Cranfield
UNIVERSITY

Motion of a rigid body in an unsteady non uniform heavy fluid

P.G.Thomasson

COA report No.9501
July 1995

College of Aeronautics
Cranfield University
Cranfield
Bedford MK43 0AL
England

College of Aeronautics Report No.9501
July 1995

Motion of a rigid body in an unsteady non uniform heavy fluid

P.G.Thomasson

ISBN 1 871564 859

£8

College of Aeronautics
Cranfield University
Cranfield
Bedford MK43 0AL
England

Motion of a Rigid Body in an Unsteady Non Uniform Heavy Fluid

by
P.G.Thomasson
College of Aeronautics
Cranfield University

1.0 Introduction

The Lagrangian formulation of the equations of motion of a body immersed in a steady but heavy perfect fluid is outlined in Lamb [1] and a resulting set of equations for the unsteady fluid case, is given in Lewis et al [2] in a form suitable for the flight dynamics of underwater vehicles. These equations have also been used to model the motion of airships in a steady uniform atmosphere, Cook et al [4]. Recently however the author has had some difficulty in applying these equations to the motion of other vehicles. In principle they should be applicable to not only underwater vehicles but also to airships, parafoils and aircraft. Two major problems of the equations in [2] is that they do not reduce to the small perturbation equations that are used for aircraft in gusts [3],[13], plus as will be seen later, the fluids inertial velocity causes difficulty. This paper identifies the source of the problems as being the conventional approach of lumping together the inertial and added masses. It provides an alternative formulation that keeps them separate and so avoids the difficulties.

2.0 Original equations

The form of the equations given in [2] is an expression of Newton's second law of motion expressed in body axes,

$$\mathbf{M}\dot{\mathbf{x}} = \mathbf{F_d} + \mathbf{F_f} + \mathbf{A} + \mathbf{F} \tag{1}$$

where

$\mathbf{x} = \begin{bmatrix} u & v & w & p & q & r \end{bmatrix}^T$ the inertial body axis velocities of the vehicle

$\mathbf{M}$ = the 6x6 mass matrix including added masses and inertias

$\mathbf{F_d}$ = the dynamics vector arising from body fixed rotating axes (2)

$\mathbf{F_f}$ = the vector of forces and moments due to the fluids inertial motion

$\mathbf{A}$ = the vector of the fluid dynamic forces and moments due to relative velocity

$\mathbf{F}$ = the vector of other external forces and moments

From [2], we define effective masses and inertia's as, (the bar refers to the displaced fluid, and conventional derivative terms are used to represent the added masses and inertia's),

$$
\begin{aligned}
&m_x = m - X_{\dot{u}} && \overline{m}_x = \overline{m} - X_{\dot{u}} \\
&m_y = m - Y_{\dot{v}} && \overline{m}_y = \overline{m} - Y_{\dot{v}} \\
&m_z = m - Z_{\dot{w}} && \overline{m}_z = \overline{m} - Z_{\dot{w}} \\
&J_x = I_x - L_{\dot{p}} && J_{yz} = I_{yz} + M_{\dot{r}} = I_{zy} + N_{\dot{q}} \\
&J_y = I_y - M_{\dot{q}} && J_{zx} = I_{zx} + N_{\dot{p}} = I_{xz} + L_{\dot{r}} \\
&J_z = I_z - N_{\dot{r}} && J_{xy} = I_{xy} + L_{\dot{q}} = I_{yx} + M_{\dot{p}}
\end{aligned}
\tag{3}
$$

If the c.g. has body axis co-ordinates of (a_x, a_y, a_z) then the mass matrix is,

$$
\mathbf{M} = \begin{bmatrix}
m_x & 0 & 0 & -X_{\dot{p}} & a_z m - X_{\dot{q}} & -a_y m - X_{\dot{r}} \\
0 & m_y & 0 & -a_z m - Y_{\dot{p}} & -Y_{\dot{q}} & a_x m - Y_{\dot{r}} \\
0 & 0 & m_z & a_y m - Z_{\dot{p}} & a_x m - Z_{\dot{q}} & -Z_{\dot{r}} \\
-L_{\dot{u}} & -a_z m - L_{\dot{v}} & a_y m - L_{\dot{w}} & J_{xx} & -J_{xy} & -J_{zx} \\
a_z m - M_{\dot{u}} & -M_{\dot{v}} & -a_x m - M_{\dot{w}} & -J_{xy} & J_{yy} & -J_{yz} \\
-a_y m - N_{\dot{u}} & a_x m - N_{\dot{v}} & -N_{\dot{w}} & -J_{zx} & -J_{yz} & J_{zz}
\end{bmatrix}
\tag{4}
$$

and if the co-ordinates of the centre of buoyancy are (b_x, b_y, b_z) then the dynamics vector is given by,

$$
\mathbf{F_d} = \begin{bmatrix}
-m_z wq + m_y rv + m[a_x(q^2 + r^2) - a_y pq - a_z rp] \\
-m_x ur + m_z pw + m[-a_x pq + a_y(p^2 + r^2) - a_z rq] \\
-m_y vp + m_x qu + m[-a_x rp - a_y rq + a_z(q^2 + p^2)] \\
-(J_z - J_y)rq + J_{yz}(q^2 - r^2) + J_{zx}pq - J_{xy}pr + m[-a_y(vp - qu) + a_z(ur - pw)] \\
-(J_x - J_z)pr - J_{yz}pq + J_{zx}(r^2 - p^2) + J_{xy}qr + m[a_x(vp - qu) - a_z(wq - rv)] \\
-(J_y - J_x)qp + J_{yz}pr - J_{zx}qr + J_{xy}(p^2 - q^2) + m[-a_x(ur - pw) + a_y(wq - rv)]
\end{bmatrix}
\tag{5}
$$

whilst the fluid motion vector is,

$$
\mathbf{F_f} = \begin{bmatrix}
\overline{m}_x \dot{u}_f + \overline{m}_z w_f q - \overline{m}_y r v_f \\
\overline{m}_y \dot{v}_f + \overline{m}_x u_f r - \overline{m}_z p w_f \\
\overline{m}_z \dot{w}_f + \overline{m}_y v_f p - \overline{m}_x q u_f \\
-L_{\dot{u}}\dot{u}_f - (b_z \overline{m} + L_{\dot{v}})\dot{v}_f + (b_y \overline{m} - L_{\dot{w}})\dot{w}_f + \overline{m}[b_y(v_f p - q u_f) - b_z(u_f r - p w_f)] \\
(b_z \overline{m} + M_{\dot{v}})\dot{u}_f - M_{\dot{v}}\dot{v}_f - (b_x \overline{m} - M_{\dot{w}})\dot{w}_f + \overline{m}[-b_x(v_f p - q u_f) + b_z(w_f q - r v_f)] \\
-(b_y \overline{m} + N_{\dot{u}})\dot{u}_f + (b_x \overline{m} - N_{\dot{v}})\dot{v}_f - M_{\dot{w}}\dot{w}_f + \overline{m}[b_x(u_f r - p w_f) - b_y(w_f q - r v_f)]
\end{bmatrix}
\tag{6}
$$

In the derivation of the equations given in [2] some added mass terms have been assumed to be zero (Lambs A', B' and C') and some perfect fluid terms have been moved to the right hand side of the equation and absorbed into the vector **A**. In addition of course, the vehicle mass and inertia's have been combined with the added mass and inertia terms to give 'effective inertia's'.

2.1 Difficulties

Several difficulties arise with the above equations. The most obvious is if the fluid is unsteady, then the fluid motion vector $\mathbf{F_f}$ is a function of the fluid inertial velocity as well as its inertial acceleration and this is counter intuitive. This can be clearly seen by giving the body the mass and inertia properties of the fluid that it displaces. In that case the relative acceleration between the body and the fluid should be zero, but the above equations do not reduce to this. A less obvious difficulty arises with the mass matrix **M** if we attempt to apply the equations to a vehicle such as a partially constrained dynamic wind tunnel model. Then it would be expected that the forces and moments due to the vehicles inertia would depend upon its inertial acceleration whilst the forces and moments due to the fluid acceleration (the added mass and inertia terms) would depend upon the relative acceleration of the fluid and the vehicle and so we might rearrange the equations to the form,

$$\mathbf{M_i}\dot{\mathbf{x}} + \mathbf{M_r}\dot{\mathbf{x}}_\mathbf{r} = -\mathbf{M_r}\dot{\mathbf{x}}_\mathbf{f} + \mathbf{F_d} + \mathbf{F_f} + \mathbf{A} + \mathbf{F} \tag{7}$$

where

$$\mathbf{x_r} = \mathbf{x} - \mathbf{x_f} = \text{ relative velocities of vehicle and fluid, and} \tag{8}$$

$$\mathbf{M_i} = \begin{bmatrix} m & 0 & 0 & 0 & a_z m & -a_y m \\ 0 & m & 0 & -a_z m & 0 & a_x m \\ 0 & 0 & m & a_y m & -a_x m & 0 \\ 0 & -a_z m & a_y m & I_{xx} & -I_{xy} & -I_{zx} \\ a_z m & 0 & -a_x m & -I_{xy} & I_{yy} & -I_{yz} \\ -a_y m & a_x m & 0 & -I_{zx} & -I_{yz} & I_{zz} \end{bmatrix} \tag{9}$$

$$\mathbf{M_r} = \begin{bmatrix} -X_{\dot{u}} & 0 & 0 & -X_{\dot{p}} & -X_{\dot{q}} & -X_{\dot{r}} \\ 0 & -Y_{\dot{v}} & 0 & -Y_{\dot{p}} & -Y_{\dot{q}} & -Y_{\dot{r}} \\ 0 & 0 & -Z_{\dot{w}} & -Z_{\dot{p}} & -Z_{\dot{q}} & -Z_{\dot{r}} \\ -L_{\dot{u}} & -L_{\dot{v}} & -L_{\dot{w}} & -L_{\dot{p}} & -L_{\dot{q}} & -L_{\dot{r}} \\ -M_{\dot{u}} & -M_{\dot{v}} & -M_{\dot{w}} & -M_{\dot{p}} & -M_{\dot{q}} & -M_{\dot{r}} \\ -N_{\dot{u}} & -N_{\dot{v}} & -N_{\dot{w}} & -N_{\dot{p}} & -N_{\dot{q}} & -N_{\dot{r}} \end{bmatrix} \tag{10}$$

For completeness the $\mathbf{M_r}$ matrix should be full but for compatibility with [2] some elements have been left zero.

