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The Time to Achieve Peak Output With Special Reference to Aircraft Production

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P.J. Stanley, B.A. (Cantab.), F.S.S.

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SUMMARY

During the war, the rate of production which could be expected from an organisation setting out to manufacture a new type of aircraft was of great interest from the planning point of view. It was found that a time of the order of twelve to eighteen months elapsed between the delivery of the first production type and the achievement of the peak rate of production for which the factory was planned. This report analyses how this growth takes place and uses the theoretical growth curve, known as the Logistic, for fitting smooth curves to a large amount of data from several countries. From the Logistic formula an index of the rate of growth for judging the relative performances in different projects has been evolved. Its uses include also the planning of production schedules. The factors determining the rate of growth are investigated but these would appear to be more in the intangible category rather than among the simple physical conditions such as size of programme or of aircraft.

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1. Introduction

Considerable interest is always aroused by the question of the development of a new aircraft type and great efforts are made to shorten this time by as much as possible in each individual case. The reasons for this are obvious, being based mainly on strategy in the case of service types and on economy in the case of civil types. For the purpose of the present report this development time is looked upon as consisting of two separate parts. One is the time taken for the original conception to materialise in the form of the production prototype and the other is the period between this materialisation and the time when the manufacturing organization is producing at the rate for which it was designed and planned at the time when production commenced. It is with this latter period only that this report is concerned. It attempts to analyse how this growth of production to programme peak takes place and to discover whether some general or "average" growth curve can be evolved. Some suggestions concerning a possible form of this curve are made as also are its possible uses. These include the planning of production schedules, the setting of a standard in this respect and use as a "yard-stick" against which actual deliveries in any individual project can be judged.

This investigation follows that in the same field by J.V. Connolly in which he investigated the form of the growth curve. Continuing on lines suggested by his approach it has been found advisable to treat each project separately and in considerably more detail. This has entailed an investigation into the history and use of growth curves generally and work which has been done previously in spheres far removed from aeronautics. The theoretical curve which has been chosen as the most useful is the Logistic or, as it is sometimes known, the Pearl-Reed curve. From the formula for this an index of the steepness of the curve or the rate of growth has been evolved and is put forward as the best means of judging the relative performances in different projects.

The actual data which has been used in this investigation and to which the formula is applied is extensive having been gathered from the war-time production statistics of four countries, Great Britain, the U.S.A., Germany and Japan. In all, the statistics for 86 projects were available. One point of interest is that there is no evidence of any association between the rate of build up and either the size of the production programme or the size of the aircraft. It would, therefore, appear that the skill and managerial efficiency of the organization and the design of the aircraft are the determining factors.

2. The Nature of the Data

An example of the type of data characteristic of growth in production rate is presented in Fig. 1 where the total production during each month is plotted against time.

At first the rate of production is small, only a few aircraft per month, but it is not only increasing but increasing at an increasing rate. This goes on until the output is about half that planned as the peak output of the plant when the increase achieved each month becomes smaller as time goes by until peak output is reached.

When the general manufacturing conditions are studied it is easy to see that such a result is to be expected. In the very early stages production methods will be very similar to those used for the production of prototypes and components will be produced individually. However, when the decision to produce the aircraft in quantity is made, orders for the design and manufacture of tools, jigs, gauges, etc. for quantity production will be issued. Thus, in due course, when these items come forward and labour has been trained in their use, the rate of production will increase. This process proceeds with ever increasing momentum as all the factors tend in this direction. The more trained labour that is available, the more capacity for training there will be. More tools will be produced and the effect of the first tooling up in the manufacture of components will be felt later when the larger quantities flowing from these departments reach the production line.

If this was a complete description of affairs it might be expected that the behaviour of the resulting rate of production curve would be similar to that of the exponential function, at least in its earlier stages. In the case of several of the factors it could be argued that the rate of increase is proportional to the absolute value of the quantity and this is one of the chief characteristics of the exponential function.

But, when the decision to produce aircraft is made, the actual quantities required are also decided upon. This entails fixing a figure for the maximum or peak output of the plant and from this the whole manufacturing unit is planned. With this figure as a basis, the floor area, the number of each type of machine tool, the quantity and type of labour, etc., in short, all the characteristics that can be brought together under the heading of manufacturing capacity, have to be calculated.

This maximum capacity fixes an upper ceiling above which the output cannot rise and so a factor is introduced which tends to limit the output, which works in a direction opposite to those discussed previously. Thus the second half of the curve is obtained with the output still increasing but with less rapidity until the upper limit fixed by the total capacity of the plant is reached.

Obviously, an object for every management is to get the plant working at maximum capacity in as short a time as possible and it is the time taken to do this that is studied here. A rough indication of the period can be obtained from the time that elapses between the production of the first aircraft and the attainment of peak output. This measure has certain disadvantages which arise primarily over difficulties with definition. Firstly, there are obvious difficulties about the definition of "the first aircraft to be produced" even when this is considered only from

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the production point of view. This is because there are many more disturbing influences affecting the start of a production run than are found at later stages. The effect of these on the figures can be relatively large owing to the smallness of the actual output. Also, as output statistics are usually quoted monthly, the time taken to reach peak output can be expressed only in units of one month. Consequently, if the fluctuations are such that the start of the run is wrongly estimated by one unit, an error of the order of 6 - 7% is immediately introduced. Similar observations apply to the estimation of the point when peak output is reached. The points do not follow a smooth curve precisely, there are fluctuations due to a variety of external causes such as weather, shortage of supplies, etc. As the points approach the peak figure comparatively slowly it sometimes happens that the fluctuations bring the output figure above the peak earlier or later than the smooth 'average' curve through the points. Thus the choice of this point loses some objectivity.

This is a further reason for the attempt to find some general curve which can be applied to this type of data, a curve which would facilitate a more objective comparison of times to achieve peak output.

3. Previous Work

As far as is known, the only previous similar approach to this problem was made by J.V. Connolly in an unpublished paper.*, He took his data from war-time British sources and the same figures have been used to represent this country in this report. He recalls that as long ago as 1940 it was found that in practice all factories attained peak output approximately between one year and one year and four months after the first delivery of the production type was made. He noted that the size of the programme and of the aircraft seemed to have relatively small effects and that the time to peak output seemed to be more directly a function of the skill of the company and the type of aeroplane from the design point of view.

From the elementary studies made at that time, (1940), a rough empirical formula was derived for planning programmes. The data for 25 British aircraft programmes was analysed in 1946 and an attempt made to find the 'average' curve of best fit by first summing the results for all the programmes (by adding all the outputs for Month 1, then for Month 2, etc.), smoothing by taking a 3 months moving average and then fitting theoretical curves of both the Gompertz and the Logistic types. In both cases the formula fitted the observations to a high degree of accuracy over the middle portion of the data.

Thus it was concluded that the statistics relating to the building up of output were of the same nature as those found in other types of build up. These theoretical curves have been applied with success to data on consumption of commodities, rate of growth of populations, etc.

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* One of a series of reports to D.Air P. at the Ministry of Supply, 1946

This was as far as the investigation was carried in Connolly's paper but, since then, more efforts have carried the work further. These efforts were made when more production statistics from other countries came to hand and also when it was thought desirable to treat the original figures in greater detail.

After working through the data for a number of projects it became obvious that the method of summing the numbers of aircraft produced each month is not entirely satisfactory. A brief survey of some of the possibilities readily supports this. Firstly, if a curve is fitted to the summed data there still remains only the rough method of inspection of the original data for judging the effect of the different conditions on the final result. Secondly, even in any particular category of aircraft, the different projects give different times of build up to the first programme maximum. Then, after the first peak has been reached, several alternatives can occur. The factory may continue to produce at the same peak output. On the other hand, there may be a change of policy which involves a further increase in output through the installation of new capacity or, perhaps, a decline or even a cessation of production. Therefore, the form of the summed data will be affected by the variations in output for each project, each contribution being weighted by the absolute size of the output of that project. This cannot be considered anything but an undesirable situation.

From this it was concluded that, in order to place the work on a sure foundation, it would be better to treat each programme separately and to form some idea of the differences between them before attempting any combination of the observations. That this view is correct was borne out by the German data which, when summed, gave a curve of most peculiar shape as shown in Fig.2. It seems that the combination of British results to give a smooth curve of roughly the same shape as its constituent parts must be looked upon as quite fortuitous.

4. Growth Curves

It has been mentioned already that theoretical curves exist which have, in the past, found great use in fitting data which has a pattern similar to that discussed here. These curves are known generally as Asymptotic Growth Curves since the growth represented is not limitless but approaches some upper limit asymptotically. At least, so much can be said of the mathematical picture of the process but adjustments must be made to suit this to the practical finite examples with which we deal. The subject will be discussed in greater detail later when some problems of fitting smooth curves to discrete data arise.

There are two main curves available. One of these is that discovered by Gompertz in the early part of the 19th century (Ref.No.2) and was used by him in some researches into human mortality. It is now known under

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his name and can be written in the form

$$y = ka^{b^x}$$

where 'y' represents the quantity whose growth is being considered, e.g. the output of aircraft per month, and x represents the time against which this quantity is being plotted. 'k','a' and 'b' are constants for each particular growth curve. In words it can be described as a series in which the growth increments of the logarithms of 'y' are declining by a constant percentage. It has been used extensively by business statisticians to describe the growth of new industries and a typical example, that of the rayon industry in the U.S.A., is given by Croxton and Cowden (Ref. No.1) who also quote Prescott (Ref. No. 5) who has concluded that the Gompertz curve expresses a law of growth. Prescott divides the growth of an industry into four stages as follows:

(1) Period of experimentation

(2) Period of growth into the social fabric

(3) Period through the point where growth increases but at a diminishing rate

(4) Period of stability

These can be readily associated with various portions of the Gompertz curve and, also, such a breakdown of the process of growth can be seen to have certain similarities with the conditions applying at different times in an aircraft manufacturing unit.

