REPORT NO. 11

September, 1947.

THE COLLEGE OF AERONAUTICS

CRANFIELD.

Application of the Linear Perturbation Theory to Compressible Flow about Bodies of Revolution.

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A.D. Young, M.A., and S. Kirkby, B.Sc., Ph.D.,

of the Department of Aerodynamics.

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- SUMMARY -

The linearised theory is developed in some detail in order to clarify the difference between two-dimensional and axi-symmetric flow. In agreement with other authors it is concluded that the perturbation velocity on a thin body of revolution in compressible flow is $1/\beta^2$ times the perturbation velocity in incompressible flow on a thinner body at reduced incidence obtained by reducing the lateral dimensions of the original body in the ratio $\beta : 1$.

This result is applied to a representative family of streamline bodies of revolution at zero incidence. It is found that, without an undue loss of accuracy, the results of the calculations can be presented in a relatively simple form in a diagram showing the variation of velocity with Mach number for a range of values of velocity on the surface of a streamline body in incompressible flow (fig.6). This variation is always less than that predicted by the Glauert law but approaches it with increase in the basic incompressible flow velocity, being very close to it for basic incompressible velocity ratios, u/U_0, of 1.10 and higher.

It is shown that the blockage factor for a body of revolution in a wind tunnel is increased in compressible flow in the ratio $1/\beta^3$ and not $1/\beta^4$ as quoted in reference 1.

/ Notation ...

Notation

x	-	ordinate measured in direction of undisturbed stream
y, z	-	ordinates measured normal to each other and to the x-axis
r	-	radial ordinate measured normal to the axis of the body
U_o	-	undisturbed stream velocity
α	-	incidence of body (assumed small)
u, v, w	-	components of velocity measured in x, y, z directions ($u - U_o$, v and w are assumed small compared with U_o)
M_o	-	Mach number of undisturbed stream
β	-	$\sqrt{1 - M_o^2}$
T	-	maximum diameter of body
2ℓ	-	length of body
ϕ	-	velocity potential
ψ	-	stream function

The suffix i refers to quantities measured in incompressible flow, the suffix c refers to quantities measured in compressible flow, the suffix s refers to quantities measured at the surface of the body.

/Introduction ...

1. Introduction.

A number of authors (refs. 2, 3, 4) have at various times pointed out an error in the application of the linearised perturbation theory of compressible flow to bodies of revolution given by Goldstein and Young in R. and M. 1909 (ref.1). The error undoubtedly exists, but it is felt that a certain degree of confusion and, in some cases, inaccuracy is present in the papers discussing it. There is therefore a need for a simple and clear exposition of the source of the error and of the correct solution. The latter has been applied to a family of streamline bodies of revolution as well as spheroids, and curves have been derived from which the effect of compressibility on the velocity distribution on a body of the former family can be readily obtained with reasonable accuracy.

2. Theory.

To appreciate the source of the error referred to above, it is necessary to recapitulate some of the theory of R. and M. 1909 (ref.1) both for two-dimensional and three-dimensional flow.

Consider first two-dimensional flow. If $f(x,y)$ is the perturbation potential for a thin cylinder (i.e., its y ordinates, which we will write as y_s , are small), then, to the order of approximation involved in the linearised theory, the difference between $f(x,y_s)$ and $f(x, 0)$ is of the second order in y_s and may be neglected when compared with $f(x, 0)$. The same is true for any of the derivatives of $f(x, y)$ and in particular for $v = f_y(x, y)$. Hence, we may write that, on the surface of the cylinder,

$$v_s = f_y (x, 0),$$

and since to the order of our approximation

$$\frac{v_s}{U_o} = \frac{dy_s}{dx},$$

the slope of the tangent at any point to the body is given by

$$\frac{dy_s}{dx} = \frac{f_y (x, 0)}{U_o} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now, if we consider the compressible flow about a thin cylinder, we know that the linearised perturbation potential ϕ_c satisfies (see ref.1)

$$\left.\begin{aligned} \beta^2 \frac{\partial^2 \phi_c}{\partial x^2} + \frac{\partial^2 \phi_c}{\partial y^2} &= 0 \\ \text{where} \qquad \beta^2 &= 1 - M_o^2. \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

Hence, any function of the form

$$\phi_c = (1/\beta^n)\, f(x, \beta y) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

will satisfy the equation (2) if

$$\phi_i = f(x,y) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

satisfies the incompressible flow potential equation (Laplace equation)

$$\frac{\partial^2 \phi_i}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial y^2} = 0.$$

It follows from equation (3) that the lateral velocity at the surface of the disturbing body is given by

$$v_{cs} = (1/\beta^{n-1}) f_y(x, \beta y_s).$$

Hence, by the above argument,

$$v_{cs} = (1/\beta^{n-1}).\ f_y(x,0) = (1/\beta^{n-1})\ v_{is} \quad \ldots\ldots\ldots\ldots(5)$$

where v_{is} is the lateral velocity at the surface of the body for which ϕ_i is the incompressible flow perturbation potential.