The above fluid motion and fluid acceleration vectors can be combined into a new fluid motion vector. This results in most of the acceleration terms cancelling but not all and the fluid motion vector remains a function of the inertial fluid velocity. The source of this is party due to the fact that in the derivation used in [2] some but not all perfect fluid relative velocity terms have been absorbed into the vector **A**. As a result a full set of equations involving all the perfect fluid terms is required prior to any rearrangement such as that above.

3.0 Alternative Derivation

Lamb [2] derives the general expression for the kinetic energy of a body moving in a steady fluid but does not give the general expressions for the forces and moments. Imlay [5] carries out the necessary differentiation's of the full energy equation and presents the general expressions for the forces and moments. Lipscombe [6] attributes to Burnett the addition of the bulk fluid motion terms to Lambs original analysis and uses a partially complete energy equation to derive the forces and moments. However the present author has been unable to trace Burnett's work and [6] only quotes a set of equations of motion for the zero velocity gradient case. Taylor [8] derives expresions for the forces on a stationary ridgid body in a fluid with velocity gradients

Due to the complexity of the real fluid case the approximate approach based upon the ideas of Lamb and Taylor is used as the starting point. The development of the equations is done in several stages. First the equations of motion of a rigid body in a non uniform unsteady perfect fluid are derived for the case in which the undisturbed fluid velocities do not change significantly over distances comparable to the dimensions of the vehicle. Secondly the viscous forces and moments are added to the above equations and combined with the perfect fluid terms that are a function of relative velocity alone. Finally gust penetration effects are grafted on so as to represent the variation of the undisturbed moving fluid velocities over the vehicle.

3.1 Perfect Fluid Equations

If we consider a rigid body moving in a perfect fluid that is circulating in a multiply connected region and use Taylors [8] approximation that the circulating fluid velocities do not change significantly over the length of the vehicle, we can write the Lagrangian of the system including the bulk translation of the multiply connected fluid, as twice the total kinetic energies of,

1. The undisturbed circulating fluid moving with velocity $\overline{u}_f, \overline{v}_f, \overline{w}_f$,
2. The fluid disturbed by the presence of the body. This is due to the relative velocity, $u - u_f, v - v_f, w - w_f$, plus
3. a uniform density, neutrally buoyant vehicle moving at the relative velocities $u - u_f, v - v_f, w - w_f$,
4. The actual non uniform density, non neutrally buoyant vehicle moving with the vehicles speed u,v,w,. Minus,
5. a uniform density, neutrally buoyant vehicle moving with the vehicles speed u,v,w. Plus
6. the steady circulating fluid in the multiply connected region

The second and third items are essentially those due to Lamb but modified so as to be in terms of the relative velocity, whilst the remaining ones are required to account for the fluids own motion and that of the vehicle.

In equation form this is,

$$
\begin{aligned}
&2T \\
&= M_f\left(\bar{u}_f^2 + \bar{v}_f^2 + \bar{w}_f^2\right) \\
&-X_{\dot{u}}\left(u-u_f\right)^2 - Y_{\dot{v}}\left(v-v_f\right)^2 - Z_{\dot{w}}\left(w-w_f\right)^2 \\
&-2Y_{\dot{w}}\left(v-v_f\right)\left(w-w_f\right) - 2X_{\dot{w}}\left(w-w_f\right)\left(u-u_f\right) - 2X_{\dot{v}}\left(u-u_f\right)\left(v-v_f\right) \\
&-L_{\dot{p}}p^2 - M_{\dot{q}}q^2 - N_{\dot{r}}r^2 - 2M_{\dot{r}}qr - 2L_{\dot{r}}rp - 2L_{\dot{q}}pq \\
&-2p\left(X_{\dot{p}}\left(u-u_f\right) + Y_{\dot{p}}\left(v-v_f\right) + Z_{\dot{p}}\left(w-w_f\right)\right) \\
&-2q\left(X_{\dot{q}}\left(u-u_f\right) + Y_{\dot{q}}\left(v-v_f\right) + Z_{\dot{q}}\left(w-w_f\right)\right) \\
&-2r\left(X_{\dot{r}}\left(u-u_f\right) + Y_{\dot{r}}\left(v-v_f\right) + Z_{\dot{r}}\left(w-w_f\right)\right) \\
&+\bar{m}\left(\left(u-u_f-rb_y+qb_z\right)^2 + \left(v-v_f-pb_z+rb_x\right)^2 + \left(w-w_f+pb_y-qb_x\right)^2\right) \\
&+\bar{I}_{xx}^b p^2 + \bar{I}_{yy}^b q^2 + \bar{I}_{zz}^b r^2 - 2\bar{I}_{yz}^b qr - 2\bar{I}_{xz}^b pr - 2\bar{I}_{xy}^b pq \\
&+m\left(\left(u-ra_y+qa_z\right)^2 + \left(v-pa_z+ra_x\right)^2 + \left(w+pa_y-qa_x\right)^2\right) \\
&+I_{xx}^g p^2 + I_{yy}^g q^2 + I_{zz}^g r^2 - 2I_{yz}^g qr - 2I_{xz}^g pr - 2I_{xy}^g pq \\
&-\bar{m}\left(\left(u-rb_y+qb_z\right)^2 + \left(v-pb_z+rb_x\right)^2 + \left(w+pb_y-qb_x\right)^2\right) \\
&-\bar{I}_{xx}^b p^2 - \bar{I}_{yy}^b q^2 - \bar{I}_{zz}^b r^2 + 2\bar{I}_{yz}^b qr + 2\bar{I}_{xz}^b pr + 2\bar{I}_{xy}^b pq \\
&+T_0
\end{aligned}
\tag{11}
$$

Where M_f is the mass of the volume of fluid circulating in the multiply connected space and the region is moving with velocity $\bar{u}_f, \bar{v}_f, \bar{w}_f$, T_0 is the kinetic energy of the undisterbed circulating fluid. The superscripts b and g refer to inertias derived with the axis origin at the centre of buoyancy and the centre of gravity respectively.

The symmetries in the equations can be exposed by using matrix notation so that the Lagrangian becomes,

$$
2T = 2T_0 + \begin{bmatrix} u_f \\ v_f \\ w_f \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \times \begin{bmatrix} M_f & 0 & 0 & 0 & 0 & 0 \\ 0 & M_f & 0 & 0 & 0 & 0 \\ 0 & 0 & M_f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} u_f \\ v_f \\ w_f \\ 0 \\ 0 \\ 0 \end{bmatrix} -
$$

$$\begin{bmatrix} u-u_f \\ v-v_f \\ w-w_f \\ p \\ q \\ r \end{bmatrix}^T \times \begin{bmatrix} X_{\dot u} & X_{\dot v} & X_{\dot w} & X_{\dot p} & X_{\dot q} & X_{\dot r} \\ Y_{\dot u} & Y_{\dot v} & Y_{\dot w} & Y_{\dot p} & Y_{\dot q} & Y_{\dot r} \\ Z_{\dot u} & Z_{\dot v} & Z_{\dot w} & Z_{\dot p} & Z_{\dot q} & Z_{\dot r} \\ L_{\dot u} & L_{\dot v} & L_{\dot w} & L_{\dot p} & L_{\dot q} & L_{\dot r} \\ M_{\dot u} & M_{\dot v} & M_{\dot w} & M_{\dot p} & M_{\dot q} & M_{\dot r} \\ N_{\dot u} & N_{\dot v} & N_{\dot w} & N_{\dot p} & N_{\dot q} & N_{\dot r} \end{bmatrix} \times \begin{bmatrix} u-u_f \\ v-v_f \\ w-w_f \\ p \\ q \\ r \end{bmatrix} + \qquad (12)$$

$$\begin{bmatrix} u-u_f \\ v-v_f \\ w-w_f \\ p \\ q \\ r \end{bmatrix}^T \times \begin{bmatrix} \bar m & 0 & 0 & 0 & \bar m b_z & -\bar m b_y \\ 0 & \bar m & 0 & -\bar m b_z & 0 & \bar m b_x \\ 0 & 0 & \bar m & \bar m b_y & -\bar m b_x & 0 \\ 0 & -\bar m b_z & \bar m b_y & \bar I^b_{xx} + \bar m\left(b_y^2 + b_z^2\right) & -\bar I^b_{xy} - \bar m b_x b_y & -\bar I^b_{xz} - \bar m b_x b_z \\ \bar m b_z & 0 & -\bar m b_x & -\bar I^b_{yx} - \bar m b_y b_x & \bar I^b_{yy} + \bar m\left(b_x^2 + b_z^2\right) & -\bar I^b_{yz} - \bar m b_y b_z \\ -\bar m b_y & \bar m b_x & 0 & -\bar I^b_{zx} - \bar m b_z b_x & -\bar I^b_{zy} - \bar m b_z b_x & \bar I^b_{zz} + \bar m\left(b_x^2 + b_y^2\right) \end{bmatrix} \times \begin{bmatrix} u-u_f \\ v-v_f \\ w-w_f \\ p \\ q \\ r \end{bmatrix} +$$

