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On some problems of unsteady supersonic aerofoil theory.

– By –

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SUMMARY

Unsteady supersonic flow round an aerofoil of infinite span is considered in the first part of the paper. It is shown that the pressure at any given point of an aerofoil under forward acceleration can be analysed into three components, one of which is the steady (Ackeret) pressure due to the instantaneous velocity, while of the other two, one depends directly on the acceleration, and one on the square of the velocity, during a limited time interval preceding the instant under consideration. However, the difference between the total pressure and the "steady pressure component" is such that it can be neglected in all the definitely supersonic conditions which are likely to occur in practice.

The oscillatory supersonic flow round a Delta wing inside the Mach cone emanating from its apex is considered in the second part of the paper. Particular "normal" solutions are obtained by means of a special system of curvilinear coordinates. It is shown that the velocity potentials corresponding to vertical and pitching oscillations of the wing can be represented by a series of such normal solutions.

The assumptions of linearised theory are adopted throughout.

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1. INTRODUCTION.

1.1. In the present paper, the linearised theory of compressible flow will be applied to some problems of unsteady supersonic aerofoil theory. Two specific topics will be dealt with under this heading, viz. (i) unsteady supersonic flow round an aerofoil in two dimensions, with particular reference to accelerated motion, and (ii) oscillatory motion of a Delta wing at supersonic speeds.

1.2. In the first part of the paper (Section 2), we consider in the first instance two dimensional accelerated flow round a symmetrical aerofoil at zero incidence. The velocity potential for this type of flow cam be represented by a distribution of elementary solutions as given by

$$\frac{a}{\sqrt{a^2(t-t_o)^2-(x-x_o)^2-(y-y_o)^2}}$$

in (x,y,t) space, where t denotes the time and a the velocity of sound. This distribution is of a type which has been used in connection with steady supersonic aerofoil theory in three dimensions (Refs. 1 and 2). However, in that application t represents the third spatial dimension and a is the non-dimensional constant $1/\sqrt{M^2-1}$ where M is the Mach number of the flow. Thus, if the direction of flight coincides with the direction of increasing x while the chord of the aerofoil always lies on the x-axis, $\Phi(x_o, y_o, t_o)$ at any point outside the aerofoil is given by

$$\Phi(x_o, y_o, t_o) = a \iint \frac{\sigma(x,t)\, d\,x\, dt}{\sqrt{a^2(t-t_o)^2-(x-x_o)^2-y_o^2}}$$

In the above formula, the "source density" $\sigma(x,t)$ is related to the normal velocity component by

$$v_o = \lim_{y_o \to 0} \left(\frac{\partial \Phi}{\partial y_o}\right) = \pi\sigma(x_o, to),$$

and the integration extends over values of x and t which corresponds to points on the aerofoil, and which satisfy the conditions $t < t_o$ and $a^2(t-t_o)^2 - (x-x_o)^2 - y_o^2 > 0$.

The vertical velocity v_o in turn can be expressed in terms of the kinematic conditions at the aerofoil.

Using the above representation it is shown that the pressure at any given point of the aerofoil can be analysed into three parts, one of which is the steady pressure due to the instantaneous velocity, while of the other two one depends directly on the acceleration, and one on the square of the velocity, during a limited time interval preceding the instant under consideration. The component depending on the acceleration gives rise to an expression for the apparent mass of an aerofoil at supersonic speeds, which is calculated for various cases of uniform acceleration.

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However, it is shown that the difference between the total pressure and the "steady pressure component" is such that under definitely supersonic conditions ($M > 1.15$, say) it can be neglected in all cases which are likely to occur in practice. This statement does not apply to transonic speeds, but the conclusions for such speeds reached on the basis of linearised theory are of doubtful validity in any case.

In view of the fact that conditions above and below the aerofoil are independent of one another under two-dimensional supersonic conditions, the methods and results mentioned above also apply to aerofoils at incidence.

1.3. The second part of the paper (Section 3) deals with unsteady supersonic conditions in three dimensions, in particular with the oscillatory motion of a Delta wing whose leading edges are inside the Mach cone emanating from the apex. The alternative problem (leading edges of wing outside Mach cone emanating from apex) has already been solved by Garrick and Rubinow (Ref. 3).

It is assumed that the free stream velocity is parallel to the positive direction of the x-axis, while the Delta wing lies (approximately) in the (x,y) plane, its apex coinciding with the origin. A special system of co-ordinates (r, ρ, σ) is then introduced by

$$x = r \quad \mathrm{ns}\,(\rho, k')\,\mathrm{nd}\,(\sigma, k)$$

$$y = \frac{1}{\beta} r \quad \mathrm{ds}\,(\rho, k')\,\mathrm{sd}\,(\sigma, k)$$

$$z = \frac{1}{\beta} r \quad \mathrm{cs}\,(\rho, k')\,\mathrm{cd}\,(\sigma, k)$$

In these formulae, $k = \beta \cdot \tan \gamma$, $k^2 + k'^2 = 1$, $k > 0$, $k' > 0$, $\beta = \sqrt{M^2 - 1}$, where M is the Mach number of the flow, γ is the apex semi-angle of the Delta wing, and ns, nd, etc., are the well known Jacobian elliptic functions in Glaisher's notation. Particular "normal" solutions for the velocity potential are then given by

$$\Phi = \frac{1}{\sqrt{r}} J_{n+\frac{1}{2}}(\lambda r)\, F_n^m(\mathrm{ns}(\rho, k'))\, E_n^m(k'\mathrm{nd}(\sigma, k))\, \exp\left[i \frac{\omega}{V}(Vt - x \sec^2 \mu) \right]$$

where $\lambda = \dfrac{\omega}{V\beta} \sec \mu$, $\mu = \operatorname{cosec}^{-1} M$, and the $J_{n+\frac{1}{2}}$, and E_n^m and F_n^m are Bessel functions, and Lamé functions of the first and second kind respectively. It is shown that the velocity potentials corresponding to the oscillation of a Delta wing in vertical motion and in pitch can be represented by series of normal solutions as mentioned above.

/Acceleration.....

2. ACCELERATION EFFECTS IN SUPERSONIC FLOW.

2.1. Consider the two-dimensional rectilinear unsteady flow round a symmetrical aerofoil moving at zero incidence. We choose a system of coordinates which is at rest relative to the fluid in regions far away from the aerofoil , such that the chord of the aerofoil always coincides with the x - axis, with the leading edge pointing in positive direction. The following analysis includes the possibility that the surface of the aerofoil be deformable, provided the aerofoil remains symmetrical throughout.

The linearised equation for the velocity potential is

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} - \frac{1}{a^2} \frac{\partial^2 \Phi}{\partial t^2} = 0 \qquad - - - (1)$$

where t is the time coordinate and a the velocity of sound. We now apply the theory developed for dealing with steady flow round aerofoils at zero incidence in three dimension (see Refs. 1 and 2). Taking into account that the equation of motion in that theory is $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} - (M^2-1) \frac{\partial^2 \Phi}{\partial z^2} = 0$

where z is the direction of motion of the aerofoil and M the Mach number, we have to replace the quantity $\beta = \sqrt{M^2 - 1}$ everywhere in that theory by $\frac{1}{a}$. We then find that in our present case the velocity potential Φ at a point $P = (x_o, y_o, t_o)$, cam be represented by

$$\Phi\,(x_o, y_o, t_o) = a \iint_R \frac{\sigma\,(x,t)\; d\,x\; d\,t}{\sqrt{a^2(t-t_o)^2 - (x-x_o)^2 - y_o^2}} \qquad - - - (2)$$

where the integration extends over values (x, t) which correspond to points on the aerofoil and which satisfy the conditions

$$a^2\,(t - t_o)^2 - (x - x_o)^2 > 0 \qquad \text{and } t < t_o$$

Conditions in the (x,t) plane are sketched in Fig. 1.

