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The drag of source distributions in linearized supersonic flow

-by-

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1. Introduction and summary

Most bodies whose wave drags in supersonic flow have been calculated by linearised theory are such that the flows can be represented by a suitable distribution of sources. The present paper extends this idea by considering the drag problem for the general class of thin and slender bodies which can be replaced by sources only. The flow due to sources is necessarily acyclic, so there can be no local cross-stream forces on such bodies, since their presence would require the introduction of vortex elements.

As a preliminary, an expression for the drag force due to a volume distribution of sources is calculated from the momentum flux through a closed surface surrounding the sources. This general problem has been considered briefly by Hayes (1), but the development given here, while yielding results equivalent to those obtainable by extending Hayes's ideas, follows somewhat different lines.

Next it is shown that the flow past a body of the class under consideration can be derived from a distribution of sources in the surface of the body, whose surface density is proportional to the local slope of the surface relative to the

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direction of the undisturbed stream. Expressions for the drag force on thin and slender bodies are then obtained from the general result for volume distributions, the method of reduction being different for thin and slender bodies.

Finally, the results are applied to the problem of determining the shape of a slender fuselage carrying a given wing-system so that the drag of the combination may be a minimum. This problem is determinate only if suitable conditions on the fuselage shape are specified, and, since these conditions can take many forms, no explicit solutions are given. Instead, it is shown how the problem may be reduced to a similar problem for a slender body alone; such problems have been treated in detail by other authors.

2. Volume distributions of sources

It is assumed that the sources under consideration cause a small steady disturbance to an otherwise uniform supersonic stream of density ρ_0, flowing with speed U and Mach number M relative to a co-ordinate frame in which the sources are fixed, and that the total source strength is zero. Rectangular cartesian co-ordinates, x,y,z, are chosen so that the z-axis is parallel to the undisturbed stream and z increases downstream; $\underset{\sim}{k}$ is the unit vector parallel to the z-axis; $\underset{\sim}{R}$ denotes the position vector of the point (x,y,z) relative to the origin of co-ordinates; $\underset{\sim}{r}$ denotes the vector distance from the z-axis, so that it has rectangular components (x,y,0), and dS and dV denote elements of surface area and volume respectively. Corresponding quantities with suffices 1 and 2 are defined similarly. Differentiations with respect to z, z_1, and z_2 are denoted by primes.

If $\underset{\sim}{v}$ is the perturbation particle velocity, $\underset{\sim}{w}$ is defined by

$$\underset{\sim}{w} = \underset{\sim}{v} - M^2 \underset{\sim}{k}\, \underset{\sim}{k}.\underset{\sim}{v}\,, \qquad \text{............(1)}$$

and $\rho_0 Q$ is the source density, then the linearized equations for irrotational flow are (2)

$$\nabla \wedge \underset{\sim}{v} = 0 \quad \text{and} \quad \nabla . \underset{\sim}{w} = Q. \qquad \text{.............(2)}$$

When the sources are contained in a bounded volume of space, so that they lie inside a closed surface, S, of finite extent, then the drag force, D, associated with the sources is given by the momentum-flux integral (2)

$$D = \rho_0 \underset{\sim}{k}. \int_S^{*} (\tfrac{1}{2}\underset{\sim}{v}.\underset{\sim}{w}\,\underset{\sim}{n} - \underset{\sim}{v}\,\underset{\sim}{w}.\underset{\sim}{n})dS, \qquad \text{.............(3)}$$

where $\underset{\sim}{n}$ is the unit outward normal to S. The divergence theorem applied to the surface integral in (3) gives

$$D = \rho_0 \underset{\sim}{k}. \int \left[\underset{\sim}{w} \wedge (\nabla \wedge \underset{\sim}{v}) - \underset{\sim}{v}\,\nabla . \underset{\sim}{w}\right] dV, \qquad \text{..........(4)}$$

where the integration is over the interior of S, and, on using the equations of motion (2), this becomes

$$D = - \rho_0 \underset{\sim}{k}. \int Q(\underset{\sim}{R}_1)\, \underset{\sim}{v}(\underset{\sim}{R}_1)\, dV_1. \qquad \text{.............(5)}$$