The other formula which produces a curve of very similar shape is the Logistic

$$y = \frac{k}{1 + e^{(a+bx)}} \quad \text{or} \quad \frac{1}{y} = K + AB^x$$

where $y = k$ is the upper asymptote which is approached as $x \longrightarrow +\infty$ for negative b. For ease of computation this is fitted in the form

$$y = \frac{k}{1 + (10)^{a+bx}}$$

in the numerical work concerned with fitting the curve to the available data but this will be referred to again at the appropriate point.

The Logistic is sometimes known as the Pearl-Reed curve due to its extensive use by these two writers (Ref. No.3). Originally it was propounded as a law of population growth by a Belgian mathematician, Verhulst. Pearl and Reed have found that it can be used to describe the growth of an albino rat, a tadpole's tail, the number of yeast cells in nutritive solution and, of more general interest, the number of humna beings in a geographical area.

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Pearl (Ref. No. 4) states that the law of growth expressed by the Logistic curve is one in which, at any time, the rate of growth is proportional to two things, (a) the absolute size already attained and (b) the amount still unused or unexpended of the actual or potential resources for support of growth.

Mathematically speaking, this statement can be expreseed as

$$\frac{dy}{dx} = -\frac{b}{k}y(k-y)$$

In the same way, for the Gompertz curve

$$\frac{dy}{dx} = (\log b)\,y\,(\log y - \log k) = (\log b)\,y\,\log\left(\frac{y}{k}\right)$$

which is certainly the more complicated expression of the two and cannot be expressed in words so simply as it involves logarithms of the quantities.

Now it must be insisted that the use of both these formulae is based on empirical development only. It is important to realise that it is only in this sense that the phrase "law of growth" can be used in connection with them. Indeed, it may be argued that the use of the word 'law' is definitely misleading in this case. There is at present no theory expressible mathematically that will lead to a formula for a growth curve of this nature. Both the Gompertz and Logistic forms are only mathematical expressions which happen to produce curves similar in shape to those given by the data when plotted. There is no reason to suppose that they bear any relationship to the physical conditions which produce the results. In the above analysis it has been shown that, in the case of the Logistic, the rate of increase is proportional to certain quantities but it would be very difficult, if not impossible, to prove or to disprove that this is the true picture of reality.

In fact, in the cases of human populations and manufacturing enterprises, both of which are extremely complex entities, the situation at any one time is the result of the influence of an enormous number of complex forces and interactions. Taking this into account it becomes doubtful that anything but a rough approximation can be made by the use of one formula. This situation may be likened to the use of theoretical frequency distributions expecially the Gaussian or Normal distribution, when treating the properties of a group statistically. We are hardly ever sure that the theoretical formula has any connection with the cause system producing the distribution but this does not prevent us from producing valuable results by this method.

Therefore, while realising the shortcomings of this empirical approach, it is not necessary to avoid it altogether as it may prove to be quite profitable especially in the absence of any better technique.

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Here it may be as well to note that another even more empirical method of fitting a curve to growth data has been used in the past. It depends on the fact that the S - shape of the growth curve is very similar to that of the Ogive whichis obtained when the Gaussian distribution is plotted as a cumulative frequency distribution. Hence it has been suggested that a method of smoothing this data would be to plot it, after a suitable change in scale by dividing by a factor to be determined, on arithmetic probability paper when it should be possible to fit a straight line which, when reconverted to the original units, would give a smooth curve. This method achieves no more than the smoothing of the data and, even as a method of doing this, cannot be considered particularly easy or one which produces a standard form of curve amenable to further treatment and analysis. So it is not used in the present investigation.

5. Methods of Fitting the Gompertz and Logistic Curves

With one exception, all available references to the question of fitting growth curves describe some comparatively rough method. None of these methods is amenable to the application of tests of significance by which real differences between various practical series may be recognised. In curve fitting generally this is usually achieved by using the method of Least Squares, a technique giving equations which can be solved directly in the case of polynomials and certain other classes of curves.

However, in the cases of the Gompertz and Logistic, the 'normal' equations which are produced by the least squares solution are not explicit and cannot be solved directly. This problem has been studied by Stoner (Ref. No. 6) who has described in detail an approximating procedure for solving the 'normal' equations in the fitting of the Gompertz curve. He admits that the method is tedious and, in fact, the amount of computational effort would be very large. This, combined with the fact that the equations themselves are complicated necessitating much computation in their formation, makes the method impracticable if a large number of curves have to be fitted as in the present investigation. The extra precision has to be sacrificed in order to bring the amount of calculation within reasonable limits.

Perhaps this is the appropriate place to mention that the Logistic can be fitted by the method of least squares if a certain assumption is made, namely, that the value of the upper limit of the quantity (y) is known, i.e. if 'k' is known in the formula

$$y = \frac{k}{1 + e^{a+bx}} \quad .$$

Then this is reducible to

$$Y = \log_e \left(\frac{k-y}{y}\right) = a + bx \quad ,$$

which is in the form of a straight line. This can be easily fitted by least squares. However, it will be seen later that it would not always be safe to assume a value for 'k' so this method was abandoned as a general approach. But it might prove useful in special cases where there was more general information about the situation.

The Logistic is a symmetrical curve; the point of inflexion occurs when 'y' is 50% of the peak value. But the curve may be made more general by the addition of a further term and the possibility of skewness is admitted. The formula is then

$$y = \frac{k}{1 + e^{a+bx+cx^2}} .$$

The data of the present investigation has not necessitated the use of a skew curve as it has all been reasonably symmetrical and, if this were necessary, difficulties of fitting would have to be faced. There is no simple method equivalent to that for the Logistic described below, the only method so far discovered being by least squares on the assumption of a value for 'k' when it could be treated in the form

$$Y = \log_e \left(\frac{k-y}{y}\right) = a + bx + cx^2 .$$

So the usual methods of fitting described by Croxton and Cowden (Ref. No. 1) have been adopted. The details of the method for the Logistic which involves the use of three selected points are set out in Appendix 'A'.

Generally speaking, it was found that there was nothing to choose between the fit of the Gompertz and that of the Logistic in any particular case so, rather than embark on a programme of curve fitting larger than necessary, it was decided that only one of these curves should be used. The Logistic was the one chosen and the reasons for this were many and varied. Among them were the greater ease of fitting the Logistic, the fact that its basis was simpler than that of the Gompertz (see Section 4, where the possible theoretical basis of the curves is discussed) and the possibility of obtaining from the Logistic a useful index of the steepness of the curve or its comparative rate of growth.

In the numerical work it is necessary to take logarithms and because of the availability of more comprehensive tables, as is usually the case, these were taken to the base 10. Hence it is essential to state and important to stress that from this point onwards the Logistic is treated as

$$y = \frac{k}{1 + (10)^{a+bx}} .$$

All the values quoted for the constants 'a', 'b' and 'k' refer to the equation in this form.

6. The Parameters of the Logistic

Before proceeding to consider the actual production figures it will assist in their appraisal to consider the meaning of the parameters of the Logistic, 'k', 'a' and 'b' which are calculated when the curve is fitted to the data.

Firstly there is the value of 'k' which represents the upper limit beyond which the curve never rises. Mathematically it is known as an asymptote because the shape of the curve is such that it is always approaching this limit steadily but never actually reaches it. In the same way the line $y = 0$ is also an asymptote.

The parameter 'a' determines the proportion of the peak output which is represented by the curve at the origin of the time scale. In practice this will be greatly affected by the absolute value of the peak output as there will be no data until at least one complete aircraft has been produced. Actual production can only be represented by discrete steps and not by a smooth curve.

Put in another way, the effect of a change in 'a' can be thought of as a uniform shift of the whole curve in the direction of the time axis without a change of shape. This shows that the value of 'a' is not important in judging the relative rates of build up of different projects but it does suggest a way of using the curve. This way will be described in a later section when more about the properties of parameter 'b' is known.

Lastly there is the effect of the value of 'b' on the curve to consider. This is such that a change in 'b' results in an alteration in the steepness of the curve. As it is this property which determines the time in which peak production is reached, it is in the parameter 'b' that the greatest interest centres. It can be shown (see Appendix 'B') that the time taken to increase from one given value of the quantity to another given value is inversely proportional to '-b'. Hence 'b' could be used as an index of the rate of build up to peak but, as it would be more convenient to have an index which is directly proportional to time, the proposed index of rate of growth is

$$B = -1/b \quad .$$

This quantity has been calculated for all the fitted curves and it is suggested that comparisons of the various achievements be made on the basis of these figures. The use of this index would avoid all the objections to the other suggested measure of time taken to reach peak which were detailed in Section 2. This index takes into account the whole shape of the curve and is not influenced unduly by the data at the beginning and at the end of the series.

These facts have prompted some ideas on how the Logistic curve and the index 'B' could be used in practice both as a standard against which past achievements could be measured and as the basis for planning production schedules. These will be presented later, however, after the actual data has been discussed.

/7. British ...

7. British Data

The available figures on British production cover 25 separate programmes in 18 factories and involving 16 different types of aircraft. Each programme reached peak at various times in the period from 1939 to 1946. Some selection of the data was inevitable but this was only to ensure that the production occurred in a situation which could be expected to produce the effect under consideration. Conditions which were considered suitable, for instance, were the production of one major programme, not a complex mixture of types and the absence of any setback of a serious nature such as bombing, change of policy or serious technical failure of the type. The figures representing all the usual manufacturing contingencies remained.

The figures for each programme, together with the Logistic curve fitted in each case, are presented in Fig.3. The parameters of the Logistic and certain other statistics are given in Table I. It will be seen that the curve gives a good fit in nearly all cases. The figure for the programme peak was not always easy to determine as this often had to be done solely from the appearance of the figures and these could be complicated by a decision to extend the programme at a later date which meant that the figures continued to increase. However, the guiding principle used was that the programme peak figure should be the planned capacity of the factory at the time the first production plans were made.