It follows from equation (1) that if we scale the ordinates (and therefore the slopes and incidence) of the body about which we require the compressible flow by the factor β^{n-1}, and then find the incompressible flow perturbation potential function for the transformed body, that function is the function ϕ_i. We see that we have in fact an infinite choice of bodies in incompressible flow with their associated perturbation potentials, from any one of which we can obtain ϕ_c. For example, if we take $n = 0$, then we scale the body ordinates and incidence up in the ratio $1/\beta$, and hence on the original body

$$u_c = f_x(x, \beta y) = f_x(x, 0)$$

$$= u_i \text{ on the fattened body} \quad \ldots\ldots\ldots\ (6)$$

This is Method II of R. and M. 1909 (ref.1).

Again, if we take $n = 1$, we do not alter the body shape and incidence, but then

$$u_c = (1/\beta)\ f_x(x, \beta y) = (1/\beta)\ f_x(x, 0)$$

$$= (1/\beta)\ u_i. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (7)$$

This is Method I of R. and M. 1909 (ref.1). And so on.

This infinite choice is associated with the fact that in two-dimensions the velocity perturbation on the surface of a thin body varies linearly with the thickness. In every case we arrive at the well known Glauert law, viz., the perturbation velocity is increased in the ratio $1/\beta$ in two-dimensional compressible flow.

The above argument can be validly extended to the case of flow about a three-dimensional body provided the spanwise or z ordinate of the body is large compared with the y ordinate. This extension is discussed in general in R. and M. 1909 (ref.1) and its particular application to swept back wings is discussed in ref. 5.

However, when we come to consider bodies such as bodies of revolution for which the y and z ordinates are of the same order, an essential difference appears. It is then no longer true to say that, if $f(x, y, z)$ is the perturbation potential, near the axis

$$f(x, y, z) - f(x, 0, 0)$$

is of the second order in y (or z) and may be neglected compared

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with $f(x, 0, 0)$. The perturbation potential function and its derivatives are in fact infinite along the portion of the axis of the body where their singularities lie. As a corollary to this we may note that it is no longer true that the velocity perturbation on a thin body of revolution varies linearly with thickness, it varies more nearly like the square of the thickness (ref.6). It was this essential distinction of axi-symmetric flow that was overlooked in R. and M. 1909 and led to the error referred to in paragraph 1. In R. and M. 1909 (ref. 1) it was assumed that the argument given above for two-dimensional bodies could be applied unchanged to bodies of revolution and hence it was deduced that the Glauert law applied. But it will now be clear that in the case of bodies of revolution we must be careful to match our boundary conditions on the actual boundary in compressible flow, and not on the axis, when deciding on an appropriate incompressible flow. As will be seen in what follows the Glauert law does not apply, although the actual law does not lead to results as different numerically from the Glauert law as some writers (refs. 2, 7, 8) have surmised.

It is still true to say that if

$$\phi_i = f(x, y, z)$$

satisfies the Laplace equation, then

$$\phi_c = (1/\beta^n)\, f(x, \beta y, \beta z)$$

satisfies the linearised perturbation potential equation for compressible flow. For the latter flow the corresponding values of the lateral velocity components are

$$\left.\begin{aligned} v_c &= (1/\beta^{n-1}).f_y(x, \beta y, \beta z), \\ \text{and} \quad w_c &= (1/\beta^{n-1}).f_z(x, \beta y, \beta z) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(8)$$

and hence the slopes of the tangents to the body, in planes parallel to the xy and zx planes, are given by

$$\left.\begin{aligned} \frac{v_{cs}}{U_o} &= \frac{1}{\beta^{n-1}} \frac{f_y(x_s, \beta y_s, \beta z_s)}{U_o} \\ \text{and} \quad \frac{w_{cs}}{U_o} &= \frac{1}{\beta^{n-1}} \frac{f_z(x_s, \beta y_s, \beta z_s)}{U_o} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(9)$$

where the ordinates of the body are given by x_s, y_s, z_s. Now the points x_s, βy_s, βz_s define a body derived from the one we are considering by scaling its lateral ordinates and incidence down in the ratio β :1. If v_i, w_i are the lateral velocities about this thinner body in incompressible flow, it follows that

$$\left.\begin{aligned} v_{cs} &= U_o \frac{\partial y_s}{\partial x_s} = \frac{U_o}{\beta} \frac{\partial(\beta y_s)}{\partial x_s} = \frac{v_{is}}{\beta} \\ \text{and similarly} \quad w_{cs} &= \frac{w_{is}}{\beta} \end{aligned}\right\} \quad \ldots\ldots\ldots(10)$$

/ It ...