Since the perturbations are due to sources only, it can be shown (2) that the value of $\underset{\sim}{k}.\underset{\sim}{v}$ is given by

$$\underset{\sim}{k}.\underset{\sim}{v}(\underset{\sim}{R}_1) = - \frac{1}{2\pi} \int^{*} Q(\underset{\sim}{R}_2)\, \frac{\partial}{\partial z_2}\left(\frac{1}{R_B}\right) dV_2, \text{............(6)}$$

where

$$R_B^2 = (z_1 - z_2)^2 - B^2 (\underset{\sim}{r}_1 - \underset{\sim}{r}_2)^2, \quad B^2 - M^2 - 1, \quad \text{......(7)}$$

the integration is over that part of the domain $R_B^2 \geqslant 0$, $z_1 \geqslant z_2$ for which $Q \neq 0$, and the star indicates that the finite part of the divergent integral is to be taken in Hadamard's sense. On substituting (6) in (5), the expression for the drag becomes

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* If the total source strength is not zero, then an extra term has to be added to this expression for the drag.

$$D = -\frac{\rho_0}{2\pi}\int Q(\underset{\sim}{R}_1)dV_1 \int^{*} Q(\underset{\sim}{R}_2)\ \frac{\partial}{\partial z_2}\left(\frac{1}{R_B}\right) dV_2, \quad \ldots(8)$$

or, on integrating by parts with respect to z_1 and z_2, and since $Q = 0$ outside a bounded volume of space,

$$D = -\frac{\rho_0}{2\pi}\int Q'(\underset{\sim}{R}_1)dV_1 \int Q'(\underset{\sim}{R}_2)\cosh^{-1}\left(\frac{z_1-z_2}{B|\underset{\sim}{r}_1-\underset{\sim}{r}_2|}\right) dV_2. \quad \ldots\ldots\ldots\ldots(9)$$

Finally, this repeated integral can be written as the double integral

$$D = -\frac{\rho_0}{4\pi}\iint Q'(\underset{\sim}{R}_1)Q'(\underset{\sim}{R}_2)\cosh^{-1}\left(\frac{|z_1-z_2|}{B|\underset{\sim}{r}_1-\underset{\sim}{r}_2|}\right) dV_1 dV_2, \quad \ldots\ldots\ldots\ldots(10)$$

in which the integration is over the domain for which $|z_1-z_2| \geqslant B|\underset{\sim}{r}_1-\underset{\sim}{r}_2|$ and $Q \neq 0$. When Q is a discontinuous function of z, then (10) is to be interpreted as a double Stieltjes integral.

3. The representation of bodies by sources

The bodies under consideration are supposed to be thin or slender, so that the slopes of their surfaces relative to the direction of the undisturbed stream are small compared with unity everywhere, and they are also supposed to experience no local cross-stream forces, in which case the flow past them is acyclic and can be represented by distributions of sources in their surfaces.*

For any body of the above class, let Σ^* be the part of its surface which lies inside any closed surface which intersects the body; let S be the part of the closed surface which

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* A proof of this statement for incompressible flow is given in Lamb's 'Hydrodynamics' §58, 6th Edition (Cambridge, 1932). The corresponding result for linearized compressible flow can be proved in an analogous way.

lies inside the body; let $\underset{\sim}{n}$ be the unit normal to the new closed surface $\Sigma^{*} + S$ so formed, and let $\rho_0 q$ be the surface density of sources on Σ^{*}, which are assumed to lie just inside $\Sigma^{*} + S$. Then the law of conservation of mass for the interior of $\Sigma^{*} + S$ gives

$$\int_{\Sigma^{*}+S} \underset{\sim}{w}.\underset{\sim}{n}\, dS = \int_{\Sigma^{*}} q\, dS. \qquad \text{............(11)}$$

By using the linearized boundary condition on Σ^{*}, namely

$$\underset{\sim}{v}.\underset{\sim}{n} = \underset{\sim}{w}.\underset{\sim}{n} = -U\, \underset{\sim}{k}.\underset{\sim}{n} = U\eta \text{ say}, \qquad \text{............(12)}$$

where η is the slope of the surface relative to the direction of the undisturbed stream, (11) becomes

$$\int_{\Sigma^{*}} (q - U\eta)\, dS = \int_{S} \underset{\sim}{w}.\underset{\sim}{n}\, dS. \qquad \text{............(13)}$$

If the thickness ratio of the body is denoted by t, and η is $O(t)$ uniformly over the body surface, then q and $\underset{\sim}{w}$ are $O(t)$, and if S can be chosen so that the ratio of the area of S to the area of Σ^{*} is $O(t)$, then (13) can be written

$$\int_{\Sigma^{*}} \left[q - U\eta + O(t^2) \right] dS = 0 . \qquad \text{............(14)}$$

Since Σ^{*} is an arbitrary part of the body surface, it follows from (14) that q is given by

$$q = U\eta + O(t^2). \qquad \text{............(15)}$$

A little reflexion will show that the surface S can be chosen in the way described whenever the body is slender, or when its local thickness is small and $O(t)$, or for combinations of these. But in the case of the external flow past quasicylindrical bodies, for example, S cannot be chosen in this way, and (15) is not true in general for bodies of this kind.