Instead of listing the upper asymptote, i.e. the value of 'k', in absolute terms in Table I this figure has been quoted as a percentage of the programme peak. It is interesting to note that the great majority of values are greater than 100% and are mainly in the interval 100% - 110%, but this feature will be dealt with in more detail later.

8. U.S.A. Data

The figures on production in the U.S.A. cover 35 separate programmes in 32 plants and involving 27 different types of aircraft. They originated in the period between 1940 and 1945 inclusive. The statistics were drawn from a comprehensive publication of the Civil Aeronautics Administration (Ref. No. 7) and again some selection had to be made. The remarks that were made on the selection of the British data apply in this case also except that less was known from other sources about the histories of the individual projects, firms and types. However, the same guiding principles were used as the object was the same in both instances.

The original data is presented in Fig. 4 and the parameters of the fitted Logistics, etc., are given in Table 2. A discussion of the results will de deferred until later as it will be more convenient to treat all the data together.

9. German Data

Here the same remarks that were made under British and American headings apply but there are a few extra points to note. Here it was more difficult to find even a fairly smooth series as they were, in general, much more erratic. The reasons for this are not hard to find and examples such as bombing and serious shortage of materials need only be quoted here.

In all, the figures for 10 programmes at 10 separate factories producing 5 types of aircraft were selected. The data, the fitted curves and their parameters are presented in Table 3 and Fig. 5.

10. Japanese Data

This data covered 16 programmes undertaken by 8 manufacturers and involving 16 types of aircraft. It is presented in Fig. 6. It is obvious from the diagrams that this data is of little use for the purpose at hand, The main reason is that in nearly all cases production was interrupted before peak output had been attained. The growth of production was apparently slower than in the other three countries and most of the larger programmes were started at a relatively late date, hence, the war ended and production ceased before the peaks were reached.

So it was decided that no further work should be done, at least for the present, on this data and attention was concentrated on the results from the other three countries.

11. Discussion of Results

11.1. Upper Limit of Curve

First, consideration will be given to the upper limit of growth, i.e. the value of 'k' estimated from the data. It has already been mentioned in Section 7 that this was nearly always greater than the planned programme peak. This means that the curve actually crosses the 100% line at a definite time so that, for some purposes, this point could be looked upon as the end of the build up time. The data does not always show this as often there were policy changes and also, in certain cases, deliveries were constrained to the programme figure. This is obviously so in American projects Nos. 8 and 17 (see Fig. 4).

Despite this, it is reasonable to suppose that such a tendency exists on grounds other than the rather dangerous appeal to the shape of the fitted curve. Such an improvement in output is to be expected, other things being equal, from the known facts about the efficiency of manufacturing operations. In a variety of human activities it has been found that the increase in efficiency following from the acquisition of skill by practice in the activity follows a definite pattern. For example, if, in

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manufacturing, efficiency is measured by the number of man-hours required to produce one unit, it is found that when man-hours per unit is plotted against accumulated total of units produced on log-log paper, i.e. each co-ordinate is plotted on a logarithmic, not arithmetic, scale, a straight line relationship results. Symbolically this relation can be expreseed as

$$\log m \;=\; A + B \log n \quad ,$$

where m = No. of man-hours to produce one unit,
n = accumulated total of units produced
and A and B are constants for any one straight line.
It can be shown that, if this relation holds, each time the accumulated number produced is increased to a given multiple of its previous value the man-hours per unit reduce to a constant fraction of the previous figure.

In a large number of varied cases from the aircraft industry the data has plotted as a straight line on log-log paper. The slope of these lines has been always approximately the same and such that, each time the total number of aircraft is doubled the man-hours per aircraft is reduced to 80% of the previous figure. Consequently, this effect has sometimes been referred to as the "80% law".

Therefore it does seem to be an acceptable proposition that, even when the planned capacity of a factory has been reached, further increases in production in line with the "80% law" are possible owing to the increasing skill and practice of the operatives.

11.2. Value of 'a'

As regards a comparison of past results the value of 'a' is not important. Its effects on the curve has already been discussed in Section 6 and it was seen that this parameter had no relationship with the steepness of the curve or the speed with which peak is reached.

But if a curve has to be defined as a standard for future production it will be essential to specify all three parameters, 'k', 'a' and 'b' (or 'B'). The value of 'a' will have to be chosen to suit each particular programme and will be irrespective of the other two. It will depend on the actual size of the programme and the times at which the first few aircraft are produced. It is at the start of the series where the point about there being no effect until one complete aircraft is delivered is important. Probably here the smooth curve represents a truer picture of the situation than the actual data as that part of it over the negative time axis can be thought of as representing the efforts being made then which result in the delivery of the first aircraft some time afterwards. Since, then, it will be difficult to predict just how the series will commence, it will be difficult to fix the value of 'a' beforehand.

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However, as this quantity is not related to the speed with which peak output is attained this is not so important and a means of solving the problem is suggested later.

11.3. Production Growth Index, B.

The values of 'B' vary considerably and there is no obvious connection between them and either the structure weight of the aircraft or the peak output planned. To test this more objectively a correlation analysis was carried out on the British data and there was definitely no statistically significant evidence of any association between either of these pairs of variables. Hence this further work has upheld Connolly's suggestion that the major contribution to the causes operating is from the skill and technical and managerial efficiency of the organization and the design of the aircraft. So probably the fixing of indices to provide the basis of future programmes will be done best by considering an analysis of past results at the same factory when building aircraft with not extremely different characteristics.

But when consideration is given to the overall picture presented by each country and a comparison is made several interesting features are obvious. There can be no doubt that there are decided differences between them. These are best illustrated by the histograms of the Growth Index, 'B', for each country displayed in Fig. 7. The interpretation of these figures is easy when it is remembered that 'B' is proportional to the time taken to reach capacity production. For example, if we imagine two plants, designated by P and Q, and Plant P has an index 4 and Plant Q has an index 8, then Plant P will achieve any particular increase, e.g. from 5% to 95% of planned peak, in half the time the Plant Q will take to achieve the same increase.

The diagram shows that on average America achieved a faster introduction than Great Britain and that few results from Germany indicate that the introduction there was faster than the other two countries. As always, though, the complete story is not told by the average figures. The variations in the American figures are greater than those in the British, the latter being much more consistent. While the best figure is better than any of the examples from this country, it is also true that the worst American figures are worse than any from either here or Germany. Put another way, the longest British introduction was twice as long as the best while the worst American introduction was $3\frac{1}{2}$ times as long as their best.

It is unwise to place too much emphasis on individual results but the general trend is unmistakeable.

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12. The Use of the Logistic Curve and the Production Growth Index

Enough has already been written on the use of the Logistic curve for judging past achievements through the Production Growth Index so this section will deal solely with their application to the planning of production programmes. First of all it is necessary to decide on the values of 'k', the upper limit of the curve, and the Growth Index, 'B'. As regards 'k' it would seem from the data studied here that it would be convenient and suitable to make 'k' equal to 105% of the planned programme peak. This is a manageable round figure and also approximates to the central tendency of the practical values. The errors in 'B' introduced by this procedure will only be of the same order as those arising from the estimation of this index by the fitting of a Logistic to previous data, the method which must be used to choose 'B'. Practical experience will have to be drawn upon to decide upon the appropriate precedent to follow.

The difficulties and considerations involved in the choice of the parameter 'a' have already been dwelt upon in Section 11.2. These suggest that it would not be practicable to fix definitely the position of the curve with respect to the time variable before production has commenced and that it is probable that a horizontal adjustment will be necessary when the figures for the first few months production come to hand. An objection that may be raised here is that this introduces too much laxity into the scheme and may be compared with moving the target to suit the achievement. This has a certain amount of force but not sufficient to condemn the practice as it must be remembered that the scope of the problem has been limited to cover only the production aspects of the development time. So we are not concerned with the actual time when the first aircraft appears as this is something that depends on the overall process of development.

The theoretical curve is extremely useful even if it has to be adjusted along the time scale to suit the early data. It can show, once production has started, whether the rate of growth is as anticipated. One way of doing this would be to plot always on charts of the same size and with the same scales. One suitable scale would be percentage of programme peak and then the actual numbers could be given on a subsidiary scale in each case if necessary. Standard curves for various values of 'B' could be prepared and drawn on some transparent material. Examples of such curves are shown in Fig. 8. By placing the transparency over a chart of the actual data and adjusting so that its position suited the first few points it would be easy to see whether the standard of growth was being maintained. If it was not, the other standard curves could be used to estimate the actual index of the rate of growth and the discrepancy could be estimated in terms of time.

For the sake of those who may wish to practice this technique, full details of the fitting of growth curves and a little of the mathematics omitted from the main body of the report are given in appendices.

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6.	Stoner, P.M.	Fitting the Exponential Function and the Gompertz Function by the Method of Least Squares. Journal of the American Statistical Association, Vol. 36, 1941, p.515.
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TABLE I.

BRITISH PRODUCTION DATA

PARAMETERS OF LOGISTIC CURVES FITTED, ETC.