It follows from equations (9) and (10) that if we take $f(x, y, z)$ to be the perturbation potential about the thinner body and take $n = 2$, ϕ_c will then be the correct compressible flow perturbation potential about the body we are considering. Hence, the perturbation velocity on the latter is $1/\beta^2$ times the incompressible flow velocity on the similar body obtained by reducing the lateral dimensions and incidence of the original body in the ratio $\beta : 1$. This was of course true in two-dimensions, but there is no longer an infinity of possible bodies and their associated incompressible flows with which to correlate the compressible flow. In this case there is only the one body and its incompressible flow that we can take for this purpose.

3. Application and Range of Calculations.

From the foregoing it will be clear that to determine the velocity distribution over a body of fineness ratio, say, $T/2\ell$ at a Mach number M and incidence α, we require the incompressible flow velocity distribution over the body scaled down to the thickness $\beta T/2\ell$ and incidence $\beta\alpha$, and we then multiply the perturbation velocity by $1/\beta^2$ to derive the required perturbation velocity. Hence, to cover a range of Mach numbers, we require the incompressible flow velocity distributions over a family of shapes derived by scaling the lateral ordinates of the body down by a range of factors between 1.0 and 0. It will be clear that in general the precise change in local velocity brought about by a change of Mach number will depend on the shape of the body, the position considered and the magnitude of the incompressible value; and a simple universal law of the Glauert type can no longer apply. However to investigate the effect for a typical family of similar streamline bodies of revolution at zero incidence, calculations have been made for one of the families of similar shapes developed in ref.9. This family was defined by the parameters $a/b = 1/2$, $Z' = 0.4$, using the notation of ref. 9, i.e., they have a moderate velocity gradient ahead of the position of maximum velocity, which occurs at approximately 40% of the body length aft of the nose. The incompressible flow velocity distributions for members of this family of fineness ratios $T/2\ell$ of 0.05, 0.1, 0.15, 0.2, 0.25 and 0.3 are shown in fig.1. These distributions were calculated by the method of ref.6.

4. Analysis.

4.1. Variation of maximum velocity with Mach number

The curve of maximum velocity as a function of fineness ratio is shown in fig.2. It is there compared with the corresponding curve for spheroids at zero incidence. The departure of these curves from linearity through the origin is a measure of how far the variation of maximum velocity with Mach number may be expected to depart from the Glauert law. Thus we may expect a behaviour closer to the Glauert law on streamline bodies of revolution in the region of maximum velocity than on spheroids. This is confirmed by figs. 3 and 4 where are shown the variations with Mach number of the maximum velocity on streamline shapes and spheroids of various fineness ratios at zero incidence, and the corresponding curves derived from the Glauert law are shown for comparison.

The variation of the critical Mach number with fineness ratio for streamline shapes and spheroids is shown in fig.5, and again

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the corresponding Glauert law curves are shown for comparison.

4.2. Variation with Mach number of velocity at any point on streamline body.

By using the curves of fig.1 and the results of the above discussion it is possible to deduce the variation with Mach number of the velocity at any point on a streamline body of revolution of the typical family, for any fineness ratio within the range considered. Systematic sets of curves were derived showing this variation for various fineness ratios, each set relating to a given position aft of the nose. An analysis of these sets of curves then showed that with little loss of accuracy the variation of velocity with Mach number from a given incompressible value might be taken to be independent of position and fineness ratio. This permitted a single family of curves to be drawn and reproduced in fig.6 showing the variation of velocity with Mach number for a range of values of the incompressible velocity. In so far as the streamline shapes for which these curves were obtained are reasonably typical, the curves are valid for general use except where very great accuracy is required. In the latter case complete and accurate calculations would be required.

5. Wind Tunnel Interference.

According to the linearised theory the equivalent source-sink distribution of a body of revolution is proportional to the square of the fineness ratio (see ref. 6). The perturbation potential varies linearly with the equivalent source-sink distribution (ref.6). It readily follows from the above that the equivalent source-sink distribution of a body remains independent of Mach number. This was demonstrated by a rather different argument by Lees (ref.2). But the velocity in the x-direction induced at the point (x, r) in compressible flow due to a given source-sink distribution is the same as that induced at the point $(x, \beta r)$ in incompressible flow. When βr is large this velocity varies inversely as $\beta^3 r^3$. Treating the wind tunnel interference on a body of revolution arising from blockage as due to the induced velocities of its series of images in the walls, it follows that the interference factor should be $1/\beta^3$ and not $1/\beta^4$ as stated in R. and M. 1909 (ref.1).

/References ...

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VELOCITY DISTRIBUTIONS ON FAMILY OF STREAMLINE BODIES OF REVOLUTION AT ZERO INCIDENCE.

