{\mu' \Delta'} - 2\overline{\mu' \frac{\partial u'}{\partial x}} \right] + \frac{\partial}{\partial y} \left[\overline{\mu' \left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \right)} \right]$$

$$- U \frac{\partial \rho}{\partial t} - \overline{u' \frac{\partial \rho'}{\partial t}} ,$$

and using (6) this becomes

$$\rho \frac{DU}{Dt} + \frac{\partial}{\partial t} (\overline{\rho' u'}) = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial U}{\partial y} \right) \right] - \frac{\partial}{\partial x} \left[\overline{\rho u'^2} + \overline{\rho' u'}\, U \right]$$

$$- \frac{\partial}{\partial y} \left[\overline{\rho u'v'} + \overline{\rho' u'v'} \right] - 2\overline{\rho' u'} \frac{\partial U}{\partial x} - \overline{\rho' v'} \frac{\partial U}{\partial y}$$

$$- \frac{\partial}{\partial x} \left[\frac{2}{3} \overline{\mu' \Delta'} - 2\overline{\mu' \frac{\partial u'}{\partial x}} \right] + \frac{\partial}{\partial y} \left[\overline{\mu' \left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \right)} \right] \ldots (7)$$

If we now assume

(1) $\overline{\rho' v'}$ and $\overline{\rho' u'}$ are of the same order so that $2\overline{\rho' u'} \frac{\partial U}{\partial x}$ can be neglected compared with $\overline{\rho' v'} \frac{\partial U}{\partial y}$,

(2) $\rho' u' v'$ is small compared with $\overline{\rho u'v'}$,

(3) $\frac{\partial}{\partial x} \left[\frac{2}{3} \overline{\mu' \Delta'} - 2\overline{\mu' \frac{\partial u'}{\partial x}} \right]$ and $\frac{\partial}{\partial y} \left[\overline{\mu' \left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \right)} \right]$ can be neglected on the grounds that they are small compared with the corresponding terms arising from mean quantities, which in turn are small compared with the eddy fluctuation terms except very near the wall.

(4) $\frac{\partial}{\partial x} (\rho\, \overline{u'^2})$ is small compared with $\frac{\partial}{\partial y} \left[\overline{\rho u'v'} \right]$ and $\frac{\partial}{\partial x} (\overline{\rho' u'} U)$ is small compared with $\overline{\rho' v'} \frac{\partial U}{\partial y}$,

then equation (7) becomes

$$\rho \frac{DU}{Dt} + \frac{\partial}{\partial t} (\overline{\rho' u'}) = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left[\mu \frac{\partial U}{\partial y} \right] - \frac{\partial}{\partial y} \left[\overline{\rho u'v'} \right] - \overline{\rho' v'} \frac{\partial U}{\partial y} \quad \ldots\ldots (8)$$

/ and if the ...

and if the motion is steady this becomes

$$\rho \frac{DU}{Dt} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[\mu \frac{\partial U}{\partial y} - \rho\overline{u'v'}\right] - \overline{\rho'v'}\,\frac{\partial U}{\partial y} \, . \qquad \ldots\ldots\ldots\ldots (9)$$

Equation (5) yields the usual result that if the curvature of the wall is not large the pressure change across the boundary layer is small and may be neglected.

5. Equation of energy*

The equation of motion (1) can be written

$$\rho \frac{\partial v_\alpha}{\partial t} = - \frac{\partial p}{\partial x_\alpha} - \frac{2}{3}\frac{\partial}{\partial x_\alpha}(\mu\Delta) + \frac{\partial}{\partial x_\beta}\left[\mu\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)\right] - \frac{\partial}{\partial x_\beta}(\rho v_\beta v_\alpha) - v_\alpha \frac{\partial \rho}{\partial t}$$

or

$$\rho \frac{Dv_\alpha}{Dt} = - \frac{\partial p}{\partial x_\alpha} - v_\alpha\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_\beta}(\rho v_\beta)\right] - \frac{2}{3}\frac{\partial}{\partial x_\alpha}(\mu\Delta) + \frac{\partial}{\partial x_\beta}\left[\mu\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)\right]$$

$$= - \frac{\partial p}{\partial x_\alpha} - \frac{2}{3}\frac{\partial}{\partial x_\alpha}(\mu\Delta) + \frac{\partial}{\partial x_\beta}\left[\mu\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)\right], \qquad \ldots\ldots\ldots\ldots (10)$$

using the equation of continuity.

