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Note on the velocity and temperature distributions attained with suction on a flat plate of infinite extent in compressible flow

-by-

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-SUMMARY-

The problem considered by Griffith and Meredith[1] for incompressible flow is here considered for compressible flow, it being assumed that there is no heat transfer by conduction at the plate. Essentially, the method consists of establishing a correspondence between the velocity and temperature profiles for incompressible flow and those for compressible flow, the lateral ordinates being scaled by factors which are functions of the ordinates and of Mach number.

The results of calculations covering a range of Mach numbers up to 5.0 are shown in Figs. 1 and 2.

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1. Notation

x	distance measured parallel to the plate in direction of main stream upstream of plate
y	distance measured normal to the plate from the surface of the plate
u	velocity component in x direction
v	velocity component in y direction
ρ	density
T	temperature
μ	coefficient of visocity
k	thermal conductivity
c_v	specific heat at constant volume
c_p	specific heat at constant pressure
σ	$\mu c_p / k$ (Prandtl number, assumed constant)
J	mechanical equivalent of heat
γ	c_p/c_v (assumed constant)
i	$J\, c_p\, T$,
τ	$\mu \frac{du}{dy}$ (shear stress)

suffix 1 refers to quantities measured at large normal distances from the plate ($y \to \infty$),
suffix w refers to quantities measured at the plate

ω	defined by $\left(\frac{\mu}{\mu_1}\right) = \left(\frac{T}{T_1}\right)^{\omega}$
η	$\frac{y\, v_1}{\nu_1}$
ζ	$\int_0^{\eta} \left(\frac{\mu_1}{\mu}\right) d\eta$
M_1	$\frac{u_1}{a_1} = \sqrt{\frac{u_1^{\,2}}{(\gamma - 1)\, J\, c_p\, T_1}}$, (a_1 is the speed of sound in the main stream)
θ	i/i_1
b	$(\gamma - 1)\, M_1^{\,2}$

/2. ....

2. Introduction

The classic solution due to Griffith and Meredith[1] of the velocity distribution attained with suction on a flat plate of infinite extent in incompressible flow is of special interest, since it is a solution of the general equations of motion and does not depend on the usual assumptions of boundary layer theory. The corresponding problem for compressible flow is by no means as simple in its most general form, However, if the usual assumptions of boundary layer theory are made, it permits of an exact solution which is easily obtained. This solution may have no practical importance at the moment, but it was felt to have sufficient intrinsic interest to be worth recording.

3. Analysis

The equation of motion in the boundary layer of a flat plate at zero incidence in steady compressible flow is

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \text{. ....(1)}$$

The equation of continuity is

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \text{ ....(2)}$$

The energy equation is

$$J c_p u \frac{\partial T}{\partial x} + J c_p v \frac{\partial T}{\partial y} = \frac{J}{\rho} \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)^2 \text{ ....(3)}$$

We are interested in the problem of the final velocity and temperature profiles far downstream from the plate leading edge when

$$\frac{\partial}{\partial x} \equiv 0.$$

/Hence.......

Hence, the above equations become

$$\rho v \frac{du}{dy} = \frac{d}{dy}\left(\mu \frac{du}{dy}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\rho v = \text{const.} = \rho_1 v_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$\rho v \frac{di}{dy} = \frac{d}{dy}\left(\frac{\mu}{\sigma} \frac{di}{dy}\right) + \mu\left(\frac{du}{dy}\right)^2 \quad \ldots\ldots\ldots(6)$$

where $i = J c_p T$, and $\sigma = \frac{\mu c_p}{k}$ (Prandtl's number assumed constant).

The gas equation leads to

$$\frac{\rho}{\rho_1} = \frac{T_1}{T} = \frac{i_1}{i} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

It will be assumed that the variation of μ with T is given by

$$\frac{\mu}{\mu_1} = \left(\frac{T}{T_1}\right)^{\omega} = \left(\frac{i}{i_1}\right)^{\omega} \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

where ω = const. For air at normal temperatures ω is about 0.76, but it increases slightly with T_1.

The boundary conditions are

$$u = u_1,\ \rho = \rho_1,\ v = v_1,\ i = i_1,\ \frac{du}{dy} = 0 \text{ at } y = \infty,$$

$u = 0$, $\frac{di}{dy} = 0$ at $y = 0$, if no heat transfer by conduction is assumed to occur at the plate.

If in equation (6) we change the independent variable from y to u, writing $\tau(u) = \mu \frac{du}{dy}$, $i = i(u)$ and eliminate ρv by means of equation (4), we obtain

$$(1 - \sigma) \frac{d\tau}{du} \frac{di}{du} + \tau\left(\frac{d^2 \tau}{du^2} + \sigma\right) = 0 \quad \ldots\ldots\ldots(9)$$

/From......

