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VISCOUS FLUX LIMITERS

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Abstract

We present **Numerical Viscosity Functions**, or NVFs, for use with Riemann-problem based shock-capturing methods as applied to viscous flows. In particular, **viscous flux limiters** are derived. The analysis pertain to a linear convection-diffusion model equation. Our NVFs combine the physical viscosity, the role of which is maximised, with numerical viscosity, whose role is minimised, to capture TVD solutions to viscous flows.

1 Introduction.

The development of Riemann-problem based shock-capturing methods for hyperbolic problems has been a significant contribution to Computational Fluid Dynamics (CFD) in the last decade. These methods are high-order extensions of the first-order Godunov method [1]. Two main features are: they use solutions to local Riemann problems and introduce implicit numerical viscosity to capture oscillation-free inviscid shocks. Applications of these methods to a wide variety of inviscid flows have proved very successful.

Quite recently these methods have also been used to solve viscous flow problems such as the Navier-Stokes equations (eg. [2], [3]). The basic strategy is to deploy these methods for the convective terms (the Euler equations) and use central differences, for example, to discretise the viscous terms. The approach has lead to satisfactory results. This is not surprising since, in the absence of strong diffusion, it is the discretisation of the convective terms what is really crucial. Riemann-problem based methods are in many ways optimal for treating convective terms, particularly, for the way the implicit numerical dissipation to capture oscillation-free shocks is controlled. In despite of this property some workers (e.g. [2]) have reported the presence of excessive artificial viscosity in the computed solutions. The explanation for this is that numerical viscosity functions, in the form of flux limiters for example, introduce numerical viscosity as if the equations were actually inviscid. These limiters are blind to the fact that the equations do have physical viscosity which is (correctly) superimposed on the added (excessive) numerical viscosity. In this paper we present numerical viscosity functions that incorporate fully the physical viscosity so as to introduce the absolute minimum of artificial viscosity for capturing oscillation-free shocks. The analysis is based on a model convection-diffusion equation and we select a particular Riemann-problem based method, namely the Weigthed Average Flux (or WAF) Method [4]. The principles however apply to other methods of this kind.

The resulting NVFs can immediately be translated to more conventional functions such as flux limiters and slope limiters ([5], [6], [7]). Numerical experiments for the linear model and for Burgers' equation show the advantage of using the new viscous NVFs over traditional limiters.

The remaining part of this paper is divided as follows: section 2 contains a succinct description of the WAF method, its application to the model equation and a derivation of TVD regions. Section 3 deals with construction of numerical viscosity functions; section 4 contains numerical experiments and conclusions are drawn in section 5.

2 TVD regions for a model equation.

We consider the model convection-diffusion equation

$$u_t + au_x = \alpha u_{xx} \tag{1}$$

where a is a wave propagation speed and α is a viscosity coefficient. Both a and α are assumed constant. We interpret the left-hand side of Eq. 1 as a conservation law with flux $F(u) = au$. Applying an explicit conservative method to the convective terms in (1) and central differences to the viscous term gives

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x}(F_{i-1/2} - F_{i+1/2}) + d(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \tag{2}$$

where $d = \frac{\alpha \Delta t}{\Delta x^2}$ is the diffusion number and $F_{i+1/2}$ is the intercell numerical flux for the pair of cells $(i, i+1)$. In this paper we use the WAF [4] numerical flux

$$F_{i+1/2} = \frac{1}{2}(1 + A_{i+1/2})F_i + \frac{1}{2}(1 - A_{i+1/2})F_{i+1} \tag{3}$$

where $A_{i+1/2}$ is a numerical viscosity function yet to be constructed. If $A_{i+1/2} = c$, where $c = \frac{a\Delta t}{\Delta x}$ is the Courant number, the WAF method reduces identically to the Lax-Wendroff method, which is second-order accurate in space and time. This identity between WAF and Lax-Wendroff is only true for the linear pure convection part of Eq. 1 (left-hand side). The quantities Δx and Δt define the mesh size in space and time respectively. In the first part of the analysis we assume $a > 0$.