The equation of energy for a perfect gas with constant specific heats is

$$\rho J c_p \frac{DT}{Dt} - \frac{Dp}{Dt} = J \frac{\partial}{\partial x_\alpha}\left(k \frac{\partial T}{\partial x_\alpha}\right) + \Phi \qquad \ldots\ldots\ldots\ldots (11)$$

where Φ is the dissipation function

$$= \mu\left\{-\frac{2}{3}\Delta^2 + \frac{1}{2}\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)\cdot\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)\right\}.$$

Multiplying (10) by v_α, we get

$$\rho \frac{D}{Dt}\left(\frac{q^2}{2}\right) = - v_\alpha \frac{\partial p}{\partial x_\alpha} - \frac{2}{3} v_\alpha \frac{\partial}{\partial x_\alpha}(\mu\Delta) + v_\alpha \frac{\partial}{\partial x_\beta}\left[\mu\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)\right]$$

/where ...

* The treatment here is based on that of Squire in Appendix I of his draft of Chap. XIV of Modern Developments in Fluid Dynamics, Vol. III. [2]

where q is the magnitude of the resultant velocity.
Adding this last equation to (1), we get

$$\rho \frac{D}{Dt}\left(J c_p T + \frac{q^2}{2}\right) - \frac{\partial p}{\partial t} = \frac{\partial}{\partial x_\alpha}\left[Jk \frac{\partial T}{\partial x_\alpha} + \mu \frac{\partial}{\partial x_\alpha}\left(\frac{q^2}{2}\right)\right] + \chi, \quad \ldots\ldots\ldots (12)$$

$$\text{where } \chi = \Phi - \frac{2}{3} v_\alpha \frac{\partial}{\partial x_\alpha}(\mu\Delta) + v_\alpha \frac{\partial}{\partial x_\beta}\left[\mu\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)\right] - \frac{\partial}{\partial x_\alpha}\left[\mu v_\beta \frac{\partial v_\beta}{\partial x_\alpha}\right]$$

$$= \frac{\mu}{3}\Delta^2 + \frac{v_\alpha}{3}\frac{\partial}{\partial x_\alpha}(\mu\Delta) - \mu\Delta^2 - v_\alpha \frac{\partial}{\partial x_\alpha}(\mu\Delta)$$

$$+ \mu v_\alpha \frac{\partial}{\partial x_\beta}\left[\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right] - \mu \frac{\partial}{\partial x_\alpha}\left[v_\alpha \frac{\partial v_\beta}{\partial x_\alpha}\right] + \frac{\mu}{2}\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)\cdot\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)$$

$$+ v_\alpha\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)\frac{\partial \mu}{\partial x_\beta} - v_\beta \frac{\partial v_\beta}{\partial x_\alpha}\frac{\partial \mu}{\partial x_\alpha}.$$

$$= \frac{\mu}{3}\Delta^2 + \mu\frac{v_\alpha}{3}\frac{\partial}{\partial x_\alpha}(\Delta) + \mu\left[\frac{\partial v_\alpha}{\partial x_\beta}\cdot\frac{\partial v_\beta}{\partial x_\alpha} - \frac{\partial v_\alpha}{\partial x_\alpha}\cdot\frac{\partial v_\beta}{\partial x_\beta}\right]$$

$$+ \frac{\partial \mu}{\partial x_\beta}\left[-v_\beta\Delta + v_\alpha \frac{\partial v_\beta}{\partial x_\alpha}\right] \quad \ldots\ldots\ldots\ldots (13)$$

An examination of the orders of magnitude of the various quantities in the above expression for χ when the usual assumptions of boundary layer theory are accepted, will readily reveal that $\chi = O(\mu)$, at the most, i.e. $\chi = O(\delta^2)$ at the most and is therefore small.

6. Boundary layer energy equation

We can write (12) in the form

$$\rho \frac{D}{Dt}(J c_p T_H) = \frac{\partial}{\partial x_\alpha}\left\{\frac{k}{c_p}\frac{\partial}{\partial x_\alpha}\left[J c_p T_H + (\sigma-1)\frac{q^2}{2}\right]\right\} + \frac{\partial p}{\partial t} + \chi,$$

where $T_H = T + \frac{q^2}{2 J c_p}$ (the total temperature),

and $\sigma = \mu c_p/k$ (Prandtl Number).