The "source density" $\sigma\,(x,t)$ is related to the normal velocity component by

$$\lim_{y_o \to o} \left(\frac{\partial \Phi}{\partial y_o} \right) = \pi\, \sigma\,(x_o,\ t_o)$$

Let the position of the surface of the aerofoil at any time be given by $y = F(x,\ t)$, then the boundary condition at the aerofoil is

$$v = u\, \frac{\partial F}{\partial x} + \frac{\partial F}{\partial t}$$

/where .....

where $u = \dfrac{\partial \Phi}{\partial x}$, $v = \dfrac{\partial \Phi}{\partial y}$. Now assume that the position of the leading edge as a function of the time is $x_{\ell} = f(t)$, while the normal coordinate of the aerofoil at a distance x' aft of the leading edge is given by $y = g\ (x', t)$. We have $x' = x_{\ell} - x = f(t) - x$, by definition, and so $y = g\ (f\ (t) - x,\ t)$, or

$$F\ (x,\ t) = g\ (f\ (t) - x,\ t)$$

so that the boundary condition becomes

$$v = u\ \left(- \frac{\partial g}{\partial x'} \right) + f'(t)\ \frac{\partial g}{\partial x'} + \frac{\partial g}{\partial t} = (f'(t) - u)\ \frac{\partial g}{\partial x'} + \frac{\partial g}{\partial t}$$

Now u may be supposed to be small compared with $f'(t)$ which is the forward velocity of the aerofoil and so can be neglected, in accordance with the simplifying assumptions of linearised theory. Hence, at the aerofoil

$$\frac{\partial \Phi}{\partial y} = v = f'(t)\ \frac{\partial g}{\partial x'} + \frac{\partial g}{\partial t}$$

Thus, finally, the source density σ at a point x, t of the aerofoil is given by

$$\sigma(x,t) = \frac{1}{\pi}\ \left(f'\ (t)\ \frac{\partial g}{\partial x'} + \frac{\partial g}{\partial t}\right) = \frac{1}{\pi}\ (f'(t)\ g_{x'} + g_t) \qquad - - - (3)$$

and

$$\Phi\ (x_o, y_o, t_o) = \frac{a}{\pi} \iint \frac{(f'(t)\ g_{x'} + g_t)\ d\ x\ dt}{\sqrt{a^2\ (t - t_o)^2 - (x - x_o)^2 - y_o^2}} \qquad - - - (4)$$

Denoting the free stream pressure by p_o, we obtain for the pressure p at any finite point,

$$p = p_o - \rho \left[\tfrac{1}{2}\ (u^2 + v^2) + \frac{\partial \Phi}{\partial t} \right]$$

or

$$p = p_o - \rho\ \frac{\partial \Phi}{\partial t} = p_o + \Delta\ p \qquad - - - (5)$$

after linearisation.

Calculating $\dfrac{\partial \Phi}{\partial t} = \Phi_t$ as given by (4), we obtain

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$$\Phi_t(x_o,y_o,t_o) = \frac{a}{\pi}\iint_R \frac{\frac{\partial}{\partial t}(f'(t)g_{x'}+g_t)\,d\,x\,dt}{\sqrt{a^2\,(t-t_o)^2 - (x-x_o)^2 - y_o^2}}$$

$$+\frac{a}{\pi}\int_C \frac{(f'(t)\,g_{x'} + g_t)\,d\,x}{\sqrt{a^2(t-t_o)^2-(x-x_o)^2-y_o^2}} \qquad ---(6)$$

where the second integral on the right hand side is taken along those parts of the boundary of the aerofoil (in (x,t) plane) which satisfy $a^2(t-t_o)^2 - (x-x_o)^2 - y_o^2 \geqslant 0$, $t \leqslant t_o$ (e.g. In Fig. 1. C is the curvilinear segment L'L"). Also,

$$\frac{\partial}{\partial t}(f'(t)\,g_{x'} + g_t) = f''(t)\,g_{x'} + \left[f'(t)\right]^2 g_{x'x'} + f'(t) g_{x't} + g_{tt} \qquad ---(7)$$

Thus, taking into account that $\frac{dx}{dt} = f'(t)$ everywhere on C,

$$\Phi_t(x_o,y_o,t_o) = \frac{a}{\pi}\left[\iint_R \frac{f''g_{x'} + (f')^2 g_{x'x'} + f'\,g_{x't} + g_{tt}}{\sqrt{a^2\,(t-t_o)^2-(x-x_o)^2 - y_o^2}}\,d\,x\,dt \right. \qquad ---(8)$$

$$\left. + \int_C \frac{(f')^2\,g_{x'} + f'\,g_t}{\sqrt{a^2(t-t_o)^2-(x-x_o)^2-y_o^2}}\,dt\right]$$

This formula is valid on the assumption that $g_{x'}$ is continuous and differentiable everywhere. In the case that $g_{x'}$ is discontinous at a number of fixed points on the aerofoil, Φ_t must be evaluated separately for the different regions in which $g_{x'}$ is continuous, and the results added. This is of practical importance for aerofoils with polygonal boundaries, e.g. with double wedge section.

Taking the particular case of a rigid aerofoil (symmetrical with respect to the y-axis, as before), we see that g is now independent of t, and so

$$\Phi_t = \frac{a}{\pi}\left[\iint_R \frac{f''\,g_{x'}}{r}\,d\,x\,dt + \iint_R \frac{(f')^2 g_{x'x'}}{r}\,d\,x\,dt + g_{x'}\int_C \frac{(f')^2 dt}{r}\right] \qquad ---(9)$$

where $r^2 = a^2\,(t-t_o)^2 - (x-x_o)^2 = y_o^2$, $r > 0$.

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It will be seen that the first integral on the right hand side of (9) depends directly on the forward acceleration $f''(t)$, while the other two integrals depend on the acceleration only through the intermediary of the velocity change. Thus, only the aerodynamic force corresponding to the first integral may be said to be a genuine apparent mass effect.

Considering conditions at the aerofoil, we may transform the expression for Φ_t still further in the following way. We have

$\dfrac{D(x, t)}{D(x', t)} = -1$, and so

$$\Phi_t = -\frac{a}{\pi}\left[\iint_{R'} \frac{f'' g_{x'}}{r}\, dx'\, dt + \iint_{R'} \frac{(f')^2 g_{x'x'}}{r}\, dx' dt + g_{x'}(0)\int_{C'} \frac{(f')^2}{r}\, dt\right] \qquad ---(10)$$

where R' and C' are the transforms of R and C in the (x', t) plane (Fig. 2).

