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Nose Controls on Delta Wings at Supersonic Speeds

-by-

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SUMMARY

Expressions are derived for l_ξ and a_2 of nose ailerons and nose elevators on a delta wing, as depicted in Fig. 1, in supersonic flight. Nose and trailing edge controls on delta wings in supersonic flight are compared.

Conclusions

On delta wings of moderate aspect ratio (say >4) nose controls are comparable with trailing edge controls. Nose controls are ineffective on delta wings of very small aspect ratio (say $\leqslant 1$).

For the same effects, the controls are deflected upwards when trailing edge controls would be deflected downwards and vice versa.

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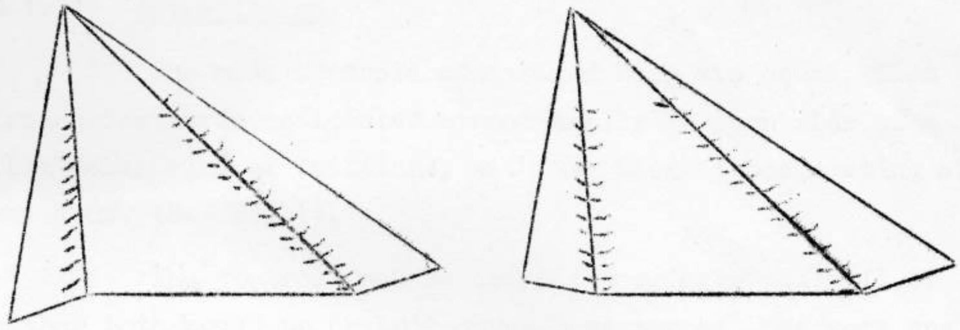
(Thesis presented for the College Diploma, June, 1949)

MEP



NOSE AILERON NOSE ELEVATOR

FIG. 1.

LIST OF CONTENTS

§ 1. Introduction

The nose controls considered here are equal, flat triangular surfaces located symmetrically on each side of a flat delta wing or tailplane, with the hinge lines meeting at the apex. (See Fig.1).

The controls may be deflected symmetrically (i.e. either both moved up or both moved down through the same angle) to produce a lift force. The controls then act as elevators. Alternatively, the controls may be deflected anti-symmetrically (i.e. one moved up and the other moved down through the same angle) to produce a rolling moment, the controls then acting as ailerons.

In § 5 the lift force and rolling moment are calculated on the assumptions of linearised theory. These results yield expressions for (i) ℓ_ξ for nose ailerons and (ii) a_2 for nose elevators.

Two kinds of supersonic flow over the wing or tailplane are possible, depending on the Mach number (M) and the apex angle (2γ). They are:

(i) A flow in which the leading edges lie outside the Mach cone of the apex*. This type of flow occurs at higher speeds, corresponding to the analytic condition $M \geqslant \operatorname{cosec} \gamma$.

(ii) A flow in which the leading edges lie inside the Mach cone. This type of flow occurs at lower speeds, i.e. when $M \leqslant \operatorname{cosec} \gamma$.

Physically, these flows are different - in the first flow the pressure distribution on either the upper or the lower surface is unaffected by the shape of the other surface, while in the second flow the pressure on either surface is affected by the shape of both upper and lower surfaces. (This follows from a property of supersonic flow, viz. that a small disturbance at a point in the field can only be communicated to the region within the Mach cone of that point).

/ In the ...

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* Hereafter, the Mach cone of the apex will be referred to simply as 'the Mach cone'.

§ 1. (Contd.)

In the Analysis these two kinds of flow are treated separately and yield different formulae. Flow (i) can be further subdivided into two cases in which the hinge lines lie (a) outside and (b) inside the Mach cone. This distinction is important when performing certain of the integrations, but the method of solution is fundamentally the same in both cases and the formulae that are derived for ℓ_ξ and a_2 are the same.

§ 2. NOTATION

a	speed of sound
a_1	lift slope of delta tailplane
a_2	rate of change of lift coefficient of delta tailplane with elevator angle $= \frac{\partial L}{\partial \eta} \Big/ \frac{1}{2} \rho U_o^2 S$
A	aspect ratio of delta wing or tailplane $= 4 \tan \gamma$
b	overall span of delta wing or tailplane
$B =$	$\sqrt{M^2-1} \tan \gamma$
c	maximum chord of delta wing or tailplane
C_L	lift coefficient
$C_{\bar{L}}$	rolling moment coefficient
$E(u)$	complete elliptic integral of the second kind $= \int_0^{\frac{\pi}{2}} \left[1 - u^2 \sin^2 \phi\right]^{\frac{1}{2}} d\phi$
$E'(u)$	complementary complete elliptic integral of the second kind $= \int_0^{\frac{\pi}{2}} \left[1 - (1-u^2) \sin^2 \phi\right]^{\frac{1}{2}} d\phi$
$E' =$	$E'(B)$
$k_1 =$	$\cot \gamma$
$k_2 =$	$\cot \Theta$
$K(u)$	complete elliptic integral of the first kind $= \int_0^{\frac{\pi}{2}} \left[1 - u^2 \sin^2 \phi\right]^{-\frac{1}{2}} d\phi$
$K'(u)$	complementary complete elliptic integral of the first kind $= \int_0^{\frac{\pi}{2}} \left[1 - (1-u^2) \sin^2 \phi\right]^{-\frac{1}{2}} d\phi$
ℓ_p	non-dimensional derivative of rolling moment with rate of roll $= \frac{\partial \bar{L}}{\partial p} \Big/ \frac{1}{4} \rho U_o S b^2$
ℓ_ξ	non-dimensional derivative of rolling moment with aileron angle $= \frac{\partial \bar{L}}{\partial \xi} \Big/ \frac{1}{2} \rho U_o^2 S b$
L	lift
$\bar{L}$	rolling moment (+ ve when starboard tip tends to dip)
M	Mach number
n_1	k_1/β
n_2	k_2/β
p	rise of pressure above pressure at infinity
p	rate of roll
$r =$	$\tan \Theta / \tan \gamma = 1 - \frac{S_c}{S}$
S	area of delta wing or tailplane $= c^2 \tan \gamma$
S_c	sum of areas of port and starboard controls $= c^2 (\tan \gamma - \tan \Theta)$

/ t = ...

§ 2. (Contd.)

$t = \dfrac{k_1 y}{x}$	
$\bar{t} = \dfrac{k_2 y}{x}$	
u	induced component of velocity in x direction
U_o	free stream velocity
v	induced component of velocity in y direction
w	induced component of velocity in z direction
x	chord-wise coordinate (measured from the apex in the direction of flow)
y	spanwise coordinate (+ ve to starboard)
z	normal coordinate (+ ve above wing)
α	incidence of wing (or tailplane)
$\beta = \sqrt{M^2 - 1}$	
γ	apex semi-angle
η	elevator deflection (+ ve when an elevator is deflected up)
θ	control deflection (+ ve when starboard control is deflected up)
Θ	semi-angle included between control hinge lines
λ	slope of wing or tailplane surface in x direction
$\mu =$	y/x
$\Pi(n,u)$	complete elliptic integral of the third kind

$$= \int_0^{\frac{\pi}{2}} (1 + n \sin^2 \phi)^{-1} \left[1 - u^2 \sin^2 \phi\right]^{-\frac{1}{2}} d\phi$$

$$\Pi = \Pi\left(B^2 r^2 - 1, \sqrt{1 - B^2}\right) = K'(B) + \frac{1}{B^2 r} \sqrt{\frac{1 - B^2 r^2}{1 - r^2}}$$

$$\left\{\frac{\pi}{2} + \left[K'(B) - E'(B)\right] sn^{-1}(r,B) - K'(B).E(\sin^{-1} r, B)\right\}$$

ρ	air density
ξ	aileron deflection (+ ve when starboard aileron is deflected up)
ϕ	induced velocity potential

§ 3. RESULTS

(See § 2. for explanation of symbols).

Nose Ailerons

	Leading Edges Outside Mach Cone ($B \geq 1$; i.e. $M \geq \operatorname{cosec} \gamma$)	Leading Edges Inside Mach Cone ($B \leq 1$; i.e. $M \leq \operatorname{cosec} \gamma$)
ℓ_ξ	$-\frac{2}{3} \frac{(1-r^2)}{B} \sin \Theta \tan \gamma$	$-\frac{2}{3} \frac{(1-r^2)^{\frac{3}{2}}}{\sqrt{1-B^2r^2}} \sin \Theta \tan \gamma$

Nose Elevators

	Leading Edges Outside Mach Cone ($B \geq 1$; i.e. $M \geq \operatorname{cosec} \gamma$)	Leading Edges Inside Mach Cone ($B \leq 1$; i.e. $M \leq \operatorname{cosec} \gamma$)
a_2	$\frac{4(1-r)}{B} \sin \Theta \tan \gamma$	B≠0: $4r\left(\frac{B^2}{E'} \Pi - 1\right)\sqrt{\frac{(1-r^2)}{(1-B^2r^2)}} \sin \Theta \tan \gamma$ B=0 (i.e. M=1): $4\left(\cos^{-1} r - r\sqrt{1-r^2}\right) \sin \Theta \tan \gamma$
Position of Centre of Pressure	On centre line, $\frac{2}{3}$ c from apex	On centre line, $\frac{2}{3}$ c from apex

§ 4. Discussion of Results

4.1. General Remarks and Conclusions

At supersonic speeds, nose controls are unsuitable for delta wings or tailplanes of very low aspect ratio (say <1). This is because the slope λ of a deflected nose control surface in the direction of the main stream is proportional to $\sin \Theta$ (where 2Θ is the angle between the control hinge lines.) The lift force or rolling moment produced, being proportional to λ, is then proportional to $\sin \Theta$. At very small aspect ratios Θ is also very small and the controls are therefore relatively ineffective.

It is possible that in a viscous fluid nose controls may have some advantages over trailing edge controls, such as greater maximum aileron (or elevator) power. This, however, remains to be investigated.

At moderate aspect ratios (say >4) the effectiveness at supersonic speeds of nose controls (as measured by ℓ_ξ and a_2) is comparable with, although less than, the effectiveness of trailing edge controls.

It should be noted that nose controls must be deflected up instead of down and down instead of up in order to produce the same effects as conventional (i.e. trailing edge) controls.

4.2. Nose Ailerons

In Fig.11, $\dfrac{-\ell_\xi}{\sin \Theta \tan \gamma}$ is plotted against B ($=\sqrt{M^2-1}\ \tan \gamma$) for several values of r ($= \tan \Theta/\tan \gamma$). On all the curves of constant r, $-\ell_\xi$ is a maximum at B=1, i.e. when M = cosec γ, i.e. when the Mach cone just touches the leading edges. For a given wing (i.e. γ and Θ, and therefore r, given) the curves show the variation of ℓ_ξ with $\sqrt{M^2-1}$.

Curves of $-\ell_\xi$ against aileron area for several Mach numbers between 1 and 3 are plotted for aspect ratios of 2·3, 4 and 6.9 in. Figs. 12 and 13.

In practice $\frac{S_c}{S}$ would probably not exceed 0.3. With this limitation, it will be seen that except for wings of higher aspect ratios at Mach numbers near 1, $-\ell_\xi$ is considerably less than for conventional ailerons in incompressible flow.

/Rate of Roll.

Rate of Roll

It is readily shown that the steady rate of roll p of a wing is given by:

$$p = \frac{2a\,\xi M}{b} \frac{\ell_\xi}{\ell_p}$$

In ref. 1 it is shown in Fig. 2 that $-\ell_p$ decreases with M. If $M < \operatorname{cosec}\gamma$, $-\ell_\xi$ increases with M. (See Fig.11). Hence by the above equation p increases with M. If $M > \operatorname{cosec}\gamma$ it is proved in ref. 1 that ℓ_p varies as $(M^2-1)^{-\frac{1}{2}}$ and it is proved in this report that ℓ_ξ also varies as $(M^2-1)^{-\frac{1}{2}}$. Hence p varies as M and increases with M. Thus at all supersonic speeds the steady rate of roll produced by the ailerons increases with increase of speed, and is directly proportional to speed when $M > \operatorname{cosec}\gamma$.

Comparison of Nose and Trailing Edge Ailerons

Using the approximate formula for trailing edge ailerons derived in Appendix V, a comparison between the effectiveness of nose and trailing edge ailerons is made in Table 1, on the basis that the speed and the ratio, control area/wing area, are the same in both cases. From this table it appears that with moderate aspect ratios (4 to 7) nose elevators are, very approximately, two thirds as effective as trailing edge ailerons at supersonic speeds, the discrepancy increasing as aspect ratio decreases.

CONDITION	$\frac{(\ell_\xi)_{nose}}{(\ell_\xi)_{T/E}}$
$A = 6.9$, $\frac{S_c}{S} = 0.2$	0.73
$A = 4$, $\frac{S_c}{S} = 0.2$	0.56

TABLE 1.

COMPARISON OF NOSE AILERONS WITH TRAILING EDGE AILERONS

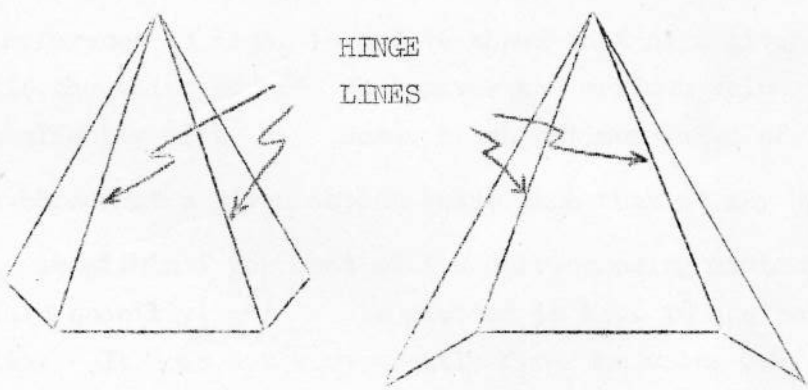
/ Note ...

Note. These figures are based on the assumptions that the leading edges of the wing lie outside the Mach cone, that the aspect ratio of the trailing edge ailerons is large compared with $\frac{1}{\beta}$ and that $\frac{b_o}{b} = \frac{2}{3}$. (See Appendix V).

4.3. Nose Elevators

It is shown in 5.123 and 5.22 that the force produced by nose elevator deflections acts always on the centre-line, at two thirds of the maximum chord from the apex. This point is also the centre of pressure of a delta wing, so that nose elevators fitted to a delta wing cannot trim the wing, i.e. cannot act as elevators.

It would be possible, however, to trim a wing by means of nose controls fitted to it provided the wing plan form is similar to either of the two types shown below:



With either of these plan forms, the centre of pressure of the force produced by control deflections would differ from the centre of pressure of the wing at incidence. Deltas of this type, with bent trailing edges, are not dealt with in this report, however.

The analysis of nose elevators of this report is applicable to delta tailplanes on supersonic aircraft. The remarks in the remainder of 4.3 refer to such a tailplane.

In Fig. 14, $\frac{a_2}{\sin \Theta \tan \gamma}$ is plotted against B for several values of r. For a given tailplane these curves show the variation of a_2 with $\sqrt{M^2-1}$.

Curves of a_2 against elevator area for four Mach numbers between 1 and 3 are plotted for aspect ratios of 2.3, 4 and 6.9 in Figs. 15 and 16. On all these curves a_2 rises to a maximum value, $a_{2\ max.}$, at a certain value of

$/ \frac{S_c}{S}$ ...

$\frac{S_c}{S}$, usually about 0.6.

The quantity $\frac{a_{2\,max}}{a_1}$ is plotted against aspect ratio in Fig. 17, where a_1 is the lift slope of the delta tailplane. In general $\frac{a_{2\,max}}{a_1}$ is a function of both M and A, but when $M > \operatorname{cosec}\gamma$, $a_{2\,max}$ and a_1 both vary as $(M^2-1)^{-\frac{1}{2}}$, and $\frac{a_{2\,max}}{a_1}$ is thus a function only of A. Only two curves, viz. those for M = 1 and $M \geqslant \operatorname{cosec}\gamma$, are therefore shown in Fig. 17. At Mach numbers between 1 and cosec γ the value of $\frac{a_{2\,max}}{a_1}$ lies between the values of $\frac{a_{2\,max}}{a_1}$ at M = 1 and at M = cosec γ, and may be found approximately by interpolation between the two curves. It will be seen that the values of $\frac{a_{2\,max}}{a_1}$ are less than conventional values of $\frac{a_2}{a_1}$ for trailing edge elevators in low-speed flow, particularly at small aspect ratios.

Reference to Figs. 15 and 16 shows that at a given aspect ratio the value of $\frac{S_c}{S}$ that gives the maximum value of a_2 varies slightly with M. However, an optimum value of $\frac{S_c}{S}$ may be chosen at a given aspect ratio such that at any Mach number a_2 is within 1 per cent of its corresponding maximum value. This quantity $\left(\frac{S_c}{S}\right)_{OPT}$ is plotted in Fig. 18 against aspect ratio. It does not vary greatly from the value 0.6.

Comparison of Nose and Trailing Edge Elevators

Using the approximate formula for trailing edge elevators derived in Appendix V, a comparison between the effectiveness of nose and trailing edge elevators is made in Table 2, on the basis that speed and the ratio control area/ tailplane area are the same in both cases. From this table it appears that with moderate aspect ratios (4 to 7) nose elevators are, very approximately, half as effective as trailing edge elevators at supersonic speeds, the discrepancy increasing as aspect ratio decreases.

CONDITION	$\frac{(a_2)_{nose}}{(a_2)_{T/E}}$
$A = 6.9$, $\frac{S_c}{S} = 0.5$	0.65
$A = 4$, $\frac{S_c}{S} = 0.5$	0.45

TABLE 2.

COMPARISON OF NOSE ELEVATORS WITH TRAILING EDGE ELEVATORS

Note. These figures are based on the assumptions that the leading edges of the tailplane lie outside the Mach cone and that the aspect ratio of the trailing edge elevators is large compared with $\frac{1}{\beta}$.

§ 5. Analysis

As stated in the Introduction there are two different conditions of flow to consider, viz. (i) the leading edges lying outside the Mach cone and (ii) the leading edges lying inside the Mach cone.

5.1. Pressure Distributions with Leading Edges Lying Outside Mach Cone

With the assumption of small perturbation of flow, the equation giving the induced velocity potential ϕ in three dimensional, inviscid, isentropic, steady flow past a body is:

$$-\beta^2 \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \text{...........(1)}$$

Consider $$\phi = \frac{-q}{\sqrt{(x-\xi)^2-\beta^2\left\{(y-\eta)^2+z^2\right\}}} \qquad \text{...........(2)}$$

It is readily verified that ϕ as given by equation (2) satisfies equation (1). It may be shown that equation (2) gives the velocity potential of a supersonic source of strength q at $(\xi,\eta,0)$.

/ By ...

By superposition,

$$\phi = -\iint \frac{q\ (\xi,\eta)\ d\xi d\eta}{\sqrt{(x-\xi)^2-\beta^2\left[(y-\eta)^2+z^2\right]}} \qquad \ldots\ldots\ldots\ldots(3)$$

is also a solution of equation (1). Equation (3) gives the velocity potential of a continuous distribution of elementary sources $q\ (\xi,\eta)d\xi d\eta$. We shall investigate whether with correct choice of the distribution function q, equation (3) will give the flow past the delta wing.

From equation (3), the normal velocity w is given by:

$$w = \frac{\partial\phi}{\partial z} = -\beta^2 z\iint\frac{q\ (\xi,\eta)\ d\xi d\eta}{\left[(x-\xi)^2-\beta^2\left\{(y-\eta)^2+z^2\right\}\right]^{\frac{3}{2}}}$$

$$\therefore (w)_{z=-o} = -(w)_{z=+o} \qquad \ldots\ldots\ldots\ldots(4)$$

To the accuracy of the linearised theory it is correct to assume that the velocity at a point on the wing is the velocity at the projection of that point on the plane $z = o$. Therefore from equation (4),

$$(w)_{L/S} = -(w)_{U/S}\ , \qquad \ldots\ldots\ldots\ldots(5)$$

(where the suffices 'L/S' and 'U/S' refer to the lower and upper surfaces of the wing respectively).

Actually the condition represented by equation (5) is not satisfied in our problem since

$$(w)_{L/S} = U_o\lambda = (w)_{U/S}\ ,$$

(where λ is the slope of the surface in the x direction at any point of the surface.) However, the flow above the wing is independent of the flow below it, because the leading edges lie outside the Mach cone. We are therefore justified, when confining our attention to one surface, in assuming that equation (5) is satisfied. Equation (3) therefore gives the velocity potential correctly when considering one surface.