Expressing each term as the sum of its mean and fluctuating components and taking means we get

$$\rho \frac{D}{Dt}(J c_p T_H) + \overline{\rho' \frac{\partial}{\partial t}(J c_p T_H')} + \overline{\rho v_\alpha' \frac{\partial}{\partial x_\alpha}(J c_p T_H')} + \overline{\rho' v_\alpha'}\frac{\partial}{\partial x_\alpha}(J c_p T_H)$$

$$+ V_\alpha \overline{\rho' \frac{\partial}{\partial x_\alpha}(J c_p T_H')} + \overline{\rho' v_\alpha' \frac{\partial}{\partial x_\alpha}(J c_p T_H')} = \frac{\partial}{\partial x_\alpha}\left\{\frac{k}{c_p}\frac{\partial}{\partial x_\alpha}\left[J c_p T_H + (\sigma-1)\frac{V_\beta^2}{2}\right]\right\}$$

$$+ \frac{\partial}{\partial x_\alpha}\overline{\left\{\frac{k'}{c_p}\frac{\partial}{\partial x_\alpha}\left(J c_p T_H' + (\sigma-1)v_\beta' V_\beta + (\sigma-1)\frac{v_\beta'^2}{2}\right)\right\}} + \frac{\partial p}{\partial t} + \chi + \chi'.$$

/Here ...

Here $J c_p T_H$ denotes the mean total energy per unit mass and includes the mean kinetic energy associated with the turbulent velocity fluctuations. The term χ' represents the non-vanishing components of χ arising from the fluctuations when the mean is taken. It has also beenassumed that fluctuations in viscosity and conductivity follow directly from the temperature fluctuations so that $\frac{\mu'}{\mu} = \frac{k'}{K}$, and $\frac{\mu' c_p}{k'} = \sigma$.

The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_\alpha}(\rho V_\alpha) + \frac{\partial}{\partial x_\alpha}(\rho' V_\alpha) + \frac{\partial}{\partial x_\alpha}(\rho v'_\alpha) + \frac{\partial}{\partial x_\alpha}(\rho' v'_\alpha) = 0$$

Hence $$\overline{T'_H \frac{\partial}{\partial x_\alpha}\left[\rho' V_\alpha + \rho v'_\alpha + \rho' v'_\alpha\right]} = -\overline{T'_H \frac{\partial \rho'}{\partial t}}.$$

The left hand side of the above energy equation can therefore be written

$$\rho \frac{D}{Dt}(J c_p T_H) + \frac{\partial}{\partial t}\left[\overline{\rho' J c_p T'_H}\right] + \frac{\partial}{\partial x_\alpha}\left\{\overline{T'_H\left(\rho v'_\alpha + \rho' V_\alpha + \rho' v'_\alpha\right)}\right\} + \overline{\rho' v'_\alpha}\frac{\partial}{\partial x_\alpha}(J c_p T_H).$$

Now, making the usual boundary layer assumptions and further assuming that χ (and presumably χ') involve small terms which can be neglected, we obtain

$$\rho \frac{D}{Dt}(J c_p T_H) = \frac{\partial}{\partial y}\left\{\frac{k}{c_p}\frac{\partial}{\partial y}\left[J c_p T_H + (\sigma - 1)\frac{U^2}{2}\right] + \overline{\frac{k'}{c_p}\frac{\partial}{\partial y}\left[J c_p T'_H + (\sigma-1) u'U\right]} - \rho \overline{v'(J c_p T'_H)}\right\} - \overline{\rho' v'}\frac{\partial}{\partial y}(J c_p T_H) + \frac{\partial p}{\partial t} - \frac{\partial}{\partial t}\left[\overline{\rho' J c_p T'_H}\right].$$

If we assume also that the term involving k' can be neglected compared with the term involving k, and that the mean motion is steady, then

$$\rho \frac{D}{Dt}(J c_p T_H) = \frac{\partial}{\partial y}\left\{\frac{k}{c_p}\frac{\partial}{\partial y}\left[J c_p T_H + (\sigma-1)\frac{U^2}{2}\right] - \rho \overline{v'(J c_p T'_H)}\right\} - \overline{\rho' v'}\frac{\partial}{\partial y}(J c_p T_H). \qquad \text{............(14)}$$

7. Some deductions

If $\sigma = 1$, equation (14) becomes

$$\rho \frac{D}{Dt} (J c_p T_H) = \frac{\partial}{\partial y} \left\{ \mu \frac{\partial}{\partial y} (J c_p T_H) - \rho \overline{v' (J c_p T_H')} \right\} - \overline{\rho' v'} \frac{\partial}{\partial y} (J c_p T_H) \quad \ldots\ldots\ldots\ldots (15)$$

For zero external pressure gradient, the equation of motion (9) becomes

$$\rho \frac{DU}{Dt} = \frac{\partial}{\partial y} \left\{ \mu \frac{\partial U}{\partial y} - \rho \overline{u'v'} \right\} - \overline{\rho' v'} \frac{\partial U}{\partial y} . \quad \ldots\ldots\ldots\ldots (16)$$

Comparing equations (15) and (16) we see that they are similar and since $T_H - T_{Hw}$ (suffix w denotes the value at the wall) and U have analogous boundary conditions they permit the solution

$$T_H - T_{Hw} = K_1 U, \; T_H' = K_1 u' , \quad \ldots\ldots\ldots\ldots (17)$$

where K_1 is a constant determined by the boundary values of T_H and U.