As it is well known, the Lax-Wendroff method is upwind biased, for the coefficients (or weights) $W_1 = \frac{1+c}{2}$ and $W_2 = \frac{1-c}{2}$ in Eq. (3) control the upwind F_i and downwind F_{i+1} contributions to the intercell numerical flux $F_{i+1/2}$, with $W_1 > W_2$ for positive speed a. TVD solutions are ensured by increasing the upwind contribution and decreasing the downwind contribution. The objetive here is to construct numerical viscosity functions $A = A(r, c, d)$ that depend on a flow parameter r, the Courant number c and the diffusion number d.

Insertion of (3) into (2) and after rearranging gives

$$H = \frac{u_i^{n+1} - u_i^n}{u_{i-1}^n - u_i^n} = \frac{1}{2}c[\frac{1}{r_i}(1 - A_{i+1/2}) + A_{i-1/2} + 1] + d(1 - \frac{1}{r_i}) \tag{4}$$

where the flow parameter r_i is given by

$$\frac{u_i^n - u_{i-1}^n}{u_{i+1}^n - u_i^n} = \frac{\Delta_{upw}}{\Delta_{loc}} \tag{5}$$

and is the ratio of the upwind jump Δ_{upw} to the local jump Δ_{loc} across the wave of speed a. We now impose the sufficient TVD condition $0 \leq H \leq 1$ in eq. (4). After rearranging we obtain

$$-1-\frac{2}{R_{ce}}+\frac{2}{R_{ce}r_i}\leq\frac{1}{r_i}(1-A_{i+1/2})+A_{i-1/2}\leq\frac{2-c}{c}-\frac{2}{R_{ce}}+\frac{2}{R_{ce}r_i} \tag{6}$$

where $R_{ce}=\frac{c}{d}$ is the cell Reynold's number.

So far we have assumed that the speed a is positive. For negative a the result is the same as that of inequialities (6) but c is changed to $|c|$ and the upwind jump in (5) is changed to $\Delta_{upw}=u^n_{i+2}-u^n_{i+1}$.

Hence in what follows we adopt the general case

$$-1-\frac{2}{R_{ce}}+\frac{2}{R_{ce}r_i}\leq\frac{1}{r_i}(1-A_{i+1/2})+A_{i-1/2}\leq\frac{2-|c|}{|c|}-\frac{2}{R_{ce}}+\frac{2}{R_{ce}r_i} \tag{7}$$

where now the cell Reynolds number is $R_{ce}=\frac{|c|}{d}$.

We now select two inequalities such that (7) holds automatically. These are

$$\frac{2}{R_{ce}r_i}-K\leq\frac{1}{r_i}(1-A_{i+1/2})\leq\frac{2-|c|}{|c|}-1+\frac{2}{R_{ce}r_i} \tag{8}$$

$$-1-\frac{2}{R_{ce}}+K\leq A_{i-1/2}\leq 1-\frac{2}{R_{ce}} \tag{9}$$

For given c and d, or equivalently R_{ce}, inequalities (8)-(9) lead to TVD regions for A as a function of the flow parameter r (subscripts omitted) as shown in Fig. 1; dotted lines illustrate the boundaries of these regions for the inviscid case $d=0$ which apply to the pure convection problem. The viscous TVD region is bounded by the thick full lines. There are two horizontal lines that define the upper and lower bounds A_M and L respectively. The lateral bounds for the left and right sides are given by the straight lines designated A_L and A_R. In the present study we keep the upper bound A_M fixed while all other bounds are allowed to vary.