Project No.	Type	Approximate Structure Weight lb.	Assumed Programme Peak	Parameters of Logistic fitted: Upper Asymptote %	a	b	B
1	Lancaster)		100	100	1.204	-0.120	8.33
2	")		40	108	2.037	-0.180	5.56
3	")		25	111	1.716	-0.199	5.03
4	Halifax)	18 000	45	108	1.369	-0.161	6.21
5	")		20	100	1.121	-0.130	7.69
6	")		25	103	1.080	-0.232	4.32
7	Wellington)	10 000	60	105	1.526	-0.179	5.59
8	Hampden)		50	105	1.122	-0.164	6.10
9	Mosquito)		60	118	2.132	-0.167	5.99
10	")		50	125	1.936	-0.193	5.18
11	")	6 000	25	127	1.266	-0.141	7.10
12	Beaufighter)		65	101	1.743	-0.186	5.38
13	Barracuda)		35	101	1.501	-0.201	4.99
14	")	5-4 000	40	99	1.407	-0.152	6.62
15	Blenheim)		60	108	1.423	-0.169	5.92
16	Tempest)		70	108	1.453	-0.169	5.92
17	Typhoon)		100	101	2.088	-0.202	4.96
18	Battle)		70	98	1.433	-0.207	4.83
19	Oxford)		45	100	1.025	-0.145	6.90
20	")	4 000-	60				
21	Defiant)	-2 500	50	105	1.898	-0.202	4.96
22	Vampire)		20	108	1.733	-0.185	5.41
23	Hurricane)		150	115	1.653	-0.220	4.55
24	Spitfire)		140	89	1.312	-0.223	4.48
25	")		50	102	1.234	-0.180	5.56

TABLE II.

U.S.A. PRODUCTION DATA

PARAMETERS OF LOGISTIC CURVES FITTED, ETC.

Project No.	Type	Approximate Structure Weight lb.	Assumed Programme Peak	Upper Asymptote %	Parameters of fitted Logistic a	b	B
1	Superfortress		60	114	1.575	-0.126	7.94
2	"		55	114	1.991	-0.254	3.94
3	Commando	23 900	50	103	2.008	-0.307	3.26
4	Liberator	23 200	200	102	2.273	-0.288	3.47
5	"	23 200	80	100	1.929	-0.260	3.85
6	"	23 200	100	109	2.253	-0.263	3.80
7	Skymaster	app.23 000	50	102	1.864	-0.165	6.06
8	Ventura	app.15 000	100	123	2.218	-0.309	3.24
9	Invader	app.10 000	100	109	2.466	-0.184	5.43
10	Baltimore	9 900	60	99	2.169	-0.403	2.48
11	Lightning	8 300	150	102	2.394	-0.165	6.06
12	Black Widow	app.8 000	60	108	1.481	-0.149	6.71
13	Tigercat	app.8 000	25	112	1.485	-0.185	5.41
14.	Bermuda	7 200	80	105	1.613	-0.311	3.22
15	Avenger	6 900	350	100	1.953	-0.148	6.76
16	Helldiver	6 400	70	102	1.588	-0.181	5.52
17	"	6 400	120	110	2.160	-0.264	3.79
18	Thunderbolt	6 400	250	104	2.652	-0.237	4.22
19	"	6 400	350	112	1.544	-0.107	9.35
20	Hellcat	5 600	500	107	2.440	-0.206	4.85
21	Corsair	5 500	250	96	0.881	-0.180	5.56
22	"	5 500	220	107	2.028	-0.222	4.50
23	King-cobra	app.5 000	250	101	1.944	-0.229	4.37
24	Dauntless	4 500	300	101	1.446	-0.133	7.52
25	Mustang	4 500	250	102	2.040	-0.326	3.07
26	Wildcat	3 600	300	102	2.414	-0.205	4.88
27	Wichita	3 300	45	105	1.576	-0.271	3.69
28	Texan	2 700	450	100	1.391	-0.109	9.17
29	Reliant	1 900	25	112	1.725	-0.294	3.40
30	Duck	1-2 000	20	106	1.202	-0.219	4.57
31	Norseman	1-2 000	45	100	1.299	-0.156	6.41
32	Cornell	1 400	75	103	2.289	-0.405	2.47
33	"	1 400	40	100	1.909	-0.318	3.14
34	Sentinel	1 000	150	103	1.951	-0.238	4.20
35	Privateer		65	102	2.176	-0.363	2.75

TABLE III

GERMAN PRODUCTION DATA

PARAMETERS OF LOGISTIC CURVES FITTED, ETC.

Project No.	Type	Approximate Structure Weight lb.	Assumed Programme Peak	Upper Asymptote %	a	b	B
1	Hs 129		24	102	1.504	-0.325	3.08
2	He 177	24 000	16	104	1.248	-0.386	2.59
3	He 177	24 000					
4	Do 217	11 500	32	109	2.700	-0.357	2.80
5	Do 217	11 500					
6	Me 410	6 300	60	131	1.448	-0.279	3.58
7	FW 190	3 400	75	102	1.472	-0.216	4.63
8	FW 190	3 400	45	96	1.464	-0.338	2.96
9	FW 190	3 400	50	109	1.509	-0.257	3.89
10	FW 190	3 400	50	120	1.274	-0.198	5.05

APPENDIX 'A'

FITTING THE LOGISTIC CURVE BY THE METHOD OF 3 SELECTED POINTS

The equation for the Logistic curve is usually quoted as

$$y = \frac{k}{1 + e^{a+bx}} ,$$

but since, in practical computation, it is much more convenient to take logarithms to the base 10 as tables of these are usually readily available, the curve fitted to the data will be

$$y = \frac{k}{1 + (10)^{a+bx}} .$$

Select 3 points, not necessarily included in the data, through which the curve is to pass. Denote these by (x_1, y_1), (x_2, y_2) and (x_3, y_3) where $x_3 > x_2 > x_1$. A restriction placed on these points is that they must be equally spaced with respect to the x co-ordinate i.e. $(x_3 - x_2) = (x_2 - x_1) = n$.

Then the parameters of the curve are calculated from the following expressions:

$$k = \frac{2y_1y_2y_3 - y_2^2(y_1 + y_3)}{y_1y_3 - y_2^2} ,$$

$$b = \frac{1}{n} \log_{10} \frac{y_1(k - y_2)}{y_2(k - y_1)} ,$$

$$a = \log_{10}\left(\frac{k - y_1}{y_1}\right) - bx_1 .$$

Then sufficient points for the plotting of the curve can be calculated from its equation and it is suggested that this is best done by using the form

$$\log_{10}\left(\frac{k-y}{y}\right) = a + bx .$$

The expression $(k-y)/y$ can be calculated first for several values of x whence the values of y follow from a simple transformation.

APPENDIX 'B'

DERIVATION OF THE GROWTH INDEX B.

The equation of the Logistic

$$y = \frac{k}{1 + e^{a+bx}} ,$$

can also be expressed in the form

$$\log\left(\frac{k-y}{y}\right) = a + bx .$$

Let (x_1, y_1) and $x_2, y_2)$ be two points on the curve. By substituting the values in the equation and subtracting we have

$$b(x_2 - x_1) = \log\left(\frac{k-y_2}{y_2}\right) - \log\left(\frac{k-y_1}{y_1}\right)$$

$$(x_2 - x_1) = \frac{1}{b} \log \frac{y_1(k-y_2)}{y_2(k-y_1)} .$$

Hence, if y_1 and y_2 are fixed quantities, $(x_2 - x_1)$ is inversely proportional to $-b$; i.e. the time taken to increase from y_1 to y_2 is proportional to

$$-{}^1/_b = B .$$

So B is suggested as an index of the rate of increase of production.

-oOo-

As a matter of interest the same analysis has been carried out in the case of the Gompertz curve

$$y = ka^{b^x} .$$

This shows that

$$(x_2 - x_1) = \frac{1}{\log b}\left[\log\log({}^{y_2}/k) - \log\log({}^{y_1}/k)\right],$$

and hence that the time taken to increase from one fixed value y_1 to another fixed value y_2 is inversely proportional to log b. (Note that here b has no relationship with the same symbol used in connection with the Logistic above).

So a similar index could be proposed on the basis of the Gompertz curve but it would be somewhat more complicated than that which has been called B.

FIG. 1.

EXAMPLE OF AIRCRAFT PRODUCTION DATA.
(WITH FITTED LOGISTIC CURVE)