From (4) and (5)

$$\frac{d\tau}{du} = \rho v \;\; = \rho_1 v_1,$$

and hence

$$\tau = \rho_1 v_1 u + C_1 ,$$

where C_1 is a const.

If we write τ_w = value of τ at the wall,

$$C_1 = \tau_w .$$

Further, since $\tau = 0$, when $u = u$,

$$C_1 = -\rho_1 v_1 u_1 .$$

Therefore,

$$\left.\begin{aligned} \tau &= -\rho_1 v_1 (u_1 - u) \\ &= \rho_1 v_1 u + \tau_w \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (10)$$

Equation (9) can then be written

$$(1 - \sigma)\rho_1 v_1 \frac{di}{du} - \rho_1 v_1 (u_1 - u)\left(\frac{d^2 i}{du^2} + \sigma\right) = 0 \quad ..(11)$$

This equation is readily solved to give

$$i_1 - i = \frac{\sigma u_1^2}{2(2 - \sigma)}\left[\left(1 - \frac{u}{u_1}\right)^2 - \frac{2}{\sigma}\left(1 - \frac{u}{u_1}\right)^\sigma\right] ..(12)$$

satisfying the conditions $i = i_1$, when $u = u_1$, and

$$\frac{di}{du} = 0, \quad \text{when} \quad u = 0.$$

It is of interest to note that at the wall
where $u = 0$,

$$i_w = i_1 + \frac{u_1^2}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

/and.......

and hence the total energy at the wall differs from that in the main stream only by the quantity $\left(v_w^2 - v_1^2\right)$.

From (10) we have, since $\tau = \mu \frac{du}{dy}$,

$$\frac{u}{u_1} = 1 - \exp.\left[\rho_1 v_1 \int_0^y \frac{dy}{\mu}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Let $\eta = - \dfrac{v_1\, y}{\nu_1}$,

and let

$$d\zeta = d\eta \,.\, \frac{\mu_1}{\mu}$$
$$= d\eta\left(\frac{T_1}{T}\right)^{\omega} = d\eta\left(\frac{i_1}{i}\right)^{\omega} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

with $\zeta = 0$, when $\eta = 0$.

Then, from (14)

$$\frac{u}{u_1} = 1 - \exp.(\zeta) \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

and from (12)

$$i_1 - i = \frac{\sigma u_1^2}{2(2-\sigma)}\left[\exp.\,(2\zeta) - \frac{2}{\sigma}\exp.(\sigma\zeta)\right] . (17)$$

Writing $\Theta = i/i_1$, $b = (\gamma - 1)\, M_1^2$, then

$$\left.\begin{aligned} \frac{\Theta - 1}{b/2} &= \frac{\sigma}{(2-\sigma)} F(\zeta) \\ \text{where } F(\zeta) &= \left[\frac{2}{\sigma}\exp.(\sigma\zeta) - \exp.(2\zeta)\right] \end{aligned}\right\} \ldots\ldots (18)$$

From (16) and (18) we can express $\dfrac{u}{u_1}$ & $\dfrac{\Theta - 1}{b/2}$ as functions of ζ only, independent of Mach number.

To derive the actual velocity and temperature distributions for any given Mach number we need to evaluate the relation between ζ and η (or y) given by (15).

/From......

From (15)

$$\eta = \int_0^{\zeta} \left(\frac{i}{i_1} \right)^{\omega} d\zeta$$

$$= \int_0^{\zeta} \left[1 + \frac{b \ \sigma}{2(2-\sigma)} \ F(\zeta) \right]^{-\omega} d\zeta \quad \ldots\ldots\ldots (19)$$

In general, the integral on the right hand side of (19) must be evaluated either numerically or graphically, giving η as a function of ζ and M_1. Since v_1 is negative, only positive values of ζ need be considered and it will be found that values of ζ greater than 10 may be ignored. Having determined η (or y) for a comprehensive range of values of ζ and M_1 we can then, for each Mach number, replot

$\frac{u}{u_1}$ and $\frac{\theta - 1}{b/2}$ as functions of η, using the basic (or incompressible) profiles given by (16) and (18)*

For the special case $\omega = 1.0$, (19) can be integrated outright to give

$$\eta = \zeta + \frac{b}{2} \ \frac{\sigma}{(2-\sigma)} \left[\frac{2}{\sigma^2} \exp(\sigma\zeta) - \frac{\exp.\ (2\zeta)}{2} - \frac{2}{\sigma^2} + \frac{1}{2} \right] \quad \ldots\ldots\ldots (20).$$

4. Calculations and results

The velocity and temperature distributions have been calculated for $\omega = 0.76$ and $M_1 = 0$, 1.0, 2.0, 3.0, 4.0 and 5.0, σ being taken as 0.72. For comparison, calculations have also been made for $\omega = 1.0$ and $M_1 =$ 1.0, 3.0 and 5.0. The resulting velocity distributions /as....