The bounds are given by

$$A_M=1-\frac{2}{R_{ce}} \tag{10}$$

$$L=-1-\frac{2}{R_{ce}} \tag{11}$$

$$A_L=1-\frac{2}{R_{ce}}+Kr \tag{12}$$

$$A_R=1-\frac{2}{R_{ce}}-\frac{2(1-|c|)}{|c|}r \tag{13}$$

The parameter K is related to the minimum A_m by

$$K=A_m+1+\frac{2}{R_{ce}} \tag{14}$$

It is therefore the choice of K in the range

$$0\leq K\leq|c|-1-\frac{2}{R_{ce}} \tag{15}$$

that determines the value of the lower bound A_m as well as A_L. For $K = 0$ the boundary A_L coincides with A_M and the left TVD region coalesce to the single line A_M for negative r. Actually one could include the band bewteen A_M and 1 within the TVD region but this would result in excessive artificial viscosity. Note that the value $A = 1$ would give the Cole-Murmam first-order upwind method for the pure convection (inviscid) equation $u_t + au_x = 0$, which introduces too much artificial viscosity. The viscous version of this scheme is given by $A = A_M$, which reduces the amount of artificial viscosity of the Cole-Murmam scheme. The choice $A = |c|$ would give the Lax- Wendroff method for the inviscid equation which has no artificial viscosity. If $A_M \geq |c|$ then some artificial viscosity is needed to ensure TVD results. Otherwise the physical viscosity is sufficient to do this and one should select $A = |c|$ in that case. For $A \leq |c|$ one would be be adding negative artificial viscosity, that is to say the downwind contribution would be greater than that of the Lax-Wendroff scheme. The extreme case would be $A_M = A_m = -1$, which for the inviscid case leads to an unstable scheme.

The inclusion of the physical viscosity in the construction of the numerical viscosity functions A results in a reduction of the numerical viscosity introduced by the upwind contribution to the intercell flux. The physical viscosity compensates for that reduction.

3 Construction of numerical viscosity functions.

There is virtually unlimited freedom in constructing NVFs A within the TVD regions shown in Fig. 1. The choice of the lower boundary A_m is significant. Here we select two cases, namely $A_m = |c|$ ($K = |c| - 1 - \frac{2}{R_{ce}}$) and $A_m = 2|c| - 1 - 2/Rce$. The first case generates NVFs that are associated with the well known flux limiter MINMOD or MINBEE while the second choice is associated with the flux limiter SUPERBEE. We thus call the respective families of NVFs the MINAD and SUPAD families.

Once a function A has been constructed we relate it to flux limiters via

$$B = \frac{A - 1}{|c| - 1} \tag{16}$$

3.1 The MINAD family.

This family is shown in Fig. 2; it lies between the horizontal lines A_M and $A_m = |c|$ and the inclined lines

$$A_L = A_M + (|c| + 1 + \frac{2}{R_{ce}})r; A_R = A_M - (\frac{2}{|c|} + \frac{2}{R_{ce}})r \tag{17}$$

We select two members which we call MINAD1 and MINAD2. They are respectively shown by dotted and full lines. MINAD1 can be easily expressed as

$$MINAD1 = A_M + (|c| - A_M) r sign(r) for |r| \leq 1; MINAD1 = |c| \tag{18}$$

otherwise MINAD2 passes through the points r_L and r_R which are given by

$$r_L = \frac{|c| + \frac{2}{R_{ce}} - 1}{|c| + \frac{2}{R_{ce}} + 1}; r_R = \frac{|c| + \frac{2}{R_{ce}} - 1}{\frac{2}{R_{ce}} - \frac{2}{|c|}} \tag{19}$$

3.2 The SUPAD family.

This family is illustrated in Fig. 3. Only one member is selected and corresponds to the lowest boundary of the TVD region. For non-negative r and $d = 0$ this function can be identified with the inviscid flux limiter SUPERBEE. We therefore call it SUPAD.