We define a function $h(x')$ by

$$h(x') = \int_{r^2 > 0} \frac{(f')^2}{r}\, dt \qquad \text{for } 0 \leq x' \leq x_o'$$

where $x'_o = f(t_o) - x_o$. In terms of this function, the second integral on the right hand side of (10) becomes

$$\iint_{R'} \frac{(f')^2 g_{x'x'}}{r}\, dx' dt = \int_o^{x'_o} g_{x'x'} h(x') dx' = \Big[g_{x'} h(x') \Big]_o^{x'_o} - \int_o^{x'_o} g_{x'} h'(x') dx',$$

while the third integral can be written

$$\int_{C'} \frac{(f')^2}{r}\, dt = h(0)$$

Hence

$$\iint_{R'} \frac{(f')^2 g_{x'x'}}{r}\, dx' dt + g_{x'}(o) \int_{C'} \frac{(f')^2}{r}\, dt = \lim_{x' \to x'_o} g_{x'} h(x') - \int_o^{x'_o} g_{x'} h'(x')\, dx'$$

Now it can be shown that

$$\lim_{x' \to x'_o} g_{x'}\, h(x') = g_{x'}(x'_o) \cdot \Big[f'(t_o) \Big]^2 \cdot \frac{\pi}{a\sqrt{\left[\dfrac{f'(t_o)}{a}\right]^2 - 1}}$$

Substituting in (10) and putting $V(t) = f'(t)$ for the forward velocity, $\alpha(x') = g_{x'}(x')$ for the local incidence, and $M(t) = \dfrac{V(t)}{a}$ for the Mach number, we obtain

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$$\Phi_t = -\alpha(x'_o) \frac{\left[V(t_o)\right]^2}{\sqrt{\left[M(t_o)\right]^2 - 1}} + \frac{a}{\pi}\int_o^{x'_o}\left(\alpha(x')\frac{d}{dx'}\int\frac{\left[V(t)\right]^2}{r}\,dt\right)dx'$$

$$-\frac{a}{\pi}\iint_{R'}\frac{V'(t)\alpha(x')}{r}\,d\,x'\,dt \qquad - - - (11)$$

The excess pressure, Δp (see equation (5) above) is then obtained by multiplying (11) by $-\rho$.

$$\Delta p = +\rho\alpha(x'_o)\frac{\left[V(t_o)\right]^2}{\sqrt{\left[M(t_o)\right]^2-1}} - \frac{a\rho}{\pi}\int_o^{x'_o}\left(\alpha(x')\frac{d}{dx'}\int\frac{\left[V(t)\right]^2}{r}\,dt\right)dx'$$

$$+\frac{a\rho}{\pi}\iint_{R'}\frac{V'(t)\alpha(x')}{r}\,dx'\,dt \qquad - - - (12)$$

The first term on the right hand side of (12) is the steady motion term, as obtained by Ackeret's theory, while the second depends on the square of the velocity during a limited period preceding t_o. The third term depends on the acceleration and may be said to be an apparent mass effect.

2.2. We are now going to consider some special cases. It will appear that for all the practical cases of purely supersonic flow that can be envisaged at present the "unsteady terms" are negligible compared with the "steady term "in the expression for Δ p. It follows that for such cases, the expression for the drag given by Ackeret's theory is adequate.

Only cases of uniform acceleration will be considered. For such cases f(t) can be written in the form

$$f(t) = f(t_o) + V(t_o)\,(t - t_o) + \tfrac{1}{2}\,s\,(t - t_o)^2 \qquad - - - (13)$$

where t_o is an arbitary moment of time and s is a constant.

The acceleration term in the expression for Δ p in equation (12) becomes

$$\Delta p_a = \frac{a\rho}{\pi}\iint_{R'}\frac{V'(t)\alpha(x')}{r}\,d\,x'\,dt = \frac{a\rho s}{\pi}\int_o^{x'_o}\alpha(x')\,k(x')\,d\,x' \qquad - - - (14)$$

where

$$k(x') = \int_{r^2>0}\frac{dt}{r} = \int\frac{dt}{\sqrt{a^2(t-t_o)^2-(x-x_o)^2}} = \int\frac{dt}{\sqrt{a^2(t-t_o)^2-(f(t)+x'-f(t_o)-x'_o)^2}}$$

$$= \int\frac{dt}{\sqrt{a^2(t-t_o)^2 - \left[V(t_o)\,(t-t_o) + \tfrac{1}{2}\,s\,(t-t_o)^2 - (x'-x'_o)\right]^2}} \qquad - - - (15)$$

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Let M be the Mach number of the flow at time t_o
$M = M(t_o) = \frac{V(t_o)}{a}$, as before, while $K(\lambda)$ is the complete elliptic integral of the first kind, $K(\lambda) = \int_o^{\pi/2} \frac{d\phi}{\sqrt{1 - \lambda^2 \sin^2\phi}}$,

and q is the non dimensional parameter $\frac{1}{a}\sqrt{2(x'_o - x')|s|}$.
Then it can be shown that

$$k(x') = \frac{2}{a} \frac{1}{\sqrt{M^2 - (1-q)^2}} K\left(2\sqrt{\frac{q}{M^2 - (1-q)^2}}\right) \text{ when } s \geqslant 0 \qquad ---(16)$$

and

$$k(x') = \frac{2}{a} \frac{1}{\sqrt[4]{(M^2-1)^2 + 2q^2(M^2+1) + q^4}} K\left(\sqrt{\tfrac{1}{2}\left(1 - \frac{M^2 - 1 + q^2}{\sqrt{(M^2-1)^2 + 2q^2(M^2+1) + q^4}}\right)}\right)$$

$$\text{when } s \leqslant 0 \qquad ---(17)$$

The last expression may also be written in the form

$$k(x') = \frac{2}{a} \frac{1}{\sqrt[4]{(M^2-1+q^2)^2 + 2q^2}} K\left(\sqrt{\tfrac{1}{2}\left(1 - \frac{M^2 - 1 + q^2}{\sqrt{(M^2-1+q^2)^2 + 2q^2}}\right)}\right)$$

$$---(18)$$

For small q, and therefore for small $|s|$, the expression for $k(x')$ for both positive and negative s becomes equal to $\frac{\pi}{a} \cdot \frac{1}{\sqrt{M^2 - 1}}$. For all cases of accelerated supersonic flow which are likely to occur in practice, the approximation $k(x') = \frac{\pi}{a} \cdot \frac{1}{\sqrt{M^2 - 1}}$ appears to be adequate.

Accepting this approximation, Δp_a as given by equation (14) becomes

$$\Delta p_a = \frac{\rho_s}{\sqrt{M^2-1}} \int_o^{x'_o} \alpha(x')dx' = \frac{\rho_s}{\sqrt{M^2-1}}(g(x'_o) - g(0)) = \frac{\rho_s}{\sqrt{M^2-1}} g(x'_o)$$

since $g = 0$ at the leading edge. $\qquad ---(19)$

The total longitudinal force D_a due to the aerodynamic inertia effect is then obtained by multiplying Δp_a by the local incidence and integrating over the top and bottom surfaces of the aerofoil. Thus

$$D_a = 2\int_o^c \Delta p_a(x')\alpha(x')\,dx' = \frac{2\rho_s}{\sqrt{M^2-1}} \int_o^c g(x')\,g'(x')\,dx' = \frac{\rho_s}{\sqrt{M^2-1}}\left[(g(c))^2 - (g(0))^2\right] = 0$$

$$---(20)$$

/since.....

since $g(c) = g(0) = 0$ for a (closed) symmetrical aerofoil. We have therefore shown that $D_a = o(s)$, i.e. $\lim_{s \to o} \frac{D_a}{s} = o$, in other words, D_a vanishes for small s except for expressions of the second order of smallness in s. This result presumably holds even for a wedge-shaped aerofoil since the cut-off trailing edge should be considered as the limit of a trailing edge of finite shape in connection with the present problem. However, the formal expression for D_a, for aerofoils with cut-off trailing edges, is

$$D_a = \frac{\rho_s}{\sqrt{M^2-1}} \Big[g(c) \Big]^2 \qquad - - - (21)$$

Coming back to the exact expressions for $k(x')$ and p_a, we see that equation (16) is valid only provided $\mu < M - 1$. Subject to this condition, which has a single geometrical interpretation, and subject to $\mu \ll 1$, it can be shown (using equations (16) - (18)) that p_a is at any rate numerically small compared with the "steady flow term" for the pressure, $\rho \alpha(x'_o) \frac{[V(t_o)]^2}{\sqrt{[M(t_o)]^2-1}}$. Now, for any given aerofoil, μ is not greater than $\frac{1}{a}\sqrt{2\ c\ s}$, where a is the velocity of sound and c is the chord of the aerofoil, as before. Assuming c = 20 ft. and s = 100 ft/sec^2 - values which are as high as any that can be expected in practice for the time being - we see that $\frac{1}{a}\sqrt{2\ c\ s}$ is of the order of $.05 \ll 1$, the exact value depending on the altitude.