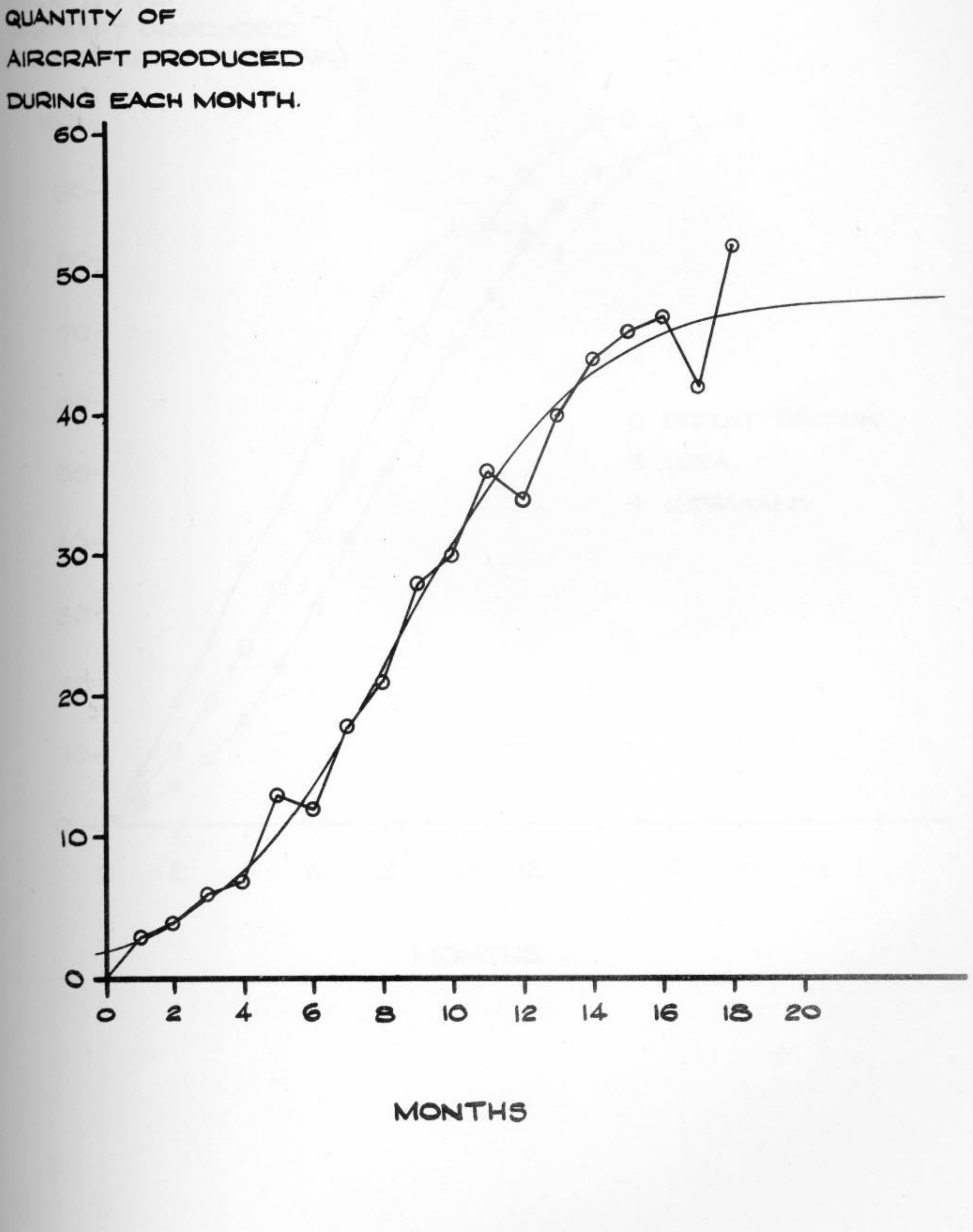
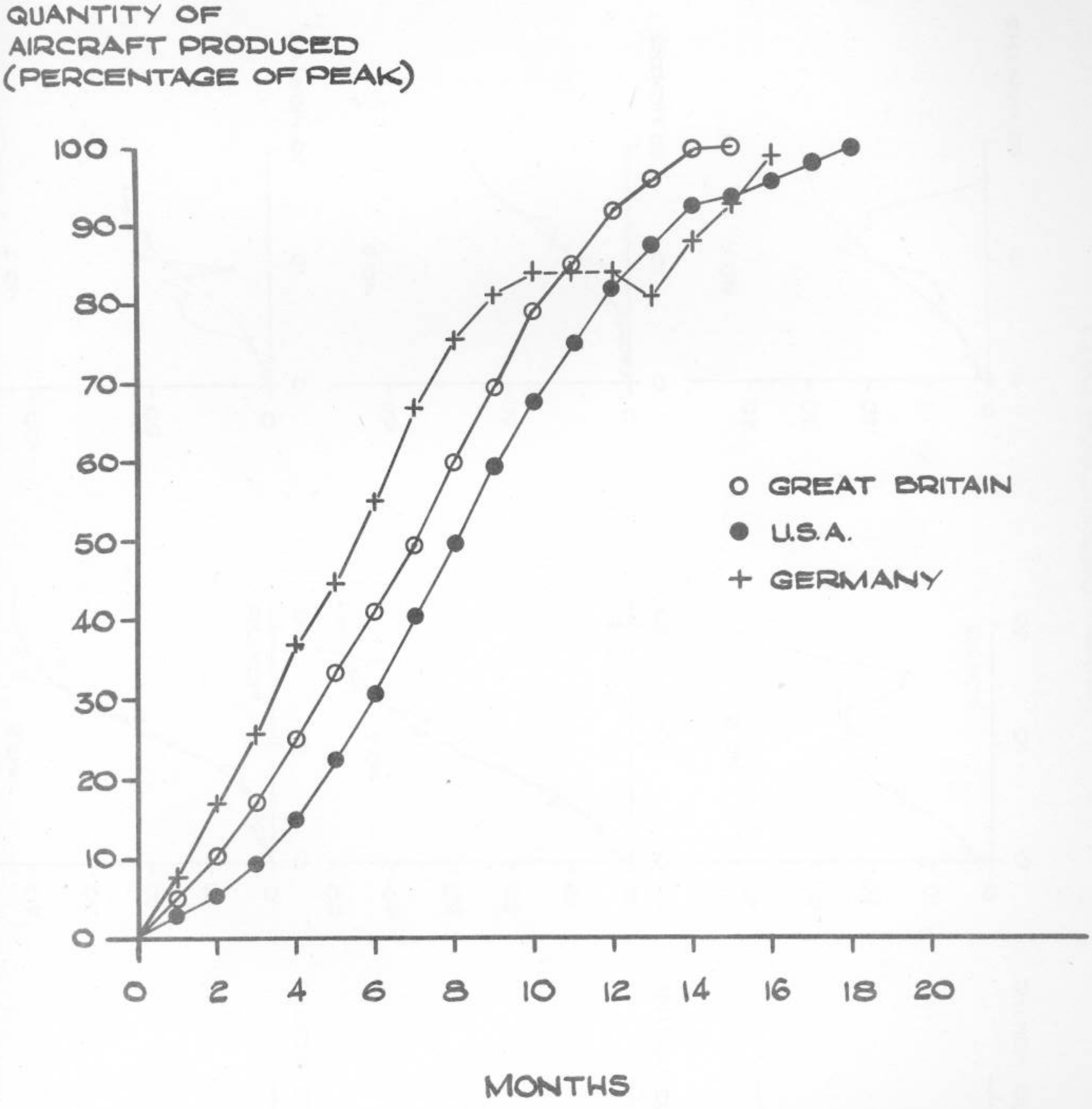


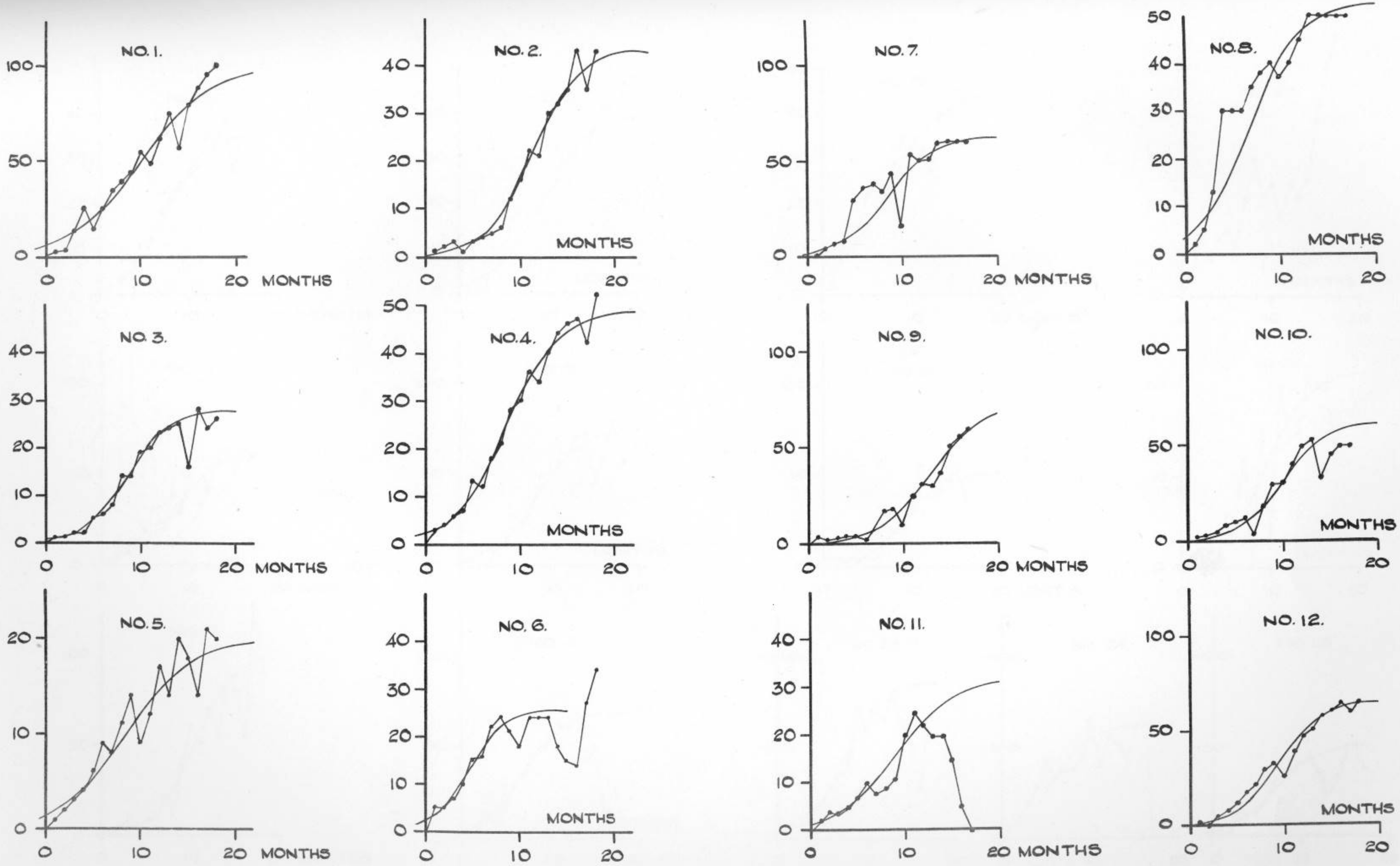
FIG. 2.

TOTALS FOR EACH COUNTRY

(3 MONTHS MOVING AVERAGE)

ILLUSTRATES METHOD OF SUMMING DATA NOT ADOPTED FINALLY.





BRITISH PRODUCTION DATA
(WITH LOGISTIC CURVES FITTED)

FIG. 3.