4 Numerical experiments.

Two test problems with exact solution are chosen for numerical experiments. Test 1 is the linear convection-diffusion equation (1) in the spacial domain $[0,2]$, initial condition $u(0,x) = sin(\pi x)$ and periodic boundary conditions. The second test problem concerns Burgers' equation

$$u_t + uu_x = \alpha u_{xx} \tag{20}$$

in the spacial domain $[0,1]$. The initial condition is $u(0,x) = 1.0$ for $x \leq 0.1$ and $u(0,x) = 0.0$ otherwise. Transmissive boundary conditions are applied. Of course the analysis of this paper is strictly valid only for the linear model equation, but empirical application to the (non-linear) Burgers' equation gives very satisfactory results. This is encouraging, since for realistic problems (eg. the Navier-Stokes equations) the extension of these ideas will necessarily be empirical. We use 100 computing cells for both tests and march the solution at a Courant number of 0.8; for α and a we take 0.001 and 1.0 respectively. We compare numerical results with the exact solution for MINMOD, MINAD2 SUPERBEE and SUPAD for positive r only; for negative r we set $A = A_M$.

Fig 4. shows results for Test 1; 4(a) shows the result of using MINMOD while 4(b) shows the result when using one of its viscous counterpart MINAD2; 4(c) and 4(d) are results obtained by SUPERBEE and its viscous counterpart SUPAD. For this case of a smooth solution one can clearly see the advantages of using the viscous functions of the MINAD type. Note that clipping is absent in 4(b). As expected SUPERBEE gives wrong results for smooth flows as seen in 4(c); its viscous counterpart does not improve matters as seen in 4(d). Note that MINMOD adds too much numerical viscosity while SUPERBEE underdiffuses the solution, which is also incorrect.

Fig. 5 shows the corresponding results for Test 2. For this case with a shock wave both families of viscous functions are superior to their inviscid counterparts.

5 Conclusions.

Numerical viscosity functions for a model equation have been presented. These incorporate the physical viscosity so that the contribution from the numerical viscosity to ensure TVD results is minimised. Numerical experiments confirm the theoretical analysis. Extension of these ideas to realistic problems is a pending task.

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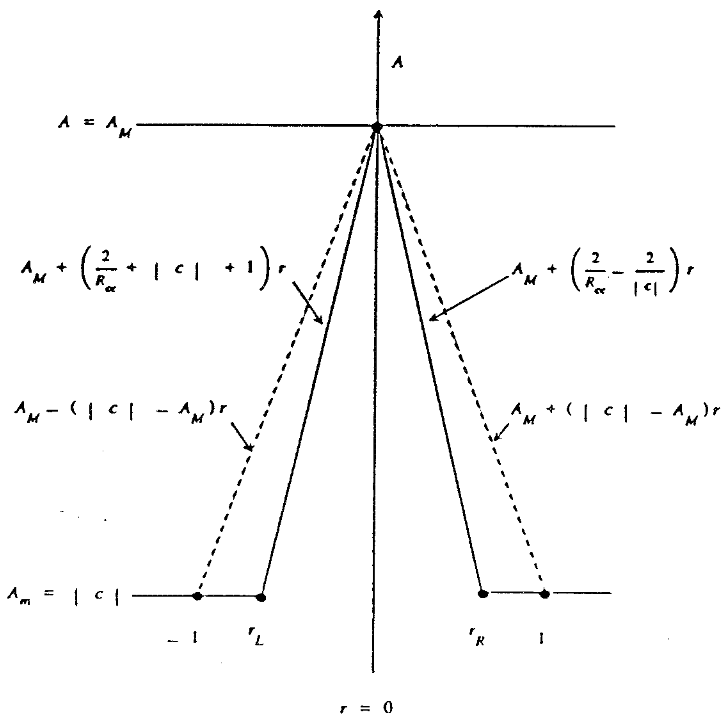

Figure 2: MINAD family of Numerical Viscosity Functions. MINAD1 is given by dotted lines and MINAD2 is given by full lines.