$$\begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix}^T \times \begin{bmatrix} m & 0 & 0 & 0 & m a_z & -m a_y \\ 0 & m & 0 & -m a_z & 0 & m a_x \\ 0 & 0 & m & m a_y & -m a_x & 0 \\ 0 & -m a_z & m a_y & I^g_{xx} + m\left(a_y^2 + a_z^2\right) & -I^g_{xy} - m a_x a_y & -I^g_{xz} - m a_x a_z \\ m a_z & 0 & -m a_x & -I^g_{yx} - m a_y a_x & I^g_{yy} + m\left(a_x^2 + a_z^2\right) & -I^g_{yz} - m a_y a_z \\ -m a_y & m a_x & 0 & -I^g_{zx} - m a_z a_x & -I^g_{zy} - m a_z a_x & I^g_{zz} + m\left(a_x^2 + a_y^2\right) \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix} -$$

$$\begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix}^T \times \begin{bmatrix} \bar m & 0 & 0 & 0 & \bar m b_z & -\bar m b_y \\ 0 & \bar m & 0 & -\bar m b_z & 0 & \bar m b_x \\ 0 & 0 & \bar m & \bar m b_y & -\bar m b_x & 0 \\ 0 & -\bar m b_z & \bar m b_y & \bar I^b_{xx} + \bar m\left(b_y^2 + b_z^2\right) & -\bar I^b_{xy} - \bar m b_x b_y & -\bar I^b_{xz} - \bar m b_x b_z \\ \bar m b_z & 0 & -\bar m b_x & -\bar I^b_{yx} - \bar m b_y b_x & \bar I^b_{yy} + \bar m\left(b_x^2 + b_z^2\right) & -\bar I^b_{yz} - \bar m b_y b_z \\ -\bar m b_y & \bar m b_x & 0 & -\bar I^b_{zx} - \bar m b_z b_x & -\bar I^b_{zy} - \bar m b_z b_x & \bar I^b_{zz} + \bar m\left(b_x^2 + b_y^2\right) \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix}$$

The terms added to the products and moments of inertia are, by the parallel axis theorem those required to give the corresponding inertias about the co-ordinate origin. As a result the superscripts can be omitted and the extra terms dropped, and all inertia's will then be taken to be referenced to the body axis origin.

The vehicle equations of motion with the extra terms for the fluids motion can be derived using Lagranges equations. However the above Lagrangian is not in terms of generalised co-ordinates and their velocities and so the convential equations cannot be used.

This problem belongs to a class that can be solved by the use of 'quasi co-ordinates'. This general class of problems is described by Whittaker [7] and Meirovitch [16].

The true co-ordinates of the current problem are

$$\mathbf{q} = (N, E, D, \phi, \theta, \psi, N_f, E_f, D_f) \tag{13}$$

while the rates of change of the quasi co-ordinates are

$$\omega = (u, v, w, p, q, r, u_f, v_f, w_f). \tag{14}$$

Lagrange's equations in matrix form are,

$$\dot{\mathbf{T}}_{\dot{\mathbf{q}}} - \mathbf{T}_{\mathbf{q}} = \mathbf{Q} \tag{15}$$

where

$$\mathbf{T}_{\mathbf{q}} = \begin{bmatrix} \frac{\partial T}{\partial N} & \frac{\partial T}{\partial E} & \frac{\partial T}{\partial D} & \frac{\partial T}{\partial \phi} & \frac{\partial T}{\partial \theta} & \frac{\partial T}{\partial \psi} & \frac{\partial T}{\partial N_f} & \frac{\partial T}{\partial E_f} & \frac{\partial T}{\partial D_f} \end{bmatrix}^T \tag{16}$$

where Q represents the generalised forces. To convert this to quasi co-ordinates first express the rates of change of the quasi co-ordinates as linear combinations of the rates of change of the true co-ordinates,

$$\omega = [\alpha]^T \dot{\mathbf{q}} \tag{17}$$

then the inverse relation is

$$\dot{\mathbf{q}} = [\beta]\omega \tag{18}$$

and Lagrange's equations can then be written as [7],[16],

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_r}\right) + \sum_s \sum_l \gamma_{rsl}\omega_l \frac{\partial T}{\partial \omega_s} - \frac{\partial T}{\partial \pi_r} = \Pi_r \qquad (r = 1,2,\ldots,n) \tag{19}$$

where the Π_r are the generalised forces associated with the quasi co-ordinates π_r and

$$\gamma_{rsl} = \sum_k \sum_m \beta_{kr}\beta_{ml}\left(\frac{\partial \alpha_{ks}}{\partial q_m} - \frac{\partial \alpha_{ms}}{\partial q_k}\right) \tag{20}$$

For the present problem we have,

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{N} \\ \dot{E} \\ \dot{D} \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & \sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{N}_f \\ \dot{E}_f \\ \dot{D}_f \end{bmatrix} + \begin{bmatrix} u_c \\ v_c \\ w_c \end{bmatrix} \quad (23)$$

Where $[u_c \quad v_c \quad w_c]^T$ are the steady circulating velocities.

The above equations define the $[\alpha]$ matrix, and after considerable algebra the indicated transformations yield the Lagrangian equations of motion.

$$\begin{aligned}
X &= \frac{d}{dt}\left(\frac{\partial T}{\partial u}\right) - r\frac{\partial T}{\partial v} + q\frac{\partial T}{\partial w} - \frac{\partial T}{\partial x} \\
Y &= \frac{d}{dt}\left(\frac{\partial T}{\partial v}\right) - p\frac{\partial T}{\partial w} + r\frac{\partial T}{\partial u} - \frac{\partial T}{\partial y} \\
Z &= \frac{d}{dt}\left(\frac{\partial T}{\partial w}\right) - q\frac{\partial T}{\partial u} + p\frac{\partial T}{\partial v} - \frac{\partial T}{\partial z} \\
L &= \frac{d}{dt}\left(\frac{\partial T}{\partial p}\right) - w\frac{\partial T}{\partial v} + v\frac{\partial T}{\partial w} - w_f\frac{\partial T}{\partial v_f} + v_f\frac{\partial T}{\partial w_f} - r\frac{\partial T}{\partial q} + q\frac{\partial T}{\partial r} \\
M &= \frac{d}{dt}\left(\frac{\partial T}{\partial q}\right) - u\frac{\partial T}{\partial w} + w\frac{\partial T}{\partial u} - u_f\frac{\partial T}{\partial w_f} + w_f\frac{\partial T}{\partial u_f} - p\frac{\partial T}{\partial r} + r\frac{\partial T}{\partial p} \\
N &= \frac{d}{dt}\left(\frac{\partial T}{\partial r}\right) - v\frac{\partial T}{\partial u} + u\frac{\partial T}{\partial v} - v_f\frac{\partial T}{\partial u_f} + u_f\frac{\partial T}{\partial v_f} - q\frac{\partial T}{\partial p} + p\frac{\partial T}{\partial q}
\end{aligned} \quad (24)$$

These equations are similar to Lamb's [1] but with fluid motion terms included.

For subsequent analysis equations (24) are written more conveniently in matrix form as,

$$\dot{\mathbf{T}}_{\mathbf{x}} = \mathbf{F} - (\mathbf{P} + \mathbf{W})\mathbf{T}_{\mathbf{x}} - \mathbf{W}_{\mathbf{f}}\mathbf{T}_{\mathbf{f}} - \mathbf{T}_{\mathbf{s}} \quad (25)$$

where

$$\begin{aligned}
\mathbf{T}_{\mathbf{x}} &= \left[\frac{\partial T}{\partial u} \quad \frac{\partial T}{\partial v} \quad \frac{\partial T}{\partial w} \quad \frac{\partial T}{\partial p} \quad \frac{\partial T}{\partial q} \quad \frac{\partial T}{\partial r}\right]^T \\
\mathbf{T}_{s} &= \left[\frac{\partial T}{\partial x} \quad \frac{\partial T}{\partial y} \quad \frac{\partial T}{\partial z} \quad 0 \quad 0 \quad 0\right]^T
\end{aligned} \quad (26)$$

$$\mathbf{F} = [\mathbf{X} \quad \mathbf{Y} \quad \mathbf{Z} \quad \mathbf{L} \quad \mathbf{M} \quad \mathbf{N}]^T \quad (27)$$

$$\mathbf{P} = \begin{bmatrix} 0 & -r & q & 0 & 0 & 0 \\ r & 0 & -p & 0 & 0 & 0 \\ -q & p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -r & q \\ 0 & 0 & 0 & r & 0 & -p \\ 0 & 0 & 0 & -q & p & 0 \end{bmatrix} \tag{28}$$

$$\mathbf{W} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -w & v & 0 & 0 & 0 \\ w & 0 & -u & 0 & 0 & 0 \\ -v & u & 0 & 0 & 0 & 0 \end{bmatrix} \tag{29}$$

$$\mathbf{W_f} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -w_f & v_f & 0 & 0 & 0 \\ w_f & 0 & -u_f & 0 & 0 & 0 \\ -v_f & u_f & 0 & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

$$\mathbf{T_f} = \begin{bmatrix} \frac{\partial T}{\partial u_f} & \frac{\partial T}{\partial v_f} & \frac{\partial T}{\partial w_f} & 0 & 0 & 0 \end{bmatrix}^T \tag{31}$$