The result is proved in ref.2, equation (45), that:

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=+o} - \left(\frac{\partial\phi}{\partial z}\right)_{z=-o} = 2\pi q$$

Since $\frac{\partial\phi}{\partial z} = w$, this becomes:

/ (w) ...

$$(w)_{z=+o} - (w)_{z=-o} = 2\pi q .$$

With equation (4), this gives:

$$q = \frac{(w)_{z=+o}}{\pi} ,$$

or if w_s denote the component of velocity in the z-direction at the upper surface,

$$q = \frac{w_s}{\pi} .$$

Hinge Lines Outside the Mach Cone

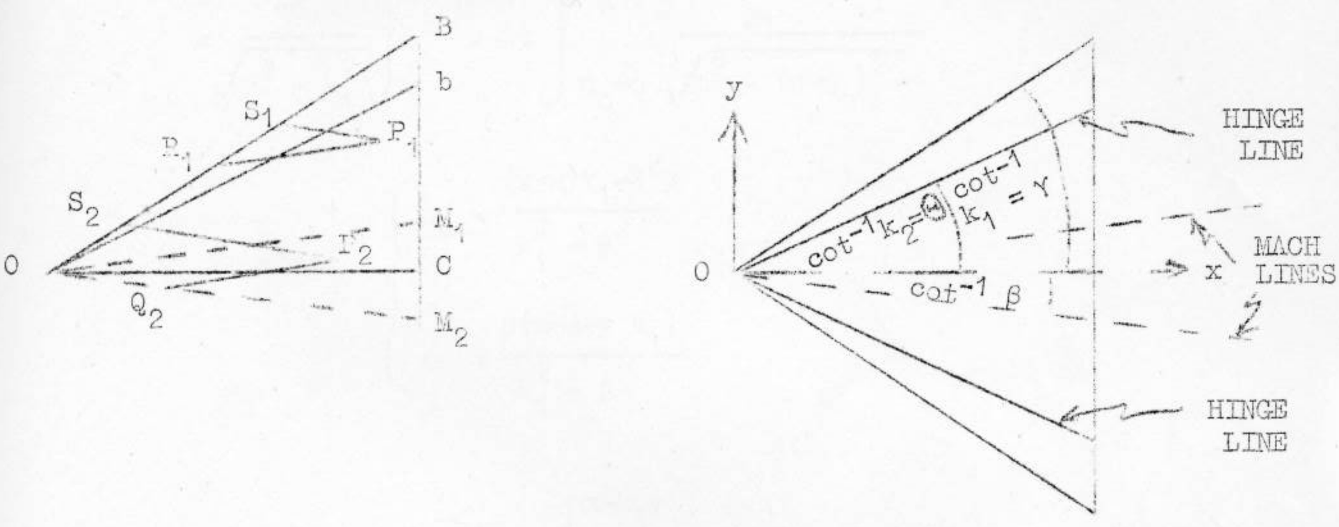


FIG. 2

It is sufficient to consider an upward deflection θ of the starboard control only, since the effect of deflecting the port control as well may be found by superposition. w_s is then zero everywhere on the wing except on the control surface, where $w_s = - U_o\, \theta \sin \Theta$. There is thus a uniform source distribution $(- \frac{1}{\pi} U_o\, \theta \sin \Theta)$ over bOB. (See Fig. 2). This is equivalent to two uniform source distributions:

(i) $q_1 = - \dfrac{U_o\, \theta \sin \Theta}{\pi}$ over OBM_2, and

(ii) $q_2 = + \dfrac{U_o\, \theta \sin \Theta}{\pi}$ over ObM_2.

It should be noted that the effects of deflecting control bOB are confined to the region M_2OB of the wing.

Let P_1 (x, y) be a point on the upper surface in the region BOM_1. (See Fig. 2).

Then due to q_1, the potential ϕ at P_1 (x, y) is

/given ...

given by:

$$\phi = -q_1 \iint \frac{d\xi\, d\eta}{\sqrt{(x-\xi)^2 - \beta^2 (y-\eta)^2}} \quad ,$$

the integration extending over the region $R_1\ P_1\ S_1$, where P_1R_1 and $P_1\ S_1$, are Mach lines through P.

Let $s = \xi - k_1\eta$

$$\therefore \phi = -q_1 \int_0^{x-k_1 y} ds \int_{\eta_{R_1P_1}}^{\eta_{S_1P_1}} \frac{d\eta}{\sqrt{(x-s-k_1\eta)^2 - \beta^2 (y-\eta)^2}}$$

$$= \frac{-q_1}{\sqrt{\beta^2 - k_1^2}} \int_0^{x-k_1 y} ds \int_{\eta_o - c}^{\eta_o + c} \frac{d\eta}{\sqrt{c^2 - (\eta - \eta_o)^2}}$$

$$\text{where} \begin{cases} \eta_o = \dfrac{(x-s)k_1 - \beta^2 y}{k_1^2 - \beta^2} \\[2ex] c = \dfrac{\beta(x-s-y\ k_1)}{k_1^2 - \beta^2} \end{cases}$$

$$\therefore \phi = \frac{-q_1}{\sqrt{\beta^2 - k_1^2}} \int_0^{x-k_1 y} \pi\ ds$$

$$\text{i.e. } \phi = \frac{-\pi q_1}{\sqrt{\beta^2 - k_1^2}} \quad (x - k_1 y)$$

$$\therefore u = \frac{\partial \phi}{\partial x} = \frac{-\pi q_1}{\sqrt{\beta^2 - k_1^2}} = \frac{-\pi q_1}{\beta\sqrt{1-n_1^2}} \ , \quad \text{where} \quad n_1 = \frac{k_1}{\beta}$$

In linearised theory the pressure is given by:

$$p = -\rho\ U_o\ u.$$

Hence at points such as P_1, the upper surface pressure due to q_1 is given by:

$$p = \frac{\pi\ \rho\ U_o q_1}{\beta\sqrt{1-n_1^2}} \qquad \text{...........(6,1)}$$

/Similarly ...

Similarly at points within bOM_1, the upper surface pressure due to q_2 is given by:

$$p = \frac{\pi \rho U_o q_2}{\beta\sqrt{1-n_2^2}} \qquad \text{..........(6.2)},$$

(where $n_2 = \frac{k_2}{\beta}$).

Let P_2 be a point on the upper surface in the region $M_1 O M_2$.

Due to q_1,

$$\phi = - q_1 \int\int \frac{d\xi d\eta}{\sqrt{(x-\xi)^2-\beta^2(y-\eta)^2}}$$, the integration extend-

over $P_2 Q_2 O S_2$.

Let $s = \xi + \beta\eta$.

$$\therefore \phi = - q_1 \int_0^{x+\beta y} ds \int_{\frac{s+\beta y-x}{2\beta}}^{\frac{s}{k_1+\beta}} \frac{d\eta}{\sqrt{(x-s+\beta\eta)^2-\beta^2(y-\eta)^2}}$$

$$= - q_1 \int_0^{x+\beta y} ds \int_{\frac{s+\beta y-x}{2\beta}}^{\frac{s}{k_1+\beta}} \frac{d\eta}{\sqrt{2\beta(x-s+\beta y)\eta+(x-s)^2-\beta^2y^2}}$$

$$= - q_1 \int_0^{x+\beta y} \left[\frac{\sqrt{2\beta(x-s+\beta y)\eta+(x-s)^2-\beta^2y^2}}{\beta\ (x-s+\beta y)}\right]_{\eta=\frac{s+\beta y-x}{2\beta}}^{\eta=\frac{s}{k_1+\beta}} ds$$

Let $\ell = x-s+\beta y$.

$$\therefore \phi = - q_1 \int_0^{x+\beta y} \left[\frac{\sqrt{2\beta\ell\eta+(\ell-\beta y)^2-\beta^2y^2}}{\beta\ell}\right]_{\eta=\frac{2\beta y-\ell}{2\beta}}^{\eta=\frac{x+\beta y-\ell}{k_1+\beta}} d\ell$$

$$= - q_1 \int_0^{x+\beta y} \frac{1}{\beta\ell}\sqrt{\frac{(k_1-\beta)\ell^2-2\beta\ell(k_1y-x)}{k_1+\beta}}\ d\ell$$

$$/ = \frac{-q_1}{\beta} \quad .....$$

$$= \frac{-q_1}{\beta}\sqrt{\frac{\beta-k_1}{\beta+k_1}}\int_o^{x+\beta y}\frac{1}{\ell}\sqrt{\frac{2\beta(x-k_1y)}{\beta-k_1}\ell-\ell^2}\;d\ell$$

Let $\ell = n^2 . \dfrac{2\beta(x-k_1y)}{\beta-k_1}$. $\therefore \dfrac{d\ell}{\ell} = 2\dfrac{dm}{m}$

$$\therefore \emptyset = -\frac{4q_1(x-k_1y)}{\sqrt{\beta^2-k_1^2}}\int_o^{\sqrt{\frac{(\beta-k_1)(x+\beta y)}{2\beta(x-k_1y)}}}\sqrt{1-m^2}\;dm$$

$$= -\frac{2q_1(x-k_1y)}{\sqrt{\beta^2-k_1^2}}\left\{\frac{\sqrt{(\beta^2-k_1^2)(x^2-\beta^2y^2)}}{2\beta(x-k_1y)} + \sin^{-1}\sqrt{\frac{(\beta-k_1)(x+\beta y)}{2\beta(x-k_1y)}}\right\}$$

i.e. $$\emptyset = -q_1\left\{\frac{2(x-k_1y)}{\sqrt{\beta^2-k_1^2}}\sin^{-1}\sqrt{\frac{(\beta-k_1)(x+\beta y)}{2\beta(x-k_1y)}} + \frac{1}{\beta}\sqrt{x^2-\beta^2y^2}\right\}$$

$$\therefore \frac{\partial\emptyset}{\partial x} = -q_1\left\{\frac{2}{\sqrt{\beta^2-k_1^2}}\sin^{-1}\sqrt{\frac{(\beta-k_1)(x+\beta y)}{2\beta(x-k_1y)}} + \frac{1}{\beta}\sqrt{\frac{x-\beta y}{x+\beta y}}\right\}$$

$$\therefore p = \rho U q_1\left\{\frac{2}{\sqrt{\beta^2-k_1^2}}\sin^{-1}\sqrt{\frac{(\beta-k_1)(x+\beta y)}{2\beta(x-k_1y)}} + \frac{1}{\beta}\sqrt{\frac{x-\beta y}{x+\beta y}}\right\}$$

Let $t = \dfrac{k_1y}{x}$

$$\therefore p = \frac{\rho U_o q_1}{\beta}\left\{\frac{2}{\sqrt{1-n_1^2}}\sin^{-1}\sqrt{\frac{(1-n_1)(n_1+t)}{2n_1(1-t)}} + \sqrt{\frac{n_1-t}{n_1+t}}\right\}$$

$$\ldots\ldots\ldots\ldots (7.1)$$

at points such as P_2, due to q_1.

Similarly, at points such as P_2, the pressure due to q_2 is given by:

$$p = \frac{\rho U_o q_2}{\beta}\left\{\frac{2}{\sqrt{1-n_2^2}}\sin^{-1}\sqrt{\frac{(1-n_2)(n_2+\bar{t})}{2n_2(1-\bar{t})}} + \sqrt{\frac{n_2-\bar{t}}{n_2+\bar{t}}}\right\} \quad (7.2)$$

(where $\bar{t} = \dfrac{k_2y}{x}$).

Hinge Lines Inside Mach Cone

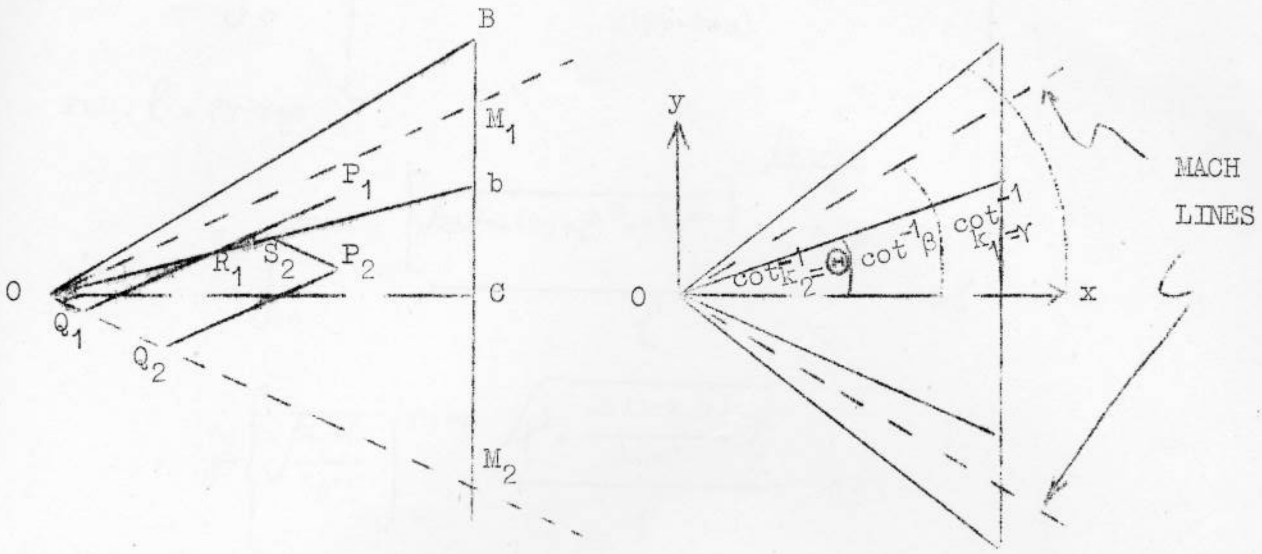


FIG. 3

Due to the source distribution q_1 over OBM_2 the pressure is given still by equations (7.1) and (6.1), but the equations for pressure due to q_2 are different.

The effects of q_2 are confined to the triangle M_1OM_2.

Let P_1 be a point on the upper surface in the region bOM_1. (See Fig.3).

Due to q_2, the potential ϕ at P_1 is given by:

$$\phi = - q_2 \iint \frac{d\xi\, d\eta}{\sqrt{(x-\xi)^2-\beta^2(y-\eta)^2}} \quad ,$$

the integration extending over the region $OR_1\, Q_1$.

Put $s = \xi - \beta y$.

$$\therefore\ \phi = - q_2 \int_{o}^{x-\beta y} ds \int_{\frac{-s}{2\beta}}^{\frac{s}{k_2-\beta}} \frac{d\eta}{\sqrt{(x-s-\beta\eta)^2-\beta^2(y-\eta)^2}}$$

$$= - q_2 \int_{o}^{x-\beta y} ds \int_{\frac{-s}{2\beta}}^{\frac{s}{k_2-\beta}} \frac{d\eta}{\sqrt{2\beta(\beta y-x+s)\eta+(x-s)^2-\beta^2 y^2}}$$

$$= - q_2 \int_0^{x-\beta y} \left\{ \frac{\left[\left\{ 2\beta(\beta y-x+s)\eta + (x-s)^2 - \beta^2 y^2 \right\}^{\frac{1}{2}} \right]_{\eta=\frac{-s}{2\beta}}^{\eta=\frac{s}{k_2-\beta}}}{\beta(\beta y-x+s)} \right\} ds$$

Put $\ell = \beta y-x+s$

$$\therefore \phi = \frac{q_2}{\beta} \int_0^{\beta y-x} \frac{\left[\sqrt{2\beta\ell\eta + (\beta y-\ell)^2 - \beta^2 y^2} \right]_{\eta=\frac{\beta y-x-\ell}{2\beta}}^{\eta=\frac{\ell+x-\beta y}{k_2-\beta}}}{\ell} d\ell$$

$$= \frac{q_2}{\beta} \left\{ \sqrt{\frac{k_2+\beta}{k_2-\beta}} \int_0^{\beta y-x} \frac{\sqrt{\ell^2 + \frac{2\beta(x-k_2 y)}{k_2+\beta}\ell}}{\ell} d\ell \right.$$

$$\left. + \sqrt{x+\beta y} \int_0^{\beta y-x} \frac{d\ell}{\sqrt{-\ell}} \right\}$$

$$= \frac{q_2}{\beta} \left\{ \sqrt{\frac{k_2+\beta}{k_2-\beta}} \int_0^{\beta y-x} \frac{\sqrt{\ell^2 - \frac{2\beta(k_2 y-x)}{k_2+\beta}\ell}}{\ell} d\ell \right.$$

$$\left. - 2\sqrt{x^2-\beta^2 y^2} \right\}$$

To evaluate $\displaystyle\int_0^{\beta y-x} \frac{\sqrt{\ell^2 - \frac{2\beta(k_2 y-x)}{k_2+\beta}\ell}}{\ell} d\ell$,

put $\ell = \dfrac{-2\beta(k_2 y-x)}{k_2+\beta} m^2$. $\quad \therefore \dfrac{d\ell}{\ell} = 2\dfrac{dm}{m}$

$$\therefore \int_0^{\beta y-x} \frac{\sqrt{\ell^2 - \frac{2\beta(k_2 y-x)}{k_2+\beta}\ell}}{\ell} d\ell$$

$$= \frac{4\beta(k_2 y-x)}{k_2+\beta} \int_0^{\sqrt{\frac{(k_2+\beta)(x-\beta y)}{2\beta(k_2 y-x)}}} \sqrt{m^2+1} \quad dm$$

$$= \frac{2\beta(k_2y-x)}{k_2+\beta}\left[m\sqrt{m^2+1} + \sinh^{-1}m\right]_0^{\sqrt{\frac{(k_2+\beta)(x-\beta y)}{2\beta(k_2y-x)}}}$$

$$= \frac{2\beta(k_2y-x)}{k_2+\beta}\left\{\frac{\sqrt{(x^2-\beta^2y^2)(k_2^2-\beta^2)}}{2\beta(k_2y-x)} + \sinh^{-1}\sqrt{\frac{(k_2+\beta)(x-\beta y)}{2\beta(k_2y-x)}}\right\}$$

$$=\sqrt{\frac{k_2-\beta}{k_2+\beta}}\;(x^2-\beta^2y^2) + \frac{2\beta(k_2y-x)}{k_2+\beta}\sinh^{-1}\sqrt{\frac{(k_2+\beta)(x-\beta y)}{2\beta(k_2y-x)}}$$

$$\therefore\ \phi = q_2\left\{\frac{2(k_2y-x)}{\sqrt{k_2^2-\beta^2}}\sinh^{-1}\sqrt{\frac{(k_2+\beta)(x-\beta y)}{2\beta(k_2y-x)}} - \frac{\sqrt{x^2-\beta^2y^2}}{\beta}\right\}$$

$$\therefore\ \frac{\partial\phi}{\partial x} = -q_2\left\{\frac{1}{\beta}\sqrt{\frac{x-\beta y}{x+\beta y}} + \frac{2}{\sqrt{k_2^2-\beta^2}}\sinh^{-1}\sqrt{\frac{(k_2+\beta)(x-\beta y)}{2\beta\ (k_2y-x)}}\right\}$$

$$= -q_2\left\{\frac{1}{\beta}\sqrt{\frac{n_2-\bar{t}}{n_2+\bar{t}}} + \frac{2}{\beta\sqrt{n_2^2-1}}\sinh^{-1}\sqrt{\frac{(n_2+1)(n_2-\bar{t})}{2n_2(\bar{t}-1)}}\right\}$$

$$\therefore\ p = \frac{\rho\ Uq_2}{\beta}\left\{\sqrt{\frac{n_2-\bar{t}}{n_2+\bar{t}}} + \frac{2}{\sqrt{n_2^2-1}}\sinh^{-1}\sqrt{\frac{(n_2+1)(n_2-\bar{t})}{2n_2(\bar{t}-1)}}\right\} \quad \ldots(8)$$

Finally, let P_2 be a point in the region bOM_2. (See Fig. 3).

Due to q_2, the potential ϕ at P_2 (x,y) is given by:

$$\phi = -q_2\iint\frac{d\xi\,d\eta}{\sqrt{(x-\xi)^2-\beta^2(y-\eta)^2}},$$

the integration extending over $P_2\,Q_2\,O\,S_2$.