The results of the previous section can now be used to obtain the drags of bodies for which (15) is true, by allowing Q to become very large inside thin layers of sources located on the body surfaces and to vanish outside these layers, and then proceeding to the limit of a surface distribution of sources with density $U\eta$. The volume integrals then become surface integrals, and if dS denotes an element of the body surface, then Q dV is to be replaced by $U\eta\, dS$. If the body is closed, then it follows from (15) that the total source strength is zero.

4. The drag of thin bodies

In the case of a thin (wing-like) body, the expression (10) can be modified directly, because differentiation with respect to z on the body surface is the same, within a factor $1 + O(t)$, as partial differentiation with respect to z, and $Q'dV$ in (10) can be replaced directly by $U\eta' dS$. The expression for the drag then becomes

$$D = -\frac{\rho_0 U^2}{4\pi}\iint \eta'(\underset{\sim}{R}_1)\eta'(\underset{\sim}{R}_2)\cosh^{-1}\left(\frac{|z_1 - z_2|}{B|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|}\right) dS_1 dS_2, \qquad \text{............(16)}$$

where the integrations are over those parts of the body surface for which $|z_1 - z_2| \geqslant B|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|$.

Such a body possesses a mean surface*, Σ, and the source distributions on the actual body surface can be combined to give a source distribution on Σ. If $T(\underset{\sim}{R})$ is the thickness of the body at any point $\underset{\sim}{R}$ on the mean surface, measured normally to the mean surface, then, since the actual and mean

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* The mean surface is a developable surface, with generators parallel to the z-axis, to which the body reduces as $t \to 0$.

surfaces are separated by a distance of magnitude $O(t)$, and the sum of the slopes of the body surfaces on the two sides of Σ is $T'(\underset{\sim}{R})$, (16) can be replaced by

$$D = -\frac{\rho_o U^2}{4\pi} \iint T''(\underset{\sim}{R}_1) T''(\underset{\sim}{R}_2) \cosh^{-1}\left(\frac{|z_1 - z_2|}{B|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|}\right) dS_1 dS_2 \quad ..(17)$$

where the integration is over those parts of Σ for which $|z_1 - z_2| \geqslant B|r_1 - r_2|$.

In general, η and T' are discontinuous functions of z (for example, at leading edges of finite angle), and the integrations in (16) and (17) must be taken in Stieltjes sense. A very similar result was given by Lighthill (3) for the special case of plane wings with polygonal streamwise sections. The expressions (16) and (17) are not valid when the body has rounded leading or trailing subsonic edges at which $T'(\underset{\sim}{R})$ becomes infinite.

5. The drag of slender bodies.

When the body under consideration is slender, so that all points on its surface lie at distances of magnitude $O(t)$ from a mean axis parallel to the z-axis, then, although $Q\,dV$ may still be replaced by $U\eta\,dS$, it is no longer possible to replace $Q'dV$ in (10) by $U\eta'dS$, and an alternative limiting process must be sought. The simplest procedure in this case is to express the inverse cosh in (10) as a logarithm, and to rewrite (10) in the form

$$\begin{aligned} D = &-\frac{\rho_o}{4\pi} \iint Q'(\underset{\sim}{R}_1) Q'(\underset{\sim}{R}_2) \log(|z_1 - z_2|)\, dV_1 dV_2 \\ &+ \frac{\rho_o}{4\pi} \iint Q'(\underset{\sim}{R}_1) Q'(\underset{\sim}{R}_2) \log(\tfrac{1}{2}B|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|)\, dV_1 dV_2 \\ &- \frac{\rho_o}{4\pi} \iint Q'(\underset{\sim}{R}_1) Q'(\underset{\sim}{R}_2) \log\left(\frac{|z_1 - z_2| + R_B}{2|z_1 - z_2|}\right) dV_1 dV_2, \quad (18) \end{aligned}$$

where the integrations are over the domain $|z_1 - z_2| \gg B|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|$ in each integral. Some of the integrations can now be performed before proceeding to the limiting forms for surface distributions.