The differentiation gives,

$$\frac{\partial T}{\partial u} = -X_{\dot u}\left(u-u_f\right) - X_{\dot w}\left(w-w_f\right) - X_{\dot v}\left(v-v_f\right) - X_{\dot p}p - X_{\dot q}q - X_{\dot r}r - \overline{m}u_f + m\left(u - a_y r + a_z q\right)$$

$$\frac{\partial T}{\partial v} = -Y_{\dot v}\left(v-v_f\right) - Y_{\dot w}\left(w-w_f\right) - Y_{\dot u}\left(u-u_f\right) - Y_{\dot p}p - Y_{\dot q}q - Y_{\dot r}r - \overline{m}v_f + m\left(v - a_z p + a_x r\right)$$

$$\frac{\partial T}{\partial w} = -Z_{\dot w}\left(w-w_f\right) - Z_{\dot v}\left(v-v_f\right) - Z_{\dot u}\left(u-u_f\right) - Z_{\dot p}p - Z_{\dot q}q - Z_{\dot r}r - \overline{m}w_f + m\left(w + a_y p - a_x q\right)$$

$$\frac{\partial T}{\partial p} = (I_{xx} - L_{\dot p})p + (-I_{xz} - L_{\dot r})r + (-I_{xy} - L_{\dot q})q$$
$$-\left(L_{\dot u}\left(u-u_f\right) + L_{\dot v}\left(v-v_f\right) + L_{\dot w}\left(w-w_f\right)\right) + \overline{m}b_z v_f - \overline{m}b_y w_f - m a_z v + m a_y w$$

$$\frac{\partial T}{\partial q} = (-I_{xy} - M_{\dot{p}})p + (I_{yy} - M_{\dot{q}})q + (-I_{yz} - M_{\dot{r}})r$$
$$-(M_{\dot{u}}(u - u_f) + M_{\dot{v}}(v - v_f) + M_{\dot{w}}(w - w_f)) - \overline{m}b_z u_f + \overline{m}b_x w_f + ma_z u - ma_x w$$

$$\frac{\partial T}{\partial r} = \left(-I_{xz} - N_{\dot{p}}\right)p + \left(-I_{yz} - N_{\dot{q}}\right)q + \left(I_{zz} - N_{\dot{r}}\right)r$$
$$-\left(N_{\dot{u}}(u - u_f) + N_{\dot{v}}(v - v_f) + N_{\dot{w}}(w - w_f)\right) + \overline{m}b_y u_f - \overline{m}b_x v_f - ma_y u + ma_x v \tag{32}$$

$$\frac{\partial T}{\partial u_f} = X_{\dot{p}}p + \left(-\overline{m}b_z - X_{\dot{q}}\right)q + \left(\overline{m}b_y + X_{\dot{r}}\right)r + \left(X_{\dot{u}}\left(u - u_f\right) + X_{\dot{v}}\left(v - v_f\right) + X_{\dot{w}}\left(w - w_f\right)\right)$$
$$-\overline{m}u + \left(\overline{m} + M_f\right)u_f$$

$$\frac{\partial T}{\partial v_f} = \left(\overline{m}b_z + Y_{\dot{p}}\right)p + Y_{\dot{q}}q + \left(-\overline{m}b_x + Y_{\dot{r}}\right)r + \left(Y_{\dot{u}}\left(u - u_f\right) + Y_{\dot{v}}\left(v - v_f\right) + Y_{\dot{w}}\left(w - w_f\right)\right)$$
$$-\overline{m}v + \left(\overline{m} + M_f\right)v_f$$

$$\frac{\partial T}{\partial w_f} = \left(-\overline{m}b_y + Z_{\dot{p}}\right)p + \left(\overline{m}b_x + Z_{\dot{q}}\right)q + Z_{\dot{r}}r + \left(Z_{\dot{u}}\left(u - u_f\right) + Z_{\dot{v}}\left(v - v_f\right) + Z_{\dot{w}}\left(w - w_f\right)\right)$$
$$-\overline{m}w + \left(\overline{m} + M_f\right)w_f$$

$$\frac{\partial T}{\partial x} =$$

$$\frac{\partial T}{\partial y} =$$

$$\frac{\partial T}{\partial z} =$$

and again more convenient matrix forms of these are,

$$\mathbf{T_x} = \mathbf{M_r x_r} + \mathbf{M_i x} - \overline{\mathbf{M}}_\mathbf{i}\mathbf{x_f} = \left(\mathbf{M_r} + \mathbf{M_i}\right)\mathbf{x} - \left(\mathbf{M_r} + \overline{\mathbf{M}}_\mathbf{i}\right)\mathbf{x_f}$$
$$\mathbf{T_f} = -\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\left(\mathbf{M_r} + \overline{\mathbf{M}}_\mathbf{i}\right)\mathbf{x_r} + M_\mathbf{f}\mathbf{x_f} \tag{33}$$
$$\text{and } \mathbf{x_r} = \mathbf{x} - \mathbf{x_f} \tag{34}$$

where

$$\mathbf{M_r} = \begin{bmatrix} -X_{\dot u} & -X_{\dot v} & -X_{\dot w} & -X_{\dot p} & -X_{\dot q} & -X_{\dot r} \\ -Y_{\dot u} & -Y_{\dot v} & -Y_{\dot w} & -Y_{\dot p} & -Y_{\dot q} & -Y_{\dot r} \\ -Z_{\dot u} & -Z_{\dot v} & -Z_{\dot w} & -Z_{\dot p} & -Z_{\dot q} & -Z_{\dot r} \\ -L_{\dot u} & -L_{\dot v} & -L_{\dot w} & -L_{\dot p} & -L_{\dot q} & -L_{\dot r} \\ -M_{\dot u} & -M_{\dot v} & -M_{\dot w} & -M_{\dot p} & -M_{\dot q} & -M_{\dot r} \\ -N_{\dot u} & -N_{\dot v} & -N_{\dot w} & -N_{\dot p} & -N_{\dot q} & -N_{\dot r} \end{bmatrix} \tag{35}$$

$$\mathbf{x_r} = \begin{bmatrix} u-u_f & v-v_f & w-w_f & p & q & r \end{bmatrix}^T \tag{36}$$

$$\mathbf{M_i} = \begin{bmatrix} m & 0 & 0 & 0 & ma_z & -ma_y \\ 0 & m & 0 & -ma_z & 0 & ma_x \\ 0 & 0 & m & ma_y & -ma_x & 0 \\ 0 & -ma_z & ma_y & I_{xx} & -I_{xy} & -I_{xz} \\ ma_z & 0 & -ma_x & -I_{xy} & I_{yy} & -I_{yz} \\ -ma_y & ma_x & 0 & -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \tag{37}$$

$$\mathbf{x} = \begin{bmatrix} u & v & w & p & q & r \end{bmatrix} \tag{38}$$

$$\overline{\mathbf{M}}_\mathbf{i} = \begin{bmatrix} \overline{m} & 0 & 0 & 0 & \overline{m}b_z & -\overline{m}b_y \\ 0 & \overline{m} & 0 & -\overline{m}b_z & 0 & \overline{m}b_x \\ 0 & 0 & \overline{m} & \overline{m}b_y & -\overline{m}b_x & 0 \\ 0 & -\overline{m}b_z & \overline{m}b_y & 0 & 0 & 0 \\ \overline{m}b_z & 0 & -\overline{m}b_x & 0 & 0 & 0 \\ -\overline{m}b_y & \overline{m}b_x & 0 & 0 & 0 & 0 \end{bmatrix} \tag{39}$$

$$\mathbf{x_f} = \begin{bmatrix} u_f & v_f & w_f & 0 & 0 & 0 \end{bmatrix}^T \tag{40}$$

Substituting in the equations of motion gives

$$\begin{aligned} \dot{\mathbf{T}}_\mathbf{x} &= \mathbf{M_r\dot{x}_r} + \mathbf{M_i\dot{x}} - \overline{\mathbf{M}}_\mathbf{i}\mathbf{\dot{x}_f} = (\mathbf{M_r} + \mathbf{M_i})\mathbf{\dot{x}} - (\mathbf{M_r} + \overline{\mathbf{M}}_\mathbf{i})\mathbf{\dot{x}_f} \\ &= \mathbf{F} - (\mathbf{P}+\mathbf{W})(\mathbf{M_r x_r} + \mathbf{M_i x} - \overline{\mathbf{M}}_\mathbf{i}\mathbf{x_f}) + \mathbf{W_f}(\mathbf{M_r} + \overline{\mathbf{M}}_\mathbf{i})\mathbf{x_r} \\ &= \mathbf{F} - (\mathbf{P}+\mathbf{W})\{\mathbf{M_r x_r} + \mathbf{M_i x} - \overline{\mathbf{M}}_\mathbf{i}\mathbf{x} + \overline{\mathbf{M}}_\mathbf{i}\mathbf{x_r}\} + \mathbf{W_f}(\mathbf{M_r} + \overline{\mathbf{M}}_\mathbf{i})\mathbf{x_r} \\ &= \mathbf{F} - (\mathbf{P}+\mathbf{W})(\mathbf{M_i} - \overline{\mathbf{M}}_\mathbf{i})\mathbf{x} - (\mathbf{P}+\mathbf{W_r})(\mathbf{M_r} + \overline{\mathbf{M}}_\mathbf{i})\mathbf{x_r} \end{aligned} \tag{41}$$

where $\mathbf{W_r} = \mathbf{W} - \mathbf{W_f}$

3.2 Viscous Effects

The term $(\mathbf{P}+\mathbf{W_r})(\mathbf{M_r} + \overline{\mathbf{M}}_\mathbf{i})\mathbf{x_r}$ in the above equation is a function of the relative velocities only and can therefore be absorbed into the vector **A** as the perfect fluid component of the aerodynamic forces and moments due to the relative velocity between the fluid and the vehicle. In many cases the elements of the vector **A** will be derived empirically from wind tunnel or tank facilities and will include the perfect fluid effects. The acceleration dependant terms or 'added masses' arise from the work done in accelerating the perfect fluid, in a real fluid additional acceleration effects come into play such as the increase in vorticity and its convection past tailplanes etc. It is assumed that all such effects are added into the corresponding perfect fluid added mass terms. This is a reasonable assumption for streamlined vehicles supported by bouyancy, but is less exact for vehicles with substantial lift such as aircraft. The unsteady aspects of lift generation and the nature of the $\dot{\alpha}$ 'derivatives' is discussed by Etkin [14] and Hancock [15] and whilst the above assumption is not stricly correct, it is still a working approximation for many aircraft situations.