Put $s = \xi + \beta\eta$.

$$\therefore \phi = -q_2 \int_0^{x+\beta y} ds. \int_{\frac{s+\beta y-x}{2\beta}}^{\frac{s}{k_2+\beta}} \frac{d\eta}{\sqrt{(x-s+\beta\eta)^2-\beta^2(y-\eta)^2}}$$

$$= -q_2 \int_0^{x+\beta y} ds. \int_{\frac{s+\beta y-x}{2\beta}}^{\frac{s}{k_2+\beta}} \frac{d\eta}{\sqrt{2\beta(x-s+\beta y)\eta+(x-s)^2-\beta^2y^2}}$$

$$= -q_2 \int_0^{x+\beta y} \left[\frac{\sqrt{2\beta(x-s+\beta y)\eta+(x-s)^2-\beta^2y^2}}{\beta(x-s+\beta y)} \right]_{\eta=\frac{s+\beta y-x}{2\beta}}^{\eta=\frac{s}{k_2+\beta}} ds$$

Let $\ell = x-s+\beta y.$

$$\therefore \phi = -q_2 \int_0^{x+\beta y} \left[\frac{\sqrt{2\beta\ell\eta+(\ell-\beta y)^2-\beta^2y^2}}{\beta\ell} \right]_{\eta=\frac{2\beta y-\ell}{2\beta}}^{\eta=\frac{x+\beta y-\ell}{k_2+\beta}} d\ell$$

$$= -\frac{q_2}{\beta} \int_0^{x+\beta y} \frac{1}{\ell} \sqrt{\frac{(k_2-\beta)\ell^2-2\beta\ell(k_2y-x)}{k_2+\beta}}\ d\ell$$

$$= -\frac{q_2}{\beta}\sqrt{\frac{k_2-\beta}{k_2+\beta}} \int_0^{x+\beta y} \frac{1}{\ell}\sqrt{\ell^2+\frac{2\beta\ell(x-k_2y)}{k_2-\beta}}\ d\ell$$

Put $\ell = \frac{2\beta(x-k_2y)}{k_2-\beta}\ m^2 \ ; \quad \therefore \quad \frac{d\ell}{\ell} = 2\ \frac{dm}{m}$

$$\therefore \phi = -4q_2 . \frac{(x-k_2y)}{\sqrt{k_2^2-\beta^2}} . \int_0^{\sqrt{\frac{(k_2-\beta)(x+\beta y)}{2\beta(x-k_2y)}}} \sqrt{m^2+1}\ dm$$

$$= -\frac{2q_2(x-k_2y)}{\sqrt{k_2^2-\beta^2}} . \left[m\sqrt{1+m^2} + \sinh^{-1} m \right]_0^{\sqrt{\frac{(k_2-\beta)(x+\beta y)}{2\beta(x-k_2y)}}}$$

$$= -\frac{2q_2(x-k_2y)}{\sqrt{k_2^2-\beta^2}}\left\{\sqrt{\frac{(k_2^2-\beta^2)(x^2-\beta^2y^2)}{2\beta(x-k_2y)}} + \sinh^{-1}\sqrt{\frac{(k_2-\beta)(x+\beta y)}{2\beta(x-k_2y)}}\right\}$$

$$= -q_2\left\{\frac{\sqrt{x^2-\beta^2y^2}}{\beta} + \frac{2(x-k_2y)}{\sqrt{k_2^2-\beta^2}}\sinh^{-1}\sqrt{\frac{(k_2-\beta)(x+\beta y)}{2\beta(x-k_2y)}}\right\}$$

$$\therefore \frac{\partial\phi}{\partial x} = -q_2\left\{\frac{1}{\beta}\sqrt{\frac{x-\beta y}{x+\beta y}} + \frac{2}{\sqrt{k_2^2-\beta^2}}\sinh^{-1}\sqrt{\frac{(k_2-\beta)(x+\beta y)}{2\beta(x-k_2y)}}\right\}$$

$$= -\frac{q_2}{\beta}\left\{\sqrt{\frac{n_2-\bar{t}}{n_2+\bar{t}}} + \frac{2}{\sqrt{n_2^2-1}}\sinh^{-1}\sqrt{\frac{(n_2-1)(n_2+\bar{t})}{2n_2(1-\bar{t})}}\right\}$$

$$\therefore p = \frac{\rho U_0 q_2}{\beta}\left\{\sqrt{\frac{n_2-\bar{t}}{n_2+\bar{t}}} + \frac{2}{\sqrt{n_2^2-1}}\sinh^{-1}\sqrt{\frac{(n_2-1)(n_2+\bar{t})}{2n_2(1-\bar{t})}}\right\} \quad \ldots(9)$$

5.11 Controls Used as Ailerons - Calculation of ℓ_ξ

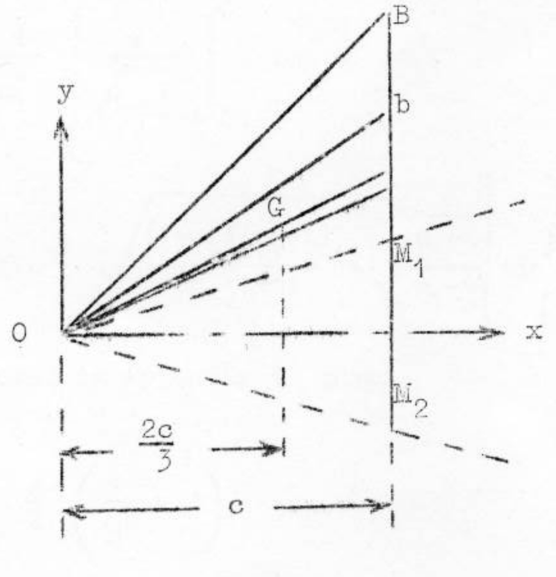


FIG. 4

Along a radial line through the apex O the pressure, being only a function of $\frac{y}{x}$, is constant. Therefore the resultant force on a thin triangular strip of the wing, of area dA,

/with ...

with vertex at O acts at a point G on $x = \frac{2c}{3}$.

Remembering that the effects of q_1 are confined to M_2OB, the rolling moment due to q_1 is given by:

$$\bar{L}_1 = \int_{M_2OB} \frac{4c}{3k_1} \, tp\,dA$$

Now $dA = \frac{S}{2}\,dt = \frac{c^2}{2k_1}\,dt = \frac{c^2}{2\beta n_1}\,dt$

$$\therefore \bar{L}_1 = \frac{2c^3}{3\beta^2 n_1^2} \int_{-n_1}^{1} tp\,dt$$

$$= \frac{2c^3 \rho U_o q_1}{3\beta^3 n_1^2} \left\{ \frac{\pi}{\sqrt{1-n_1^2}} \int_{n_1}^{1} t\,dt + \int_{-n_1}^{n_1} t \left[\frac{2}{\sqrt{1-n_1^2}} \sin^{-1} \sqrt{\frac{(1-n_1)(n_1+t)}{2n_1(1-t)}} + \sqrt{\frac{n_1-t}{n_1+t}} \right] dt \right\},$$

by equations (6.1) and (7.1)

$$\text{i.e.} \quad \bar{L}_1 = \frac{2c^3 \rho U_o q_1}{3\beta^3} \, \Phi(n_1)$$

where $$\Phi(n_1) = \frac{1}{n_1^2} \left\{ \frac{\pi}{\sqrt{1-n_1^2}} \int_{n_1}^{1} t\,dt + \int_{-n_1}^{n_1} \left[\frac{2}{\sqrt{1-n_1^2}} \sin^{-1} \sqrt{\frac{(1-n_1)(n_1+t)}{2n_1(1-t)}} + \sqrt{\frac{n_1-t}{n_1+t}} \right] t\,dt \right\}.$$

It is proved in Appendix I that:

$$\Phi(n) = \frac{\pi}{2} \left(\frac{1}{n^2} - 1 \right)$$

Hence $$\bar{L}_1 = \frac{\pi c^3 \rho U_o q_1}{3\beta^3} \left(\frac{1}{n_1^2} - 1 \right)$$

/ 5.111 ...

5.111 Hinge Lines Lying Outside the Mach Cone

Similarly, using the pressure formulae (6.2) and (7.2), the rolling moment $\bar{L}_2$ due to q_2 is given by:

$$\bar{L}_2 = \frac{2c^3 \rho U_o q_2}{3\beta^3} \, \Phi\,(n_2)$$

i.e.
$$\bar{L}_2 = \frac{\pi c^3 \rho U_o q_2}{3\beta^3} \left(\frac{1}{n_2^2} - 1 \right)$$

Hence the rolling moment due to the deflection of the starboard control, is:

$$\bar{L}_s = \bar{L}_1 + \bar{L}_2 = \frac{\pi c^3 \rho U_o}{3\beta^3} \left\{ q_1 \left(\frac{1}{n_1^2} - 1 \right) + q_2 \left(\frac{1}{n_2^2} - 1 \right) \right\}$$

With starboard aileron deflection $\theta = \xi$, q_1 and q_2 are given by:

$$q_1 = - q_2 = - \frac{U_o \xi \sin \Theta}{\pi}$$

$$\therefore \quad \bar{L}_s = - \frac{c^3 \rho U_o^2 \xi \sin \Theta}{3\beta^3} \left\{ \frac{1}{n_1^2} - \frac{1}{n_2^2} \right\}$$

i.e.
$$\bar{L}_s = - \frac{c^3 \rho U_o^2 \xi \sin \Theta}{3\beta} \left\{ \frac{1}{k_1^2} - \frac{1}{k_2^2} \right\}$$

An equal rolling moment is produced by the port aileron. Hence the total rolling moment $\bar{L}$ is given by:

$$\bar{L} = 2\bar{L}_s = - \frac{2c^3 \rho U_o^2 \xi \sin \Theta}{3\beta} \left\{ \frac{1}{k_1^2} - \frac{1}{k_2^2} \right\}$$

$$\therefore \quad C_{\bar{L}} = \frac{k_1^2 \bar{L}}{\rho U_o^2 c^3} = - \frac{2\xi \sin \Theta}{3\beta} \left\{ 1 - \frac{k_1^2}{k_2^2} \right\}$$

$$= - \frac{2\xi \sin \Theta}{3\beta} \left(1 - \frac{\tan^2 \Theta}{\tan^2 \gamma} \right)$$

$$\therefore \ \ell_\xi = \frac{\partial \bar{C}_L}{\partial \xi} = - \frac{2\sin\Theta}{3\beta}\left(1 - \frac{\tan^2\Theta}{\tan^2\gamma}\right)$$

Put $r = \dfrac{\tan\Theta}{\tan\gamma}$, $B = \beta \tan\gamma$

$$\therefore \ \ell_\xi = - \frac{2}{3} \frac{(1-r^2)}{B} \sin\Theta \tan\gamma$$

5.112 <u>Hinge Lines Lying Inside Mach Cone</u>

The pressure is given by equations (8) and (9) and therefore, due to q_2,

$$\bar{L}_2 = \frac{2c^3\rho U_o q_2}{3\beta k_2^2}\left\{\int_{-n_2}^{n_2} \bar{t}\sqrt{\frac{n_2-\bar{t}}{n_2+\bar{t}}}\, d\bar{t}\right.$$

$$\left. + \frac{2}{\sqrt{n_2^2-1}}\left[\int_{-n_2}^{1} \bar{t}\sinh^{-1}\sqrt{\frac{(n_2-1)(n_2+\bar{t})}{2n_2(1-\bar{t})}}\, d\bar{t} + \int_{1}^{n_2} \bar{t}\sinh^{-1}\sqrt{\frac{(n_2+1)(n_2-\bar{t})}{2n_2(\bar{t}-1)}}\, d\bar{t}\right]\right\}$$

$$= \frac{\pi\rho U_o q_2 c^3}{3\beta k_2^2}\left(1-n_2^2\right),$$ by the result of Appendix II.

$$= \frac{\pi c^3 \rho U_o q_2}{3\beta^3}\left(\frac{1}{n_2^2} - 1\right)$$

i.e. $\bar{L}_2$ is the same as when the hinge lines lie outside the Mach Cone. $\bar{L}_1$ remains unchanged. Hence ℓ_ξ is given as before by:

$$\ell_\xi = - \frac{2}{3} \frac{(1-r^2)}{B} \sin\Theta \tan\gamma \ .$$

5.12 <u>Controls used as Elevators - Calculations of a_2</u>

The lift due to q_1 is given by:

$$L_1 = - \int_{M_2OB} 2p\,dA \qquad \text{(See Fig.4)}$$

$$= - \frac{c^2}{\beta n_1}\int_{-n_1}^{1} p\, dt$$

$$= - \frac{c^2 \rho U_o q_1}{\beta^2 n_1} \left\{ \frac{\pi}{\sqrt{1-n_1^2}} \int_{n_1}^{1} dt \; + \right.$$

$$\left. \int_{-n_1}^{n_1} \left[\frac{2}{\sqrt{1-n_1^2}} \sin^{-1} \sqrt{\frac{(1-n_1)(n_1+t)}{2n_1(1-t)}} + \sqrt{\frac{n_1-t}{n_1+t}} \right] dt \right\},$$

by equations (6.1) and (7.1)

i.e. $L_1 = - \dfrac{c^2 \rho U_o q_1}{\beta^2} \chi (n_1)$,

$$\text{where } \chi (n_1) = \frac{1}{n_1} \left\{ \frac{\pi}{\sqrt{1-n_1^2}} \int_{n_1}^{1} dt + \int_{-n_1}^{n_1} \left[\frac{2}{\sqrt{1-n_1^2}} \sin^{-1} \right. \right.$$

$$\left. \left. \sqrt{\frac{(1-n_1)(n_1+t)}{2n_1(1-t)}} + \sqrt{\frac{n_1-t}{n_1+t}} \right] dt \right\}$$

It is proved in Appendix III that

$$\chi (n) = \pi \left(\frac{1}{n} + 1 \right)$$

$$\therefore \; L_1 = - \frac{\pi c^2 \rho U_o q_1}{\beta^2} \left(\frac{1}{n} + 1 \right)$$

5.121 Hinge Lines Lying Outside the Mach Cone

Similarly the lift L_2 due to q_2 is given by:

$$L_2 = - \frac{\pi c^2 \rho U_o q_2}{\beta^2} \left(\frac{1}{n_2} + 1 \right)$$

Hence due to the deflection of the starboard control, the lift is given by:

$$L_S = L_1 + L_2 = - \frac{\pi c^2 \rho U_o}{\beta^2} \left\{ q_1 \left(\frac{1}{n_1} + 1 \right) + q_2 \left(\frac{1}{n_2} + 1 \right) \right\}$$

With elevator deflection $\theta = \eta$, q_1 and q_2 are given by:

$$q_1 = - q_2 = - \frac{U \eta \sin \Theta}{\pi}$$

$$\therefore \; L_S = \frac{c^2 \rho U_o^2 \eta \sin \Theta}{\beta^2} \left(\frac{1}{n_1} - \frac{1}{n_2} \right)$$

/ i.e. ...

i.e. $L_S = \frac{c^2\rho U_o^2 \eta \sin\Theta}{\beta}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)$

An equal lift is produced by the port elevator. Hence the total lift is given by:

$$L = 2L_S = \frac{2\rho U_o^2 c^2 \eta \sin\Theta}{\beta}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)$$

$$\therefore \; C_L = \frac{2k_1 L}{\rho U_o^2 \; c^2} = \frac{4\eta \sin\Theta}{\beta}\left(1 - \frac{k_1}{k_2}\right) = \frac{4\eta\sin\Theta}{\beta}\left(1 - \frac{\tan\Theta}{\tan\gamma}\right)$$

$$\therefore \; a_2 = \frac{\partial C_L}{\partial \eta} = \frac{4\sin\Theta}{\beta}\left(1 - \frac{\tan\Theta}{\tan\gamma}\right)$$

i.e. $a_2 = \frac{4(1-r)}{B} \; \sin\Theta \tan\gamma$

5.122 Hinge Lines Lying Inside Mach Cone

By equations (8) and (9),

$$L_2 = - \frac{c^2\rho U_o q_2}{\beta^2 n_2}\left\{\int_{-n_2}^{n_2}\sqrt{\frac{n_2-\bar{t}}{n_2+\bar{t}}}\,d\bar{t} + \right.$$

$$\left.\frac{2}{\sqrt{n_2^2-1}}\left[\int_{-n_2}^{1}\sinh^{-1}\sqrt{\frac{(n_2-1)(n_2+\bar{t})}{2n_2(1-\bar{t})}}\,d\bar{t} + \int_{1}^{n_2}\sinh^{-1}\sqrt{\frac{(n_2+1)(n_2-\bar{t})}{2n_2(\bar{t}-1)}}\,d\bar{t}\right]\right\}$$

$$= - \frac{\pi c^2 \rho U_o q_2}{\beta^2 n_2}(n_2+1), \quad \text{by Appendix IV.}$$

$$= - \frac{\pi c^2 \rho U_o q_2}{\beta^2}\left(\frac{1}{n_2} + 1\right)$$

i.e. L_2 is the same as when the hinge lines lie outside the Mach cone. L_1 remains unchanged. Hence a_2 is given as before by:

$$a_2 = \frac{4(1-r)}{B} \; \sin\Theta \tan\gamma$$

5.123 Position of Centre of Pressure due to Elevator Deflection

/The ...

The pressure is constant over elementary triangular strips of the wing with vertex at the wing apex. The resultant force on such a strip acts at a point whose abscissa is $\frac{2}{3}$ c. The centre of pressure must therefore lie on the line $x = \frac{2}{3}c$, and by symmetry it lies on the centre line of the wing. Thus the centre of pressure due to deflection of the elevators lies on the centre line of the wing, distant $\frac{2}{3}$ c from the apex.

5.2 Solution with Leading Edges Inside Mach Cone

The solution depends on the fact that the velocity at any point upstream of the trailing edge is of degree zero in x, y and z. This is proved as follows:

Let P (x,y,z) be any such point. By dimensional theory, a typical velocity component $\bar{u}$ is given by:

$$\frac{\bar{u}}{U_o} = f\left(\frac{y}{x}, \frac{z}{x}, \frac{x}{c}\right).$$

The flow at P is uninfluenced by conditions downstream of P so that if the wing is replaced by a similar wing of larger chord c_1, the velocity at P will be unaltered,

$$\text{i.e.}\quad f\left(\frac{y}{x}, \frac{z}{x}, \frac{x}{c_1}\right) = \frac{\bar{u}}{U_o} = f\left(\frac{y}{x}, \frac{z}{x}, \frac{x}{c}\right), \quad \text{where } c_1 \neq c.$$

Hence $\bar{u}$ must be independent of $\frac{x}{c}$, i.e. $\bar{u}$ is of degree zero in x, y and z.

u, v and w are therefore of degrees zero in x,y,z.

Now u, v and w all satisfy the equation:

$$-\beta^2 \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0,$$

whose most general solution of zero degree may be written:

$$f = f_1\left(\frac{\beta[y+iz]}{x+r}\right) + f_2\left(\frac{\beta[y-iz]}{x+r}\right),$$

$$\text{where}\quad r = \sqrt{x^2-\beta^2y^2-\beta^2z^2}.$$

Let ω denote the complex variable:

$$\beta\frac{(y+iz)}{x+r}.$$

/Then ...

Then we may write: $u = R\,[U(\omega)]$
$v = R\,[V(\omega)]$
$w = R\,[W(\omega)]$.

Inside the Mach cone r is real, and therefore

$$|\omega|^2 = \frac{\beta^2(y^2+z^2)}{(x+r)^2} = \frac{x^2-r^2}{(x+r)^2} = \frac{x-r}{x+r}$$

$\therefore\ |\omega| < 1$ except on the Mach cone where $r = 0$ and $|\omega| = 1$. Thus the Mach cone and its interior are represented in the ω-plane by unit circle and its interior.

At the wing, $z = 0$, $\therefore\ \omega = \dfrac{\beta y}{x+\sqrt{x^2-\beta^2 y^2}}$

$$\text{i.e.}\quad \omega = \frac{\frac{\beta y}{x}}{1+\sqrt{1-\left(\frac{\beta y}{x}\right)^2}},$$

which is real and increases with $\frac{y}{x}$. At the leading edges; $y = \pm x \tan\gamma$;

$$\therefore\ \omega = \frac{\pm\beta\tan\gamma}{1+\sqrt{1-\beta^2\tan^2\gamma}} = \frac{\pm k'}{1+k},$$

where $k' = \sqrt{1-k^2} = \beta\tan\gamma$.

The aerofoil therefore becomes the portion of the real axis between $\pm\dfrac{k'}{1+k}$, in the ω-plane. (See Fig. 6).

The boundary conditions of the problem are:

(i) Component of velocity at the wing surface normal to the surface is zero;

(ii) u, v and w are all zero on the Mach cone.

Condition (ii) follows from the assumptions of linearised theory.

It is possible to find functions U, V and W that satisfy these boundary conditions by transforming from the ω-plane into a new plane, the τ-plane, using the transformation:

$$\mathrm{cn}(\tau,k) = \frac{2i\omega}{1-\omega^2},$$

(where $\mathrm{cn}(\tau,k)$ is one of the Jacobian elliptic functions of modulus k).