In the first integral of (18), the integrations with respect to x_1, y_1, and x_2, y_2 can be made. If ds is an element of the contour, C, in which the body surface is intersected by any plane z = constant, and $S(z)$ denotes the cross-sectional area of the body by this plane, then the transition from the volume distributions to the limiting surface distributions takes the form

$$\lim \iint Q'(\underset{\sim}{R})\, dxdy = \lim \frac{d}{dz} \iint Q(\underset{\sim}{R})\, dxdy$$

$$= \frac{d}{dz}\left(\lim \iint Q(\underset{\sim}{R})dx\, dy\right) = U \frac{d}{dz} \int_C \eta\, ds = U\, S''(z), \quad \ldots\ldots(19)$$

where the terms in the second line differ from those in the first only by a factor $1+O(t)$ because the surface slope is small and $O(t)$ everywhere. On using this last result, the first integral in (18) becomes

$$\frac{\rho_o U^2}{4\pi} \iint S''(z_1)S''(z_2) \log\left(\frac{1}{|z_1 - z_2|}\right) dz_1 dz_2, \quad \ldots\ldots(20)$$

and, on neglecting a further factor $1+O(t^2 \log t)$, the integrations can be taken over all values of z_1 and z_2 for which $S'' \neq 0$. If $S'(z)$ is discontinuous, then (20) is to be interpreted as a Stieltjes integral, but the set of points for which $z_1 = z_2$ is to be excluded from the domain of integration, otherwise infinite terms occur: this excluded set of points does not occur in the original domain of integration. Following Lighthill (4), this integral can be written

$$\frac{\rho_o U^2}{4\pi} {}^{*}\!\!\iint \log\left(\frac{1}{|z_1 - z_2|}\right) dS'(z_1)\, dS'(z_2), \quad \ldots\ldots(21)$$

where the star denotes the finite part of the double Stieltjes integral, obtained by rejecting the infinite terms for which $z_1 = z_2$.

In the second integral of (18), the integrations with respect to z_1 and z_2 can be made. If the integrations in this integral were over all space, then the integral would vanish because $Q = 0$ outside a bounded volume; hence this integral can be replaced by another of the same form, but of opposite sign, for which the integrations are over the domain $|z_1 - z_2| < B|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|$. The only significant contributions from this new integral come from discontinuities in Q, or, in the limit, from discontinuities in η. The ultimate form of the final expression depends on the location of the contours on the body surface at which η is discontinuous. The simplest case, and the only one considered in detail here, is when η is discontinuous on a finite number of contours, C_i $(i = 1,2,\ldots,n)$, lying in planes $z = \underline{constant}$ and separated by distances of magnitude $O(1)$. In this case, if the discontinuity on the contour C_i and at distance $\underset{\sim}{r}$ from the z-axis is denoted by $\Delta\eta_i(\underset{\sim}{r})$, then, on neglecting a factor $1+O(t^2 \log t)$, the contribution from the second integral is

$$\frac{\rho_o U^2}{4\pi} \sum_{i=1}^{n} \iint_{C_i} \Delta\eta_i(\underset{\sim}{r}_1) \Delta\eta_i(\underset{\sim}{r}_2) \log\left(\frac{2}{B|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|}\right) ds_1 ds_2, \qquad \ldots\ldots\ldots\ldots(22)$$

where the integrations in each double integral are both over the contour C_i. For more general contours of discontinuity, the integrations may be over only parts of the contours, and should the contours be separated by distances of magnitude $O(t)$, then terms involving integrations over more than one contour may occur. These more general results are not difficult to write down, but they can be complicated expressions, so their development is left to the reader.

The third integral in (18) can be shown to be less than the first by a factor of magnitude $O(t)$, and so is negligible to the present order of approximation.

Hence, from (21) and (22), the expression for the drag in the cases under consideration is

$$D = \frac{\rho_o U^2}{4\pi} \; {}^{*}\!\iint \log\left(\frac{1}{|z_1 - z_2|}\right) dS'(z_1)\, dS'(z_2)$$

$$+ \frac{\rho_o U^2}{4\pi} \sum_{i=1}^{n} \iint_{C_i} \Delta\eta_i(\underset{\sim}{r}_1)\, \Delta\eta_i(\underset{\sim}{r}_2) \log\left(\frac{2}{B|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|}\right) ds_1 ds_2 \qquad \text{............(23)}$$

which is a generalization of Lighthill's formula for the drag of a body of revolution with discontinuous meridian section (4), to which it reduces in that case. This result also applies to slender bodies with open noses and bases, provided that the body is considered to be continued upstream and downstream by cylinders whose generators are parallel to the z-axis, and the discontinuities in η at the joints are taken into account. Of course, the internal and external flows must be independent, and only the external drag is given by (23); when the nose and base areas are not equal, the total source strength is not zero, but the extra term in the drag is part of the internal drag.