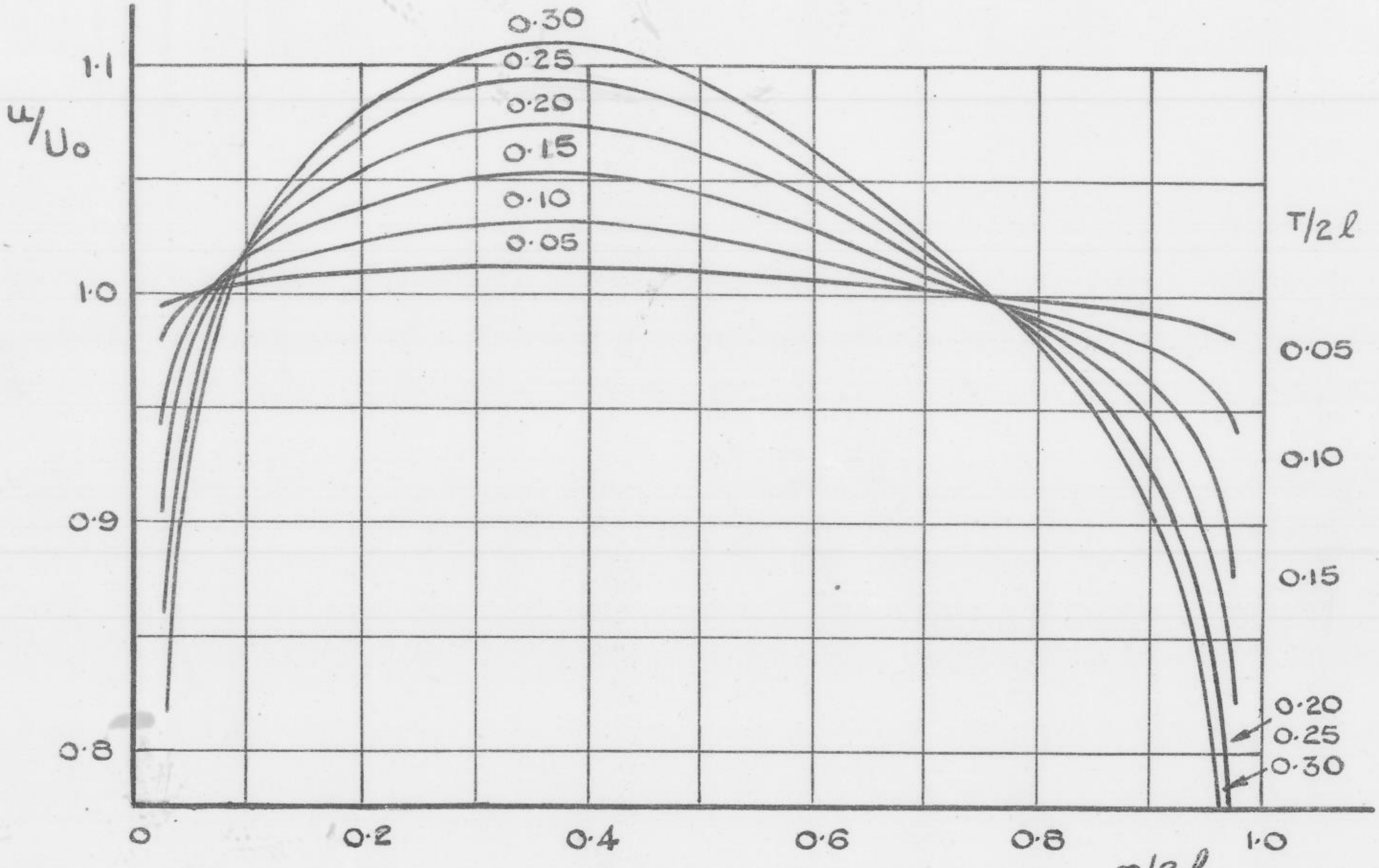

FIG.1.

VARIATION OF MAXIMUM VELOCITY ON SPHEROIDS AND STREAMLINE BODIES OF REVOLUTION WITH FINENESS RATIO.