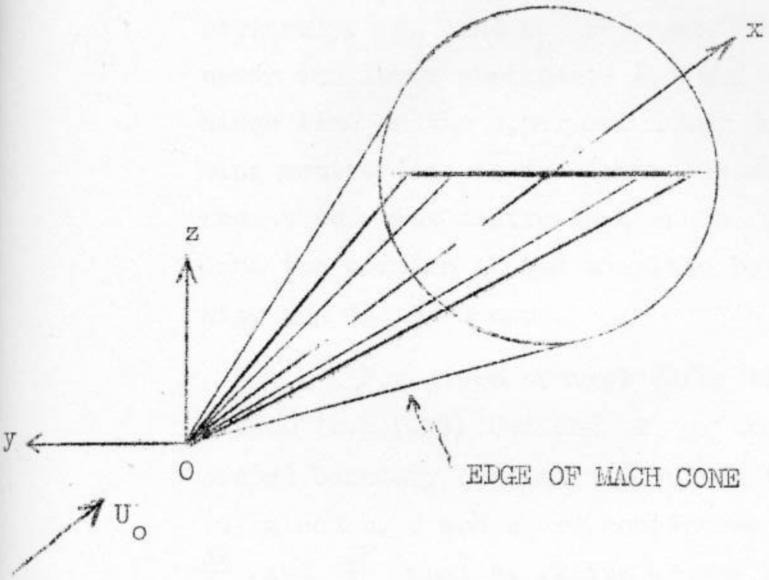


FIG. 5
THE WING IN THE (x,y,z) PLANE

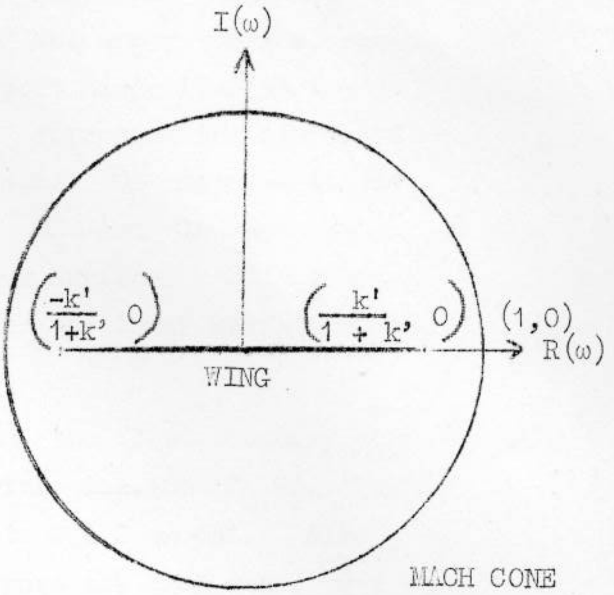


FIG. 6
THE ω-PLANE

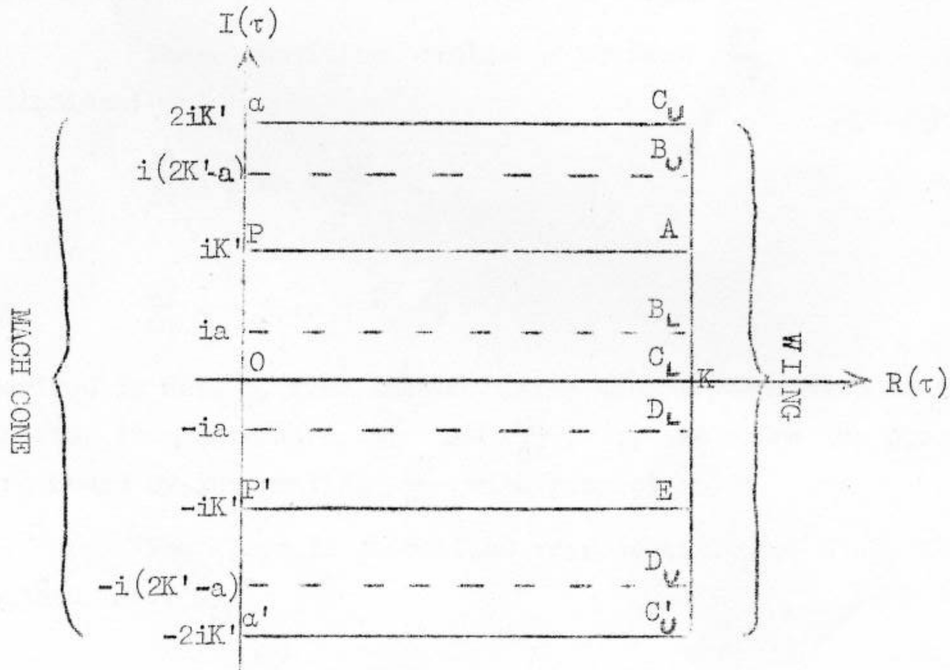


FIG. 7
THE τ-PLANE

The interior of unit circle in the ω-plane becomes the interior of the rectangle, vertices $\pm$ 2iK', K$\pm$2iK'.

In Fig. 7, the section αα' of the imaginary axis represents the Mach cone and the parallel line C_U C'_U represents

/the wing.

the wing. A and E represent port and starboard leading edges. AE represents the lower surface of the wing; AC_U and EC_U' represent the port and starboard halves of the upper surface, respectively. B_U and B_L represent the port hinge line on the upper and lower surfaces; D_U and D_L represent the starboard hinge line on the upper and lower surfaces. C_L represents the wing centre line on the lower surface; C_U and C_U' both represent the wing centre line on the upper surface. OC_L represents the portion of the xz plane between the lower surface of the wing and the Mach cone.

For given control deflections, the first boundary condition (see p.28) defines w on the wing, i.e. on $C_U C_U'$. The second boundary condition requires that $w = 0$ on $\alpha\alpha'$. Also (a) since u, v and w are continuous across the Mach cone, $\frac{dU}{d\tau}$, $\frac{dV}{d\tau}$ and $\frac{dW}{d\tau}$ must be finite at the Mach cone, (b) the aerodynamic forces must be finite, so that the integral of u with respect to area must be finite, (c) the only places where an infinite pressure is admissible are along the hinge lines and leading edges, (d) u, v and w must be single valued.

These conditions enable us to find $\frac{dW}{d\tau}$. The relations:

$$\frac{dU}{d\tau} = \frac{1}{\beta} \text{ cn } \tau \frac{dW}{d\tau}$$

$$\frac{dV}{d\tau} = - i \text{ sn } \tau \frac{dW}{d\tau} ,$$

derived in Ref. 3, from the condition that a velocity potential exists, then determine $\frac{dU}{d\tau}$ and $\frac{dV}{d\tau}$. u, and hence the pressure are found by integrating $\frac{dU}{d\tau}$ with respect to τ.

The above is a modified representation of Stewart's method (Ref. 4).

5.21 Controls Used as Ailerons

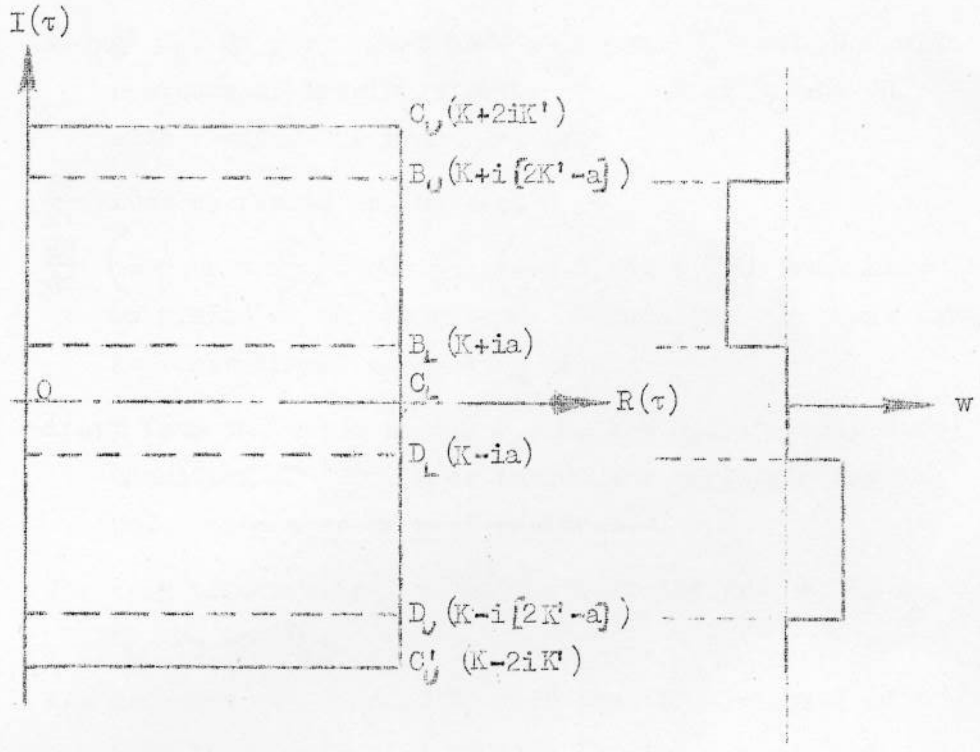


FIG. 8

The boundary condition at the wing is:

$w = -U_o \xi \sin \Theta = w_o$, over the starboard aileron,

$w = -w_o$, over the port aileron,

$w = 0$ elsewhere at the wing;

i.e. in Fig. 8,

$w = 0$ on $C_U B_U$, $B_L D_L$ and $D_U C_U'$

$w = w_o$ on $D_L D_U$

$w = -w_o$ on $B_U B_L$

Thus in integrating $\frac{dW}{d\tau}$ along $C_L C_U$, w must jump in value by an amount $(-w_o)$ at B_L and $(+w_o)$ at B_U. Hence $\frac{dW}{d\tau}$ must have simple poles at B_L and B_U with residues of imaginary parts $\frac{-w_o}{\pi}$ and $\frac{w_o}{\pi}$ respectively. Similarly $\frac{dW}{d\tau}$ must have simple poles at D_L and D_U with residues of imaginary parts $\frac{-w_o}{\pi}$ and $\frac{w_o}{\pi}$.

Except when changed by discontinuities, the value of w is constant on the wing. Also w is everywhere zero and so constant, on the Mach cone. Therefore $\frac{dW}{d\tau}$ must be real on the wing and Mach cone.

Hence $\frac{dW}{d\tau}$ must be chosen to satisfy the following conditions:

(1) $\frac{dW}{d\tau}$ must be real on the wing and Mach cone and its integral along OC_L from 0 to C_L must be zero or imaginary since w is zero at O and at C_L.

(2) Along C_U, C_U, $\frac{dW}{d\tau}$ must have poles at D_U and B_U with residues of imaginary part $\frac{w_o}{\pi}$, and at D_L and B_L with residues of imaginary part $\frac{-w_o}{\pi}$.

(3) $\frac{dW}{d\tau}$ must be finite on the Mach cone.

(4) $\frac{dU}{d\tau}$ $\left(= \frac{1}{\beta} \text{ cn } \tau \frac{dW}{d\tau}\right)$ and $\frac{dV}{d\tau}$ $\left(= - i \text{ sn } \tau \frac{dW}{d\tau}\right)$ must also be finite on the Mach cone. Therefore $\frac{dW}{d\tau}$ must have at least simple zeros at $\pm$ iK'.

(5) Apart from the poles at D_U, B_U, D_L and B_L, the only singularities of $\frac{dW}{d\tau}$ on or inside the rectangle may be poles with zero or real residues.

(6) The only places where u may be infinite are A, E, B_U, B_L, D_U and D_L.

(7) Any infinity of u must be such that the integral of u with respect to area remains finite.

(8) u, v and w must be single valued.

The required function is:

$$\frac{dW}{d\tau} = \frac{ik'^3C^2w_o}{\pi D} \text{ sn } \tau \text{ nd}^2\tau \left[\text{nc}(\tau-ai) + \text{nc}(\tau+ai)\right].$$

[Note. The symbols C, D and S (which is introduced later) are defined by:

$$C = \text{cn}(a,k')$$
$$D = \text{dn}(a,k')$$
$$S = \text{sn}(a,k') \ ]$$

$$\therefore \quad \frac{dU}{d\tau} = \frac{1}{\beta} \text{ cn } \tau \frac{dW}{d\tau} = \frac{ik'^3C^2w_o}{\pi\beta D} \text{ sd } \tau \text{ cd } \tau \left[\text{nc}(\tau-ai) + \text{nc }(\tau+ai)\right].$$

At O on the Mach cone, u = 0.

Therefore at any point (K+it) on the wing, u is given by:

$$u = R\left[\int_o^{K+it} \frac{dU}{d\tau} \, d\tau\right]$$

$$= - \frac{k'^3C^2w_o}{\pi\beta D} \, I\left[\int_o^{K+it} \text{sd } \tau \text{ cd } \tau \left[\text{nc}(\tau-ai)+ \text{nc}(\tau+ai)\right] d\tau\right]$$

$$= - \frac{k'^3C^2w_o}{\pi\beta D} \, I\left[\int_K^{K+it} \text{sd } \tau \text{ cd } \tau \left[\text{nc}(\tau-ai)+ \text{nc}(\tau+ai)\right] d\tau\right],$$

/since ...

since the omitted part of the integral, from 0 to K, is real.

$$\therefore \quad \frac{\pi\beta D}{k'^3C^2w_o} u = - I\left[\int_0^{it} \frac{1}{k'^2} \text{cn } u \text{ sn } u \left[\text{ds}(u-ai)+ \text{ds}(u+ai)\right] du\right], \quad (u = \tau - K)$$

$$= -\int_0^t \frac{1}{k'^2} \text{ sn } (v,k') \text{ nc}^2 (v,k') \times \left[\text{ds } (\overline{v-a},k')+ \text{ds}(\overline{v+a},k')\right] dv \quad (u = iv)$$

$$= \frac{2C}{k'^2}\int_0^t \frac{\text{sn}^2(v,k')\text{dn}(v,k')dv}{\text{cn}^2(v,k')\left[S^2-\text{sn}^2(v,k')\right]}$$

$$= \frac{2C}{k'^2}\int_0^{\text{sc}(t,k')} \frac{y^2dy}{S^2-C^2y^2}, \quad \left(y = \text{sc}\left[v,k'\right]\right)$$

$$= \frac{1}{Ck'^2}\int_0^{\text{sc}(t,k')} \left\{\frac{S}{S-Cy} + \frac{S}{S+Cy} - 2\right\} dy$$

$$= \frac{S}{C^2k'^2} \log_e \left|\frac{S+C\text{sc}(t,k')}{S-C\text{sc}(t,k')}\right| - \frac{2}{Ck'^2} \text{ sc } (t,k')$$

The pressure is given by:

$$p = - \rho U_o u$$

$$= - \frac{k'\rho U_o w_o}{\pi\beta D}\left\{S \log_e \left|\frac{S+C\text{sc}(t,k')}{S-C\text{sc}(t,k')}\right| - 2C \text{ sc}(t,k')\right\}$$

On the wing $\tau = K+it$ and $\omega = \dfrac{\beta y}{x+\sqrt{x^2-\beta^2y^2}}$

Substituting these values in $\text{cn}(\tau,k) = \dfrac{2i\omega}{1-\omega^2}$ gives:

$$- ik' \text{ sd}(t,k') = \frac{i\beta y}{\sqrt{x^2-\beta^2y^2}}$$

Let $\mu = \dfrac{y}{x}$.

$$\therefore \quad k'\text{sd}(t,k') = - \frac{\beta\mu}{\sqrt{1-\beta^2\mu^2}}.$$ Now $\text{dn}(t,k')$ is + ve.

/Therefore ...

$$\therefore \ \mathrm{sn}(t,k') = -\frac{\beta}{k'},\ \mu = -\frac{\mu}{\tan\gamma}$$

$$\mathrm{dn}(t,k') = \sqrt{1-\beta^2\mu^2}\ .$$

On the starboard upper surface, $\mathrm{cn}(t,k') = -\sqrt{1-\mu^2/\tan^2\gamma}$, the sign of the root being determined by $-K' \gtrless t \gtrless -2K'$.

Thus on the starboard upper surface, the pressure is given by:

$$p = -\frac{k'\rho U_o w_o}{\pi\beta D}\left\{ S\log_e\left|\frac{S\sqrt{\tan^2\gamma-\mu^2}+C\mu}{S\sqrt{\tan^2\gamma-\mu^2}-C\mu}\right| - \frac{2C\mu}{\sqrt{\tan^2\gamma-\mu^2}}\right\},$$

which again is only a function of $\frac{y}{x}$.

The rolling moment is given by:

$$\bar{L} = \frac{4c^3}{3}\int_o^{\tan\gamma} \mu\ p\ d\mu\ .$$

$$\therefore \ \bar{L} = -\frac{4k'\rho U_o w_o c^3}{3\pi\beta D}\int_o^{\tan\gamma}\left\{\mu\ S\ \log_e\left|\frac{S\sqrt{\tan^2\gamma-\mu^2}+C\mu}{S\sqrt{\tan^2\gamma-\mu^2}-C\mu}\right| - \frac{2C\mu^2}{\sqrt{\tan^2\gamma-\mu^2}}\right\}d\mu$$

$$\therefore \ \frac{3\pi\beta D}{4k'\rho U_o w_o c^3}\bar{L} = -S\mathop{\mathcal{L}t}_{\epsilon\to 0}\left\{\int_o^{(S-\epsilon)\tan\gamma}\mu\log_e\left[\frac{S\sqrt{\tan^2\gamma-\mu^2}+C\mu}{S\sqrt{\tan^2\gamma-\mu^2}-C\mu}\right]d\mu\right.$$

$$\left. + \int_{(S+\epsilon)\tan\gamma}^{o}\mu\log_e\left[\frac{S\sqrt{\tan^2\gamma-\mu^2}+C\mu}{C\mu-S\sqrt{\tan^2\gamma-\mu^2}}\right]d\mu\right\} + 2C\int_o^{\tan\gamma}\frac{\mu^2 d\mu}{\sqrt{\tan^2\gamma-\mu^2}}$$

Integrating by Parts,

$$\frac{3\pi\beta D}{4k'\rho U_o w_o c^3}\bar{L} = -\frac{S}{2}\mathop{\mathcal{L}t}_{\epsilon\to 0}\left\{\left[\mu^2\log_e\left\{\frac{S\sqrt{\tan^2\gamma-\mu^2}+C\mu}{S\sqrt{\tan^2\gamma-\mu^2}-C\mu}\right\}\right]_o^{(S-\epsilon)\tan\gamma}\right.$$

$$\left. + \left[\mu^2\log_e\left\{\frac{S\sqrt{\tan^2\gamma-\mu^2}+C\mu}{S\sqrt{\tan^2\gamma-\mu^2}-C\mu}\right\}\right]_{(S+\epsilon)\tan\gamma}^{\tan\gamma}\right\} + \frac{S}{2}\int_o^{\tan\gamma}\frac{2CS\mu^2\tan^2\gamma d\mu}{(S^2\tan^2\gamma-\mu^2)\sqrt{\tan^2\gamma-\mu^2}}$$

$$+ 2C \int_0^{\tan\gamma} \frac{\mu^2 d\mu}{\sqrt{\tan^2\gamma - \mu^2}}$$

$$= -\frac{S}{2} \mathcal{L}t_{\epsilon\to 0} \left\{ (S^2 + {}^2)\tan^2\gamma \left[\log_e \left(\frac{2SC^2 + 0(\epsilon)}{\epsilon + 0(\epsilon^2)} \right) \right.\right.$$

$$\left.\left. - \log_e \frac{2SC^2 + 0(\epsilon)}{\epsilon + 0(\epsilon^2)} \right] \right\} + S \mathcal{L}t_{\epsilon\to 0} \left\{ \epsilon \tan^2\gamma \left[\log_e \left(\frac{2SC^2 + 0(\epsilon^2)}{\epsilon + 0(\epsilon^2)} \right) \right.\right.$$

$$\left.\left. + \log_e \left(\frac{2SC^2 + 0(\epsilon)}{\epsilon + 0(\epsilon^2)} \right) \right] \right\} + S^2 C \tan^2\gamma \int_0^{\tan\gamma} \frac{\mu^2 d\mu}{(S^2\tan^2\gamma - \mu^2)\sqrt{\tan^2\gamma - \mu^2}}$$

$$+ 2C \int_0^{\tan\gamma} \frac{\mu^2 d\mu}{\sqrt{\tan^2\gamma - \mu^2}}$$

The limit terms both vanish as $\epsilon \to 0$.

$$\therefore \frac{3\pi\beta D}{4k'\rho U_o w_o c^3} \bar{L} = S^2 C \tan^2\gamma \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta d\theta}{S^2 - \sin^2\theta}$$

$$+ 2C \tan^2\gamma \int_0^{\frac{\pi}{2}} \sin^2\theta d\theta, \quad (\mu = \tan\gamma \sin\theta)$$

$$= S^2 C \tan^2\gamma \int_0^{\frac{\pi}{2}} \left[\frac{S^2}{S^2 - \sin^2\theta} - 1 \right] d\theta + \frac{\pi C}{2} \tan^2\gamma$$

$$= S^4 C \tan^2\gamma \int_0^{\frac{\pi}{2}} \frac{d\theta}{S^2 - \sin^2\theta} + \frac{\pi C}{2} \tan^2\gamma \, (1 - S^2)$$

$$= S^4 C \tan^2\gamma \int_0^{\infty} \frac{dt}{S^2 - C^2 t^2} + \frac{\pi C^3}{2} \tan^2\gamma$$

$$(t = \tan\theta)$$

$$= \frac{\pi C^3}{2} \tan^2\gamma$$

$$\therefore \bar{L} = \frac{2k'C^3\tan^2\gamma\rho U_o w_o c^3}{3\beta D}$$

$$= \frac{2C^3\tan^3\gamma\rho U_o w_o c^3}{3D}$$

$$= - \frac{2\xi C^3\tan^3\gamma\rho U_o^2 c^3\sin \Theta}{3D}$$

$$\therefore C_{\bar{L}} = \frac{\bar{L}}{\rho U_o^2 c^3\tan^2\gamma} = - \frac{2\xi C^3\tan\gamma\ \sin \Theta}{3D}$$

$$\therefore \ell_\xi = \frac{\partial C_{\bar{L}}}{\partial \xi} = - \frac{2C^3}{3D}\ \tan\gamma\ \sin \Theta$$

Using the relations of p.34, we have:

$$sn(a,k') = S = \frac{\tan \Theta}{\tan \gamma}$$

$$cn(a,k') = C = \sqrt{1-\tan^2\Theta/\tan^2\gamma}$$

$$dn(a,k') = D = \sqrt{1-\beta^2\tan^2\Theta}$$

$$\text{Hence } \ell_\xi = - \frac{2}{3}\ \frac{(1-\cot^2\gamma\tan^2\Theta)^{\frac{3}{2}}}{(1-\beta^2\tan^2\Theta)^{\frac{1}{2}}}\ \sin\Theta\tan\gamma$$

$$\text{i.e. } \ell_\xi = - \frac{2}{3}\ \frac{(1-r^2)^{\frac{3}{2}}}{(1-B^2r^2)^{\frac{1}{2}}}\ \sin\Theta\tan\gamma$$

When the Mach cone just touches the leading edges (i.e. $\beta\tan\gamma = 1$) the above formula and the formula derived in 5.112 both give the same expression for ℓ_ξ.