The perfect fluid terms that may be absorbed into the vector **A** are,

$$\left\{\begin{array}{ccc}
-rY_{\dot{u}}+qZ_{\dot{u}} & -rY_{\dot{v}}+qZ_{\dot{v}} & -rY_{\dot{w}}+qZ_{\dot{w}} \\
rX_{\dot{u}}-pZ_{\dot{u}} & rX_{\dot{v}}-pZ_{\dot{v}} & rX_{\dot{w}}-pZ_{\dot{w}} \\
-qX_{\dot{u}}+pY_{\dot{u}} & -qX_{\dot{v}}+pY_{\dot{v}} & -qX_{\dot{w}}+pY_{\dot{w}} \\
-w_rY_{\dot{u}}+v_rZ_{\dot{u}}-rM_{\dot{u}}+qN_{\dot{u}} & -w_rY_{\dot{v}}+v_rZ_{\dot{v}}-rM_{\dot{v}}+qN_{\dot{v}} & -w_rY_{\dot{w}}+v_rZ_{\dot{w}}-rM_{\dot{w}}+qN_{\dot{w}} \\
w_rX_{\dot{u}}-u_rZ_{\dot{u}}+rL_{\dot{u}}-pN_{\dot{u}} & w_rX_{\dot{v}}-u_rZ_{\dot{v}}+rL_{\dot{v}}-pN_{\dot{v}} & w_rX_{\dot{w}}-u_rZ_{\dot{w}}+rL_{\dot{w}}-pN_{\dot{w}} \\
-v_rX_{\dot{u}}+u_rY_{\dot{u}}-qL_{\dot{u}}+pM_{\dot{u}} & -v_rX_{\dot{v}}+u_rY_{\dot{v}}-qL_{\dot{v}}+pM_{\dot{v}} & -v_rX_{\dot{w}}+u_rY_{\dot{w}}-qL_{\dot{w}}+pM_{\dot{w}}
\end{array}\right.$$

$$\left.\begin{array}{ccc}
-rY_{\dot{p}}+qZ_{\dot{p}} & -rY_{\dot{q}}+qZ_{\dot{q}} & -rY_{\dot{r}}+qZ_{\dot{r}} \\
rX_{\dot{p}}-pZ_{\dot{p}} & rX_{\dot{q}}-pZ_{\dot{q}} & rX_{\dot{r}}-pZ_{\dot{r}} \\
-qX_{\dot{p}}+pY_{\dot{p}} & -qX_{\dot{q}}+pY_{\dot{q}} & -qX_{\dot{r}}+pY_{\dot{r}} \\
-w_rY_{\dot{p}}+v_rZ_{\dot{p}}-rM_{\dot{p}}+qN_{\dot{p}} & -w_rY_{\dot{q}}+v_rZ_{\dot{q}}-rM_{\dot{q}}+qN_{\dot{q}} & -w_rY_{\dot{r}}+v_rZ_{\dot{r}}-rM_{\dot{r}}+qN_{\dot{r}} \\
w_rX_{\dot{p}}-u_rZ_{\dot{p}}+rL_{\dot{p}}-pN_{\dot{p}} & w_rX_{\dot{q}}-u_rZ_{\dot{q}}+rL_{\dot{q}}-pN_{\dot{q}} & w_rX_{\dot{r}}-u_rZ_{\dot{r}}+rL_{\dot{r}}-pN_{\dot{r}} \\
-v_rX_{\dot{p}}+u_rY_{\dot{p}}-qL_{\dot{p}}+pM_{\dot{p}} & -v_rX_{\dot{q}}+u_rY_{\dot{q}}-qL_{\dot{q}}+pM_{\dot{q}} & -v_rX_{\dot{r}}+u_rY_{\dot{r}}-qL_{\dot{r}}+pM_{\dot{r}}
\end{array}\right\}\begin{bmatrix} u_r \\ v_r \\ w_r \\ p \\ q \\ r \end{bmatrix} \tag{42}$$

$$-\begin{bmatrix}
0 & -r\bar{m} & q\bar{m} & r\bar{m}b_z+q\bar{m}b_y & -q\bar{m}b_x & -r\bar{m}b_x \\
r\bar{m} & 0 & -p\bar{m} & -p\bar{m}b_y & r\bar{m}b_z+p\bar{m}b_x & -r\bar{m}b_y \\
-q\bar{m} & p\bar{m} & 0 & -p\bar{m}b_z & -q\bar{m}b_z & q\bar{m}b_y+p\bar{m}b_x \\
-r\bar{m}b_z-q\bar{m}b_y & 0 & 0 & w_r\bar{m}b_z+v_r\bar{m}b_y & 0 & 0 \\
0 & -r\bar{m}b_z-p\bar{m}b_x & 0 & 0 & w_r\bar{m}b_z+u_r\bar{m}b_x & 0 \\
0 & 0 & -q\bar{m}b_y+p\bar{m}b_x & 0 & 0 & v_r\bar{m}b_y+u_r\bar{m}b_x
\end{bmatrix}\begin{bmatrix} u_r \\ v_r \\ w_r \\ p \\ q \\ r \end{bmatrix}$$

Depending upon how the elements of the **A** vector are determined, some or all of the above terms may already be included and care should be exercised that terms are not omitted nor 'double accounted' for. The risk of double accounting can be seem by considering an axisymmetric body in a turn with w = p = q = 0. Then the perfect fluid yawing moment is from Equation (42),

$$N = -v_rX_{\dot{u}}u_r + u_rY_{\dot{v}}v_r$$

Now if the vehicle is sidesliping,

$$\sin\beta = \frac{v_r}{V_{tot}}$$
$$\cos\beta = \frac{u_r}{V_{tot}}$$

Hence,

$$N = \frac{V_{tot}^2}{2}\left(Y_{\dot{v}} - X_{\dot{u}}\right)\sin 2\beta$$

i.e. this is the classic Monk moment [17], and this may be already in the vector **A**.

With all the terms absorbed the equations of motion become,

$$\begin{aligned}
&(\mathbf{M}_r + \mathbf{M}_1)\dot{\mathbf{x}} = -(\mathbf{P}+\mathbf{W})(\mathbf{M}_1 - \overline{\mathbf{M}}_1)\mathbf{x} + (\mathbf{M}_r + \overline{\mathbf{M}}_1)\dot{\mathbf{x}}_f + \mathbf{A} + \mathbf{F} \\
&\text{or} \\
&\mathbf{M}\dot{\mathbf{x}} = \mathbf{F}_d + \mathbf{F}_f + \mathbf{A} + \mathbf{F} \\
&\text{where} \\
&\mathbf{M} = (\mathbf{M}_r + \mathbf{M}_1) \\
&\mathbf{F}_f = (\mathbf{M}_r + \overline{\mathbf{M}}_1)\dot{\mathbf{x}}_f
\end{aligned} \tag{43}$$

$$\mathbf{F}_d = -(\mathbf{P}+\mathbf{W})(\mathbf{M}_1 - \overline{\mathbf{M}}_1)\mathbf{x} = \begin{bmatrix}
(m-\overline{m})(rv - qw) + m[a_x(q^2+r^2) - a_y pq - a_z rp] \\
(m-\overline{m})(pw - ru) + m[a_y(p^2+r^2) - a_x pq - a_z rq] \\
(m-\overline{m})(qu - pv) + m[a_z(q^2+p^2) - a_y rq - a_x rp] \\
\begin{aligned} &-(I_{zz}-I_{yy})rq + I_{yz}(q^2-r^2) + I_{zx}pq - I_{xy}pr \\ &+m[-a_y(vp-qu) + a_z(ur-pw)] \\ &-\overline{m}[-b_y(vp-qu) + b_z(ur-pw)] \end{aligned} \\
\begin{aligned} &-(I_{xx}-I_{zz})pr - I_{yz}pq + I_{zx}(r^2-p^2) + I_{xy}qr \\ &+m[a_x(vp-qu) - a_z(wq-rv)] \\ &-\overline{m}[b_x(vp-qu) - b_z(wq-rv)] \end{aligned} \\
\begin{aligned} &-(I_{yy}-I_{xx})qp + I_{yz}pr - I_{zx}qr + I_{xy}(p^2-q^2) \\ &+m[-a_x(ur-pw) + a_y(wq-rv)] \\ &-\overline{m}[-b_x(ur-pw) + b_y(wq-rv)] \end{aligned}
\end{bmatrix}$$

These are more acceptable from a physical point of view since the fluid motion terms now only depend upon the fluids inertial acceleration. Giving the body the same mass and inertias as the displaced fluid results in the relative acceleration becoming zero as would be expected. If the mass of displaced fluid is negligible they revert to the conventional aircraft equations of motion.