If the wall is insulated $\left(\frac{\partial T_H}{\partial y}\right)_w = 0$, and hence since $\left(\frac{\partial U}{\partial y}\right)_w \neq 0$ in general, $K_1 = 0$, it follows that T_H = const. = T_{Hw} through the boundary layer and $T_H' = 0$. We may conclude therefore that when $\sigma = 1$, and the external pressure gradient is zero the relation between total energy and velocity in the turbulent boundary layer is precisely the same in form as that in the laminar boundary layer. Further, when $\sigma = 1$, and there is no heat transfer at the wall a solution of (15) is T_H = const. = T_{Hw}, and $T_H' = 0$, whether or not there is an external pressure gradient.

8. Turbulent boundary layer on infinite flat plate in steady parallel flow

From (9) the effective stress τ satisfies

$$\frac{\partial \tau}{\partial y} = \frac{\partial}{\partial y} \left[\mu \frac{\partial U}{\partial y} - \rho \overline{u'v'} \right] - \overline{\rho' v'} \frac{\partial U}{\partial y} . \quad \ldots\ldots\ldots\ldots (18)$$

But τ = const. = τ_o in steady parallel flow.
The equation of continuity is

$$\frac{\partial}{\partial x} (\rho U) + \frac{\partial}{\partial y} (\rho V) + \frac{\partial}{\partial x} (\overline{\rho' u'}) + \frac{\partial}{\partial y} (\overline{\rho' v'}) = 0.$$

In parallel flow the rates of change of mean quantities with

/respect to ...

respect to x are zero, and $V = 0$, hence

$$\frac{\partial}{\partial y}(\overline{\rho' v'}) = 0$$

or $\overline{\rho' v'}$ = const. = 0, since $v' = 0$ at the wall.

Hence from (18) $\tau_o = \mu \frac{\partial U}{\partial y} - \rho \overline{u'v'}$

$= - \rho \overline{u'v'}$, except near the wall.

On the basis of a mixing length theory, as in incompressible flow, we can write

$$\overline{u'v'} = -\ell^2 \left(\frac{\partial U}{\partial y}\right)^2$$

Hence

$$U_\tau^2 = \frac{\tau_o}{\rho_1} = \frac{\rho}{\rho_1} \ell^2 \left(\frac{\partial U}{\partial y}\right)^2 .$$

But from the gas law for a perfect gas $\frac{\rho}{\rho_1} = \frac{T_1}{T}$, and if we assume $\sigma = 1$, we have

$$T_H = T_{Hw} = K_1 U, \text{ or}$$

$$J c_p T_H = K_1 U + J c_p T_{Hw} = K_1 U + K_2, \text{ say,}$$

or $$J c_p T = K_1 U - \frac{U^2}{2} + K_2$$

and $$J c_p T_1 = K_1 U_1 - \frac{U_1^2}{2} + K_2 .$$

$$\left.\begin{aligned} \text{Hence } \frac{T_1}{T} &= \frac{K_1 U - \frac{U_1^2}{2} + K_2}{K_1 U - \frac{U^2}{2} + K_2}, \\ \text{where } K_1 &= \left(J c_p T_1 + \frac{U_1^2}{2} - J c_p T_w\right) \Big/ U_1 \\ \text{and } K_2 &= J c_p T_w. \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (19)$$

Hence $$U_\tau = \ell \left(\frac{dU}{dy}\right) \left[\left(K_1 U_1 - \frac{U_1^2}{2} + K_2\right) \Big/ \left(K_1 U - \frac{U^2}{2} + K_2\right)\right]^{\frac{1}{2}} .$$

If we assume, as in incompressible flow, that

$$\ell = C_1 y, \text{ say,}$$

where C_1 = const. then

$$\frac{U_\tau}{C_1} \cdot \frac{dy}{y} = \left[\left(K_1 U_1 - \frac{U_1^2}{2} + K_2\right) \Big/ \left(K_1 U - \frac{U^2}{2} + K_2\right)\right]^{\frac{1}{2}} . \; dU$$

/or ...

or

$$A + \frac{U_\tau}{C_1}\log y = \int_0^U \left[\left(K_1 U_1 - \frac{U_1^2}{2} + K_2\right)\Big/\left(K_1 U - \frac{U^2}{2} + K_2\right)\right]^{\frac{1}{2}} . \, dU \quad \ldots\ldots (20)$$

where A is a const.