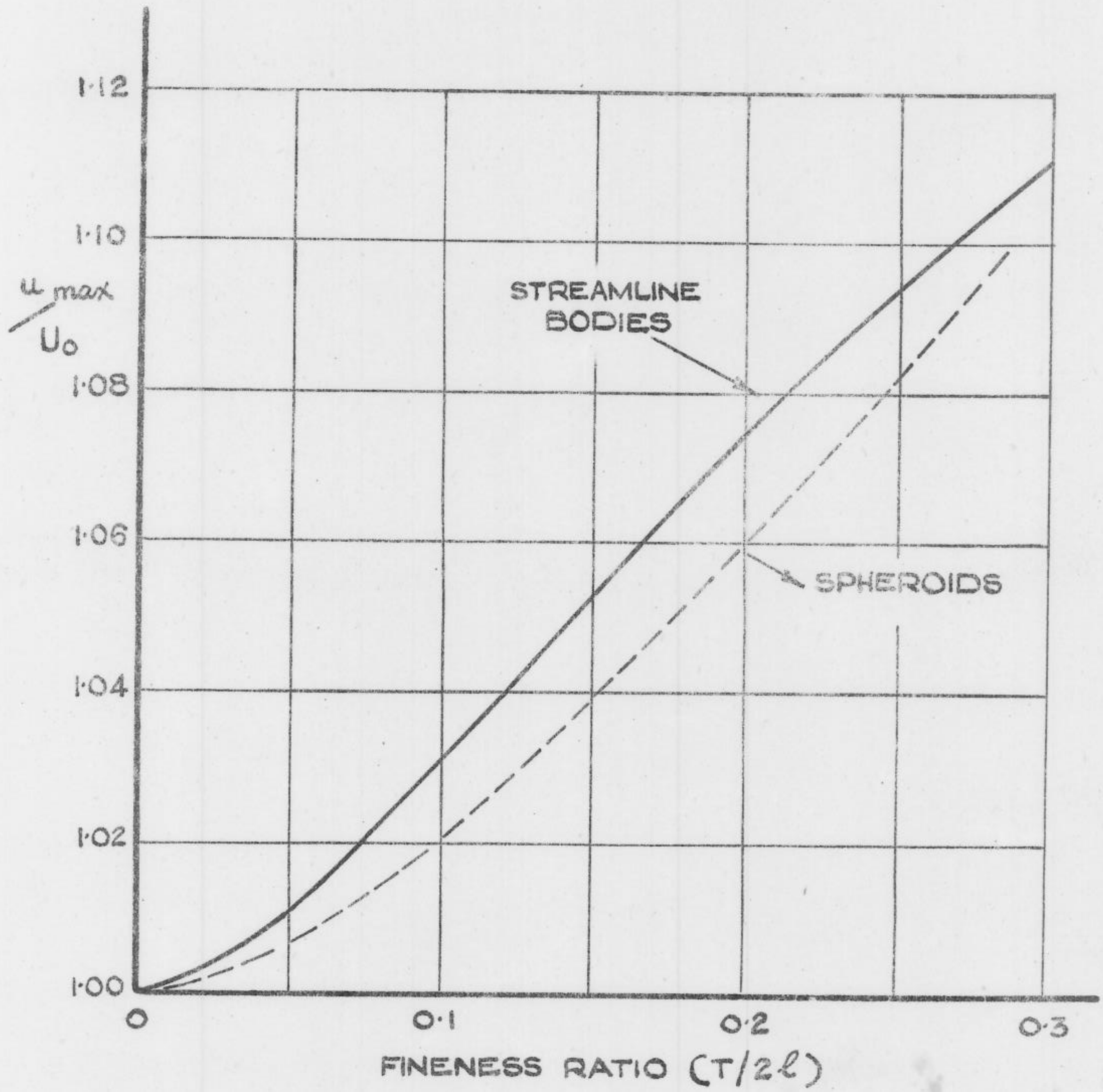

FIG. 2.

VARIATION OF MAXIMUM VELOCITY WITH MACH NUMBER ON FAMILY OF STREAMLINE BODIES OF REVOLUTION AT ZERO INCIDENCE.