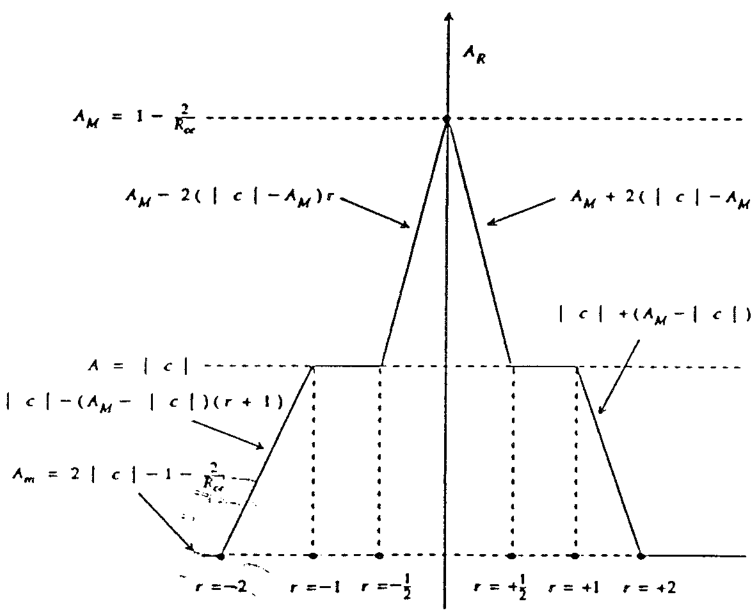

Figure 3: SUPAD family of Numerical Viscosity Functions. Only one member is selected and is given by full line.

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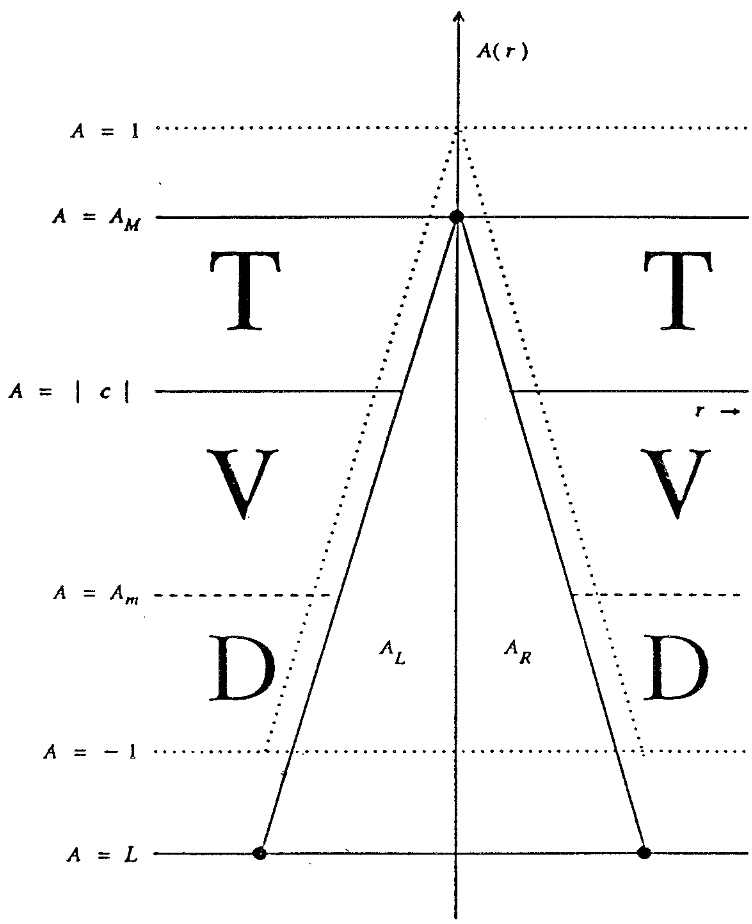

Figure 1: Viscous TVD regions for the model convection-diffusion equation.