It is interesting to notice that each double integral in (22) has the same form as the 'energy' of a two-dimensional distribution of electric charge on the contour C_i, with density proportional to $\Delta\eta_i$. Since a given charge distributes itself over the surface of a fixed conductor so that the energy is a minimum, it follows that, for a given discontinuity in $S'(z)$ over a fixed contour C_i, each term is a minimum when $\Delta\eta_i$ is proportional to the charge density on an isolated two-dimensional conductor having C_i as its boundary. Also, for a given area inside C_i, it is known that the 'capacity' is a minimum when C_i is a circle (5), so that for a given total charge, the energy is a maximum when C_i is a circle. Hence, in the present context, a circular cross-section at a section of discontinuous surface slope is the worst shape, and there is no true minimum for variations of C_i. These results have a particular application to the design of intakes (or outlets) for slender open-ended bodies when the area of the intake and the initial rate of change of cross-sectional area are given.

6. Fuselages of minimum drag

In this section, the body under consideration is taken to consist of a slender fuselage to which is attached a system of thin wings, tail surfaces, etc., called the wing-system below, on which there are no local cross-stream forces. The z-axis is chosen so that it coincides with the mean axis of the fuselage, and $S(z)$ denotes the cross-sectional area of the fuselage. For simplicity, it is assumed that $S(z)$ is an analytic function of z over that part of the z-axis occupied by the fuselage; the modifications required when $S'(z)$ is discontinuous follow the lines indicated in §5, and are very easily made. The wing-system is taken to have a thickness $T(\underset{\sim}{R})$ at any point, $\underset{\sim}{R}$, of its mean surface, as in §4. Since the surface of the fuselage is at a distance of magnitude $O(t)$ from the z-axis, the interference drag force between the fuselage and the wing-system can be calculated to a sufficient order of approximation by taking the sources on the surface of the fuselage to be concentrated on the z-axis. The drag force on the combination can then be written down at once as

$$D = -\frac{\rho_o U^2}{4\pi}\iint T''(\underset{\sim}{R}_1)T''(\underset{\sim}{R}_2)\cosh^{-1}\left(\frac{|z_1-z_2|}{B|\underset{\sim}{r}_1-\underset{\sim}{r}_2|}\right)dS_1dS_2$$

$$-\frac{\rho_o U^2}{2\pi}\iint T''(\underset{\sim}{R}_1)S''(z_2)\cosh^{-1}\left(\frac{|z_1-z_2|}{B|\underset{\sim}{r}_1|}\right)dS_1dz_2$$

$$-\frac{\rho_o U^2}{4\pi}\iint S''(z_1)S''(z_2)\log(|z_1-z_2|)\,dz_1dz_2. \quad \ldots\ldots(24)$$

In the following applications, the drag given by (24) is to be minimized for the restricted class of problems in which the wing-system is given and the area-distribution of the fuselage, $S(z)$, is varied. It is assumed that the variations of the shape of the fuselage do not affect the source distribution which represents the wing-system; this is not strictly true, and the analysis can be refined to take the change into account, but the small gain in accuracy is unlikely to justify the extra complication. An approximate method of taking the change into account is to take as part of the fuselage the small parts of the wing-system which are covered or uncovered as the fuselage varies.

These problems can be converted into similar problems of a type already familiar in slender-body theory by defining a new function, $A(z)$, satisfying the integral equation

$$\int A''(z_1)\log(|z-z_1|)dz_1 + \int T''(\underset{\sim}{R}_1)\cosh^{-1}\left(\frac{|z-z_1|}{B|\underset{\sim}{r}_1|}\right)dS_1 = 0. \qquad \ldots\ldots\ldots\ldots(25)$$