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* This process of establishing a transformation of the lateral ordinate y, which converts the temperature and velocity profiles for incompressible flow to those for compressible flow, was first used by Hantsche and Wendt in Ref.2. They then applied it to the boundary layer on a flat plate in compressible flow without suction for the special case where = 1.0. However, it seems capable of much wider application, and in a later paper it is hoped to use it for more general problems of the boundary layer on a finite flat plate both with and without suction in compressible flow.

as functions of η are shown in Fig.1, and the corresponding temperature distributions are shown in Fig.2. It will be noted that there is a thickening of the velocity and temperature boundary layer with increase of Mach number, and this process is enhanced by an increase of ω.

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- REFERENCES -

No.	Author	Title
1	Griffith and Meredith	The possible improvement in aircraft performance due to the use of boundary layer suction. R.A.E. Report No. E.3501. (A.R.C.2315). See also 'Modern Developments in Fluid Dynamics' Vol.II. p.534 (Clarendon Press).
2	Hantsche and Wendt	Zum Kompressiblitatseinflusz bei der laminaren Grenzschicht der ebenen Platte. Jahrbuch der Deutschen Luftfahrtforschung 1940. I. p.517.

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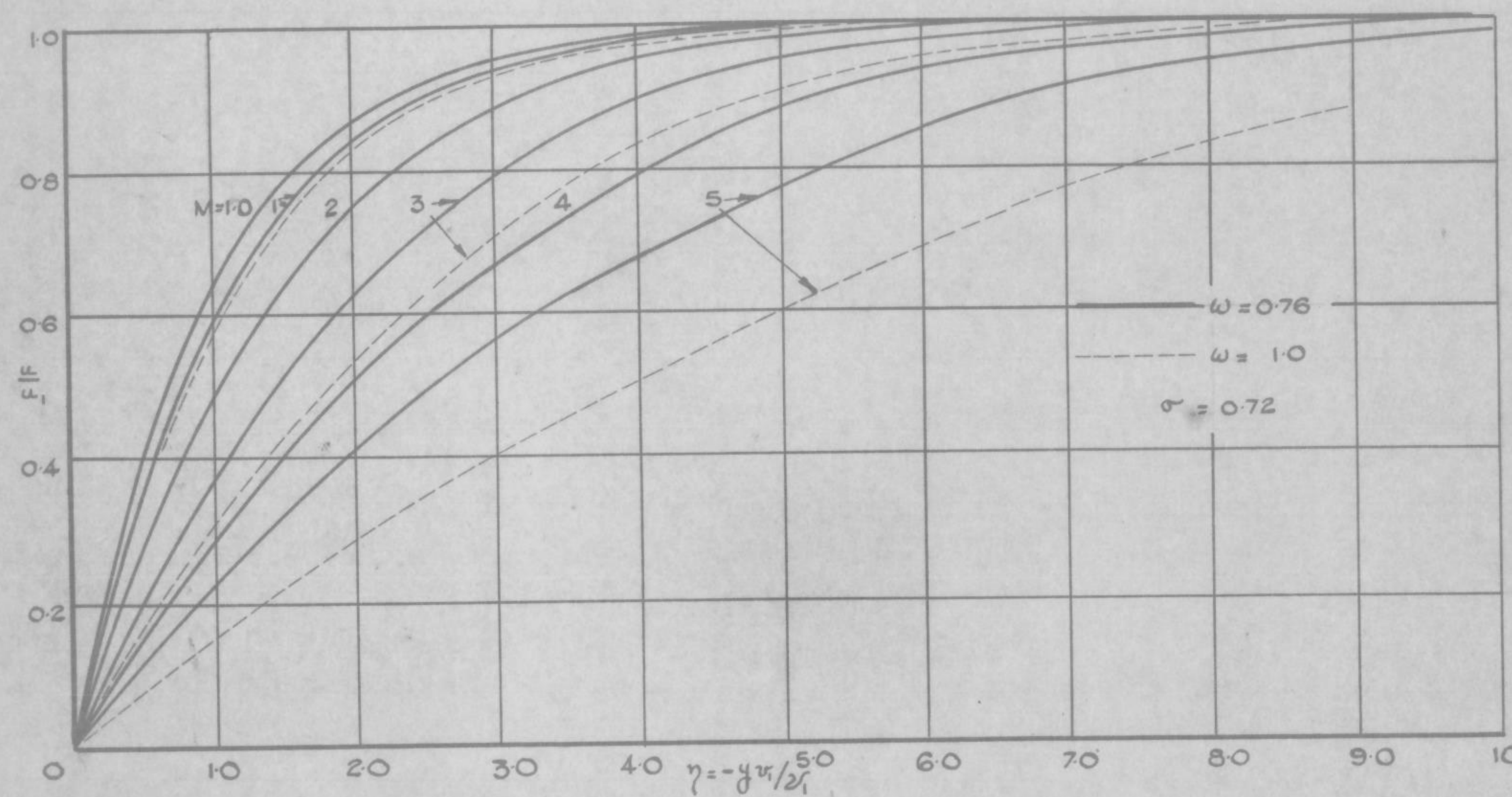
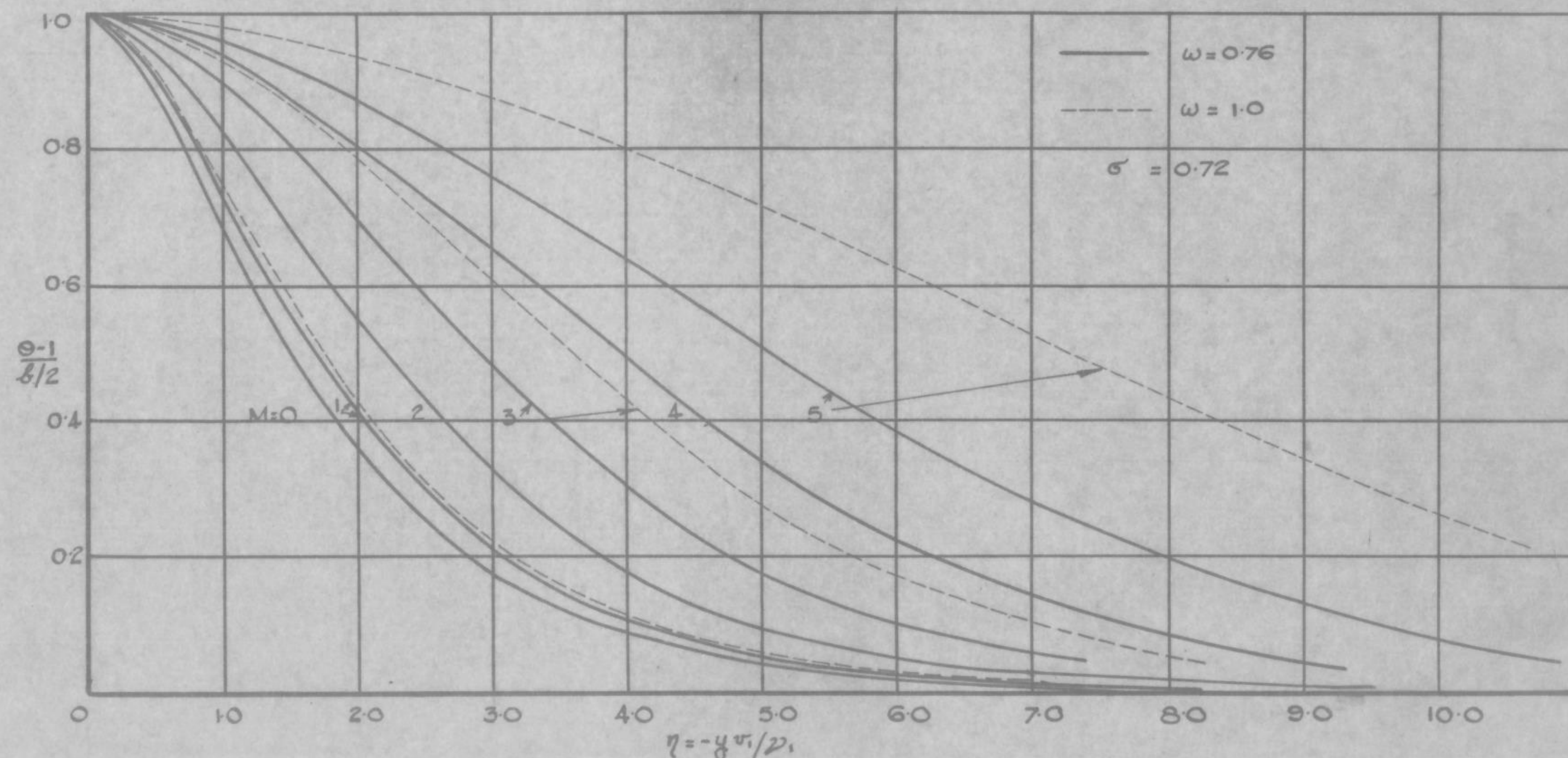
FIG. 1.



VELOCITY DISTRIBUTIONS FOR VARIOUS MACH NUMBERS

FIG. 2.



TEMPERATURE DISTRIBUTIONS FOR VARIOUS MACH NUMBERS