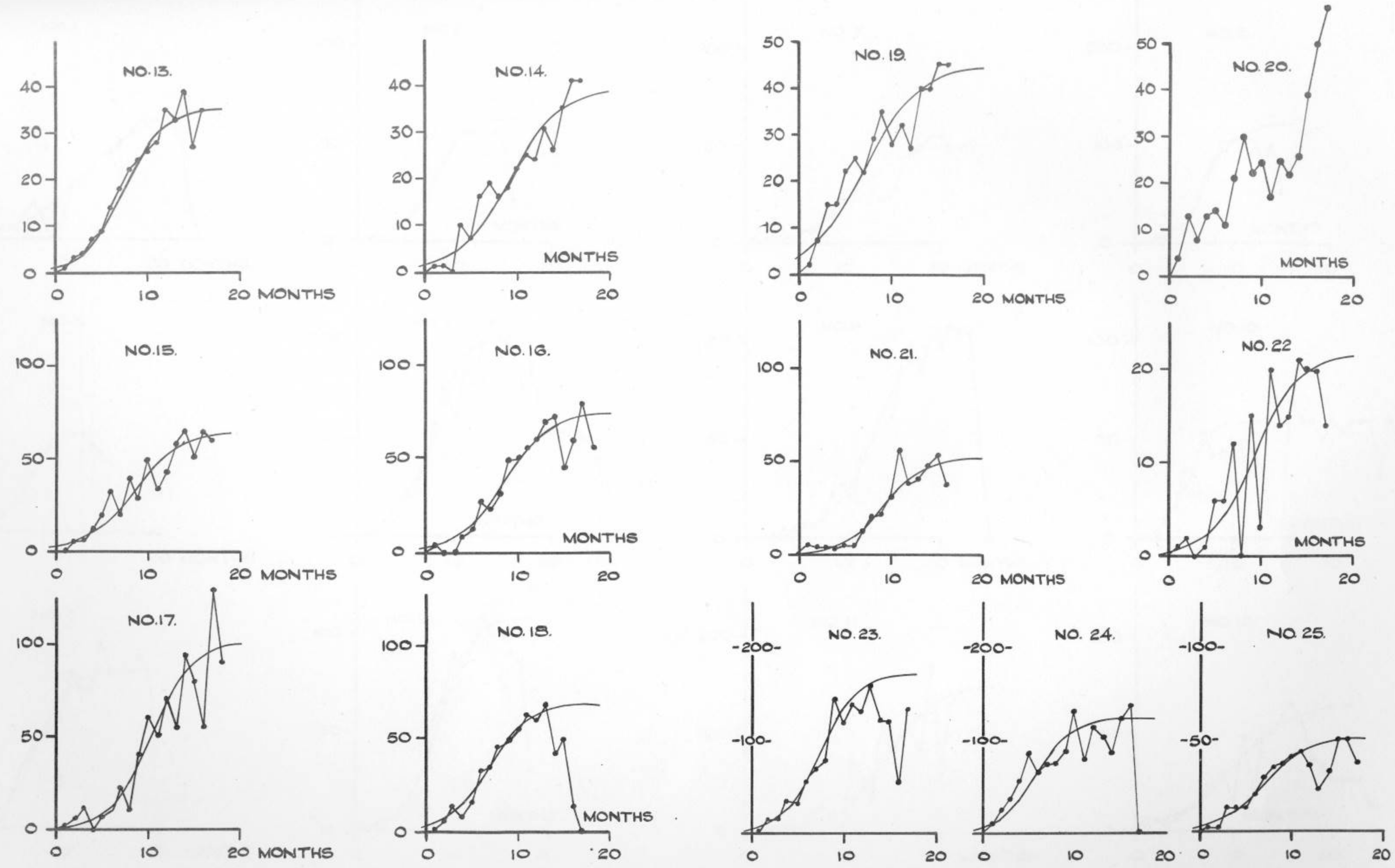
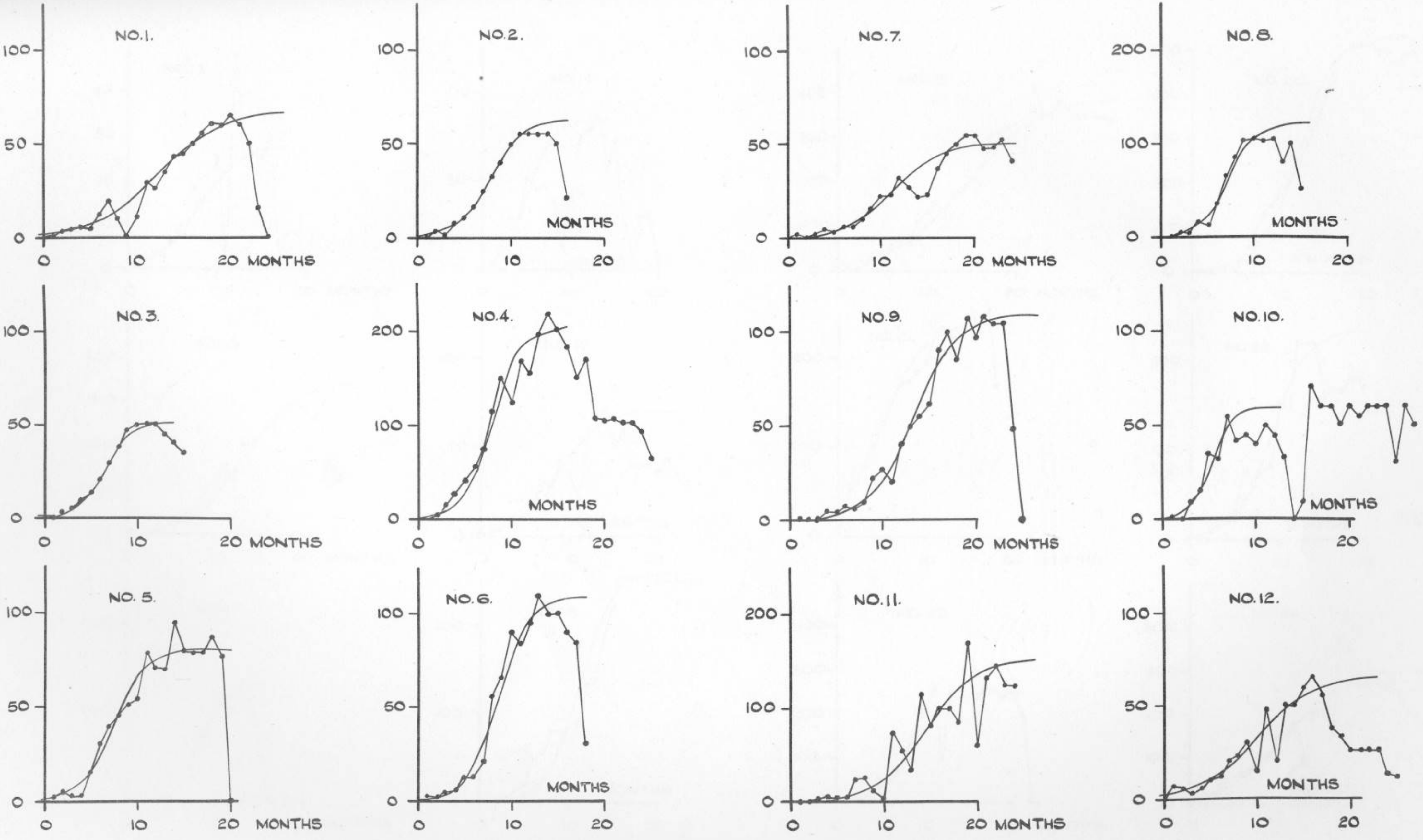


FIG. 3.
(CONTINUED)



AMERICAN PRODUCTION DATA
(WITH LOGISTIC CURVES FITTED.)

FIG. 4.

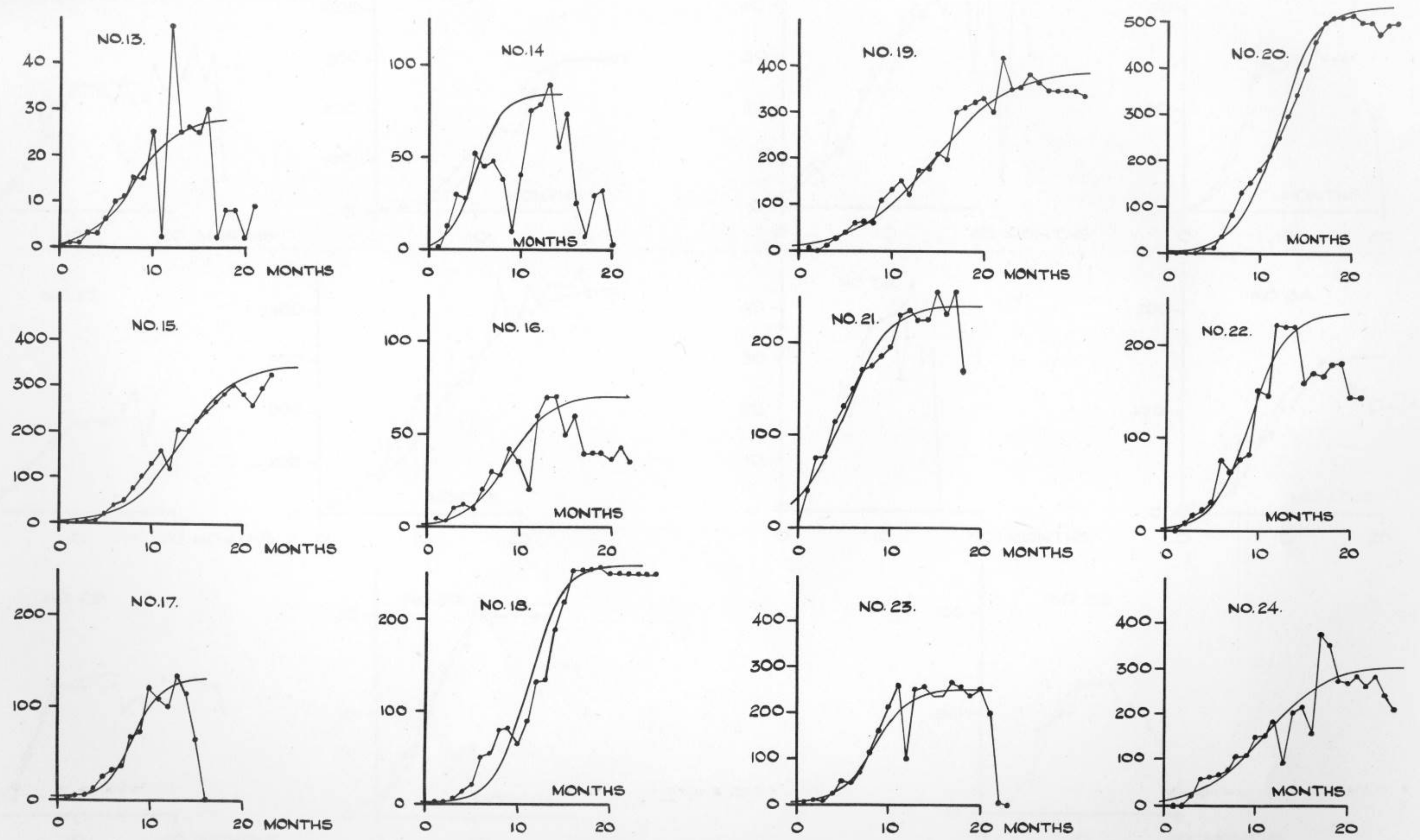


FIG. 4.
(CONTINUED)
SHEET(2)