This equation can be solved by standard methods to give the particular solution

$$A'(z) = \frac{1}{\pi}\int \frac{T'(R_1)dS_1}{\sqrt{\left[B^2\underset{\sim}{r}_1^2-(z-z_1)^2\right]}}, \qquad \ldots\ldots\ldots\ldots(26)$$

where the integrations are over that part of the mean surface of the wing-system for which $|z-z_1| \leqslant B|\underset{\sim}{r}_1|$.* On using (25), (24) can be written

$$\begin{aligned} D = &- \frac{\rho_o U^2}{4\pi}\iint T''(\underset{\sim}{R}_1)T''(\underset{\sim}{R}_2)\cosh^{-1}\left(\frac{|z_1-z_2|}{B|\underset{\sim}{r}_1-\underset{\sim}{r}_2|}\right)dS_1dS_2 \\ &+ \frac{\rho_o U^2}{4\pi}\iint A''(z_1)A''(z_2)\log(|z_1-z_2|)\,dz_1dz_2 \\ &- \frac{\rho_o U^2}{4\pi}\iint\left[S''(z_1)+A''(z_1)\right]\left[S''(z_2)+A''(z_2)\right]\log(|z_1-z_2|)dz_1dz_2, \end{aligned} \qquad \ldots\ldots\ldots\ldots(27)$$

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* This result can be obtained very easily if both integrals in (25) are integrated by parts with respect to z_1, and the result is compared with Poisson's integral connecting the real and imaginary parts of the function $\left[(z-z_1)^2 - B^2r_1^2\right]^{\frac{1}{2}}$ for real values of z.

from which it follows that the problem of determining $S(z)$ in any given problem is equivalent to that of determining $S(z) + A(z)$ so that the third integral is minimized. This problem has been investigated in detail by other authors, so it is not considered further here; it is sufficient to notice a few useful and interesting facts concerning the function $A(z)$.

By integrating (26) with respect to z, and since $T(\underline{R})dS$ is an element of volume of the wing-system, it follows immediately that

$$A(z) = \frac{1}{\pi}\int \frac{dV_1}{\sqrt{\left[B^2 \underline{r}_1^2 - (z-z_1)^2\right]}} , \quad \text{.............(28)}$$

where the integration is over that part of the domain $|z-z_1| \leq B|\underline{r}_1|$ which lies inside the wing-system. The function $A(z)$ defined by (28) is non-zero only over a finite segment of the z-axis which is such that the wing-system is just included inside the closed surface formed by the Mach cones with vertices at its ends. If $A(z)$ is the cross-sectional area of some body by planes z = <u>constant</u>, then this body can be constructed by spreading each element of volume of the wing-system over the segment of the z-axis defined by $z_1 - B|\underline{r}_1| < z < z_1 + B|\underline{r}_1|$ according to the law $dV_1/\pi\sqrt{\left[B^2\underline{r}_1^2-(z-z_1)^2\right]}$, and combining these distributions. It follows from this construction, or directly from (28) by integrating with respect to z, that the volume of this body is equal to the volume of the wing-system.

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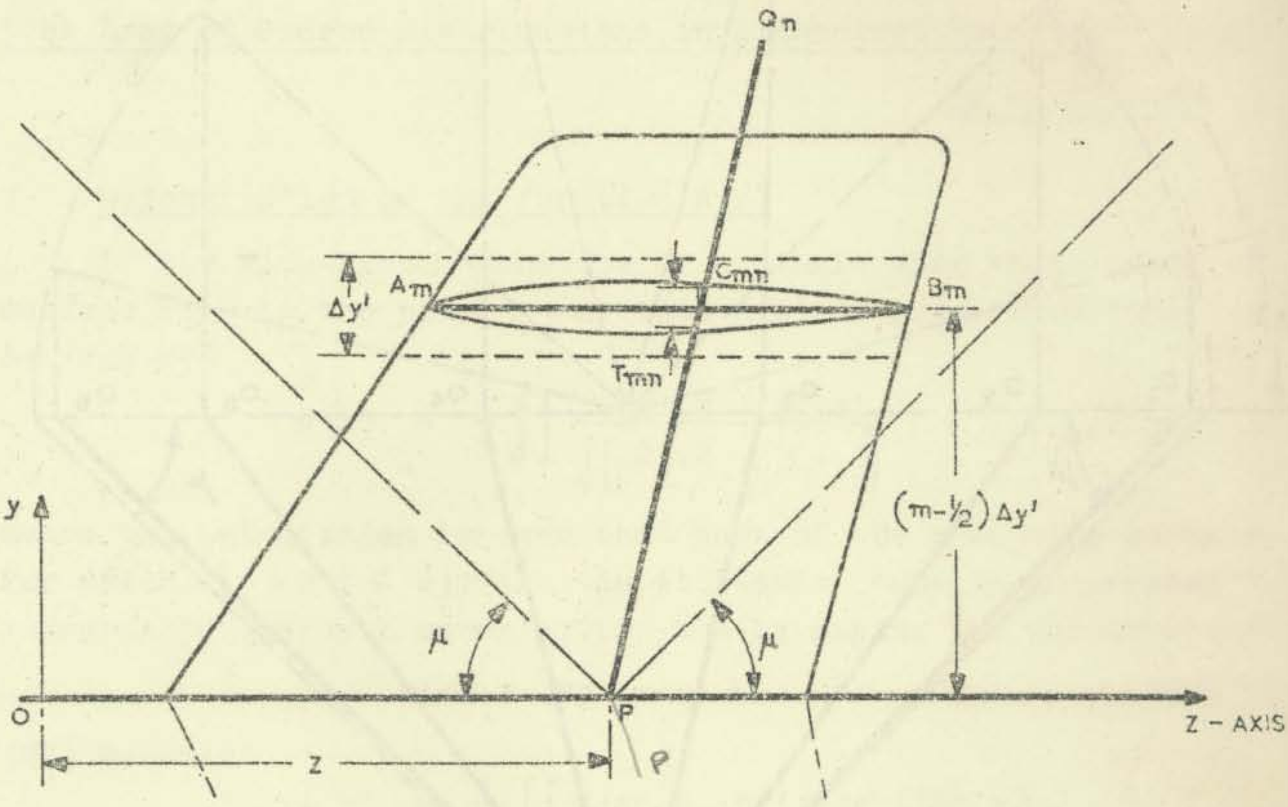