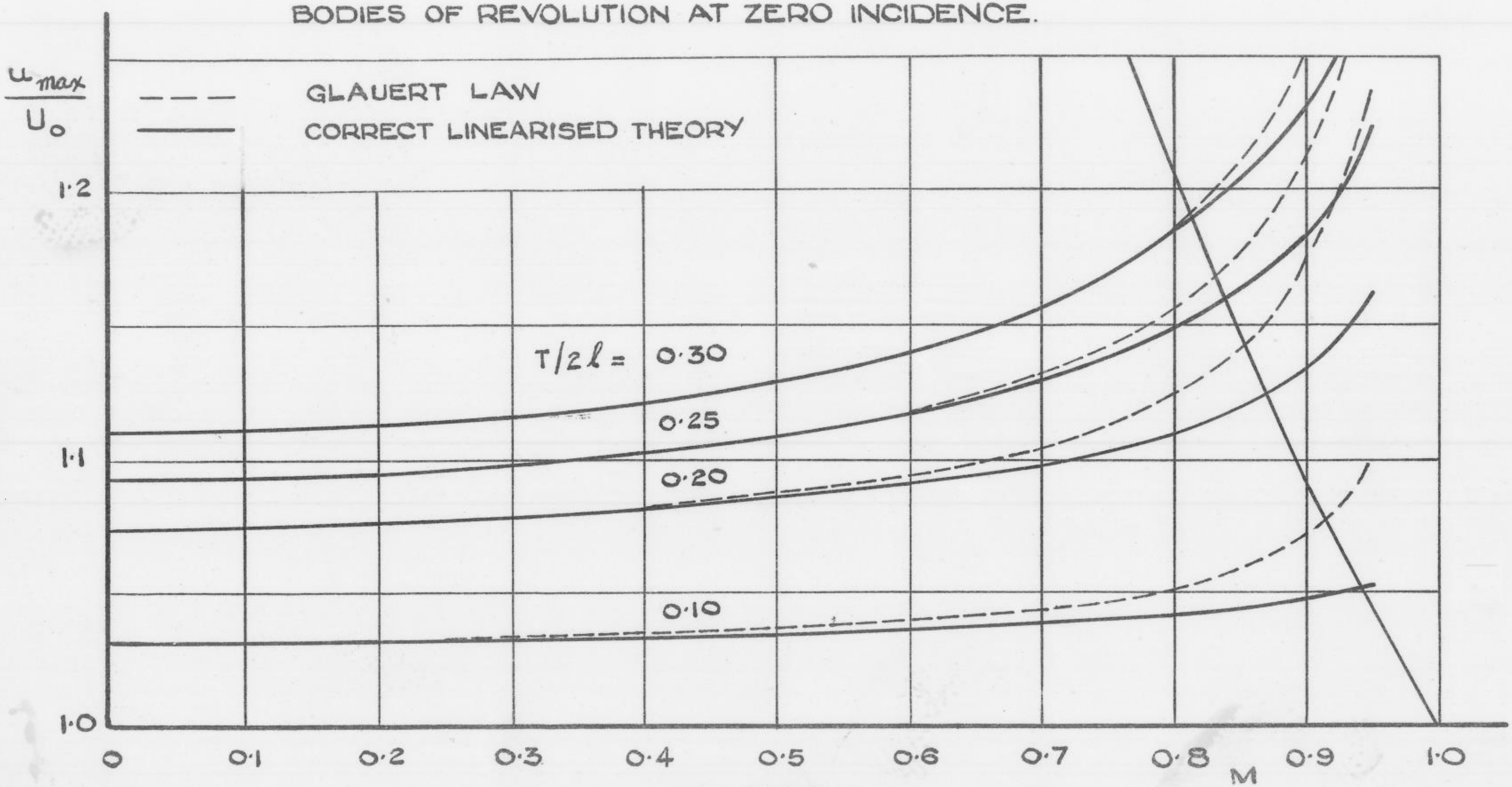

FIG. 3.

VARIATION OF MAXIMUM VELOCITY WITH MACH NUMBER ON SPHEROIDS AT ZERO INCIDENCE.