To obtain an impression of the magnitude of the "unsteady term" for the pressure which depends on the square of the velocity,

$$p_c = -\frac{a\,\rho}{\pi}\int_o^{x'_o} \alpha(x')\, \frac{d}{dx'}\left(\int \frac{[V(t)]^2}{r}\, d\,t\right) dx'$$

(compare equation (12)), we consider the particular case of a double wedge aerofoil whose maximum thickness $2\,\lambda c.\tan\beta$, at a point λ c aft of the leading edge. Then $\alpha(x') = \tan\beta$ for $x' < \lambda c$, and $\alpha(x') = -\frac{\lambda}{1-\lambda}\tan\beta$ for $x' > \lambda$ c. Hence

$$(22)\left\{\begin{array}{l} p_c = -\dfrac{a\,\rho}{\pi}\tan\beta\left[\displaystyle\int \frac{[V(t)]^2}{r}\, dt\right]_{x'=o}^{x'=x'_o} \qquad \text{for } x'_o < \lambda c \\ \text{- and} \\ p_c = -\dfrac{a\,\rho}{\pi}\tan\beta\left\{\left[\displaystyle\int\frac{[V(t)]^2}{r}dt\right]_{x'=0}^{x'=c} - \dfrac{\lambda}{1-\lambda}\left[\displaystyle\int\frac{[V(t)]^2}{r}\,dt\right]_{x'=c}^{x'=x'_o}\right\} \\ \qquad\qquad \text{for } x'_o > \lambda c. \end{array}\right.$$

Now, using a mean value theorem of the integral calculus,

$$\int_{r^2>0} \frac{\left[V(t)\right]^2}{r}\,dt = \left[V(t^*)\right]^2 \int_{r^2>0} \frac{dt}{r} = \left[V(t^*)\right]^2 k(x') \doteq \frac{\pi}{a} \frac{\left[V(t^*)\right]^2}{\sqrt{\left[M(t_o)\right]^2 - 1}}$$

where $t^* = t^*(x')$ is a specific value of t within the interval of integration, so that $t^*(x_o) = t_o$

$$\left.\begin{aligned} \Delta p_c &\doteq -\rho \frac{\tan\beta}{\sqrt{\left[M(t_o)\right]^2-1}} \left\{ \left[V(t_o)\right]^2 - \left[V(t^*(0))\right]^2 \right\} \quad \text{for } x'_o < \lambda c \\ &\text{and} \\ \Delta p_c &\doteq -\rho \frac{\tan\beta}{\sqrt{\left[M(t_o)\right]^2-1}} \left\{ \left[V(t^*(\lambda c))\right]^2 - \left[V(t^*(0))\right]^2 - \frac{\lambda}{1-\lambda}\left[\left[V(t_o)\right]^2 - \left[V(t^*(\lambda c))\right]^2\right] \right\} \quad \text{for } x'_o > \lambda c \end{aligned}\right\} \quad (23)$$

This compares with the "steady motion" term Δp_s

$$\left.\begin{aligned} \Delta p_s &= \rho \frac{\tan\beta}{\sqrt{\left[M(t_o)\right]^2-1}} \left[V(t_o)\right]^2 \quad \text{for } x'_o < \lambda c \\ &\text{and} \\ \Delta p_s &= -\rho \frac{\lambda}{1-\lambda} \frac{\tan\beta}{\sqrt{\left[M(t_o)\right]^2-1}} \left[V(t_o)\right]^2 \quad \text{for } x'_o > \lambda c \end{aligned}\right\} \quad - - - (24)$$

To prove that, in general, Δp_c is numerically small compared with Δp_s, it is sufficient to show that the difference between the squares of any two velocities V(t) within the region R' is small compared with $\left[V(t_o)\right]^2$. Indeed, the time interval involved can be no greater than $\dfrac{x'_o}{V(t_o) - a}$, and if $x'_o \leq 20$ ft. and $V(t_o) = 1.2\,a$, say, then this time interval is of the order .1 sec. Assume that $s = 100$ ft/sec^2 as before, then the variation of the velocity in the interval considered cannot be greater than 10 ft/sec., so that the variation of $\left[V(t)\right]^2$ is rather less than two per cent of $\left[V(t_o)\right]^2$.

We notice for future reference that the expression for p_c for a wedge-shaped aerofoil ($\lambda = 1$) is,

$$\Delta p_c = -\rho \tan\beta \left\{ \frac{\left[V(t_o)\right]^2}{\sqrt{\left[M(t_o)\right]^2-1}} - \frac{a}{\pi}\left[V(t^*(0))\right]^2 k(0) \right\} \quad - - - (25)$$

so that

$$\Delta p_s + \Delta p_c = \frac{a\rho}{\pi} \tan\beta \left[V(t^*(0))\right]^2 k(0) \quad - - - (26)$$

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A number of the results obtained so far can be applied to the two dimensional supersonic flow of a thin aerofoil at a small incidence. For simplicity, we shall confine our discussion to the case of an infinite flat plate.

2.3. In two-dimensional steady supersonic flow conditions on the upper and lower surface of the aerofoil are independent of one another (compare Refs. 1 and 2). This "principle of independence" also applies to certain cases of unsteady flow. More precisely, the pressure at a point x_o on the upper surface of the aerofoil, at time t_o, is independent of the geometry of the lower surface provided the angular region in the (x, t) plane, $a^2 (t - t_o)^2 - (x - x_o)^2 > 0$, $t < t_o$ does not include any part of the trailing edge. This condition is satisfied, for instance, in case the forward velocity of the aerofoil is supersonic throughout. Also, in accelerated flow it is satisfied as soon as the forward velocity exceeds the speed of sound. Again, in accelerated motion, the condition will still be satisfied, for points sufficiently close to the leading edge, even at speeds slightly below the speed of sound.

In all cases in which the principle of independence is satisfied at all points of the aerofoil, we may apply the results obtained earlier in this paper. In particular, the total pressure may be represented as the sum of three components as in equation (12). Thus, on the top surface of an aerofoil at incidence α, at a point x'_o aft of the leading edge,

$$\Delta p_a = \frac{\rho s \alpha}{\sqrt{M^2 - 1}} x'_o \qquad - - - - (27)$$

The corresponding normal force on the aerofoil then, is obtained by integrating over top and bottom surfaces

$$N_a = \frac{\rho s \alpha c^2}{\sqrt{M^2 - 1}} \qquad - - - - (28)$$

Since the acceleration normal to the plate is $s\alpha$, we may consider the ratio $\frac{N_a}{s\alpha} = \rho \frac{c^2}{\sqrt{M^2 - 1}}$ as a kind of apparent mass of a flat plate at supersonic speeds. However, as in the symmetrical case treated above, neither the acceleration term, Δp_a, nor the velocity correction term, Δp_c, are likely to be of any numerical importance for all practical purposes under definitely supersonic conditions ($M \geqslant 1.15$, say).