5.22 Controls Used as Elevators

The boundary condition at the wing is now:

$w = - U_o\ \eta\ \sin\Theta = w_o$, over the port and starboard elevators

$w = 0$ elsewhere at the wing.

The conditions that $\frac{dW}{d\tau}$ must satisfy are exactly as before in the aileron case, except that now $\frac{dW}{d\tau}$ must have poles at D_U and B_L (see Fig.9), with residues of imaginary part $\frac{w_o}{\pi}$, and at D_L and B_U with residues of imaginary part $-\frac{w_o}{\pi}$.

/ Fig. 9.

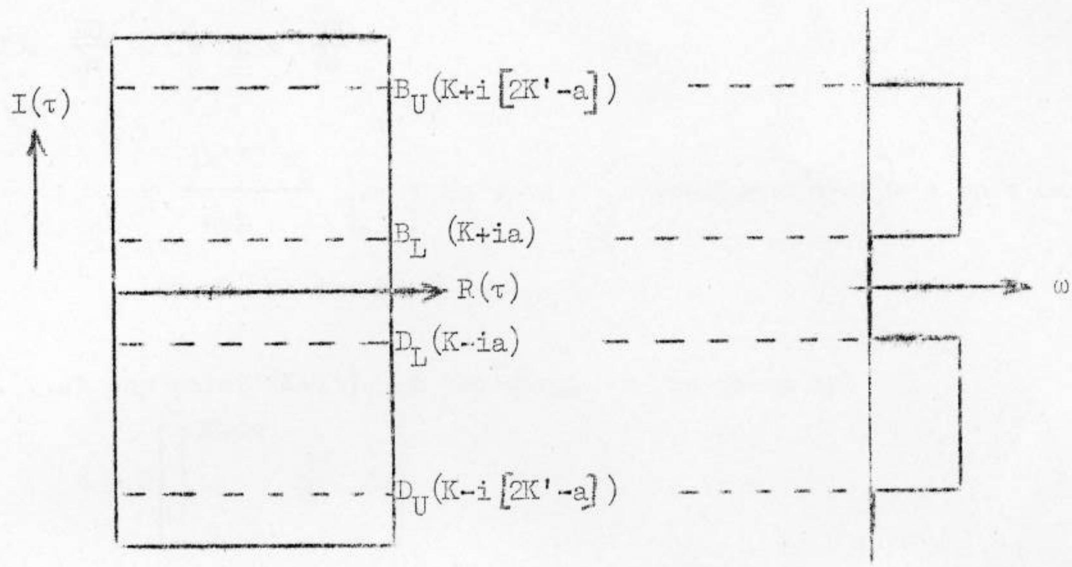


FIG. 9

The required function is:

$$\frac{dW}{d\tau} = \frac{ik'^4 C^3 w_o}{\pi D^2}\left[\operatorname{sn}\tau\ \operatorname{nd}^3\tau\left\{\operatorname{nc}(\tau+ai) - \operatorname{nc}(\tau-ai)\right\} + iA\ \operatorname{nd}^2\tau\right],$$

where A is found from the condition that $w = 0$ on the Mach cone and on the centre line of the wing,

$$\text{i.e.}\quad R\left[\int_o^K \frac{dW}{d\tau}\,d\tau\right] = 0$$

$$\text{Thus, } A\int_o^K \operatorname{nd}^2\tau d\tau = \int_o^K i\ \operatorname{sn}\tau\ \operatorname{nd}^3\tau\left\{\operatorname{nc}(\tau+ai) - \operatorname{nc}(\tau-ai)\right\}d\tau$$

$$= -\ 2SD\int_o^K \frac{\operatorname{sn}^2\tau\ d\tau}{\operatorname{dn}^2\tau(1-D^2\operatorname{sn}^2\tau)}$$

$$= \frac{2SD}{k'^2C^2}\left[\int_o^K \operatorname{nd}^2\tau\ d\tau - \int_o^K \frac{d\tau}{1-D^2\operatorname{sn}^2\tau}\right]$$

$$\text{i.e. } \frac{E}{k'^2}\ A = \frac{2SD}{k'^2C^2}\left[\frac{E}{k'^2} - \Pi(-D^2,k)\right],$$

where $\Pi(-D^2,k)$ and E are complete elliptic integrals of the third and second kinds with modulus k.

$$\therefore A = \frac{2SD}{k'^2C^2}\left[1 - \frac{k'^2}{E}\ \Pi\ (-D^2,k)\right].$$

Now $\frac{dU}{d\tau} = \frac{1}{\beta}\ \mathrm{cn}\ \tau\ \frac{dW}{d\tau}$

$$= \frac{ik'^4C^3w_o}{\pi\beta D^2}\left[\mathrm{cn}\ \tau\ \mathrm{sn}\ \tau\ \mathrm{nd}^3\tau\left\{\mathrm{nc}(\tau+ai)-\mathrm{nc}(\tau-ai)\right\}+iA\ \mathrm{nd}^2\tau\ \mathrm{cn}\ \tau\right]$$

At O on the Mach cone, $u = 0$.

$\therefore$ At any point $(K+it)$ on the wing, u is given by:

$$u = R\left[\int_o^{K+it}\frac{dU}{d\tau}\ d\tau\right]$$

$$= -\ \frac{k'^4C^3w_o}{\pi\beta D^2}\ I\left[\int_o^{K+it}\left\{\frac{2iSD\ \mathrm{sn}^2\tau\ \mathrm{cn}\ \tau}{\mathrm{dn}^2\tau(\mathrm{dn}^2\tau-k'^2C^2\mathrm{sn}^2\tau)} + iA\ \mathrm{cn}\ \tau\ \mathrm{nd}^2\tau\right\}d\tau\right]$$

$$= -\ \frac{k'^4C^3w_o}{\pi\beta D^2}\int_o^{\frac{\mathrm{nc}(t,k')}{k'}}\left\{\frac{2SDy^2}{1-k'^2C^2y^2} + A\right\}dy,\quad y = \mathrm{sd}(\tau,k)$$

$$= -\ \frac{k'^4C^3w_o}{\pi\beta D^2}\int_o^{\frac{\mathrm{nc}(t,k')}{k'}}\left\{A - \frac{2SD}{k'^2C^2} + \frac{2SD}{k'^2C^2(1-k'^2C^2y^2)}\right\}dy$$

$$= -\ \frac{k'^4C^3w_o}{\pi\beta D^2}\left\{\left(A - \frac{2SD}{k'^2C^2}\right)\frac{\mathrm{nc}(t,k')}{k'} + \frac{SD}{k'^3C^3}\log_e\left|\frac{\mathrm{cn}(t,k')+C}{\mathrm{cn}(t,k')-C}\right|\right\}$$

On the upper surface of the wing,

$$\mathrm{cn}(t,k') = -\sqrt{1-\mu^2/\tan^2\gamma}\ ,\ \text{where } \mu = \frac{y}{x}$$

$$\therefore u = \frac{k'^4C^3w_o}{\pi\beta D^2}\left\{\left(A - \frac{2SD}{k'^2C^2}\right)\frac{\tan\gamma}{k'\sqrt{\tan^2\gamma-\mu^2}}\right.$$

$$\left. + \frac{SD}{k'^3C^3}\log_e\left|\frac{C\ \tan\gamma+\sqrt{\tan^2\gamma-\mu^2}}{C\ \tan\gamma-\sqrt{\tan^2\gamma-\mu^2}}\right|\right\}$$

$\therefore$ The pressure is given by:

/ p =

$$p = \frac{k'^4 C^3 \rho U_o w_o}{\pi\beta D^2} \left\{ \left(\frac{2SD}{k'^2 C^2} - A \right) \frac{\tan\gamma}{k' \sqrt{\tan^2\gamma - \mu^2}} \right.$$

$$\left. + \frac{SD}{k'^3 C^3} \log_e \left| \frac{C \tan\gamma - \sqrt{\tan^2\gamma - \mu^2}}{C \tan\gamma + \sqrt{\tan^2\gamma - \mu^2}} \right| \right.$$

The lift is therefore given by:

$$L = - 2c^2 \int_o^{\tan\gamma} p d\mu$$

$$= - \frac{2k'^4 C^3 \rho U_o w_o c^2}{\pi\beta D^2} \int_o^{\tan\gamma} \left\{ \left(\frac{2SD}{k'^2 C^2} - A \right) \frac{\tan\gamma}{k' \sqrt{\tan^2\gamma - \mu^2}} \right.$$

$$\left. + \frac{SD}{k'^3 C^3} \log_e \left| \frac{C \tan\gamma - \sqrt{\tan^2\gamma - \mu^2}}{C \tan\gamma + \sqrt{\tan^2\gamma - \mu^2}} \right| \right\} d\mu$$

Substituting the value of A previously found,

$$\therefore \; - \frac{\pi\beta D}{2k'^3 C S \rho U_o w_o c^2} L = \int_o^{\tan\gamma} \left\{ \frac{2\Pi(-D^2, k)}{E} \frac{\tan\gamma}{\sqrt{\tan^2\gamma - \mu^2}} \right.$$

$$\left. + \frac{1}{k'^2 C} \log_e \left| \frac{C \tan\gamma - \sqrt{\tan^2\gamma - \mu^2}}{C \tan\gamma + \sqrt{\tan^2\gamma - \mu^2}} \right| \right\} d\mu$$

$$= \tan\gamma \int_o^1 \left[\frac{2\Pi(-D^2, k)}{E \sqrt{1 - s^2}} + \frac{1}{k'^2 C} \log_e \left| \frac{C - \sqrt{1 - s^2}}{C + \sqrt{1 - s^2}} \right| \right] ds$$

$$(s = \mu \cot\gamma)$$

$$= \frac{\pi \, \Pi(-D^2, k) \tan\gamma}{E} + \frac{\tan\gamma}{k'^2 C} \mathcal{L}t_{\epsilon \to 0} \left\{ \int_o^{S-\epsilon} \log_e \left(\frac{\sqrt{1 - s^2} - C}{\sqrt{1 - s^2} + C} \right) ds \right.$$

$$\left. + \int_{S+\epsilon}^1 \log_e \left(\frac{C - \sqrt{1 - s^2}}{C + \sqrt{1 - s^2}} \right) ds \right\}$$

$$= \frac{\pi\, \Pi(-D^2,k)\tan\gamma}{E} + \frac{\tan\gamma}{k'^2 C} \mathop{\mathcal{L}t}_{\epsilon\to 0} \left\{ \left[s \log_e \left(\frac{\sqrt{1-s^2}-C}{\sqrt{1-s^2}+C} \right) \right]_0^{S-\epsilon} \right.$$

$$+ \frac{2}{k'^2} \int_0^{S-\epsilon} \frac{s^2 ds}{(S^2-s^2)\sqrt{1-s^2}} + \left[s \log_e \left(\frac{C-\sqrt{1-s^2}}{C+\sqrt{1-s^2}} \right) \right]_{S+\epsilon}^{1}$$

$$\left. + \frac{2}{k'^2} \int_{S+\epsilon}^{1} \frac{s^2 ds}{(S^2-s^2)\sqrt{1-s^2}} \right\}$$; integrating by parts.

$$\therefore - \frac{\pi\beta D \cot\gamma}{2k'^3 CS\rho U_o w_o c^2} L$$

$$= \frac{\pi\, \Pi(-D^2,k)}{E} + \frac{1}{k'^2 C} \mathop{\mathcal{L}t}_{\epsilon\to 0} \left\{ S \left[\log_e \left(\frac{2C^2+O(\epsilon)}{S\epsilon+O(\epsilon^2)} \right) \right. \right.$$

$$\left. - \log_e \left(\frac{2C^2+O(\epsilon)}{S\epsilon+O(\epsilon^2)} \right) \right] + \epsilon \left[\log_e \left(\frac{2C^2+O(\epsilon)}{S\epsilon+O(\epsilon^2)} \right) \right.$$

$$\left.\left. + \log_e \left(\frac{2C^2+O(\epsilon)}{S\epsilon+O(\epsilon^2)} \right) \right] \right\} + \frac{2}{k'^2} \int_0^1 \frac{s^2 ds}{(S^2-s^2)\sqrt{1-s^2}}$$

$$= \frac{\pi}{E} \Pi(-D^2,k) + \frac{2}{k'^2} \int_0^1 \left\{ \frac{S^2}{(S^2-s^2)\sqrt{1-s^2}} \right.$$

$$\left. - \frac{1}{\sqrt{1-s^2}} \right\}$$ ds, since the limit term is zero.

$$= \pi \left(\frac{\Pi(-D^2,k)}{E} - \frac{1}{k'^2} \right) + \frac{2S^2}{k'^2} \int_0^{\frac{\pi}{2}} \frac{d\theta}{S^2-\sin^2\theta},$$

(putting $s = \sin\theta$)

$$= \pi \left(\frac{\Pi(-D^2,k)}{E} - \frac{1}{k'^2} \right) + \frac{2S^2}{k'^2} \int_0^{\infty} \frac{dt}{S^2-C^2t^2}$$

(putting $t = \tan\theta$)

$$= \pi\left(\frac{\Pi(-D^2,k)}{E} - \frac{1}{k'^2}\right)$$

Hence $$L = -\frac{2k'CS\rho U_o w_o c^2 \tan\gamma}{\beta D}\left(\frac{k'^2\Pi(-D^2,k)}{E} - 1\right)$$

$$= \frac{2\eta CS\rho U_o^2 \; c^2\tan^2\gamma \; \sin\Theta}{D}\left[\frac{k'^2\Pi(-D^2,k)}{E} - 1\right]$$

$$\therefore C_L = \frac{2L}{\rho U_o^2 c^2 \tan\gamma}$$

$$= \frac{4\eta CS\tan\gamma\,\sin\Theta}{D}\left[\frac{k'^2\Pi(-D^2,k)}{E} - 1\right]$$

$$\therefore a_2 = \frac{\partial C_L}{\partial\eta}$$

$$= 4\;\frac{CS}{D}\left[\frac{k'^2\Pi(-D^2,k)}{E(k)} - 1\right]\sin\Theta\,\tan\gamma$$

$$= 4\;\frac{CS}{D}\left[\frac{k'^2\Pi(k'^2S^2-1,k)}{E'(k')} - 1\right]\sin\Theta\,\tan\gamma$$

$$= 4\,\sin\Theta\,\tan\Theta\sqrt{\frac{1-\cot^2\gamma\tan^2\Theta}{1-\beta^2\tan^2\Theta}}\left[\frac{\beta^2\tan^2\gamma\,\Pi(\beta^2\tan^2\Theta-1,k)}{E'(\beta\tan\gamma)} - 1\right]$$

$$= 4\,r\sqrt{\frac{1-r^2}{1-B^2r^2}}\left(\frac{B^2\Pi(B^2r^2-1,\sqrt{1-B^2})}{E'(B)} - 1\right)\sin\Theta\tan\gamma\,,$$

(since $r = \tan\Theta\cot\gamma$, $B = \beta\tan\gamma$)

We may abbreviate this to:

$$a_2 = 4r\left(\frac{B^2\Pi}{E'} - 1\right)\sqrt{\frac{1-r^2}{1-B^2r^2}}\;\sin\Theta\;\tan\gamma\,.$$

Special Case: $B = 0$, i.e. $M = 1$

When $B = 0$, $k' = 0$ and $k = 1$.

Also, from § 2

$$B^2\Pi = B^2K'(B) + \frac{1}{r}\sqrt{\frac{1-B^2r^2}{1-r^2}}\left\{\frac{\pi}{2} - E'(B)\,\mathrm{sn}^{-1}(r,B)\right\}$$

$$+ \frac{1}{r}\sqrt{\frac{1-B^2r^2}{1-r^2}} \left\{ sn^{-1}(r,B) - E(\sin^{-1}r,B) \right\} K'(B)$$

In the limit as $B \to 0$, it is proved in Appendix VI that: $B^2K'(B) \to 0$, and in Appendix VII that: $\left\{sn^{-1}(r,B)-E(\sin^{-1}r,B)\right\}K'(B) \to 0$

Also as $B \to 0$, $E'(B) \to 1$ and $sn^{-1}(r,B) \to \sin^{-1}r$.
Hence $B^2 II \to \frac{1}{r\sqrt{1-r^2}}\left[\frac{\pi}{2} - \sin^{-1}r\right]$.

i.e. $B^2 II \to \frac{1}{r\sqrt{1-r^2}} \cos^{-1}r$, as $B \to 0$.

Hence the expression for a_2 becomes:

$$a_2 = 4\left(\cos^{-1}r - r\sqrt{1-r^2}\right)\sin\Theta \tan\gamma$$

Position of the Centre of Pressure due to Deflection of the Elevators

The pressure is again a function only of $\frac{y}{x}$, so that the centre of pressure due to deflection of the elevators is, as before, at $\frac{2}{3}c, 0$.

5.3 Effects of Infinite Pressure at the Leading Edge

When the leading edges lie inside the Mach cone the leading edge pressure is infinite. (See pp. 34 and 39).

It may be proved that with the controls set at an angle θ and the wing at incidence α, the total velocity q at a point P $(x_0+\epsilon, x_0\tan\gamma)$ very near the starboard leading edge is normal to the leading edge and is given by

$$q = (c_1\alpha+c_2\theta)\sqrt{\frac{x_0}{\epsilon}} + \text{bounded terms},$$

where the coefficients c_1 and c_2 are functions only of U_0, γ, Θ and β. (This is true for both the elevator and aileron cases).

By the result proved in Appendix IV of Ref.3, the suction force per unit length of leading edge in a direction normal to the leading edge equals:

$\pi\rho x_0 \cos\gamma \sqrt{1-\beta^2\tan^2\gamma}\ (c_1\alpha+c_2\theta)^2$, which is a term of second order.

/In the ...

In the aileron case the suction per unit length measured parallel to the outward normals to the leading edges is equal and opposite at corresponding points on port and starboard leading edges. The leading edge suction thus produces a side force and a yawing moment about Oz which are both second order terms.

In the elevator case the suction forces per unit length, normal to the leading edges, are equal and of the same sign at corresponding points on port and starboard leading edges. The suction thus produces a drag which is of second order, i.e. is of the same order as the drag of a delta wing at incidence and cannot be neglected.

5.4 Acknowledgements

In conclusion, I wish to express my thanks to Dr. A. Robinson for much valuable help, advice and encouragement.