On integrating the right hand side of (20) and making use of (19), we eventually obtain

$$A + \frac{U_\tau}{C_1}\log y = \frac{U_1}{\alpha}\left\{\operatorname{Sin}^{-1}\left[\frac{\alpha\left(\frac{U}{U_1} - \frac{\beta}{2}\right)}{\left(\frac{T_w}{T_1} + \frac{\alpha^2\beta^2}{4}\right)^{\frac{1}{2}}}\right] - \operatorname{Sin}^{-1}\left[\frac{-\alpha\beta}{2\left(\frac{T_w}{T_1} + \frac{\alpha^2\beta^2}{4}\right)^{\frac{1}{2}}}\right]\right\} \quad (21)$$

where $\alpha^2 = \frac{(\gamma-1)}{2} M_1^2$, and $\beta = \left[1 + \alpha^2 - \frac{T_w}{T_1}\right]\Big/\alpha^2$.

Therefore, if we write δ for the thickness of the boundary layer so that when $y = \delta$, $U = U_1$, we obtain from (21)

$$\frac{U_\tau}{C_1}\log\frac{\delta}{y} = \frac{U_1}{\alpha}\left\{\operatorname{Sin}^{-1}\left[\frac{\alpha\left(1 - \frac{\beta}{2}\right)}{\left(\frac{T_w}{T_1} + \frac{\alpha^2\beta^2}{4}\right)^{\frac{1}{2}}}\right] - \operatorname{Sin}^{-1}\left[\frac{\alpha\left(u - \frac{\beta}{2}\right)}{\left(\frac{T_w}{T_1} + \frac{\alpha^2\beta^2}{4}\right)^{\frac{1}{2}}}\right]\right\} \quad \ldots\ldots (22)$$

where we have written u for U/U_1.

For incompressible flow and no heat transfer we have similarly

$$\frac{U_\tau}{C_2}\log\frac{\delta}{y} = U_1 \ (1-u) \quad \ldots\ldots\ldots\ldots (23)$$

where C_2 is the appropriate constant linking the mixing length ℓ and the distance from the wall.

Writing Y_c for y/δ in compressible flow, $U_{\tau c}$ for the frictional velocity in compressible flow, likewise Y_i and $U_{\tau i}$ for y/δ and the frictional velocity in incompressible flow, we obtain from (22) and (23) for <u>the same value of u</u>

$$Y_c = Y_i \ \lambda(u, T_w/T_1, M_1, C_1 U_{\tau i}/C_2 U_{\tau c}) ,$$

$$\text{where } \lambda = \frac{C_1 U_{\tau i}}{C_2 U_{\tau c}} \ \frac{1}{\alpha(1-u)}\left\{\operatorname{Sin}^{-1}\left[\frac{\alpha\left(1 - \frac{\beta}{2}\right)}{\left(\frac{T_w}{T_1} + \frac{\alpha^2\beta^2}{4}\right)^{\frac{1}{2}}}\right] - \operatorname{Sin}^{-1}\left[\frac{\alpha\left(u - \frac{\beta}{2}\right)}{\left(\frac{T_w}{T_1} + \frac{\alpha^2\beta^2}{4}\right)^{\frac{1}{2}}}\right]\right\} \quad (24)$$

/Hence ...

Hence, if we have the non-dimensional velocity profile for incompressible flow, i.e. u as a function of Y_i, which is of course a function of Reynolds number, we can determine the corresponding velocity profile for any specified Mach number and ratio of wall temperature to stream temperature if we can specify or determine the corresponding value of $\frac{U_{\tau i}}{C_2} \cdot \frac{C_1}{U_{\tau c}}$.

A preliminary investigation assuming the latter quantity was unity and that no heat was transferred at the wall showed that then λ was an increasing monotonic function of u rising to unity when u = 1.0 and decreasing slowly with increase of Mach number. For M = 2.5 it was found that λ varied from 0.75 when u = 0 to 1.0 when u = 1.0, and the resulting velocity profile was barely distinguishable from the incompressible velocity profile. For M = 5.0 the value of λ rose from 0.51 when u = 0 to 1.0 when u = 1.0 and the resulting velocity profile was then appreciably different from the incompressible velocity profile. These results are illustrated in Fig. 1. The calculations demonstrated that unless the ratio $\frac{C_1}{C_2}\frac{U_{\tau i}}{U_{\tau c}}$ was considerably less than unity the resulting profile given by equation (24) was unlikely to be markedly different for Mach numbers less than about 2.5 from that for incompressible flow.