3. THE OSCILLATING DELTA WING AT SUPERSONIC SPEEDS.

3.1. Two-dimensional oscillatory aerofoil theory has been dealt with exhaustively by various authors (e.g. Refs. 4 and 5) from the point of view of linearised theory. In three dimensions we have to distinguish different physical cases, which present analytical problems of varying degrees of difficulty. The simplest case is the "definitely supersonic case", in which the principle of independence is valid, i.e. the pressures on the upper and lower surfaces are independent of the geometry of the lower and upper surfaces respectively (compare para. 2 above). This is the case which is called "purely supersonic" by Garrick and Rubinow and is considered by those authors in Ref. 3. Definitely supersonic problems can always be solved by means of single source distributions, the source density

/being .....

being related to the local incidence after the manner of para 2 above. However, Garrick and Rubinow adopt a Green's function method which has certain advantages from the point of view of uniqueness considerations.

The alternative cases, called "mixed supersonic" by Garrick and Rubinow, do not satisfy the principle of independence. The flow round a Delta wing is mixed supersonic, or as it has also been called, "quasi-subsonic", if the leading edges of the aerofoil lie inside the Mach cone emanating from the apex. In this case, the aerofoil can still be replaced by a distribution of doublets but there is no longer any simple relation between the strength of the doublets and the kinematic boundary conditions. However, particular solutions of a different kind can now be obtained by a method of pseudo-orthogonal coordinates. This method, which was originally put forward to solve the corresponding steady flow problem (Refs. 6 and 7) has also been applied to the calculation of a number of stability derivatives (Ref. 8) and to the design of a special aerofoil section (Ref. 9). In the present section we shall derive a series of normal solutions for the velocity potential in oscillatory flow and we shall determine the corresponding normal incidence and pressure distributions. To find the pressure distribution and thence the forces on a Delta wing oscillating in a given mode, we should have to determine a linear combination of normal solutions so as to satisfy the specified boundary conditions everywhere at the aerofoil. Failing the explicit determination of an exact solution, we may always adopt a collocation method, i.e. we may construct a finite linear combination of normal solutions in such a way that the boundary conditions are satisfied at least at a finite number of points.

It may be mentioned that even if linearised theory is inadequate in the purely supersonic oscillatory case, it may still provide the correct answer for the quasi-subsonic case where the second order phenomena near the leading edge are less critical.

3.2. The motion of the oscillating aerofoil is governed by the wave equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} - \frac{1}{a^2}\frac{\partial^2 \Phi}{\partial t^2} = 0 \qquad --- (29)$$

where the system of reference is at rest relative the free air, x being positive in the direction of motion of the aerofoil while y is positive to starboard, and z is positive upwards, t is the time, Φ is the velocity potential, and finally a is the velocity of sound, as before. Let x', y', z' be a system of coordinates fixed in the aerofoil, so that

$$x' = x - Vt \qquad y' = y \qquad z' = z$$

where V is the forward speed of the aerofoil, and so that the origin of coordinates coincides with its apex. Putting

$$\Phi = \Psi(x', y', z')\, e^{i\omega t} \qquad --- (30)$$

for harmonic motion, we then obtained the following differential equation for Ψ

$$(1-M^2)\frac{\partial^2 \Psi}{\partial x'^2} + \frac{\partial^2 \Psi}{\partial y'^2} + \frac{\partial^2 \Psi}{\partial z'^2} + \left(\frac{\omega}{a}\right)^2 \Psi - 2 i\omega \frac{V}{a^2}\frac{\partial \Psi}{\partial x'} = 0$$

$$--- (31)$$

/Next .....

Next we introduce $\bar{\Psi}_o$ by

$$\Psi(x',y',z') = \Psi_o(x',y',z') \exp\left(\frac{i\omega x'V}{1-M^2}\right) \qquad --- (32)$$

so that

$$\bar{\Phi} = \bar{\Psi}_o \exp\left(i \frac{\omega}{V}\left(Vt - x' \sec^2\mu \right)\right) \qquad --- (33)$$

where μ is the Mach angle, $\mu = \operatorname{cosec}^{-1} M$. Substituting (32) in (31) we find that the differential equation for $\bar{\Psi}_o$ is

$$\sin^2\mu\left(\frac{\partial^2\bar{\Psi}}{\partial y'} + \frac{\partial^2\bar{\Psi}_o}{\partial z'^2}\right) - \cos^2\mu \frac{\partial^2\bar{\Psi}_o}{\partial x'^2} - \left(\frac{\omega}{V}\right)^2 \tan^2\mu\, \bar{\Psi}_o = 0$$

or

$$\frac{\partial^2\bar{\Psi}_o}{\partial x'^2} - \frac{1}{\beta^2}\left(\frac{\partial^2\bar{\Psi}_o}{\partial y'^2} + \frac{\partial^2\bar{\Psi}_o}{z'^2}\right) + \lambda^2\bar{\Psi}_o = 0 \qquad --- (34)$$

where $\beta = \cot\mu = \sqrt{M^2-1}$ and $\lambda = \dfrac{\omega\sin\mu}{V\cos^2\mu}$.

Now (compare Ref. 10), put

$$\begin{aligned} x' &= r \operatorname{ns}(\rho, k') \operatorname{nd}(\sigma, k) \\ y' &= \frac{1}{\beta} r \operatorname{ds}(\rho, k') \operatorname{sd}(\sigma, k) \qquad --- (35) \\ z' &= \frac{1}{\beta} r \operatorname{cs}(\rho, k') \operatorname{cd}(\sigma, k) \end{aligned}$$

where $k = \beta \tan\gamma$, $k^2 + k'^2 = 1$, $k > 0$, $k' > 0$, γ is the apex semi-angle of the wing, ns, nd, etc. are the well known Jacobian elliptic functions in Glaisber's notation, and the intervals of variation of the variables r, ρ, σ are as follows

$$0 < r < \infty, \quad 0 < \rho \leq K', \quad -2K \leq \sigma \leq 2K$$

where K and K' are the complete elliptic integrals of the first kind of k and k' respectively. To every triplet r, ρ, σ within the specified interval of variation there corresponds just one point inside the cone $x'^2 - \beta^2(y'^2 + z'^2) = 0$, $x' > 0$ (except for the points of the aerofoil, which occur twice) and vice versa. The points of the aerofoil correspond to $\rho = K'$.