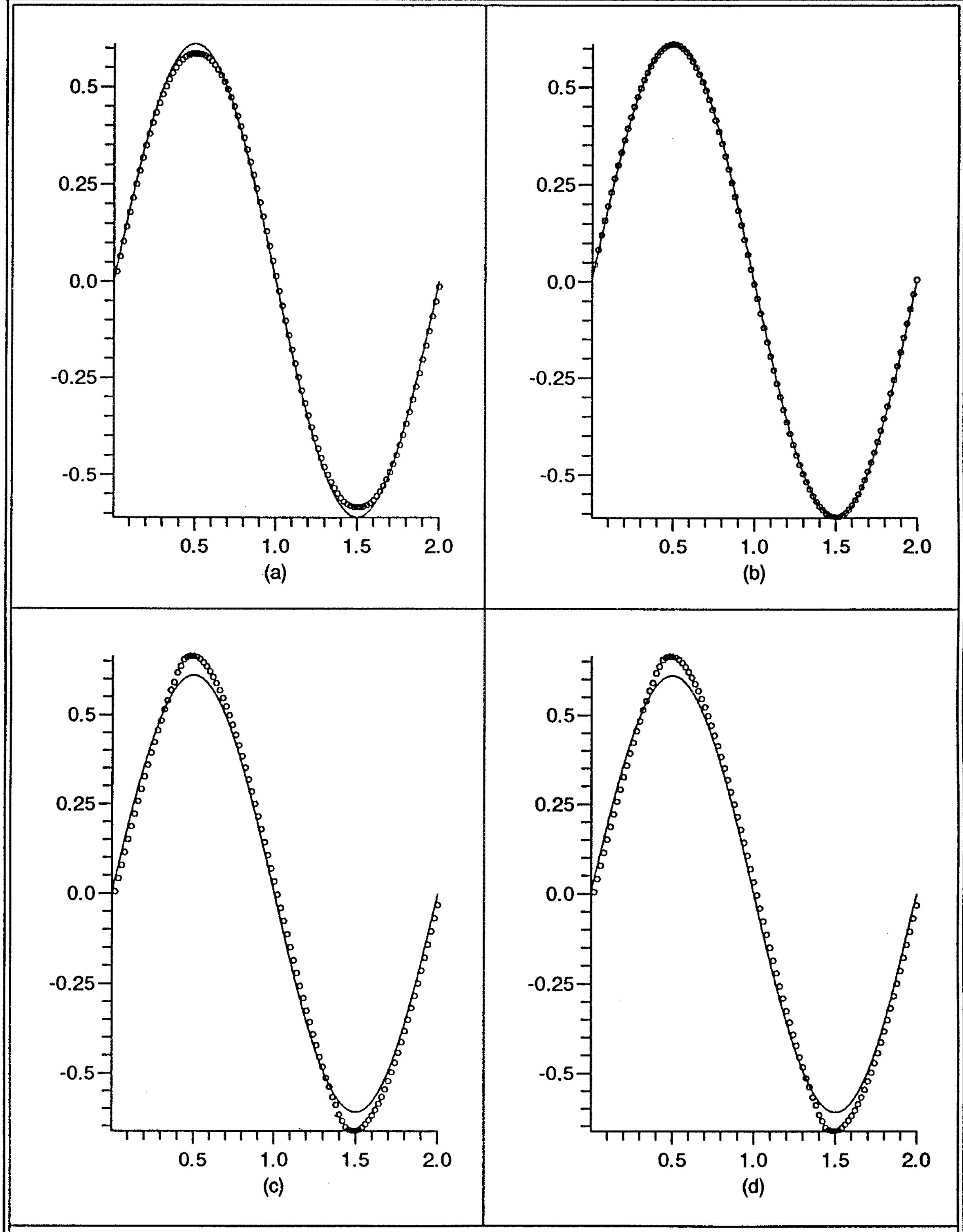

Figure 4. Numerical results for Test 1 using:
(a) MINMOD, (b) MINAD2, (c) SUPERBEE, (d) SUPAD.