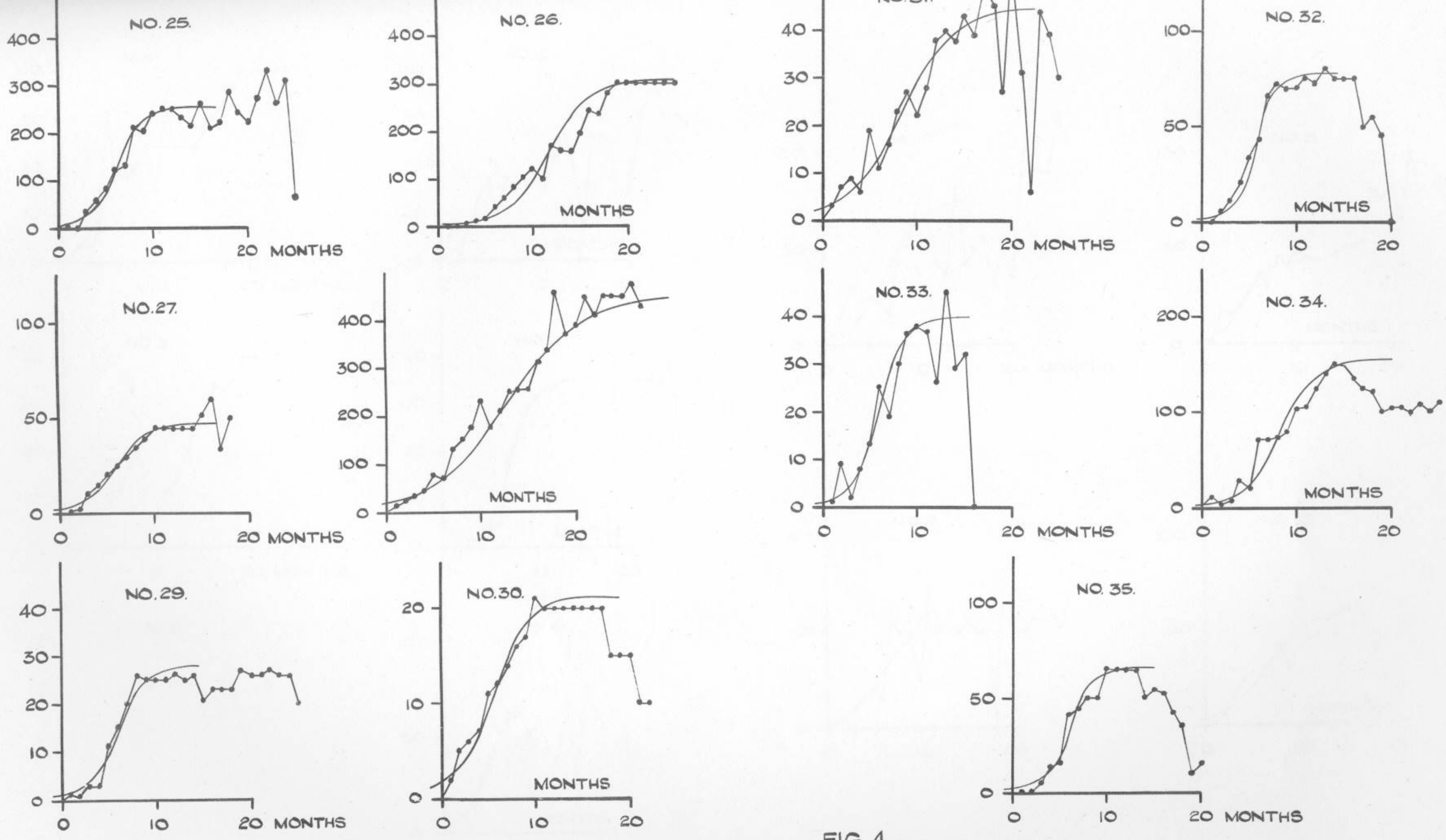
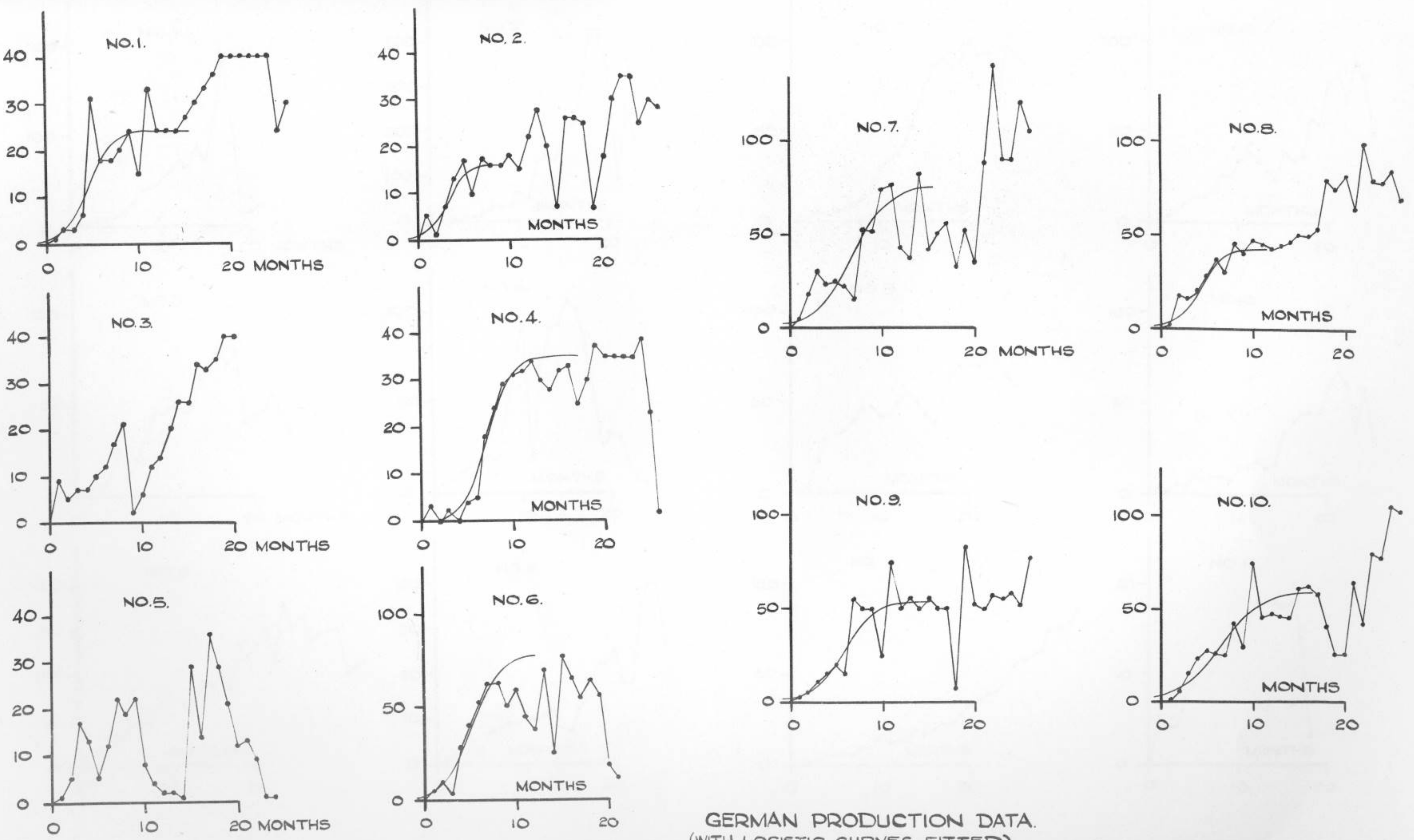
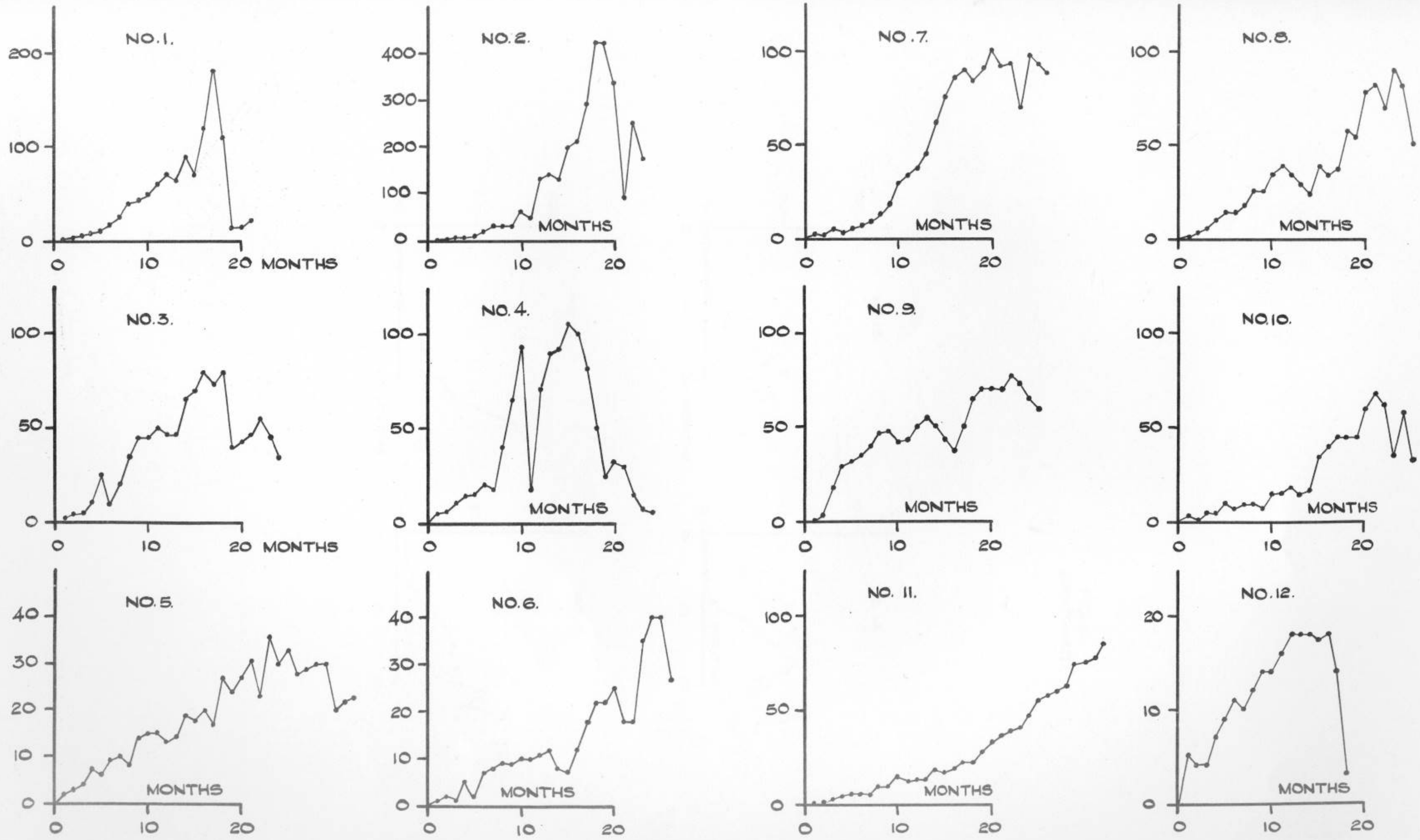