3.3 Gust Penetration Effects

In the above, all relative velocity terms were moved to the aerodynamics vector. If the vehicle is flying in a steady but non-uniform airstream there will still be varying relative velocity components due to the bodies translation through the fluid. Using Taylors theorem, the fluid velocities in the vicinity of a point x_0, y_0, z_0 can be described by the nine velocity gradients,

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \tag{44}$$

and this can be split, Prandtl [12], into symmetric and anti-symmetric parts, the former representing strain and the later representing vorticity. As a result the velocity gradients in the fluid can be treated as 'effective rotation rates' since they produce velocity distributions identical to those due to rotation,

$$\begin{bmatrix} p_f & q_f & r_f \end{bmatrix}^T = \begin{bmatrix} \frac{\partial w_f}{\partial y} - \frac{\partial v_f}{\partial z} & \frac{\partial u_f}{\partial z} - \frac{\partial w_f}{\partial x} & \frac{\partial v_f}{\partial x} - \frac{\partial u_f}{\partial y} \end{bmatrix}^T = \frac{1}{2} curl[\mathbf{V}_f] \tag{45}$$

In an aircraft the vertical dimensions are small compared to the tail arm and the span so that the variation with z can be neglected, so that,

$$\begin{bmatrix} p_f & q_f & r_f \end{bmatrix}^T = \begin{bmatrix} \frac{\partial w_f}{\partial y}, & -\frac{\partial w_f}{\partial x}, & \frac{\partial v_f}{\partial x} - \frac{\partial u_f}{\partial y} \end{bmatrix}^T \tag{46}$$

This is the linear field approximation used in [9].

If the vehicle has a velocity U relative to the fluid (along the x-axis),

$$\begin{bmatrix} p_f & q_f & r_f \end{bmatrix}^T = \begin{bmatrix} \frac{\partial w_f}{\partial y}, & -\frac{\dot{w}^*_f}{U}, & \frac{\dot{v}^*_f}{U} - \frac{\partial u_f}{\partial y} \end{bmatrix}^T \tag{47}$$

Where the starred terms are the apparent rate of change due to the motion of the turbulence field past the vehicle i.e. NOT the inertial accelerations of the undisturbed fluid. The above expression is added to the vehicle rotation rates so as to give the effective relative angular velocity for use in the aerodynamic calculations of **A** (it being solely a function of the relative velocities). This avoids one of the problems with the formulation used in [2], it had fluid velocity terms in the fluid motion vector and it was not clear how the 'effective' rotation rates should be included in the equations.

The distinction between the real and apparent fluid accelerations is important since in some derivations [13], the use of the frozen turbulence approximation leads to them being lumped together so that $Z_{\dot{w}_f} = Z_{\dot{w}} - Z_q$. In the case of a moving sea or non frozen turbulence the above treatment keeps them distinct.

More extensive gust or sea state models can be included at this point depending upon the application.

4.0 Small Perturbation Equations

The equations of motion are,

$$\begin{aligned}\mathbf{M\dot{x}} &= -(\mathbf{P}+\mathbf{W})(\mathbf{M_i}-\overline{\mathbf{M}}_\mathbf{i})\mathbf{x}+(\mathbf{M_r}+\overline{\mathbf{M}}_\mathbf{i})\mathbf{\dot{x}_f}+\mathbf{A}\{\mathbf{x}-\mathbf{x^*_f}\}+\mathbf{F} \\ \mathbf{x^*_f} &= [u_f, v_f, w_f, p_f, q_f, r_f]^T\end{aligned} \tag{48}$$

The braces {} indicating 'function of'.

We will now consider small perturbations about a steady flight condition. Let the vehicle state vector be given by,

$$\mathbf{x} = \mathbf{x}_0 + \delta\mathbf{x} \tag{49}$$

If we use stability axes the steady state is given by

$$\mathbf{x}_0^T = [U_0, \ 0, \ 0, \ 0, \ 0, \ 0] \tag{50}$$

The corresponding vectors for the fluid velocities are,

$$\begin{aligned}\mathbf{x_f} &= \mathbf{x_{f0}} + \delta\mathbf{x_f} \\ \mathbf{x^*_f} &= \mathbf{x^*_{f0}} + \delta\mathbf{x^*_f}\end{aligned} \tag{51}$$

and the steady wind components are given by,

$$\begin{aligned}\mathbf{x_{f0}} &= [u_{f0}, \ v_{f0}, \ w_{f0}, \ 0, \ 0, \ 0] \\ \mathbf{x^*_{f0}} &= [u_{f0}, \ v_{f0}, \ w_{f0}, \ p_{f0}, \ q_{f0}, \ r_{f0}]\end{aligned} \tag{52}$$

For small perturbations,

$$\mathbf{A}\{\mathbf{x}-\mathbf{x^*_f}\} = \mathbf{A}\{\mathbf{x}_0-\mathbf{x^*_{f0}}\}+\mathbf{A_e}(\delta\mathbf{x}-\delta\mathbf{x^*_f}) \tag{53}$$

where $\mathbf{A_e}$ is the small perturbation (aerodynamic derivative) matrix for the steady state flight condition.

$$\mathbf{A_e} = \begin{bmatrix} X_u & X_v & X_w & X_p & X_q & X_r \\ Y_u & Y_v & Y_w & Y_p & Y_q & Y_r \\ Z_u & Z_v & Z_w & Z_p & Z_q & Z_r \\ L_u & L_v & L_w & L_p & L_q & L_r \\ M_u & M_v & M_w & M_p & M_q & M_r \\ N_u & N_v & N_w & N_p & N_q & N_r \end{bmatrix} \tag{54}$$

It must be remembered that the $\mathbf{A_e}$ matrix contains not only the 'conventional' aerodynamic derivatives but it also contains the small perturbation part of the perfect fluid relative velocity terms that are given in equation (42) and were absorbed into the

$\mathbf{A}$ matrix. If we take the added mass matrix to be diagonal for simplicity, these components are,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & (\overline{m}+Z_{\dot{w}})w_{r0} & -(\overline{m}+Y_{\dot{v}})v_{r0} \\ 0 & 0 & 0 & -(\overline{m}+Z_{\dot{w}})w_{r0} & 0 & (\overline{m}+X_{\dot{u}})u_{r0} \\ 0 & 0 & 0 & (\overline{m}+Y_{\dot{v}})v_{r0} & -(\overline{m}+X_{\dot{u}})u_{r0} & 0 \\ 0 & (-Y_{\dot{v}}+Z_{\dot{w}})w_{r0} & (-Y_{\dot{v}}+Z_{\dot{w}})v_{r0} & \overline{m}(w_{r0}b_z+v_{r0}b_y) & -\overline{m}u_{r0}b_y & -\overline{m}u_{r0}b_z \\ (X_{\dot{u}}-Z_{\dot{w}})w_{r0} & 0 & (X_{\dot{u}}-Z_{\dot{w}})u_{r0} & -\overline{m}v_{r0}b_x & \overline{m}(w_{r0}b_z+u_{r0}b_x) & -\overline{m}v_{r0}b_z \\ (-X_{\dot{u}}+Y_{\dot{v}})v_{r0} & (-X_{\dot{u}}+Y_{\dot{v}})u_{r0} & 0 & -\overline{m}w_{r0}b_x & -\overline{m}w_{r0}b_y & \overline{m}(v_{r0}b_y+u_{r0}b_x) \end{bmatrix}$$

Under steady state we can show that,

$$\mathbf{F} = -\mathbf{A}_e\{\mathbf{x}_0 - \mathbf{x}^*_{f0}\} \tag{55}$$

As a result the small perturbation equations become,

$$\begin{aligned}(\mathbf{M}_r + \mathbf{M}_i)\delta\dot{\mathbf{x}} &= (\mathbf{M}_r + \overline{\mathbf{M}}_i)\delta\dot{\mathbf{x}}_f - (\mathbf{P}_0 + \mathbf{W}_0)(\mathbf{M}_i - \overline{\mathbf{M}}_i)\delta\mathbf{x} \\ &\quad - (\delta\mathbf{P} + \delta\mathbf{W})(\mathbf{M}_i - \overline{\mathbf{M}}_i)\mathbf{x}_0 + \mathbf{A}_e\delta\mathbf{x} - \mathbf{A}_e\delta\mathbf{x}^*_f + \delta\mathbf{F}\end{aligned} \tag{56}$$

and

$$-(\mathbf{P}_0 + \mathbf{W}_0)(\mathbf{M}_i - \overline{\mathbf{M}}_i)\delta\mathbf{x} - (\delta\mathbf{P} + \delta\mathbf{W})(\mathbf{M}_i - \overline{\mathbf{M}}_i)\mathbf{x}_0 =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (m-\overline{m})U_0 \\ 0 & 0 & 0 & 0 & -(m-\overline{m})U_0 & 0 \\ 0 & 0 & 0 & 0 & -(ma_y - \overline{m}b_y)U_0 & -(ma_z - \overline{m}b_z)U_0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta u \\ \delta v \\ \delta w \\ \delta p \\ \delta q \\ \delta r \end{bmatrix} \tag{57}$$

and if the other external forces and moments are due to just gravity

$$\delta\mathbf{F} = \begin{bmatrix} 0 & -(m-\overline{m})g \\ (m-\overline{m})g & 0 \\ 0 & 0 \\ -(a_z m - b_z\overline{m})g & 0 \\ 0 & -(a_z m - b_z\overline{m})g \\ (a_x m - b_x\overline{m})g & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \theta \end{bmatrix} \tag{58}$$