The equation for $\bar{\Psi}_o$ becomes, in terms of these coordinates,

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial\bar{\Psi}_o}{\partial r}\right) + r^2\lambda^2\bar{\Psi}_o - \frac{1}{\operatorname{ns}^2(\rho,k') - k'^2\operatorname{nd}^2(\sigma,k)}\left(\frac{\partial^2\bar{\Psi}_o}{\partial\rho^2} + \frac{\partial^2\bar{\Psi}_o}{\partial\sigma^2}\right) = 0 \qquad --- (36)$$

Introducing $\bar{\Psi}_1 = \sqrt{r}\,\bar{\Psi}_o$ as a new dependent variable, and $s = \lambda r$ to replace r as an independent variable, we obtain

/(37) .....

$$s^2 \frac{\partial^2 \Psi_1}{\partial s^2} + s \frac{\partial \Psi_1}{\partial s} + (s^2 - \frac{1}{4}) \Psi_1 - \frac{1}{ns^2(\rho, k') - k'^2\, nd^2(\sigma, k)} \cdot \left(\frac{\partial^2 \Psi_1}{\partial \rho^2} + \frac{\partial^2 \Psi_1}{\partial \sigma^2} \right) = 0 \qquad - - - (37)$$

Assuming a "normal solution" of the form

$$\Psi_1 = F(s)\, G(\rho)\, H(\sigma)$$

we obtain the following ordinary differential equations for F, G, and H,

$$\frac{d^2F}{ds^2} + \frac{1}{s} \frac{dF}{ds} + \left(1 - \frac{(n + \frac{1}{2})^2}{s^2} \right) F(s) = 0 \qquad - - - (38)$$

$$\frac{d^2G}{d\rho^2} - (n(n+1)\, ns^2(\rho, k') + q)\, G(\rho) = 0 \qquad - - - (39)$$

and

$$\frac{d^2H}{d\sigma^2} + (n(n+1)\, k'^2\, nd^2(\sigma, k) + q)\, H(\sigma) = 0 \qquad - - - (40)$$

where n and q are arbitarry constants. Equation (38) is satisfied by the Bessel function $J_{n+\frac{1}{2}}(s)$. Putting

$$\xi = ns(\rho, k'), \qquad \eta = k'\, nd(\sigma, k),$$

we obtain from (39) and (40),

$$\sqrt{\xi^2 - 1}\sqrt{\xi^2 - k'^2}\, \frac{d}{d\xi}\left(\sqrt{\xi^2 - 1}\sqrt{\xi^2 - k'^2}\, \frac{dG}{d\xi}\right) - (n(n+1)\xi^2 + q)G = 0 \qquad - - - (41)$$

and

$$\sqrt{1 - \eta^2}\sqrt{\eta^2 - k^2}\, \frac{d}{d\eta}\left(\sqrt{1 - \eta^2}\sqrt{\eta^2 - k^2}\, \frac{dH}{d\eta}\right) + \left(n(n+1)\eta^2 + q \right) H = 0 \qquad - - - (42)$$

The two equations are equivalent, except for the different ranges of the variables for which all the expressions occurring in them are real. Both are forms of Lamé's equation. They are satisfied, for appropriate q, by Lamé's functions of the first and second kind, E_n^m and F_n^m (compare Ref. 11). Bearing in mind that Φ should be continuous at and inside the cone $x'^2 - \beta^2(y'^2 + z'^2) = 0$, $x' > 0$, we find that the appropriate functions are

$$G = F_n^m(\xi) \qquad\qquad H = E_n^m(\eta)$$

so that particular solutions for Ψ_0 are given by

$$\Psi_0 = \frac{1}{\sqrt{r}}\, J_{n+\frac{1}{2}}(\lambda r)\, F_n^m(\xi)\, E_n^m(\eta) \qquad - - - (43)$$

/The .....

The corresponding expressions for $\bar{\Phi}$ are

$$\bar{\Phi} = \frac{1}{\sqrt{r}} J_{n+\frac{1}{2}}(\lambda r) \, F_n^m(\xi) \, E_n^m(\eta) \exp\left[i \frac{\omega}{V}(Vt - x' \sec^2\mu) \right] \quad \text{- - - (44)}$$

where $x' = \frac{1}{k'} r \xi \eta$.

Assume that the velocity potential corresponding to a specific case can be expressed as a linear combination of expressions of the type of (44). Then the pressure distribution and theme the forces acting on the aerofoil can be found from Bernoulli's theorem for unsteady motion,

$$\Delta p = 2 \rho_o \left(\frac{\partial \bar{\Phi}}{\partial t} + V \frac{\partial \bar{\Phi}}{\partial x'} \right) \quad \text{- - - (45)}$$

where Δp is the pressure difference between top and bottom surfaces and ρ_o is the density.

On the other hand, let the vertical coordinate of the aerofoil be given in the form

$$z' = z_o(x', y', t) = z_1(x', y') e^{i\omega t} \quad \text{- - - (46)}$$

Then the boundary condition at the aerofoil is

$$\frac{\partial \bar{\Phi}}{\partial r'} = \left[\frac{\partial \bar{\Phi}}{\partial x'} \cdot \frac{\partial z_1}{\partial x'} + \frac{\partial \bar{\Phi}}{\partial y'} \frac{\partial z_1}{\partial y'} + i\omega z_1(x', y') \right] e^{i\omega t} \quad \text{- - (47)}$$

Now, in general, $\frac{\partial \bar{\Phi}}{\partial x'}$ differs from V by a small quantity only, while $\frac{\partial \bar{\Phi}}{\partial y'}$ is itself small. Hence, in accordance with the simplifying assumptions of linearised theory (47) becomes

$$\frac{\partial \bar{\Phi}}{\partial z'} = \left[V \frac{\partial z_1}{\partial x'} + i\omega z_1(x', y') \right] e^{i\omega t} \quad \text{- - - (48)}$$

Now, if f is an arbitrary function of r, ρ, σ (and therefore of r, ξ, η), then we can express $\frac{\partial f}{\partial x'}$ and $\frac{\partial f}{\partial z'}$ as functions of r, ξ, η in the following way see Ref. 7 equations (23) and (24). Replace x, z, μ, ν, n, h, k in that reference by x', z', ξ, η, β, k', 1 respectively).

$$\frac{\partial f}{\partial x'} = -\frac{1}{\beta k'} \left[\xi \eta \frac{\partial f}{\partial r} - \frac{\eta(\xi^2 - k'^2)(\xi^2 - 1)}{r(\xi^2 - \eta^2)} \frac{\partial f}{\partial \xi} - \frac{\xi(\eta^2 - k'^2)(1 - \eta^2)}{r(\xi^2 - \eta^2)} \frac{\partial f}{\partial \eta} \right] \quad \text{- - - (49)}$$

$$\frac{\partial f}{\partial z'} = \frac{\sqrt{(\xi^2 - 1)(1 - \eta^2)}}{k} \left[-\frac{\partial f}{\partial r} + \frac{\xi(\xi^2 - k'^2)}{r(\xi^2 - \eta^2)} \frac{\partial f}{\partial \xi} - \frac{\eta(\eta^2 - k'^2)}{r(\xi^2 - \eta^2)} \frac{\partial f}{\partial \eta} \right] \quad \text{- - - (50)}$$

/Using .....