FIG. 4.
(CONTINUED)
SHEET (3)



GERMAN PRODUCTION DATA.
(WITH LOGISTIC CURVES FITTED)

FIG.5.



JAPANESE PRODUCTION DATA. FIG. 6.

FIG.6.
(CONTINUED)

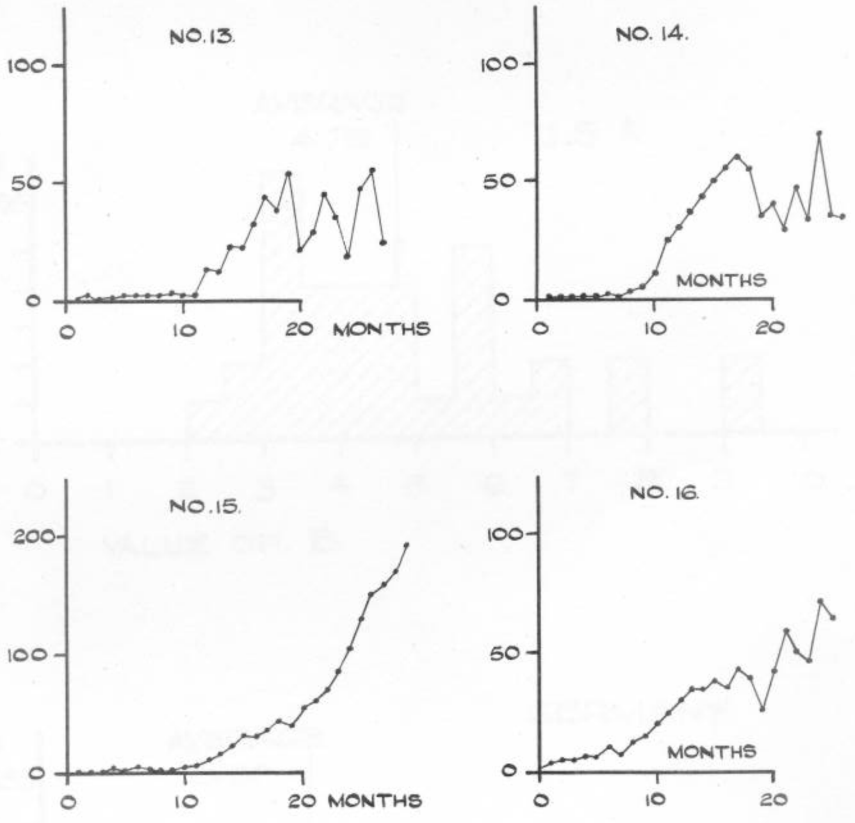


FIG.7.

PRODUCTION GROWTH INDEX, B.

HISTOGRAMS OF VALUES IN EACH COUNTRY

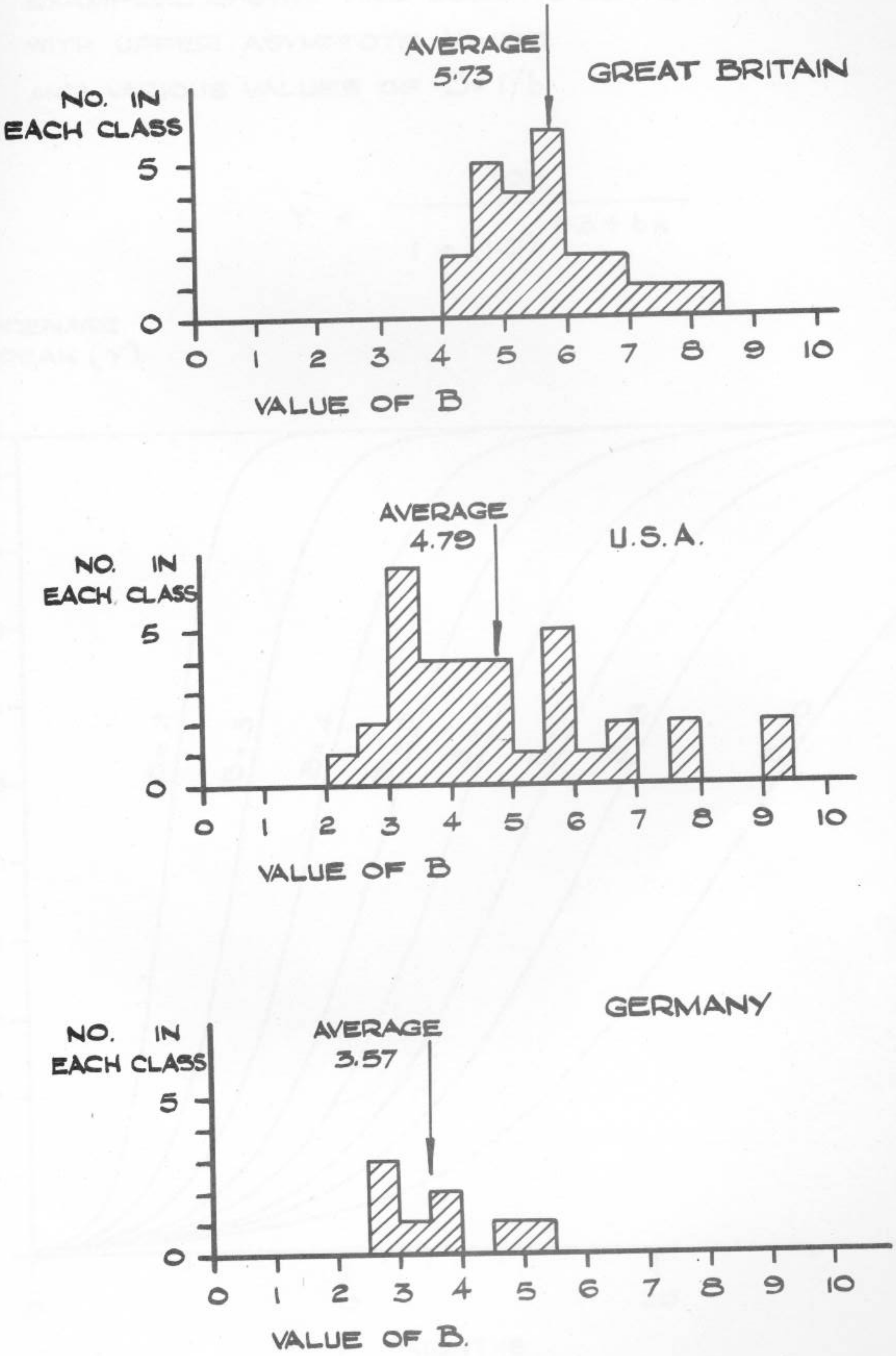


FIG.8.

EXAMPLES OF STANDARD CURVES

EXAMPLES SHOWN ARE LOGISTIC CURVES
WITH UPPER ASYMPTOTE $k = 105$
AND VARIOUS VALUES OF $B = 1/b$

$$Y = \frac{105}{1 + (10)^{2.3 + bx}}$$