FIG. I.

DETERMINATION OF THE POINTS C_{mn} AT WHICH THE WING-THICKNESSES T_{mn} ARE TO BE TAKEN FOR THE EVALUATION OF A(z) BY EQN. 32.

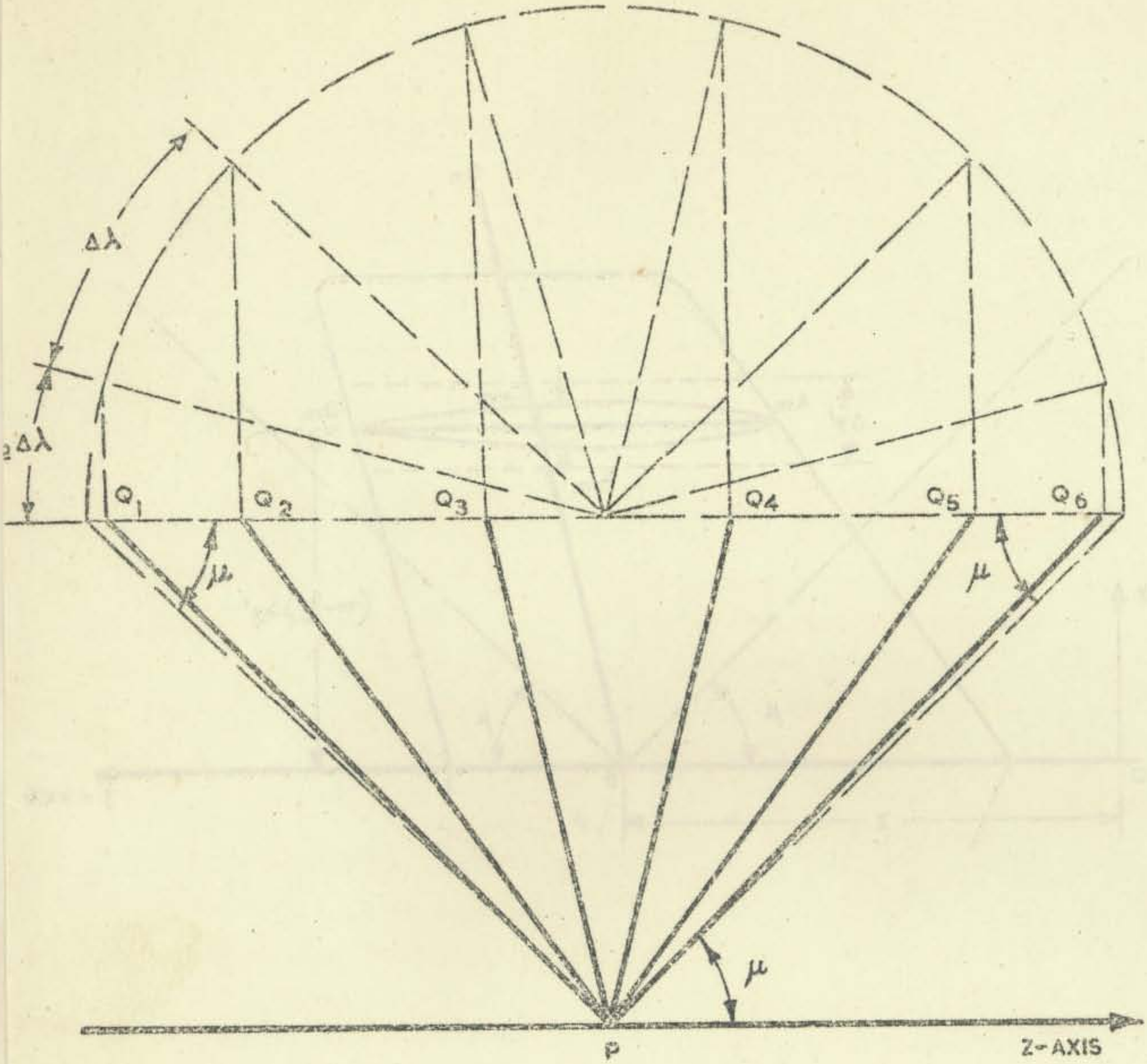

FIG. 2. GRAPHICAL CONSTRUCTION OF THE PENCIL OF LINES PQ_n FOR THE CASE $\Delta\lambda = \pi/6$

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7. Determination of the function A(z)

If the wing-system consists of a single wing whose mean surface lies in the plane $x = 0$, then $A(z)$ is given from (26) or (28) by

$$A(z) = \frac{1}{\pi} \iint \frac{T(y', z')dy'dz'}{\sqrt{\left[B^2 y'^2 - (z-z')^2\right]}}, \qquad \ldots\ldots\ (29)$$

where the integration is over that part of the mean wing surface for which $|z - z'| \leqslant B|y'|$. As it stands, this is an awkward integral to evaluate numerically, but by making the substitution

$$z' = z - By' \cos \lambda, \qquad \ldots\ldots\ (30)$$

(29) becomes

$$A(z) = \frac{1}{\pi} \iint T(y', z - By' \cos \lambda)dy'd\lambda, \qquad \ldots\ldots\ (31)$$

where λ now varies from 0 to π, and y' varies over the wing-span, and this expression for $A(z)$ can be evaluated by a simple semi-graphical method.

Let the wing-planform be divided into chord-wise strips of width $\Delta y'$ and let $A_m B_m$ be the centre-line of the m th strip as shown in Figure 1. Also let the range of λ be divided into equal increments $\Delta\lambda$, and if the point z on the axis of the fuselage at which $A(z)$ is required be denoted by P, as in Figure 1, let PQ_n correspond to the centre of the n th increment of λ, that is to $\lambda = (n - \frac{1}{2})\Delta\lambda$. If $A_m B_m$ and PQ_n intersect, let them intersect in the point C_{mn}, and let the wing-thickness at C_{mn} be denoted by T_{mn}. Then, from (31) A(z) is given by

$$A(z) = \frac{1}{\pi} \Delta y' \Delta \lambda \sum T_{mn}, \quad \text{.......} (32)$$

where the summation is over all values of m and n which correspond to points inside the wing-planform.

If the chord-wise sections corresponding to the lines $A_m B_m$ are drawn on the wing-planform, then the values of T_{mn} can be read off at each intersection, and added together to give $\sum T_{mn}$.

In practice, it will be found convenient to draw the pencil of lines PQ_n on transparent paper so that it can be moved to give A(z) for any desired value of z. The pencil of lines PQ_n can be constructed by calculation, or by a geometrical construction as shown in the self-explanatory Figure 2, for the case when $\Delta\lambda = \pi/6$.

Of course, the process described above must be carried out for both halves of the wing, or if the wing is symmetrical about the axis of the fuselage, the result for the half-wing may be doubled.

Clearly the method of evaluating (31) described above is only one of many methods which could be used, and in fact is the crudest method possible, corresponding as it does to the trapezium rule for integration. However it has the merit of being suitable for use in a drawing office, and if increased accuracy is required, this can be achieved either by decreasing the intervals $\Delta y'$ and $\Delta\lambda$, or by using Simpson's Rule, the latter probably being the more economical method.

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