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APPENDIX I

EVALUATION OF A DEFINITE INTEGRAL

The integral to be evaluated is:

$$\Phi(n) = \frac{1}{n^2}\left\{\frac{\pi}{\sqrt{1-n^2}}\int_n^1 t\,dt\right.$$

$$\left.+\int_{-n}^{n}\left[\frac{2}{\sqrt{1-n^2}}\sin^{-1}\sqrt{\frac{(1-n)(n+t)}{2n(1-t)}}+\sqrt{\frac{n-t}{n+t}}\right]t\,dt\right\}$$

Let $I_1 = \int_n^1 t\,dt$

$$I_2 = \int_{-n}^{n} 2t\sin^{-1}\sqrt{\frac{(1-n)(n+t)}{2n(1-t)}}\,dt$$

$$I_3 = \int_{-n}^{n} t\sqrt{\frac{n-t}{n+t}}\,dt$$

$$\therefore\ n^2\,\Phi(n) = \frac{\pi}{\sqrt{1-n^2}}\,I_1 + \frac{1}{\sqrt{1-n^2}}\,I_2 + I_3$$

$$I_1 = \int_n^1 t\,dt = \frac{1-n^2}{2}$$

Integrating by parts,

$$I_2 = \int_{-n}^{n} 2t\sin^{-1}\sqrt{\frac{(1-n)(n+t)}{2n(1+t)}}\,dt$$

$$= \left[t^2\sin^{-1}\sqrt{\frac{(1-n)(n+t)}{2n(1-t)}}\right]_{-n}^{n} - \int_{-n}^{n}\frac{t^2\sqrt{1-n^2}}{2(1-t)\sqrt{n^2-t^2}}\,dt$$

$$= \frac{\pi n^2}{2} + \frac{\sqrt{1-n^2}}{2}\int_{-n}^{n}\left\{\frac{t}{\sqrt{n^2-t^2}} + \frac{1}{\sqrt{n^2-t^2}} - \frac{1}{(1-t)\sqrt{n^2-t^2}}\right\}dt$$

$$= \frac{\pi n^2}{2} + \frac{\sqrt{1-n^2}}{2}\left\{\left[-\sqrt{n^2-t^2} + \sin^{-1}\frac{t}{n}\right]_{-n}^{n} - \int_{-n}^{n}\frac{dt}{(1-t)\sqrt{n^2-t^2}}\right\}$$

$$= \frac{\pi}{2}\left(n^2+\sqrt{1-n^2}\right)-\frac{\sqrt{1-n^2}}{2}\int_{-n}^{n}\frac{dt}{(1-t)\sqrt{n^2-t^2}}$$

$$= \frac{\pi}{2}\left(n^2+\sqrt{1-n^2}\right)-\frac{\sqrt{1-n^2}}{2}\int_{o}^{\frac{\pi}{2}}\frac{2d\theta}{1+n\cos2\theta} \quad \text{(putting } t = - n\cos2\theta\text{)}$$

$$= \frac{\pi}{2}\left(n^2+\sqrt{1-n^2}\right)-\sqrt{1-n^2}\int_{o}^{\infty}\frac{dv}{(1-n)v^2+(1+n)} \quad \text{(putting } v = \tan\theta\text{)}$$

$$= \frac{\pi}{2}\left(n^2+\sqrt{1-n^2}-1\right)$$

To evaluate I_3, put $t = - n\cos2\theta$.

$$\therefore I_3 = \int_{-n}^{n} t\sqrt{\frac{n-t}{n+t}}\,dt = -2n^2\int_{o}^{\frac{\pi}{2}}\cos2\theta\cot\theta\sin2\theta d\theta$$

$$= -n^2\int_{o}^{\frac{\pi}{2}}(2\cos2\theta+\cos4\theta+1)\,d\theta$$

$$= -\frac{\pi}{2}n^2$$

$\therefore$ Collecting results,

$$n^2\Phi(n) = \frac{\pi}{\sqrt{1-n^2}}I_1 + \frac{1}{\sqrt{1-n^2}}I_2 + I_3$$

$$= \frac{\pi}{2}\sqrt{1-n^2}+\frac{1}{\sqrt{1-n^2}}\cdot\frac{\pi}{2}\left(n^2+\sqrt{1-n^2}-1\right)-\frac{\pi}{2}n^2$$

$$= \frac{\pi}{2}\left(\sqrt{1-n^2}+1-\sqrt{1-n^2}-n^2\right)$$

$$= \frac{\pi}{2}(1-n^2)$$

$$\therefore \Phi(n) = \frac{\pi}{2}\left(\frac{1}{n^2}-1\right)$$

APPENDIX II

EVALUATION OF A DEFINITE INTEGRAL

The integral to be evaluated is:

$$\mathcal{I} = \left\{ \int_{-n}^{n} t\sqrt{\frac{n-t}{n+t}}\,dt + \frac{2}{\sqrt{n^2-1}} \left[\int_{-n}^{-1} t \sinh^{-1}\sqrt{\frac{(n-1)(n+t)}{2n(1-t)}}\,dt \right.\right.$$

$$\left.\left. + \int_{1}^{n} t \sinh^{-1}\sqrt{\frac{(n+1)(n-t)}{2n(t-1)}}\,dt \right] \right\}.$$

We shall first evaluate the two limits:

$$E_1 = \mathop{\mathcal{L}t}_{\epsilon \to 0} \left\{ \sinh^{-1}\sqrt{\frac{(n-1)(n+1-\epsilon)}{2n\epsilon}} - \sinh^{-1}\sqrt{\frac{(n+1)(n-1-\epsilon)}{2n\epsilon}} \right\}, \text{ and}$$

$$E_2 = \mathop{\mathcal{L}t}_{\epsilon \to 0} \epsilon \left\{ \sinh^{-1}\sqrt{\frac{(n-1)(n+1-\epsilon)}{2n\epsilon}} + \sinh^{-1}\sqrt{\frac{(n+1)(n-1-\epsilon)}{2n\epsilon}} \right\}.$$

Now

$$E_1 = \mathop{\mathcal{L}t}_{\epsilon \to 0} \left[\log_e \left\{ \frac{\sqrt{(n-1)(n+1-\epsilon)} + \sqrt{(n+1)(n-1+\epsilon)}}{\sqrt{2n\epsilon}} \right\} - \log_e \left\{ \frac{\sqrt{(n+1)(n-1-\epsilon)} + \sqrt{(n-1)(n+1+\epsilon)}}{\sqrt{2n\epsilon}} \right\} \right]$$

$$= \mathop{\mathcal{L}t}_{\epsilon \to 0} \log_e \left\{ \frac{\sqrt{(n-1)(n+1-\epsilon)} + \sqrt{(n+1)(n-1+\epsilon)}}{\sqrt{(n+1)(n-1-\epsilon)} + \sqrt{(n-1)(n+1+\epsilon)}} \right\}$$

$$= 0.$$

$$E_2 = \mathop{\mathcal{L}t}_{\epsilon \to 0} \epsilon \left[\log_e \left\{ \frac{\sqrt{(n-1)(n+1-\epsilon)} + \sqrt{(n+1)(n-1+\epsilon)}}{\sqrt{2n\epsilon}} \right\} + \log_e \left\{ \frac{\sqrt{(n+1)(n-1-\epsilon)} + \sqrt{(n-1)(n+1+\epsilon)}}{\sqrt{2n\epsilon}} \right\} \right]$$

i.e.

$$E_2 = \mathop{\mathcal{L}t}_{\epsilon \to 0} \epsilon \left[\log_e \left\{ \sqrt{(n-1)(n+1-\epsilon)} + \sqrt{(n+1)(n-1+\epsilon)} \right\} + \log_e \left\{ \sqrt{(n+1)(n-1-\epsilon)} + \sqrt{(n-1)(n+1+\epsilon)} \right\} \right.$$

$$- \log_e 2n \Big] - \mathop{\mathcal{L}t}_{\epsilon \to 0} \epsilon \log_e \epsilon$$

$$= 0$$

Returning to $\mathcal{J}$,

Let $\mathcal{J}_1 = \int_{-n}^{n} t \sqrt{\frac{n-t}{n+t}}\, dt$,

$$\mathcal{J}_2 = \int_{-n}^{1} 2t \sinh^{-1} \sqrt{\frac{(n-1)(n+t)}{2n(1-t)}}\, dt + \int_{1}^{n} 2t \sinh^{-1} \sqrt{\frac{(n+1)(n-t)}{2n(t-1)}}\, dt$$

$$\therefore \mathcal{J} = \mathcal{J}_1 + \frac{1}{\sqrt{n^2-1}} \mathcal{J}_2$$

It is proved in Appendix I that,

$$\int_{-n}^{n} t \sqrt{\frac{n-t}{n+t}}\, dt = - \frac{\pi}{2} n^2$$

$$\text{i.e. } \mathcal{J}_1 = - \frac{\pi}{2} n^2$$

$$\therefore \mathcal{J} = - \frac{\pi}{2} n^2 + \frac{1}{\sqrt{n^2-1}} \mathcal{J}_2 .$$

Now
$$\mathcal{J}_2 = \mathop{\mathcal{L}t}_{\epsilon \to 0} \left\{ \int_{-n}^{1-\epsilon} 2t \sinh^{-1} \sqrt{\frac{(n-1)(n+t)}{2n(1-t)}}\, dt + \int_{1+\epsilon}^{n} 2t \sinh^{-1} \sqrt{\frac{(n+1)(n-t)}{2n(t-1)}}\, dt \right\}$$

Integrating by Parts,

$$\mathcal{J}_2 = \mathop{\mathcal{L}t}_{\epsilon \to 0} \left\{ \left[t^2 \sinh^{-1} \sqrt{\frac{(n-1)(n+t)}{2n(1-t)}} \right]_{-n}^{1-\epsilon} - \sqrt{\frac{n^2-1}{2}} \int_{-n}^{1-\epsilon} \frac{t^2 dt}{(1-t)\sqrt{n^2-t^2}} \right.$$

$$\left. + \left[t^2 \sinh^{-1} \sqrt{\frac{(n+1)(n-t)}{2n(t-1)}} \right]_{1+\epsilon}^{n} - \sqrt{\frac{n^2-1}{2}} \int_{1+\epsilon}^{n} \frac{t^2 dt}{(1-t)\sqrt{n^2-t^2}} \right\}$$

$$= \mathop{\mathcal{L}t}_{\epsilon \to 0} \left\{ (1-\epsilon)^2 \sinh^{-1} \sqrt{\frac{(n-1)(n+1-\epsilon)}{2n\epsilon}} - (1+\epsilon)^2 \sinh^{-1} \sqrt{\frac{(n+1)(n-1-\epsilon)}{2n\epsilon}} \right\}$$

$$- \frac{\sqrt{n^2-1}}{2} \int_{-n}^{n} \frac{t^2 dt}{(1-t)\sqrt{n^2-t^2}}$$

$$= \mathop{\mathcal{L}t}_{\epsilon \to 0} \left[\sinh^{-1} \sqrt{\frac{(n-1)(n+1-\epsilon)}{2n\epsilon}} - \sinh^{-1} \sqrt{\frac{(n+1)(n-1-\epsilon)}{2n\epsilon}} \right] (1+\epsilon^2)$$

$$- 2 \mathop{\mathcal{L}t}_{\epsilon \to 0} \epsilon \left[\sinh^{-1} \sqrt{\frac{(n-1)(n+1-\epsilon)}{2n\epsilon}} + \sinh^{-1} \sqrt{\frac{(n+1)(n-1-\epsilon)}{2n\epsilon}} \right]$$

$$- \frac{\sqrt{n^2-1}}{2} \int_{-n}^{n} \frac{t^2 dt}{(1-t)\sqrt{n^2-t^2}}$$

$$= E_1 \cdot \mathop{\mathcal{L}t}_{\epsilon \to 0} (1+\epsilon^2) - 2E_2 - \frac{\sqrt{n^2-1}}{2} \int_{-n}^{n} \frac{t^2 dt}{(1-t)\sqrt{n^2-t^2}}$$

$$= - \frac{n^2-1}{2} \int_{-n}^{n} \frac{t^2 dt}{(1-t)\sqrt{n^2-t^2}}, \quad \text{since } E_1 = E_2 = 0$$

i.e. $$\mathcal{J}_2 = \frac{\sqrt{n^2-1}}{2} \left\{ \int_{-n}^{n} \frac{1+t}{\sqrt{n^2-t^2}} dt - \int_{-n}^{n} \frac{dt}{(1-t)\sqrt{n^2-t^2}} \right\}$$

i.e. $$\mathcal{J}_2 = \sqrt{n^2-1} \left\{ \int_0^{\frac{\pi}{2}} (1 - n\cos 2\theta)\, d\theta + \int_0^{\frac{\pi}{2}} \frac{d\theta}{1 + n\cos 2\theta} \right\},$$

(putting $t = -n\cos 2\theta$).

$$= \frac{\pi}{2}\sqrt{n^2-1} + \sqrt{n^2-1} \int_0^{\infty} \frac{dv}{(1+n)-(n-1)v^2}, \quad v = \tan\theta$$

$$= \frac{\pi}{2}\sqrt{n^2-1},$$ the second term vanishing because $(n-1) > 0$.

$$\therefore \mathcal{J} = - \frac{\pi}{2} n^2 + \frac{1}{\sqrt{n^2-1}} \mathcal{J}_2$$

$$= \frac{\pi}{2}(1-n^2).$$

/Appendix III)

APPENDIX III

EVALUATION OF A DEFINITE INTEGRAL

The **integral** to be evaluated is:

$$\chi(n) = \frac{1}{n}\left\{\frac{\pi}{\sqrt{1-n^2}}\int_n^1 dt + \int_{-n}^{n}\left[\frac{2}{\sqrt{1-n^2}}\sin^{-1}\sqrt{\frac{(1-n)(n+t)}{2n(1-t)}} + \sqrt{\frac{n-t}{n+t}}\right]dt\right\}$$

Let $J_1 = \int_n^1 dt$

$$J_2 = \int_{-n}^{n} \sin^{-1}\sqrt{\frac{(1-n)(n+t)}{2n(1-t)}}\,dt$$

$$J_3 = \int_{-n}^{n} \sqrt{\frac{(n-t)}{(n+t)}}\,dt$$

$$\therefore\; n\chi(n) = \frac{\pi}{\sqrt{1-n^2}}J_1 + \frac{2}{\sqrt{1-n^2}}J_2 + J_3 .$$

$$J_1 = \int_n^1 dt = (1-n)$$

Integrating by parts,

$$J_2 = \left[t\sin^{-1}\sqrt{\frac{(1-n)(n+t)}{2n(1-t)}}\right]_{-n}^{n} - \int_{-n}^{n}\frac{t\sqrt{1-n^2}}{2(1-t)\sqrt{n^2-t^2}}\,dt$$

$$= \frac{\pi n}{2} + \frac{\sqrt{1-n^2}}{2}\left\{\int_{-n}^{n}\frac{dt}{\sqrt{n^2-t^2}} - \int_{-n}^{n}\frac{dt}{(1-t)\sqrt{n^2-t^2}}\right\}$$

$$= \frac{\pi n}{2} + \frac{\pi}{2}\sqrt{1-n^2} - \sqrt{1-n^2}\int_0^{\frac{\pi}{2}}\frac{d\theta}{1+n\cos 2\theta},$$

(putting $t = -n\cos 2\theta$)

$$= \frac{\pi}{2}\left(n+\sqrt{1-n^2}\right) - \sqrt{1-n^2}\int_0^{\infty}\frac{dv}{(1-n)v^2+(1+n)},$$

(putting $v = \tan\theta$)

$$= \frac{\pi}{2}\left(n+\sqrt{1-n^2}-1\right)$$

$$J_3 = \int_{-n}^{n} \sqrt{\frac{n-t}{n+t}}\, dt$$

$$= 2n \int_{0}^{\frac{\pi}{2}} \cot\theta \sin 2\theta \, d\theta, \text{ (putting } t = -n\cos 2\theta)$$

$$= 4n \int_{0}^{\frac{\pi}{2}} \cos^2\theta \, d\theta = \pi n$$

$\therefore$ Collecting results,

$$n\chi(n) = \frac{\pi}{\sqrt{1-n^2}} J_1 + \frac{2}{\sqrt{1-n^2}} J_2 + J_3$$

$$= \pi \left[\frac{1-n}{\sqrt{1-n^2}} + \frac{(n+\sqrt{1-n^2}-1)}{\sqrt{1-n^2}} + n\right]$$

$$= \pi\ (1+n)$$

$$\therefore \chi(n) = \pi\left(\frac{1}{n}+1\right).$$

APPENDIX IV

EVALUATION OF A DEFINITE INTEGRAL

The integral to be evaluated is:

$$\mathcal{J} = \left\{ \int_{-n}^{n} \sqrt{\frac{n-t}{n+t}}\, dt + \frac{2}{\sqrt{n^2-1}} \left[\int_{-n}^{1} \sinh^{-1} \sqrt{\frac{(n-1)(n+t)}{2n(1-t)}}\, dt \right.\right.$$

$$\left.\left. + \int_{1}^{n} \sinh^{-1} \sqrt{\frac{(n+1)(n-t)}{2n(t-1)}}\, dt \right] \right\}.$$

Let $\mathcal{J}_1 = \int_{-n}^{n} \sqrt{\frac{n-t}{n+t}}\, dt$

$$\mathcal{J}_2 = \int_{-n}^{1} \sinh^{-1} \sqrt{\frac{(n-1)(n+t)}{2n(1-t)}}\, dt + \int_{1}^{n} \sinh^{-1} \sqrt{\frac{(n+1)(n-t)}{2n(t-1)}}\, dt$$

$$\therefore \mathcal{J} = \mathcal{J}_1 + \frac{2}{\sqrt{n^2-1}} \mathcal{J}_2$$

It is proved in Appendix III, that,

$$\int_{-n}^{n} \sqrt{\frac{n-t}{n+t}}\, dt = \pi n$$

$$\therefore \mathcal{J} = \pi n + \frac{2}{\sqrt{n^2-1}} \mathcal{J}_2$$

Now $\mathcal{J}_2 = \underset{\epsilon \to 0}{\mathcal{L}t} \left\{ \int_{-n}^{1-\epsilon} \sinh^{-1} \sqrt{\frac{(n-1)(n+t)}{2n(1-t)}}\, dt + \int_{1+\epsilon}^{n} \sinh^{-1} \sqrt{\frac{(n+1)(n-t)}{2n(t-1)}}\, dt \right\}$

Integrating by Parts,

$$\mathcal{J}_2 = \underset{\epsilon \to 0}{\mathcal{L}t} \left\{ \left[t \sinh^{-1} \sqrt{\frac{(n-1)(n+t)}{2n(1-t)}} \right]_{-n}^{1-\epsilon} - \frac{\sqrt{n^2-1}}{2} \int_{-n}^{1-\epsilon} \frac{t\, dt}{(1-t)\sqrt{n^2-t^2}} \right.$$

$$\left. + \left[t \sinh^{-1} \sqrt{\frac{(n+1)(n-t)}{2n(t-1)}} \right]_{1+\epsilon}^{n} - \frac{\sqrt{n^2-1}}{2} \int_{1+\epsilon}^{n} \frac{t\, dt}{(1-t)\sqrt{n^2-t^2}} \right.$$

$$= \mathop{\mathcal{L}t}_{\epsilon\to 0}\left\{(1-\epsilon)\sinh^{-1}\sqrt{\frac{(n-1)(n+1-\epsilon)}{2\epsilon n}} - (1+\epsilon)\sinh^{-1}\sqrt{\frac{(n+1)(n-1-\epsilon)}{2\epsilon n}}\right.$$

$$- \sqrt{\frac{n^2-1}{2}}\int_{-n}^{n}\frac{t\,dt}{(1-t)\sqrt{n^2-t^2}}$$

$$= \mathop{\mathcal{L}t}_{\epsilon\to 0}\left\{\left[\sinh^{-1}\sqrt{\frac{(n-1)(n+1-\epsilon)}{2\epsilon n}} - \sinh^{-1}\sqrt{\frac{(n+1)(n-1-\epsilon)}{2\epsilon n}}\right]\right.$$

$$\left. -\epsilon\left[\sinh^{-1}\sqrt{\frac{(n-1)(n+1-\epsilon)}{2\epsilon n}} - \sinh^{-1}\sqrt{\frac{(n+1)(n-1-\epsilon)}{2\epsilon n}}\right]\right\}$$

$$+\sqrt{\frac{n^2-1}{2}}\left\{\int_{-n}^{n}\frac{dt}{\sqrt{n^2-t^2}} - \int_{-n}^{n}\frac{dt}{(1-t)\sqrt{n^2-t^2}}\right\}$$

The limit term is zero from the results $E_1 = E_2 = 0$ proved in Appendix II.

Hence $$\mathcal{J}_2 = \sqrt{\frac{n^2-1}{2}}\left\{\int_{-n}^{n}\frac{dt}{\sqrt{n^2-t^2}} - \int_{-n}^{n}\frac{dt}{(1-t)\sqrt{n^2-t^2}}\right\}$$

$$= \sqrt{\frac{n^2-1}{2}}\left(\pi - 2\int_{0}^{\frac{\pi}{2}}\frac{d\theta}{1+n\cos 2\theta}\right), \text{ (putting } t = -n\cos 2\theta)$$

$$= \sqrt{\frac{n^2-1}{2}}\left(\pi - 2\int_{0}^{\infty}\frac{dv}{(1+n)-(n-1)v^2}\right), \text{ (putting } v = \tan\theta)$$

$$= \sqrt{\frac{n^2-1}{2}}\,\pi,$$ the second term vanishing because $(n-1) > 0$.

$$\therefore \mathcal{J} = \pi n + \frac{2}{\sqrt{n^2-1}}\mathcal{J}_2 = \pi n + \pi$$

i.e. $\mathcal{J} = \pi(n+1)$.