Using these formulae, we can express Δp and $\dfrac{\partial\bar{\Phi}}{\partial z'}$ in terms of the function $\bar{\Psi}_o \exp\left[i\dfrac{\omega}{V}(Vt - x'\sec^2\mu)\right]$ and of its derivatives. In particular

$$\frac{\partial\bar{\Phi}}{\partial z'} = \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{k}\left[-\frac{\partial\bar{\Psi}_o}{\partial r} + \frac{\xi(\xi^2-k'^2)}{r(\xi^2-\eta^2)}\frac{\partial\bar{\Psi}_o}{\partial\xi} - \frac{\eta(\eta^2-k'^2)}{r(\xi^2-\eta^2)}\frac{\partial\bar{\Psi}_o}{\partial\eta}\right]$$
$$\exp\left[i\frac{\omega}{V}(Vt - x'\sec^2\mu)\right] \quad - - - (51)$$

The functions $F_n^m(\xi)$ remain finite as $\xi \to 1$. Hence, at the aerofoil

$$\frac{\partial\bar{\Phi}}{\partial z'} = \lim_{\xi\to 1}\frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{k}\cdot\frac{\xi(\xi^2-k'^2)}{r(\xi^2-\eta^2)}\frac{\partial\bar{\Psi}_o}{\partial\xi}\exp\left[\frac{i\omega}{V}(Vt-x'\sec^2\mu)\right]$$
$$= \frac{k}{r\sqrt{1-\eta^2}}\lim_{\xi\to 1}\sqrt{\xi^2-1}\,\frac{\partial\bar{\Psi}_o}{\partial\xi}\exp\left[\frac{i\omega}{V}(Vt - x'\sec^2\mu)\right] \quad - - - (52)$$

The expressions $\sqrt{\xi^2-1}\,\dfrac{dF_n^m(\xi)}{d\xi}$ tend to a finite limit as ξ tends to 1, $\sqrt{\xi^2-1}\,\dfrac{dF_n^m(\xi)}{d\xi} \to f_n^m$, say (see Ref. 8 for a determinate form of this limit). Thus for any particular normal solution $\bar{\Psi}_o$,

$$\lim_{\xi\to 1}\sqrt{\xi^2-1}\,\frac{\partial\bar{\Psi}_o}{\partial\xi} = \frac{f_n^m}{\sqrt{r}}\,J_{n+\frac{1}{2}}(\lambda r)\,E_n^m(\eta) \quad - - - (53)$$

Assume that the potential can be written in the form

$$\bar{\Phi} = \left[\sum_{n=o}^{\infty}\sum_{m=1}^{2n+1}\frac{A_n^m}{\sqrt{r}}\,J_{n+\frac{1}{2}}(\lambda r)\,F_n^m(\xi)\,E_n^m(\eta)\right].$$
$$\exp\left[i\frac{\omega}{V}(Vt - x'\sec^2\mu)\right]. \quad - - - (54)$$

Then, at the aerofoil, (provided the term by term differentiation is legitimate),

$$\frac{\partial\bar{\Phi}}{\partial z'} = \frac{k}{r\sqrt{1-\eta^2}}\left[\sum_{n=o}^{\infty}\sum_{m=1}^{2n+1}\frac{A_n^m f_n^m}{\sqrt{r}}\,J_{n+\frac{1}{2}}(\lambda r)\,E_n^m(\eta)\right]$$
$$\exp\left[i\frac{\omega}{V}(Vt - x'\sec^2\mu)\right]. \quad - - - (55)$$

/On .....

On the other hand, the right hand side of the boundary condition is a known function of x', y', t, and therefore of r, η, t, say

$$\left[V \frac{\partial z_1}{\partial x'} + i\omega z_1(x', y') \right] e^{i\omega t} = g(r, \eta)\, e^{i\omega t} \qquad ---(56)$$

Hence, taking into account that $x' = r\eta$ at the aerofoil,

$$\sum_{n=0} \sum_{m=1}^{2n+1} A_n^m f_n^m \frac{1}{\sqrt{r}} J_{n+\frac{1}{2}}(\lambda r)\, E_n^m(\eta) = \frac{r\sqrt{1-\eta^2}}{k} g(r, \eta) \exp \left[i\omega \frac{\sec^2 \mu}{V} r\eta \right] \qquad ---(57)$$

$$= k(r, \eta), \text{ say,}$$

where $g(r, \eta)$ and $h(r, \eta)$ are complex. These functions are specified only for values of r, η which correspond to points on the aerofoil, i.e. $0 < r\eta \leq C$, $k' < \eta \leq 1$. Thus, we are confronted with an expansion problem which is a variant of the normal type, viz. whether it is possible to complete $h(r, \eta)$ on points aft of the trailing edge in such a way that it can be represented by an expansion as on the left hand side of (57). In the present paper, the general expansion problem will not be considered in any detail. Instead, we are going to show how, having completed $h(r, \eta)$, we may determine the coefficients A_n^m of the expansion, provided the expansion is at all possible.

For this purpose, we require two sets of relations of orthogonality, viz.,

$$\int_0^\infty J_{n+\frac{1}{2}}(\lambda r)\, J_{m+\frac{1}{2}}(\lambda r)\, \frac{dr}{r} = \begin{cases} 0 & n \neq m \\ \dfrac{1}{2n+1} & n = m \end{cases} \qquad n, m = 0, 1, 2, 3, \ldots \qquad ---(58)$$

(Compare Ref. 12, p. 388), and

$$\int E_n^m(\eta)\, E_n^p(\eta) \frac{d\eta}{\sqrt{1-\eta^2}\sqrt{\eta^2 - k'^2}} = \begin{cases} 0 & m \neq p \\ c_m^n & m = p \end{cases} \qquad ---(59)$$

$$n = 0, 1, 2, \ldots \quad m, p, = 1, 2, \ldots, 2n+1$$

(Compare Ref. 11, p. 466)

Assuming that term-by-term integration is permissible, we obtain, from (57),

/A_n^m .....

$$A_n^m f_n^m \frac{c_n^m}{2n+1} = \int_0^\infty \int_{k'}^1 h(r,\eta) \frac{dr}{\sqrt{r}} \frac{d\eta}{\sqrt{1-\eta^2}\sqrt{\eta^2-k'^2}}$$

or

$$A_n^m = \frac{2n+1}{f_n^m c_n^m} \int_0^\infty \int_{k'}^1 \frac{h(r,\eta)}{\sqrt{r(1-\eta^2)(\eta^2-k'^2)}} \, dr \, d\eta$$

$$n = 0, 1, 2, \dots \quad m = 1, 2, \dots, 2n+1. \qquad --- (60)$$

3.3. In conclusion, we are going to show that the velocity potentials corresponding to vertical and pitching oscillations can indeed be represented by expansions of the type of (54).

We have in fact, for vertical oscillations, $z_1(x', y') =$ const. $= z^*$ say, so that $g(r, \eta) = i\omega z^*$ and

$$h(r,\eta) = \frac{i\omega z^*}{k} r\sqrt{1-\eta^2} \exp\left[\frac{i\omega}{V}\sec^2\mu \cdot r\eta\right]$$

$$= i A r \sqrt{1-\eta^2}\, e^{iB\lambda r\eta} \qquad --- (61)$$

where

$$A = \frac{\omega z^*}{k} \text{ and } B = \frac{\omega}{\lambda V}\sec^2\mu = \operatorname{cosec}\mu\,, \text{ since } \lambda = \frac{\omega \sin\mu}{V\cos^2\mu}$$

(see equation (34).

Now we have (compare Ref. 12, p. 388)

$$e^{iB\lambda r\eta} = \sqrt{\frac{\pi}{2\lambda r}} \sum_{n=0}^{\infty} i^n (2n+1) P_n(B\eta) J_{n+\frac{1}{2}}(\lambda r)$$

$$--- (62)$$

where P_n is the n^{th} Legendre polynomial.