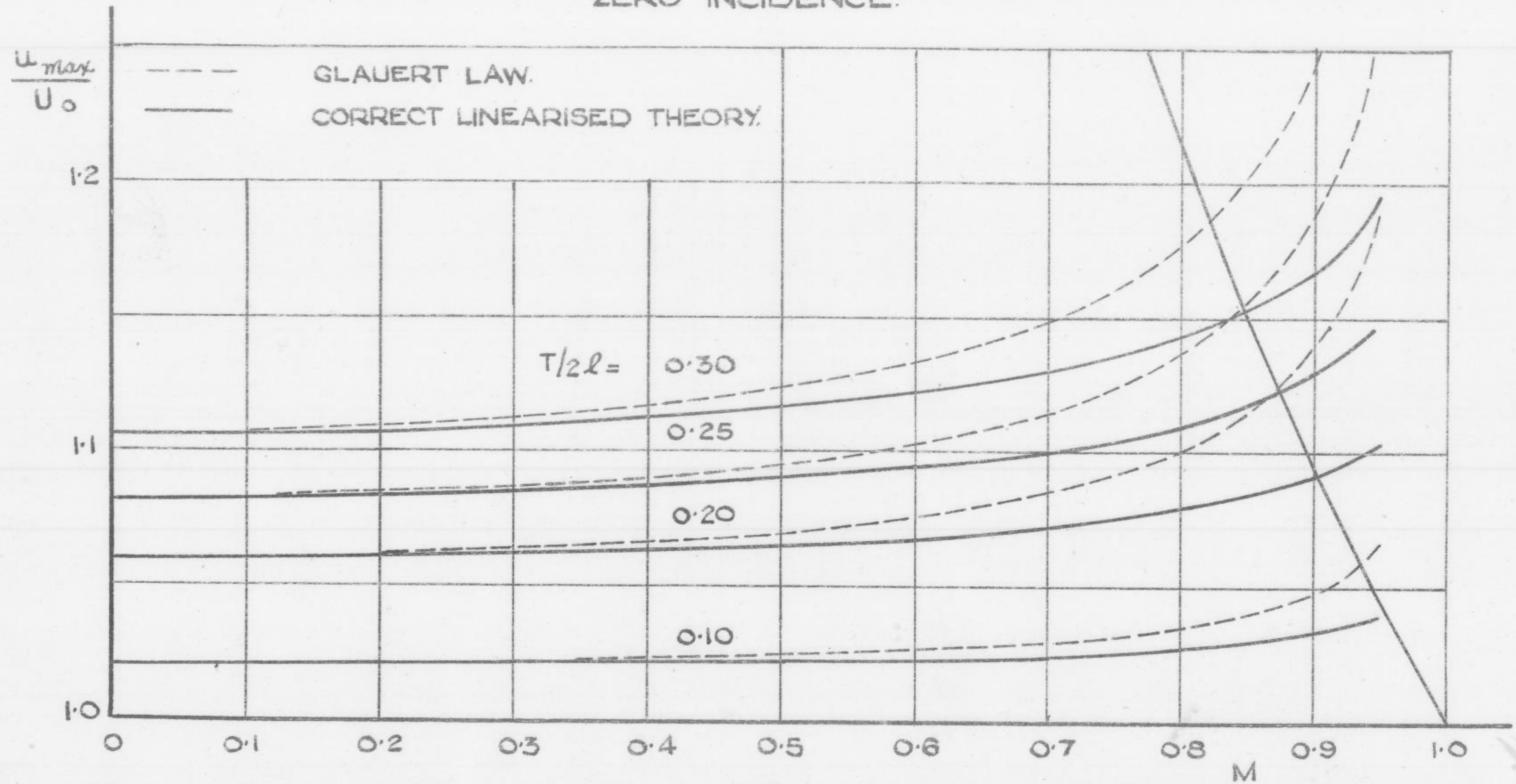

FIG. 4.

VARIATION OF CRITICAL MACH NUMBER WITH FINENESS RATIO FOR SPHEROIDS AND STREAMLINE BODIES OF REVOLUTION AT ZERO INCIDENCE.