In an attempt to allow for this ratio it was assumed that $C_2 = C_1$*. Further, the results obtained by Monoghan[3] suggested that a good approximation to the ratio of $U_{\tau i}/U_{\tau c}$ at a given Reynolds number is given by

$$\frac{U_{\tau i}}{U_{\tau c}} = \left(\frac{T_w}{T_1}\right)^{0.2} .$$

With these assumptions, the function λ becomes

$$\lambda = \left(\frac{T_w}{T_1}\right)^{0.2} \frac{1}{\alpha(1-u)} \left\{ \sin^{-1}\left[\frac{\alpha\left(1-\frac{\beta}{2}\right)}{\left(\frac{T_w}{T_1}+\frac{\alpha^2\beta^2}{4}\right)^{\frac{1}{2}}}\right] - \sin^{-1}\left[\frac{\alpha\left(u-\frac{\beta}{2}\right)}{\left(\frac{T_w}{T_1}+\frac{\alpha^2\beta^2}{4}\right)^{\frac{1}{2}}}\right] \right\} \qquad (25$$

Calculations of the resulting velocity profiles have been made for M_1 = 2.5 and 5.0 for zero heat transfer and for values of $\frac{T_w}{T_1}$ of 0.25 and 10.0, these latter values were chosen as representing somewhat extreme cases of heat transfer to and from the wall.

/The results ...

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* In incompressible flow this is Karman's constant.

The results are shown in Fig. 2. It will be seen that for $M_1 = 2.5$ the variations of the velocity profile from that for incompressible flow are small in all the cases considered except possibly in the extreme case when $T_w/T_1 = 10.0$. This agrees with the few available experimental results for velocity profile measurements in turbulent boundary layers at Mach numbers of the order of 2.5 and less under conditions approximating to zero heat transfer at the wall. These latter results show variations from incompressible flow profiles within the order of the experimental errors of measurement. For $M_1 = 5.0$, the calculations lead to variations from the incompressible velocity profile that are rather larger than for $M_1 = 2.5$, but again, except in the extreme case of $T_w/T_1 = 10.0$ they are by no means very marked.

Conclusions

It is concluded that for Mach numbers of the order of 2.5 or less and for a wide range of heat transfer conditions the velocity profile in the turbulent boundary will differ very little from that for an incompressible fluid and the same Reynolds number. For higher Mach numbers, however, small differences will become apparent particularly for case of considerable heat transfer from the surface to the fluid.