Differentiating with respect to $B\eta$,

$$\lambda r e^{iB\lambda r\eta} = \sqrt{\frac{\pi}{2\lambda r}} \sum_{n=0}^{\infty} i^{n-1} (2n+1) P'_n(B\eta) J_{n+\frac{1}{2}}(\lambda r)$$

$$--- (63)$$

Hence

$$h(r,\eta) = A\sqrt{\frac{\pi}{2\lambda^3}} \sum_{n=0}^{\infty} i^n (2n+1) P'_n(B\eta) J_{n+\frac{1}{2}}(\lambda r)$$

$$--- (64)$$

To prove that $h(r, \eta)$ can be represented by the required expansion, it is sufficient to show that the terms $\sqrt{1-\eta^2}\, P'_n(B\eta)$ can be represented as linear combinations of Lamé's functions of the first kind $E_n^m(\eta)$. Now there are just $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ Lamé functions of the first kind of order n ($\frac{1}{2}n$ or $\frac{1}{2}(n-1)$, according as n is even or odd) which are of the form $1-\eta^2 p_K(\eta)$,

/where .....

where either the $p_\kappa(\eta)$ or the $\frac{1}{\eta}p_\kappa(\eta)$ are polynomials of η^2, according as n is odd or even. For given n, all the p_κ are linearly independent, and it is therefore not difficult to see that $P'_n(B\eta)$, which is either itself a polynomial of η^2 or which is such that $\frac{1}{\eta}P'_n(B\eta)$ is a polynomial of η^2 can be represented by a linear combination of the $p_\kappa(\eta)$. It follows that $\sqrt{1-\eta^2}\,P'_n(B\eta)$ can be represented by a linear combination of functions $E^m_n(\eta)$, as required.

In the above analysis, we have assumed that the relation (61) applies to all values of r and η within the domain of definition of these variables. This fictitious assumption is acceptable as long as we are interested only in conditions at the aerofoil, but not, of course, if we wish to investigate the flow in the wake of the aerofoil.

For pitching oscillations round the apex, we have $r,(x',y') = z^*x$, say, so that $g(r,\eta) = Vz^* + i\omega z^* x$. Since we know already that the potential corresponding to $g(r,\eta) =$ const. can be represented by the required expansion, it will be sufficient to consider the case

$$g(r,\eta) = i\omega z^* x = i\omega z^* r\eta$$

We then have, for the corresponding $h(r,\eta)$,

$$h(r,\eta) = \frac{i\omega z^*}{k} r^2\eta\sqrt{1-\eta^2}\,\exp\left[\frac{i\omega}{V}\sec^2\mu\, r\,\eta\right]$$
$$= i A r^2\eta\sqrt{1-\eta^2}\,e^{iB\lambda r\eta} \qquad ---(65)$$

Differentiating (63) with respect of $B\eta$, we obtain

$$\lambda^2 r^2 e^{iB\lambda r\eta} = -\sqrt{\frac{\pi}{2\lambda r}}\sum_{n=0}^{\infty} i^n(2n+1)P''_n(B\eta)J_{n+\frac{1}{2}}(\lambda r) \qquad ---(66)$$

and so

$$h(r,\eta) = -A\sqrt{\frac{\pi}{2\lambda^5}}\sum_{n=0}^{\infty} i^{n+1}(2n+1)\sqrt{1-\eta^2}\,\eta\, P''_n(B\eta)\, J_{n+\frac{1}{2}}(\lambda r) \qquad ---(67)$$

The terms $\sqrt{1-\eta^2}\,\eta\,P''_n(B\eta)$ can be represented as linear combinations of Lamé functions of the first kind of order n, as before. This completes the argument.

The two modes of vibration considered above are rigid. Additional work on elastic modes (theoretical and numerical) may be postponed until more evidence is available on the particular problems which are likely to occur in practice.

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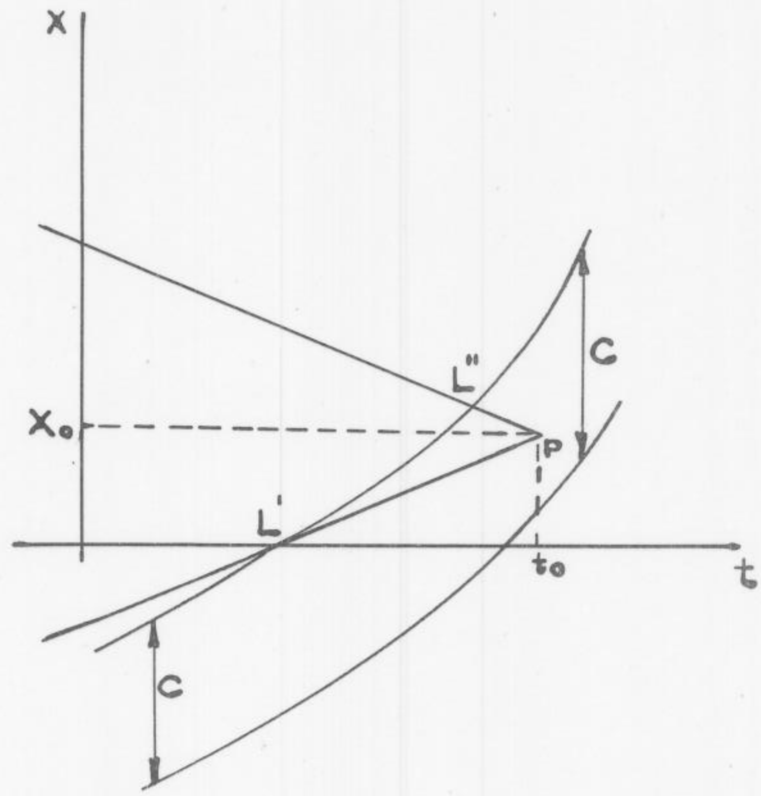
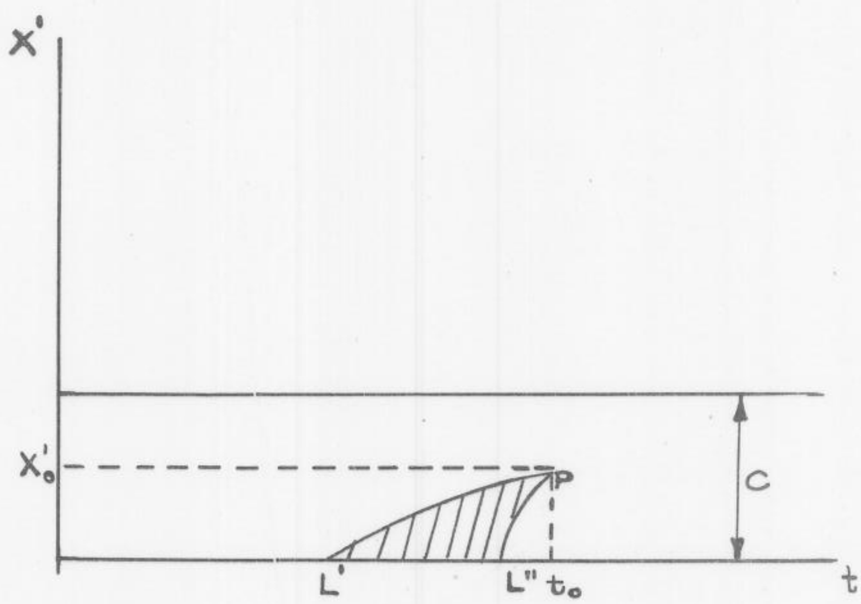


CALCULATION OF VELOCITY POTENTIAL FOR ACCELERATED MOTION

FIG. 1.



CONDITIONS IN THE (X',t) PLANE

FIG. 2.