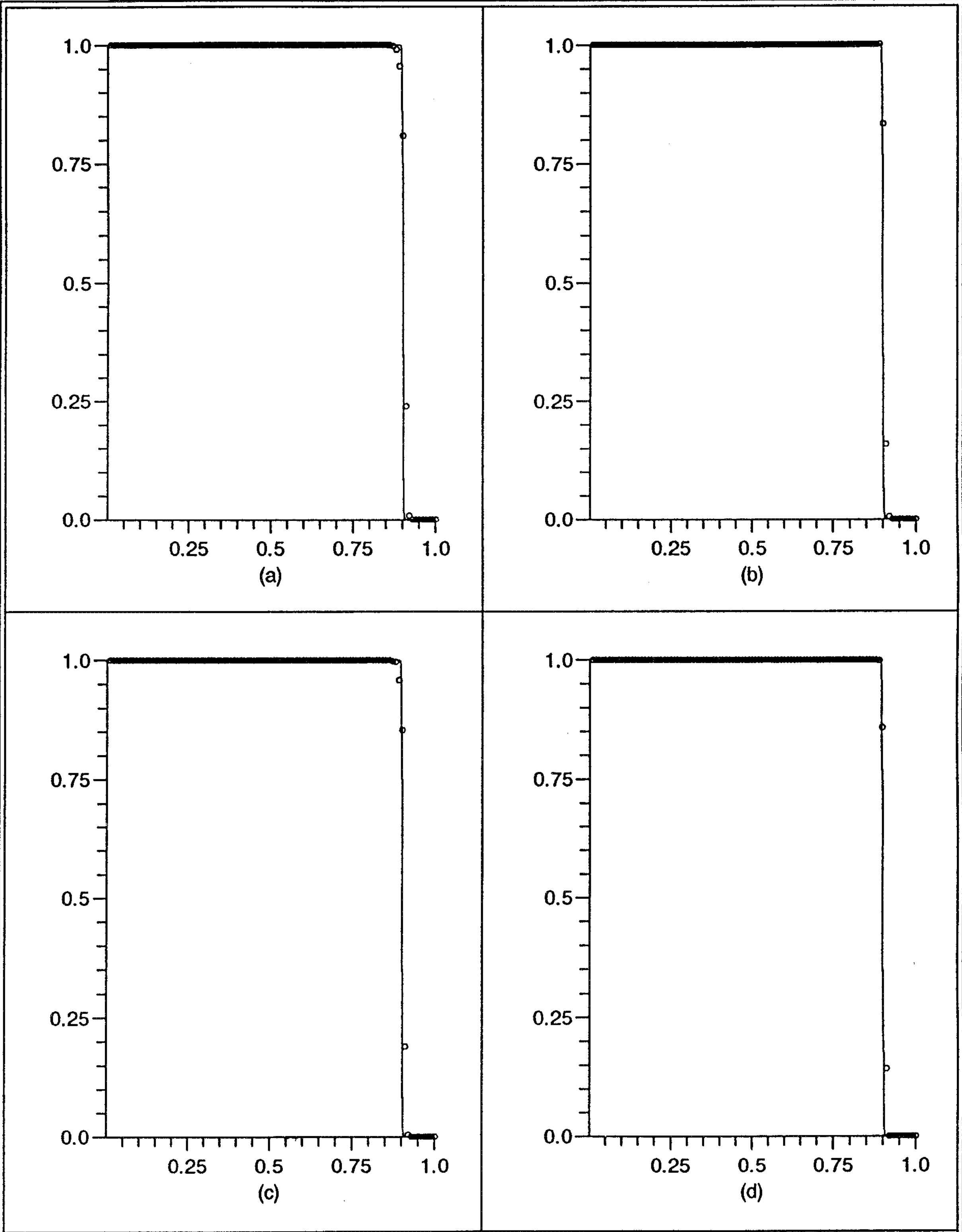

Figure 5. Numerical results for Test 2 using:

(a) MINMOD, (b) MINAD2, (c) SUPERBEE, (d) SUPAD.