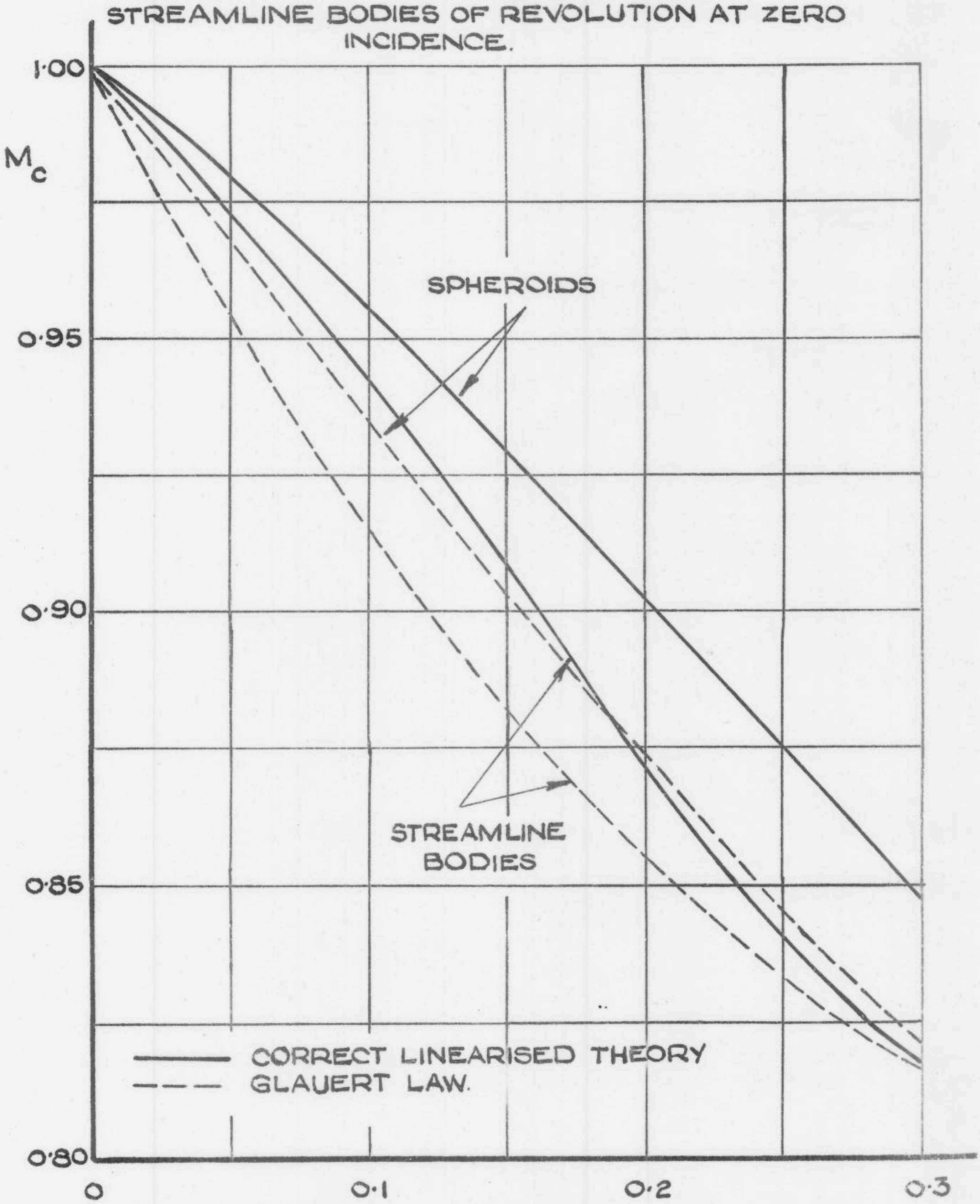

FIG. 5.

TYPICAL VARIATION WITH MACH NUMBER OF VELOCITY AT ANY POINT ON A STREAMLINE BODY OF REVOLUTION AT ZERO INCIDENCE.

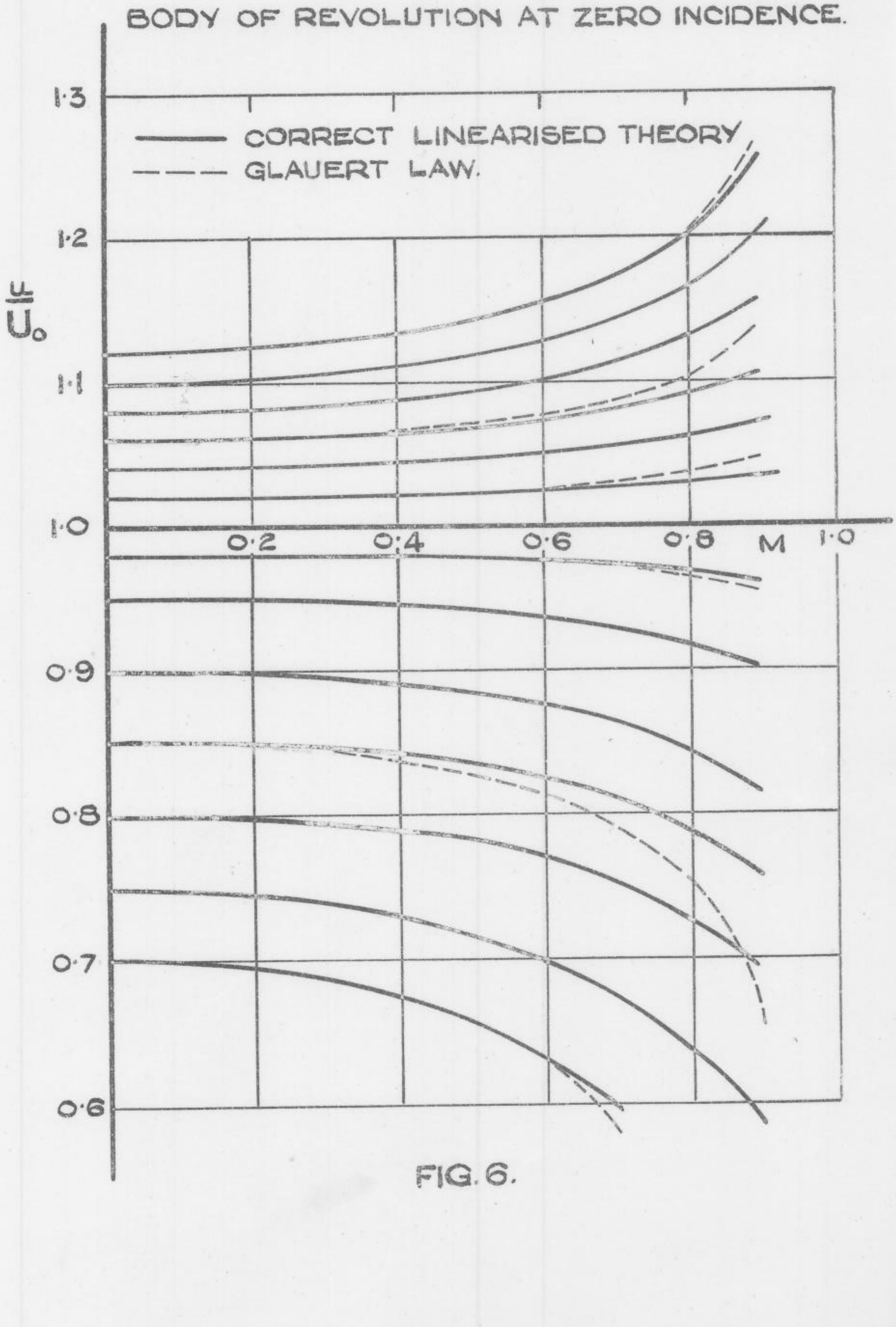

FIG. 6.