The equations become,

$$(\mathbf{M_r}+\mathbf{M_i})\delta\dot{\mathbf{x}} = (\mathbf{M_r}+\overline{\mathbf{M}}_i)\delta\dot{\mathbf{x}}_f - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (m-\overline{m})U_0 \\ 0 & 0 & 0 & 0 & -(m-\overline{m})U_0 & 0 \\ 0 & 0 & 0 & 0 & -(ma_y-\overline{m}b_y)U_0 & -(ma_z-\overline{m}b_z)U_0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix}\delta u\\ \delta v\\ \delta w\\ \delta p\\ \delta q\\ \delta r\end{bmatrix} \tag{59}$$

$$+\mathbf{A_e}\delta\mathbf{x} - \mathbf{A_e}\delta\mathbf{x}_f + \begin{bmatrix} 0 & -(m-\overline{m})g \\ (m-\overline{m})g & 0 \\ 0 & 0 \\ -(a_zm-b_z\overline{m})g & 0 \\ 0 & -(a_zm-b_z\overline{m})g \\ (a_xm-b_x\overline{m})g & 0 \end{bmatrix}\begin{bmatrix}\phi\\ \theta\end{bmatrix}$$

For conventional aircraft the displaced mass can be ignored, and making the usual assumptions regarding the added masses and the stability derivatives the above equations become the usual small perturbation equations used for aircraft.

Longitudinal Equations

$$\left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & Z_{\dot{w}} & 0 \\ 0 & M_{\dot{w}} & 0 \end{bmatrix} + \begin{bmatrix} m & 0 & a_zm \\ 0 & m & -a_xm \\ a_zm & -a_xm & I_{yy} \end{bmatrix}\right\}\begin{bmatrix}\dot{u}\\ \dot{w}\\ \dot{q}\end{bmatrix} = \begin{bmatrix}X\\ Z\\ M\end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -Z_{\dot{w}} & 0 \\ 0 & -M_{\dot{w}} & 0 \end{bmatrix}\begin{bmatrix}\dot{u}_f\\ \dot{w}_f\\ 0\end{bmatrix} + \begin{bmatrix}0\\ mU_0\\ 0\end{bmatrix}q$$
$$+\begin{bmatrix} X_u & X_w & X_q \\ Z_u & Z_w & Z_q \\ M_u & M_w & M_q \end{bmatrix}\begin{bmatrix}u-u_f\\ w-w_f\\ q-q_f\end{bmatrix} - \begin{bmatrix}mg\\ 0\\ a_zm\end{bmatrix}\theta \tag{60}$$

Lateral Equations

$$\begin{bmatrix} m & -a_zm & a_xm \\ -a_zm & I_{xx} & -I_{zx} \\ a_xm & -I_{zx} & I_{zz} \end{bmatrix}\begin{bmatrix}\dot{v}\\ \dot{p}\\ \dot{r}\end{bmatrix} = \begin{bmatrix}Y\\ L\\ N\end{bmatrix} - \begin{bmatrix}mU_0\\ -ma_zU_0\\ 0\end{bmatrix}r + \begin{bmatrix} Y_v & Y_p & Y_r \\ L_v & L_p & L_r \\ N_v & N_p & N_r \end{bmatrix}\begin{bmatrix}v-v_f\\ p-p_f\\ r-r_f\end{bmatrix} + \begin{bmatrix}mg\\ -a_zm\\ a_xm\end{bmatrix}\phi \tag{61}$$

where the δ's have been dropped and,

$$\dot{\theta} = q$$
$$\dot{\phi} = p \tag{62}$$

These are the conventional aircraft equations for flight in gusts as used in [3].

5.0 Conclusions

A new formulation of the equations of motion of a rigid body in an unsteady non uniform heavy fluid is given. This avoids the problems with an earlier set of equations

[2] with regard to the unsteady case, and in addition clearly separates out the inertias, the added masses and the relative velocity effects. In addition the small perturbation equations revert to those that are normally used for both bouyant and lifting vehicles. As a result the formulation provides a common derivation of the equations of motion for underwater vehicles, airships, parafoils and aircraft.

6.0 Acknowlegments

The author would like to thank Jim Lipscombe for intoducing him to the problem of motion in a heavy fluid and providing the initial start for this work. In addition he would like to thank Prof. Alan Simpson for his helpfull suggestions regarding an early draft of this paper.

References

[1] Lamb H. "*Hydrodynamics*", Cambridge University Press, 1932.

[2] Lewis D.J.G., Lipscombe J.M., Thomasson P.G. "*The Simulation of Remotely Operated Underwater Vehicles*", Proc. ROV84, Marine Technology Society, San Diego, May 1984.

[3] Stengel R.F. "*Wind Profile Measurement Using Lifting Sensors*", J. Spacecraft, Vol 3, No. 3, March 1966.

[4] Cook M.V, Gomes S, Thomasson P.G. "*Development of a comprehensive airship simulation for flight dynamics research*". UKSC Conference on Computer Simulation, Brighton Sept 1990.

[5] Imlay F.H. "*The Complete Expressions for Added Mass of a Rigid Body Moving in an Ideal Fluid*". David Taylor Model Basin, Report 1528 S-R009 01 01, July 1961.

[6] Lipscombe J.M. "*Left Hand Side of ROV Equations of Motion*", ROV 2.1, Private Communication, 27 April 1981.

[7] Whittaker E.T, "*A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*". Cambridge University Press 1937.

[8] Taylor G.I. "*The Forces Acting on a Body Placed in a Curved and Converging Stream.*" R&M 1166, 1928.

[9] Etkin B. "*The Turbulent Wind and its Effect on Flight.*" UTIAS Review No. 44, August 1980.

[10] Lagrange M.J.B. "*Aerodynamic Forces on an Airship Hull in Atmospheric Turbulence.*" UTIAS Report No. 277 April, 1984.

[11] Gomes S.B.V. "*Measurement of a YEZ-2A Airship Model Response to Low Altitude Turbulence.*" College of Aeronautics Report NFP8910, July,1989.

[12] Prandtl L. & Tietjens O.G. "*Fundamentals of Hydro-and Aerodynamics.*" 1934.

[13] Mulder J A & van der Vaart J C. "*Aircraft Responses to Atmospheric Turbulence.*" Lecture Notes D-47, Technical University Delft, August 1993.

[14] Etkin B. "*Dynamics of Atmospheric Flight*" John Wiley & Sons 1971.

[15] Hancock G.J. "*An Introduction to the Flight Dynamics of Rigid Aeroplanes*", Ellis Horwood, 1995.

[16] Meirovitch L. "*Methods of Analytical Dynamics*", McGraw-Hill 1970.

[17] Monk M. "*The Aerodynamic Forces on Airship Hulls*", NACA 184, 1924.

APPENDIX

The previous sections contain a number of matrix computations who's results are not obvious, the details are given below,

$$\begin{bmatrix} -X_{\dot u} & -X_{\dot v} & -X_{\dot w} & -X_{\dot p} & -X_{\dot q} & -X_{\dot r} \\ -Y_{\dot u} & -Y_{\dot v} & -Y_{\dot w} & -Y_{\dot p} & -Y_{\dot q} & -Y_{\dot r} \\ -Z_{\dot u} & -Z_{\dot v} & -Z_{\dot w} & -Z_{\dot p} & -Z_{\dot q} & -Z_{\dot r} \\ -L_{\dot u} & -L_{\dot v} & -L_{\dot w} & -L_{\dot p} & -L_{\dot q} & -L_{\dot r} \\ -M_{\dot u} & -M_{\dot v} & -M_{\dot w} & -M_{\dot p} & -M_{\dot q} & -M_{\dot r} \\ -N_{\dot u} & -N_{\dot v} & -N_{\dot w} & -N_{\dot p} & -N_{\dot q} & -N_{\dot r} \end{bmatrix} \times \begin{bmatrix} \dot u - \dot u_s \\ \dot v - \dot v_s \\ \dot w - \dot w_s \\ \dot p \\ \dot q \\ \dot r \end{bmatrix} +$$

$$\begin{bmatrix} m & 0 & 0 & 0 & ma_z & -ma_y \\ 0 & m & 0 & -ma_z & 0 & ma_x \\ 0 & 0 & m & ma_y & -ma_x & 0 \\ 0 & -ma_z & ma_y & I_{xx} & -I_{xy} & -I_{xz} \\ ma_z & 0 & -ma_x & -I_{xy} & I_{yy} & -I_{yz} \\ -ma_y & ma_x & 0 & -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \times \begin{bmatrix} \dot u \\ \dot v \\ \dot w \\ \dot p \\ \dot q \\ \dot r \end{bmatrix} =$$

$$\begin{bmatrix} \bar m & 0 & 0 \\ 0 & \bar m & 0 \\ 0 & 0 & \bar m \\ 0 & -\bar m b_z & \bar m b_y \\ \bar m b_z & 0 & \bar m b_x \\ \bar m b_y & \bar m b_x & 0 \end{bmatrix} \times \begin{bmatrix} \dot u_s \\ \dot v_s \\ \dot w_s \end{bmatrix} +$$