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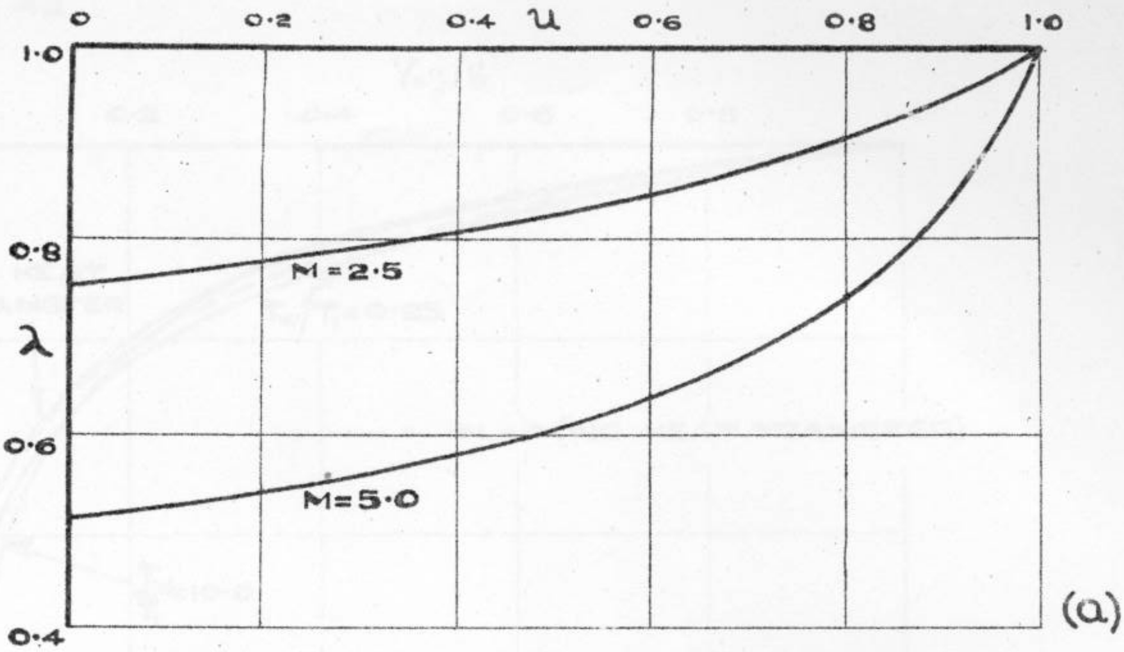
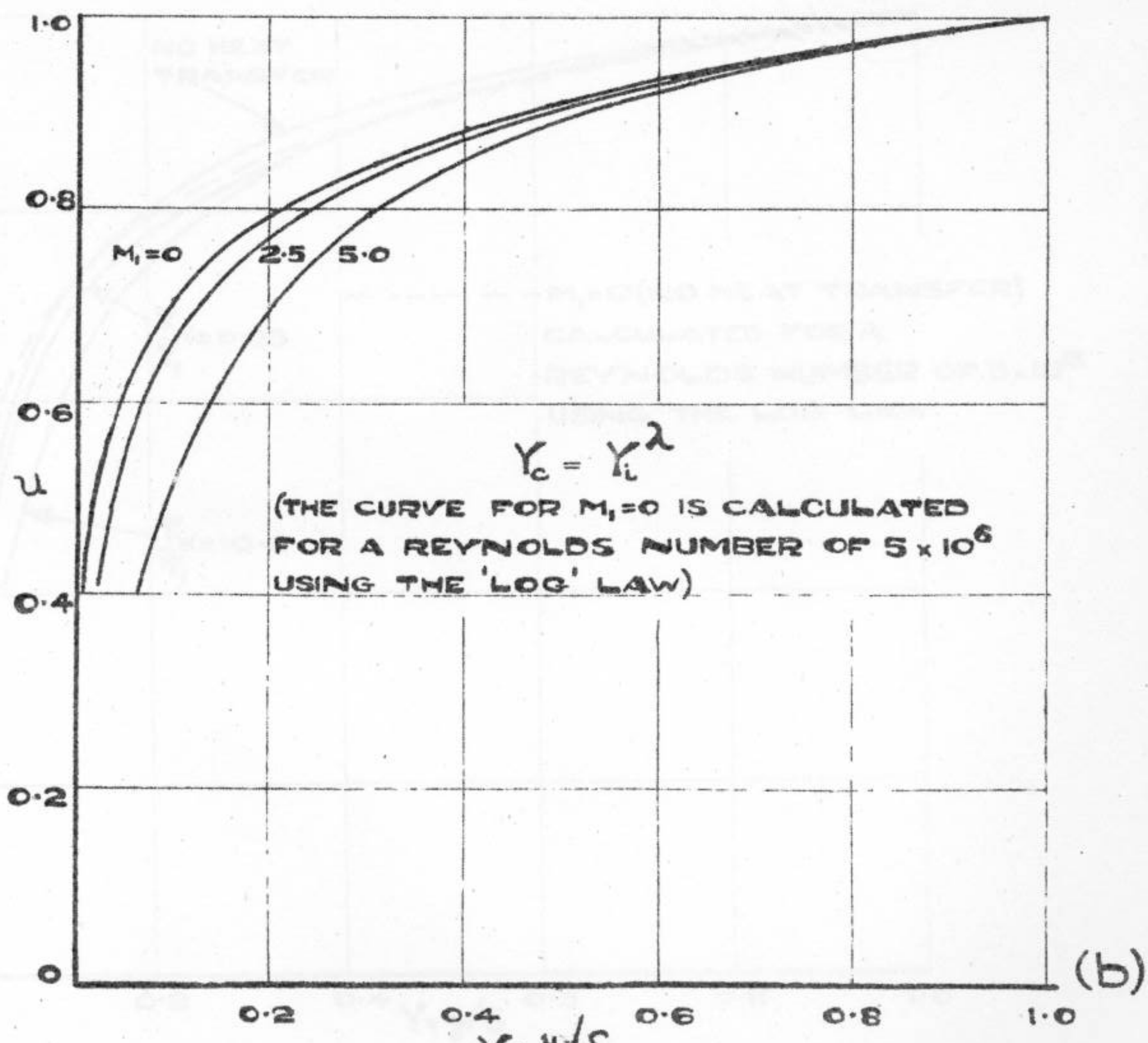
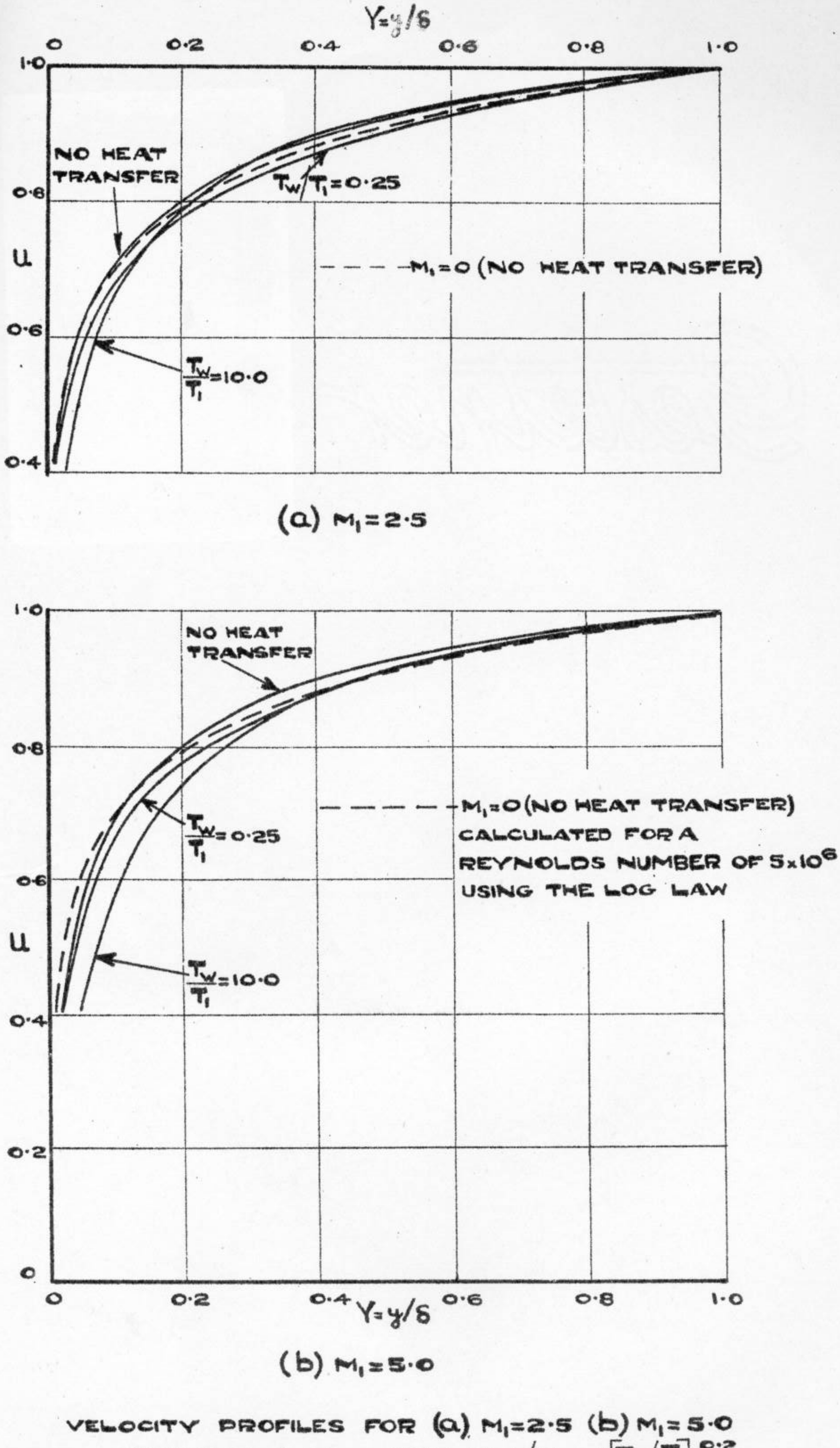


VARIATION OF λ WITH u ASSUMING $C_1 U_{Ti} / C_2 U_{Tc} = 1{\cdot}0$
AND NO HEAT TRANSFER



VELOCITY PROFILES FOR $M_1 = 0$, 2·5 & 5·0
ASSUMING $C_1 U_{Ti} / C_2 U_{Tc} = 1{\cdot}0$ AND NO HEAT TRANSFER

FIG 1



VELOCITY PROFILES FOR (a) M_1=2·5 (b) M_1=5·0
ASSUMING $C_1=C_2$, AND $U_{\tau i}/U_{\tau c}=[T_w/T_1]^{0.2}$

FIG 2