APPENDIX V

APPROXIMATE THEORY FOR TRAILING EDGE CONTROLS

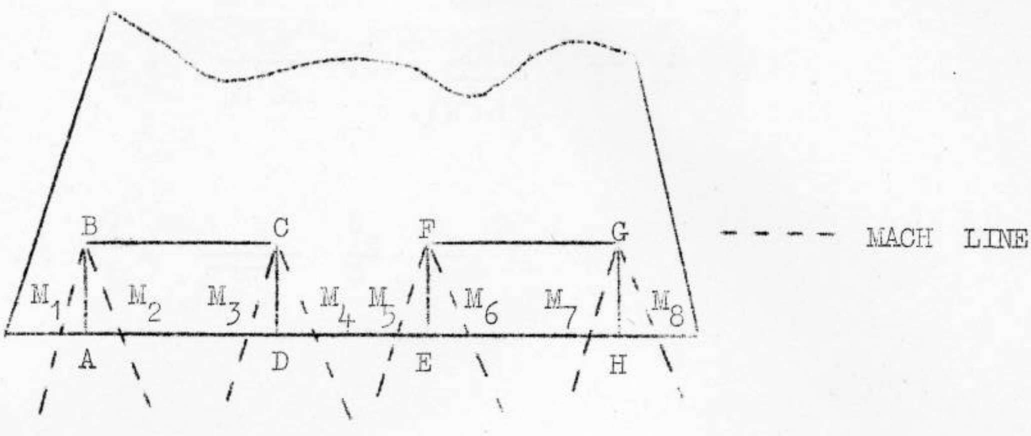


FIG. 10

The induced flow due to deflection of the control surface ABCD can only affect the area M_1BCM_4 of the wing. Over the area $M_2 M_3$ CB, flow conditions are truly two-dimensional. If the Mach angle ϕ ($= \frac{1}{2}\angle M_1BM_2$) is small or if the aspect ratio A_0 of the control is sufficiently large, it is justifiable to neglect errors introduced by assuming that the flow over M_2BA and M_3CD is also two dimensional and by neglecting the end effects over M_1BA and M_4CD when calculating forces produced by the controls.

Therefore assuming $A_0\sqrt{M^2-1} \gg 1$, the lift increment per control is given by:

$$\Delta L = \frac{\rho U^2 S_c \theta}{\sqrt{M^2-1}} .$$ (This follows from Ackeret's theory for a two dimensional wing).

If the controls are elevators, $L = \dfrac{2\rho U^2 S_c \theta}{\sqrt{M^2-1}}$,

giving: $$a_2 = \frac{4}{\sqrt{M^2-1}} \cdot \frac{S_c}{S} .$$

With our assumptions, the resultant force due to deflection of a control surface acts at its centroid. Let b_0 be the distance between the centroids of the ailerons. The rolling moment is then given by:

$$\bar{L} = b_o \, \Delta L = \frac{b_o \rho U^2 S_c \xi}{\sqrt{M^2-1}}$$

$$\therefore \; C_{\bar{L}} = \frac{2\bar{L}}{\rho U^2 S b} = \frac{2}{\sqrt{M^2-1}} \; \frac{S_c}{S} \; \frac{b_o}{b} \; \xi$$

$$\therefore \; \ell_{\xi} = \frac{2}{\sqrt{M^2-1}} \; \frac{S_c}{S} \; \frac{b_o}{b}$$

/Appendix VI ...

APPENDIX VI

EVALUATION OF A LIMIT

The limit to be evaluated is:

$$\mathop{\mathcal{L}t}_{B \to 0} \; B^2 \; K'(B) \; .$$

Now
$$K'(B) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-(1-B^2)\sin^2\theta}}$$

$$\therefore \quad \left| B^2 K'(B) \right| = \left| \int_0^{\frac{\pi}{2}} \frac{B^2 d\theta}{\sqrt{1-(1-B^2)\sin^2\theta}} \right|$$

$$= \left| \int_0^{\frac{\pi}{2}} \frac{B^2 d\theta}{\sqrt{\cos^2\theta + B^2 \sin^2\theta}} \right|$$

$$= \left| \int_0^{\infty} \frac{B^2 dt}{\sqrt{(1+t^2)(1+B^2t^2)}} \right| , \quad t = \tan\theta \; .$$

$$= \left| \int_0^{\infty} \frac{B^2 dv}{\sqrt{(B^2+v^2)(1+v^2)}} \right| , \quad v = Bt \; .$$

$$< \left| \int_0^{\infty} \frac{B^2 dv}{\sqrt{B^2(1+v^2)}} \right|$$

i.e.
$$\left| B^2 K'(B) \right| < \left| \int_0^{\infty} \frac{B \; dv}{\sqrt{1+v^2}} \right| = \frac{\pi}{2} \; B$$

Hence
$$\mathop{\mathcal{L}t}_{B \to 0} \; B^2 \; K'(B) = 0 \; .$$

/Appendix VII ...

APPENDIX VII

EVALUATION OF A LIMIT

The limit to be evaluated is:

$$\mathcal{L}t_{B\to o}\left[sn^{-1}(r,B) - E(\sin^{-1}r,B)\right]K'(B).$$

$$sn^{-1}(r,B) - E(\sin^{-1}r,B) = \int_o^{\sin^{-1}r}\left[\frac{1}{\sqrt{1-B^2\sin^2\theta}} - dn^2(\theta,B)\right]d\theta$$

$$= \int_o^{\sin^{-1}r}\left[\frac{1-\sqrt{1-B^2\sin^2\theta}}{\sqrt{1-B^2\sin^2\theta}} + B^2sn^2(\theta,B)\right]d\theta$$

$$= \int_o^{\sin^{-1}r}\left[\frac{B^2\sin^2\theta}{\sqrt{1-B^2\sin^2\theta}\left[1+\sqrt{1-B^2\sin^2\theta}\right]} + B^2sn^2(\theta,B)\right]d\theta$$

$$\therefore \mathcal{L}t_{B\to o}\left[sn^{-1}(r,B) - E(\sin^{-1}r,B)\right]K'(B)$$

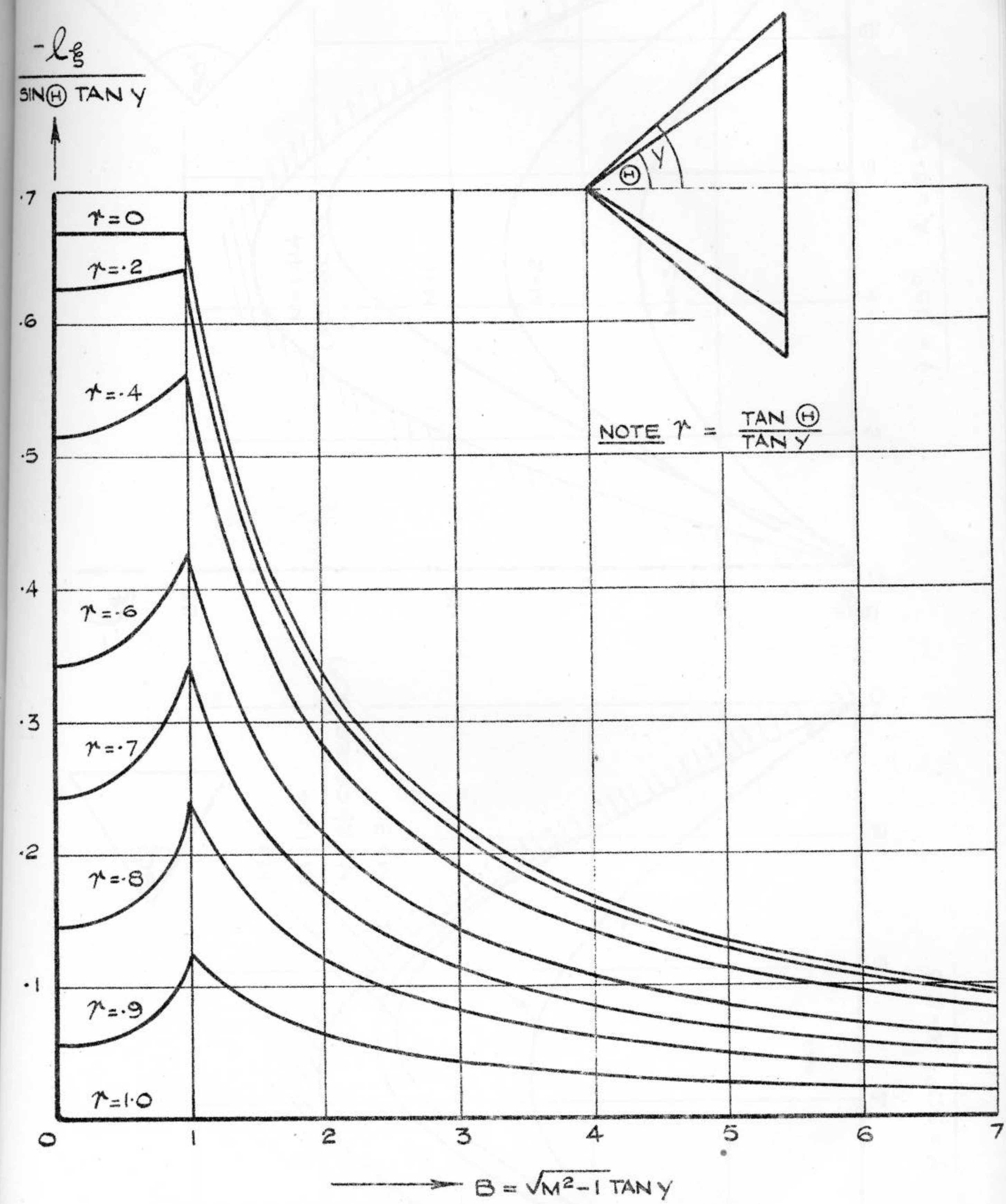
$$= \mathcal{L}t_{B\to o}\left\{\int_o^{\sin^{-1}r}\left[\frac{\sin^2\theta}{\sqrt{1-B^2\sin^2\theta}\left[1+\sqrt{1-B^2\sin^2\theta}\right]} + sn^2(\theta,B)\right]d\theta\right\} \times \mathcal{L}t_{B\to o} B^2K'(B)$$

$$= \left\{\int_o^{\sin^{-1}r}\frac{3}{2}\sin^2\theta\, d\theta\right\} \times \mathcal{L}t_{B\to o} B^2K'(B)$$

$= 0$, since $\mathcal{L}t_{B\to o} B^2K'(B) = 0$ by Appendix VI.

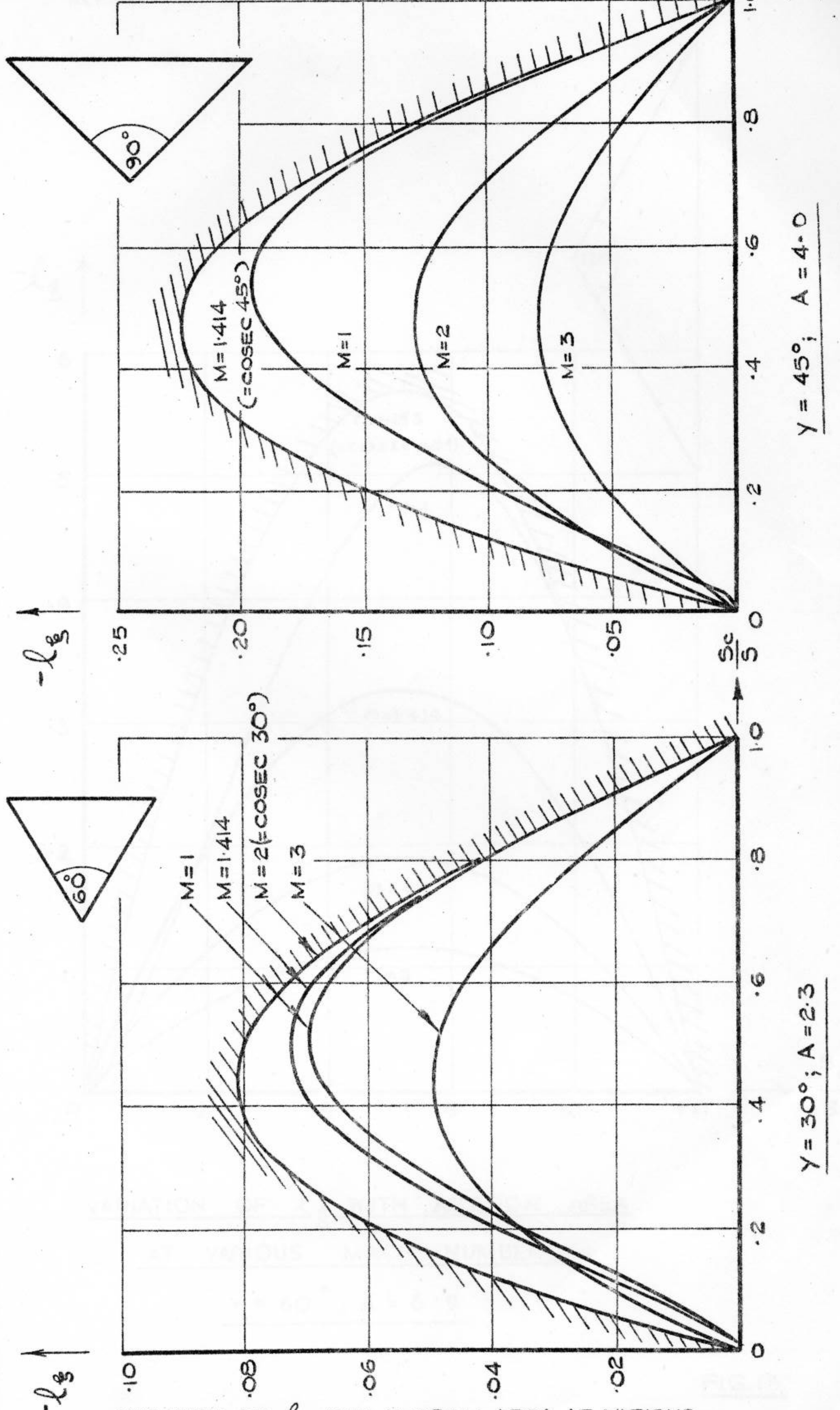
Hence $\mathcal{L}t_{B\to o}\left[\sin^{-1}(r,B) - E(\sin^{-1}r,B)\right]K'(B) = 0$.

------oOo-------



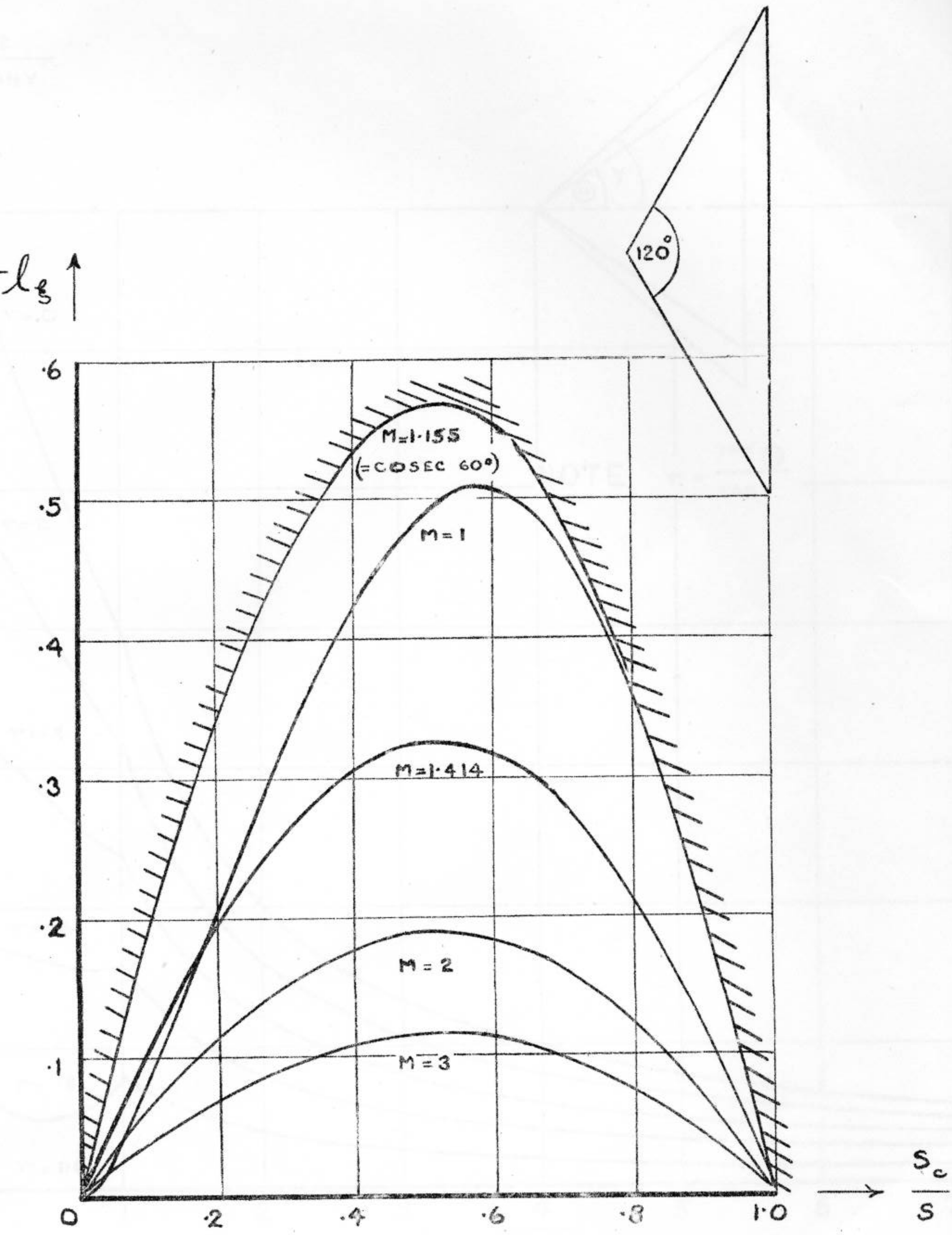
VARIATION OF $\left\{ \frac{l_\xi}{\sin \Theta \tan \gamma} \right\}$ WITH THE PARAMETERS B AND τ

FIG.11.



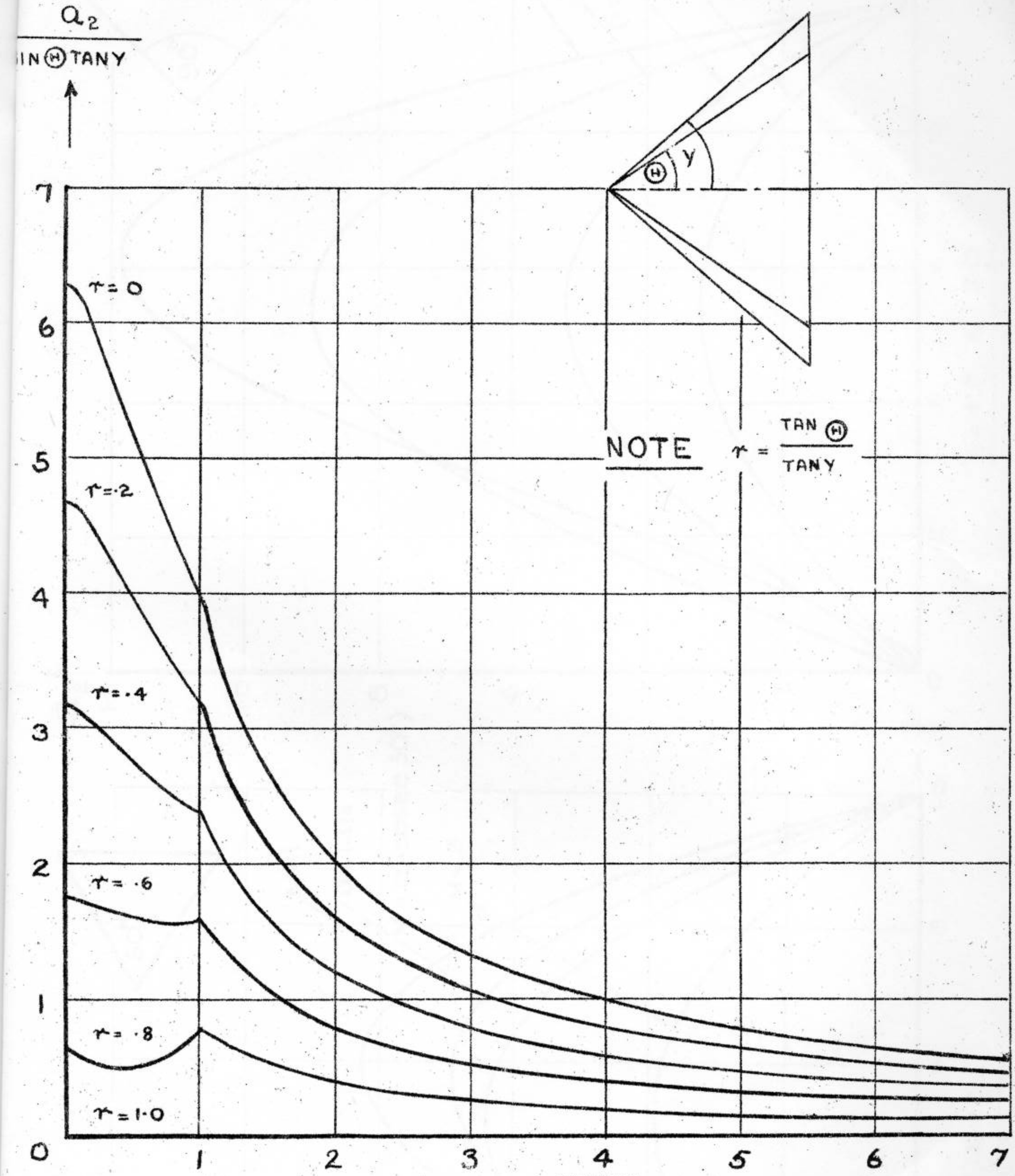
VARIATION OF l_ξ WITH AILERON AREA AT VARIOUS MACH NUMBERS

FIG.12.



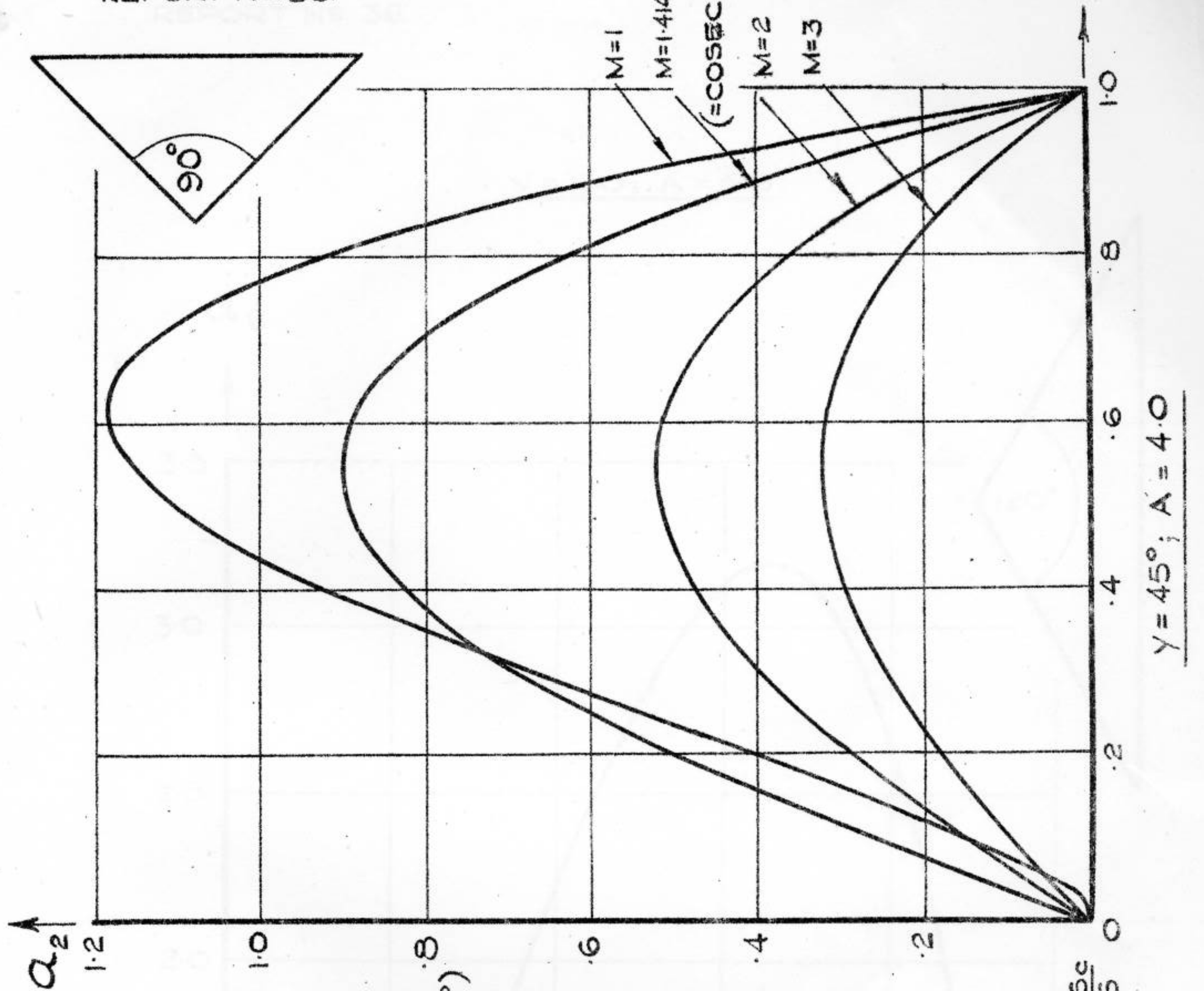
VARIATION OF l_ξ WITH AILERON AREA
AT VARIOUS MACH NUMBERS
$\gamma = 60°$; $A = 6·9$

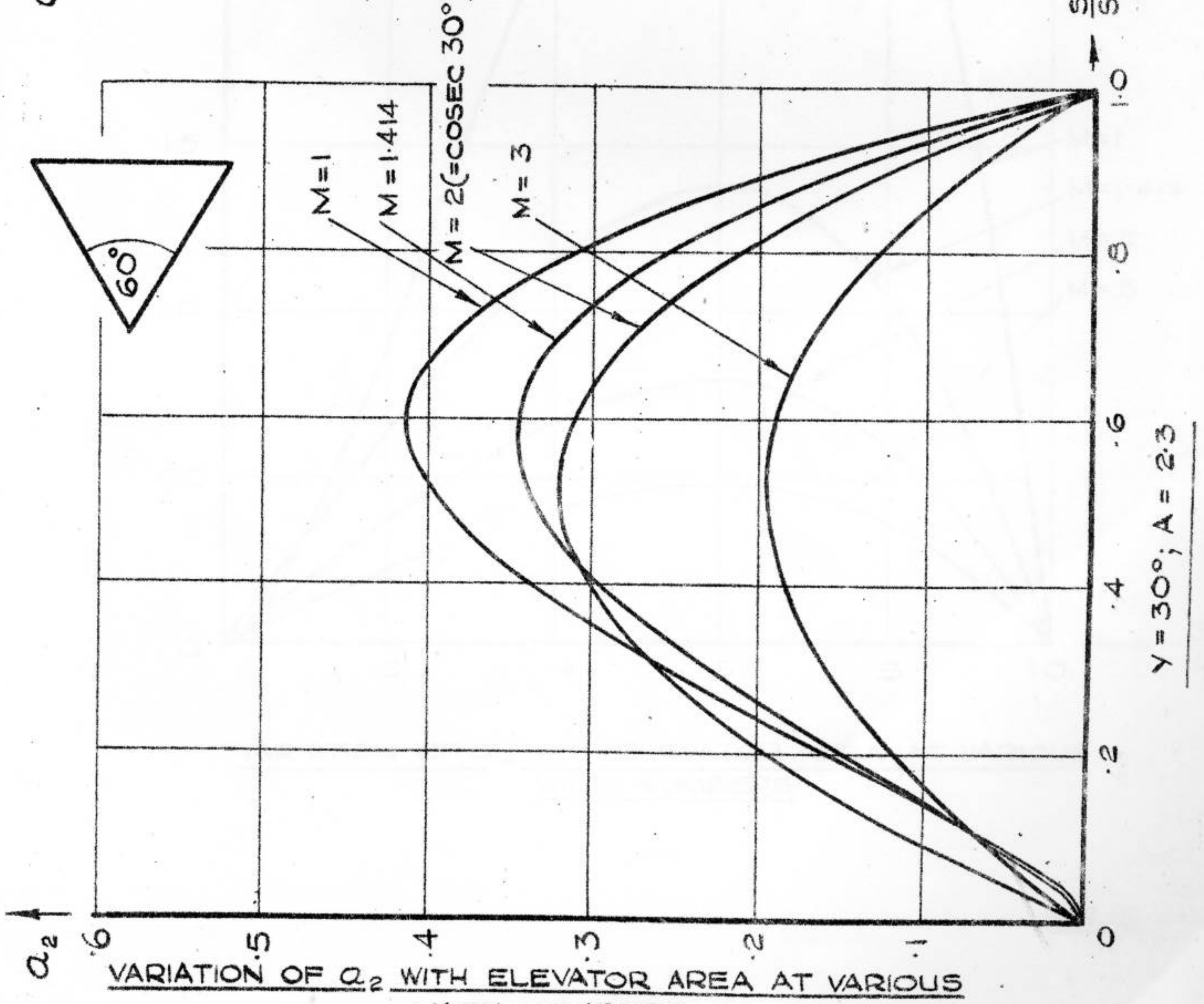
FIG. 13.



VARIATION OF $\left\{\frac{a_2}{\sin \Theta \tan \gamma}\right\}$ WITH THE PARAMETERS B AND r

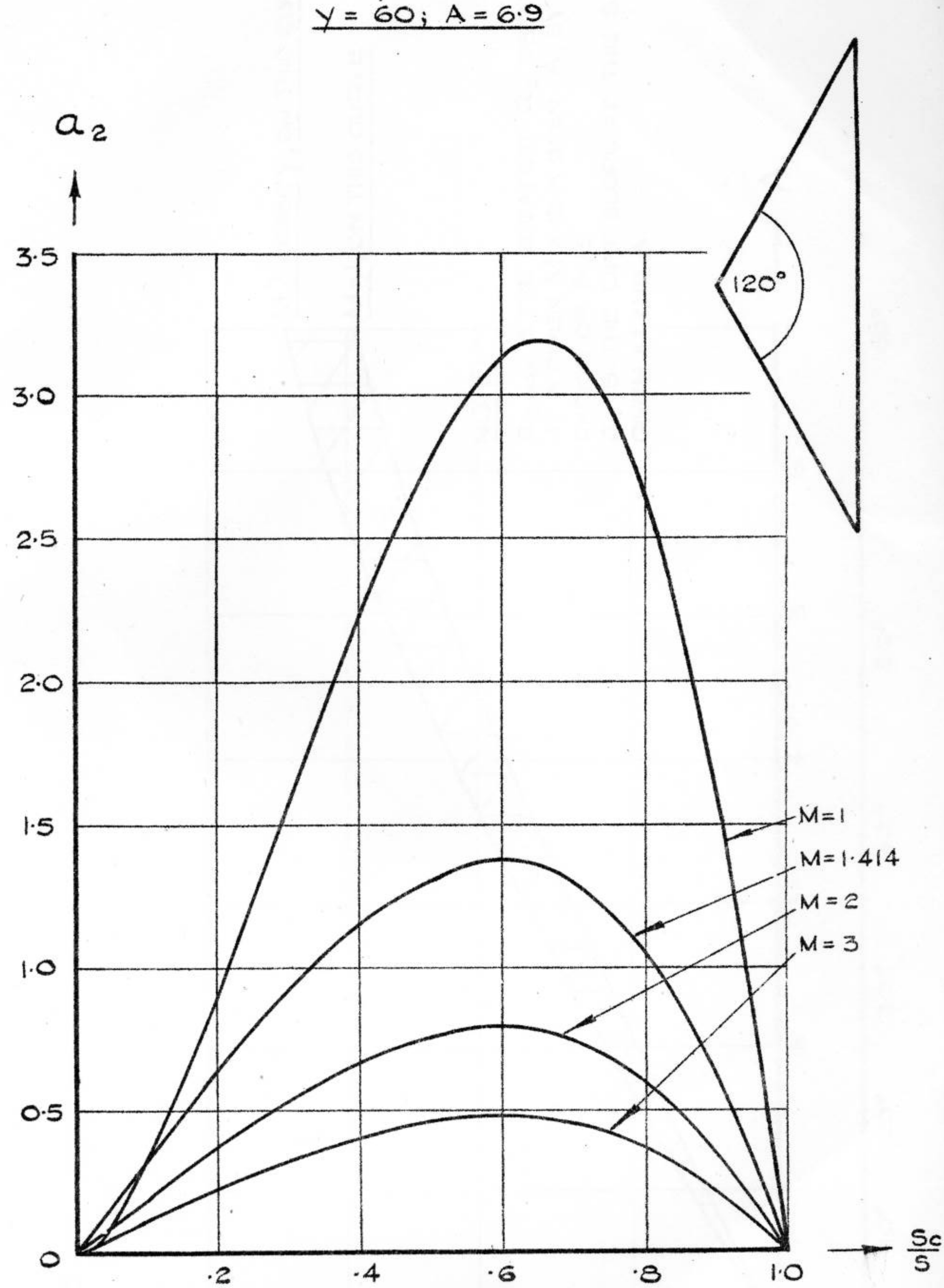
FIG 14





VARIATION OF a_2 WITH ELEVATOR AREA AT VARIOUS MACH NUMBERS

FIG.15.



VARIATION OF a_2 WITH ELEVATOR AREA AT VARIOUS MACH NUMBERS.

FIG.16.

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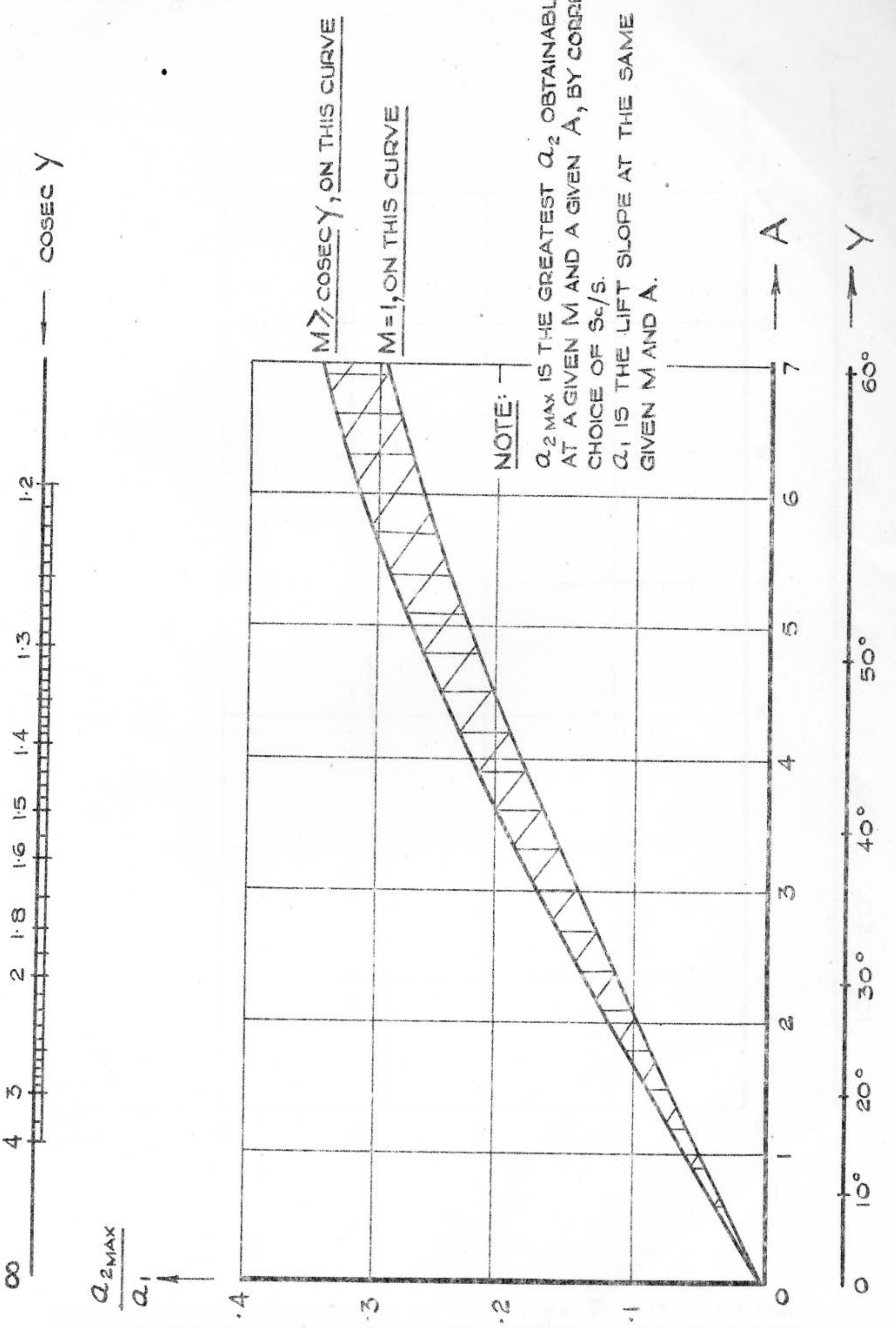
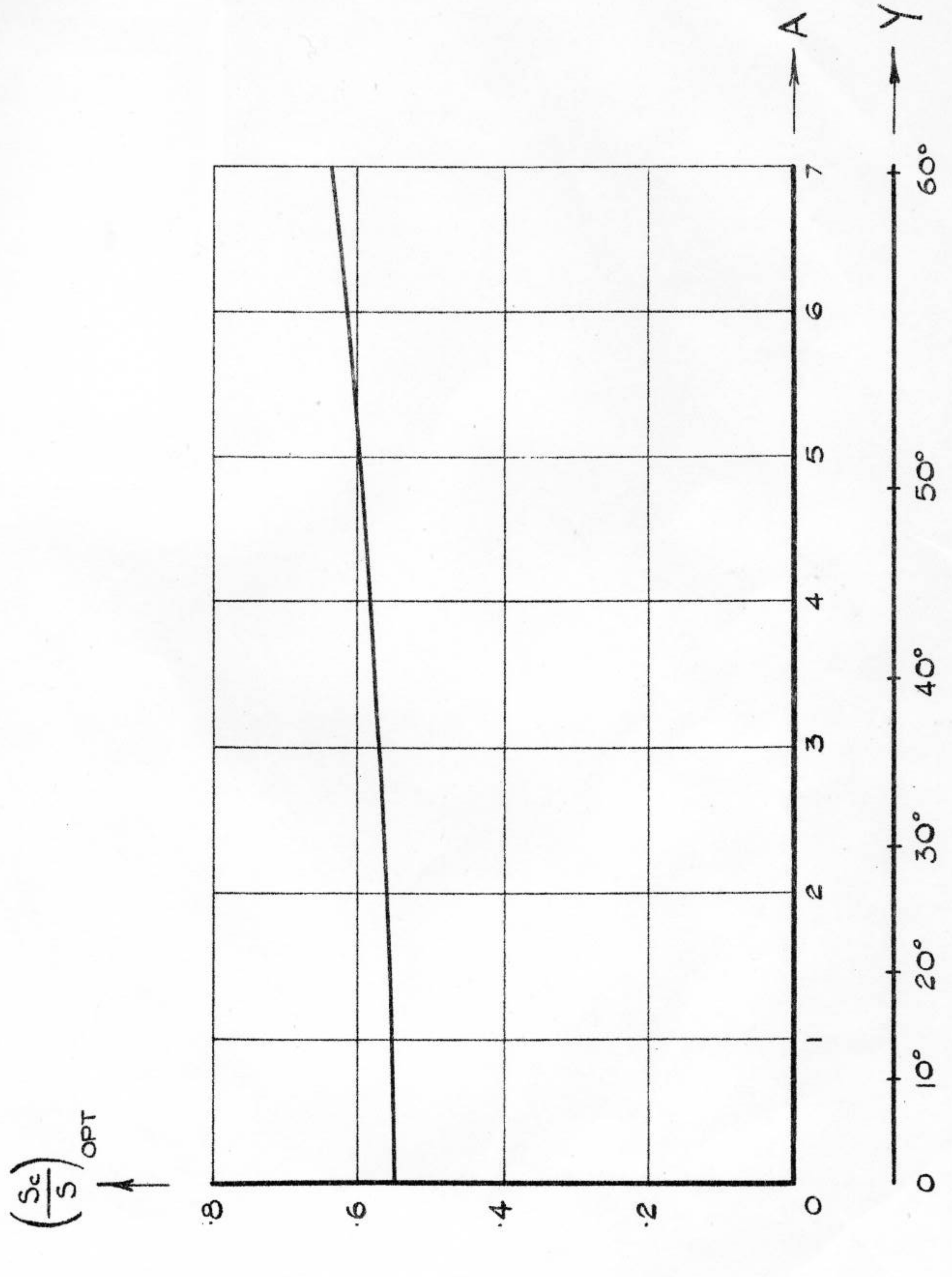
REPORT № 36.



CURVES OF $\frac{a_{2MAX}}{a_1}$ AGAINST A, SHOWING RANGE OF VARIATION OF $\frac{a_{2MAX}}{a_1}$ WITH MACH NUMBER.

FIG.17.



CURVE SHOWING VARIATION OF OPTIMUM $\frac{S_c}{S}$ WITH ASPECT RATIO.

FIG.18.