$$\begin{bmatrix} X \\ Y \\ Z \\ L \\ M \\ N \end{bmatrix} - \begin{bmatrix} 0 & -r & q & 0 & 0 & 0 \\ r & 0 & -p & 0 & 0 & 0 \\ -q & p & 0 & 0 & 0 & 0 \\ 0 & -w & v & 0 & -r & q \\ w & 0 & -u & r & 0 & -p \\ -v & u & 0 & -q & p & 0 \end{bmatrix} \times \begin{bmatrix} -X_{\dot u} & -X_{\dot v} & -X_{\dot w} & -X_{\dot p} & -X_{\dot q} & -X_{\dot r} \\ -Y_{\dot u} & -Y_{\dot v} & -Y_{\dot w} & -Y_{\dot p} & -Y_{\dot q} & -Y_{\dot r} \\ -Z_{\dot u} & -Z_{\dot v} & -Z_{\dot w} & -Z_{\dot p} & -Z_{\dot q} & -Z_{\dot r} \\ -L_{\dot u} & -L_{\dot v} & -L_{\dot w} & -L_{\dot p} & -L_{\dot q} & -L_{\dot r} \\ -M_{\dot u} & -M_{\dot v} & -M_{\dot w} & -M_{\dot p} & -M_{\dot q} & -M_{\dot r} \\ -N_{\dot u} & -N_{\dot v} & -N_{\dot w} & -N_{\dot p} & -N_{\dot q} & -N_{\dot r} \end{bmatrix} \times \begin{bmatrix} u - u_s \\ v - v_s \\ w - w_s \\ p \\ q \\ r \end{bmatrix} -$$

$$\begin{bmatrix} 0 & -r & q & 0 & 0 & 0 \\ r & 0 & -p & 0 & 0 & 0 \\ -q & p & 0 & 0 & 0 & 0 \\ 0 & -w & v & 0 & -r & q \\ w & 0 & -u & r & 0 & -p \\ -v & u & 0 & -q & p & 0 \end{bmatrix} \times \begin{bmatrix} m & 0 & 0 & 0 & ma_z & -ma_y \\ 0 & m & 0 & -ma_z & 0 & ma_x \\ 0 & 0 & m & ma_y & -ma_x & 0 \\ 0 & -ma_z & ma_y & I_{xx} & -I_{xy} & -I_{xz} \\ ma_z & 0 & -ma_x & -I_{xy} & I_{yy} & -I_{yz} \\ -ma_y & ma_x & 0 & -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix} -$$

$$\begin{bmatrix} 0 & -r & q & 0 & 0 & 0 \\ r & 0 & -p & 0 & 0 & 0 \\ -q & p & 0 & 0 & 0 & 0 \\ 0 & -w & v & 0 & -r & q \\ w & 0 & -u & r & 0 & -p \\ -v & u & 0 & -q & p & 0 \end{bmatrix} \times \begin{bmatrix} \bar m & 0 & 0 \\ 0 & \bar m & 0 \\ 0 & 0 & \bar m \\ 0 & -\bar m b_z & \bar m b_y \\ \bar m b_z & 0 & \bar m b_x \\ \bar m b_y & \bar m b_x & 0 \end{bmatrix} \times \begin{bmatrix} u_s \\ v_s \\ w_s \end{bmatrix} -$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -w_s & v_s \\ w_s & 0 & -u_s \\ -v_s & u_s & 0 \end{bmatrix} \times \left\{ \begin{bmatrix} X_{\dot u} - \bar m & X_v & X_{\dot w} & X_p & -\bar m b_z - X_{\dot q} & \bar m b_y + X_{\dot r} \\ Y_{\dot u} & Y_{\dot v} - \bar m & Y_{\dot w} & \bar m b_z + Y_{\dot p} & Y_{\dot q} & -\bar m b_x - Y_{\dot r} \\ Z_{\dot u} & Z_{\dot v} & Z_{\dot w} - \bar m & -\bar m b_y - Z_{\dot p} & \bar m b_x + Z_{\dot q} & Z_{\dot r} \end{bmatrix} \times \begin{bmatrix} u - u_s \\ v - v_s \\ w - w_s \\ p \\ q \\ r \end{bmatrix} + M_s \begin{bmatrix} u_s \\ v_s \\ w_s \end{bmatrix} \right\}$$

Expanding the matrix products gives,

$$-\begin{bmatrix} 0 & -r & q & 0 & 0 & 0 \\ r & 0 & -p & 0 & 0 & 0 \\ -q & p & 0 & 0 & 0 & 0 \\ 0 & -w & v & 0 & -r & q \\ w & 0 & -u & r & 0 & -p \\ -v & u & 0 & -q & p & 0 \end{bmatrix} \times \begin{bmatrix} -X_{\dot u} & -X_{\dot v} & -X_{\dot w} & -X_{\dot p} & -X_{\dot q} & -X_{\dot r} \\ -Y_{\dot u} & -Y_{\dot v} & -Y_{\dot w} & -Y_{\dot p} & -Y_{\dot q} & -Y_{\dot r} \\ -Z_{\dot u} & -Z_{\dot v} & -Z_{\dot w} & -Z_{\dot p} & -Z_{\dot q} & -Z_{\dot r} \\ -L_{\dot u} & -L_{\dot v} & -L_{\dot w} & -L_{\dot p} & -L_{\dot q} & -L_{\dot r} \\ -M_{\dot u} & -M_{\dot v} & -M_{\dot w} & -M_{\dot p} & -M_{\dot q} & -M_{\dot r} \\ -N_{\dot u} & -N_{\dot v} & -N_{\dot w} & -N_{\dot p} & -N_{\dot q} & -N_{\dot r} \end{bmatrix} =$$

$$\left\{ \begin{matrix} -rY_{\dot u} + qZ_{\dot u} & -rY_{\dot v} + qZ_{\dot v} & -rY_{\dot w} + qZ_{\dot w} \\ rX_{\dot u} - pZ_{\dot u} & rX_{\dot v} - pZ_{\dot v} & rX_{\dot w} - pZ_{\dot w} \\ -qX_{\dot u} + pY_{\dot u} & -qX_{\dot v} + pY_{\dot v} & -qX_{\dot w} + pY_{\dot w} \\ -w_r Y_{\dot u} + v_r Z_{\dot u} - rM_{\dot u} + qN_{\dot u} & -w_r Y_{\dot v} + v_r Z_{\dot v} - rM_{\dot v} + qN_{\dot v} & -w_r Y_{\dot w} + v_r Z_{\dot w} - rM_{\dot w} + qN_{\dot w} \\ w_r X_{\dot u} - u_r Z_{\dot u} + rL_{\dot u} - pN_{\dot u} & w_r X_{\dot v} - u_r Z_{\dot v} + rL_{\dot v} - pN_{\dot v} & w_r X_{\dot w} - u_r Z_{\dot w} + rL_{\dot w} - pN_{\dot w} \\ -v_r X_{\dot u} + u_r Y_{\dot u} - qL_{\dot u} + pM_{\dot u} & -v_r X_{\dot v} + u_r Y_{\dot v} - qL_{\dot v} + pM_{\dot v} & -v_r X_{\dot w} + u_r Y_{\dot w} - qL_{\dot w} + pM_{\dot w} \end{matrix} \right.$$

$$\left. \begin{matrix} -rY_{\dot p} + qZ_{\dot p} & -rY_{\dot q} + qZ_{\dot q} & -rY_{\dot r} + qZ_{\dot r} \\ rX_{\dot p} - pZ_{\dot p} & rX_{\dot q} - pZ_{\dot q} & rX_{\dot r} - pZ_{\dot r} \\ -qX_{\dot p} + pY_{\dot p} & -qX_{\dot q} + pY_{\dot q} & -qX_{\dot r} + pY_{\dot r} \\ -w_r Y_{\dot p} + v_r Z_{\dot p} - rM_{\dot p} + qN_{\dot p} & -w_r Y_{\dot q} + v_r Z_{\dot q} - rM_{\dot q} + qN_{\dot q} & -w_r Y_{\dot r} + v_r Z_{\dot r} - rM_{\dot r} + qN_{\dot r} \\ w_r X_{\dot p} - u_r Z_{\dot p} + rL_{\dot p} - pN_{\dot p} & w_r X_{\dot q} - u_r Z_{\dot q} + rL_{\dot q} - pN_{\dot q} & w_r X_{\dot r} - u_r Z_{\dot r} + rL_{\dot r} - pN_{\dot r} \\ -v_r X_{\dot p} + u_r Y_{\dot p} - qL_{\dot p} + pM_{\dot p} & -v_r X_{\dot q} + u_r Y_{\dot q} - qL_{\dot q} + pM_{\dot q} & -v_r X_{\dot r} + u_r Y_{\dot r} - qL_{\dot r} + pM_{\dot r} \end{matrix} \right\}$$

$$\begin{bmatrix} 0 & -r & q & 0 & 0 & 0 \\ r & 0 & -p & 0 & 0 & 0 \\ -q & p & 0 & 0 & 0 & 0 \\ 0 & -w & v & 0 & -r & q \\ w & 0 & -u & r & 0 & -p \\ -v & u & 0 & -q & p & 0 \end{bmatrix} \times \begin{bmatrix} m & 0 & 0 & 0 & ma_z & -ma_y \\ 0 & m & 0 & -ma_z & 0 & ma_x \\ 0 & 0 & m & ma_y & -ma_x & 0 \\ 0 & -ma_z & ma_y & I_{xx} & -I_{xy} & -I_{xz} \\ ma_z & 0 & -ma_x & -I_{xy} & I_{yy} & -I_{yz} \\ -ma_y & ma_x & 0 & -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & -rm & qm & rma_z + qma_y & -qma_x & -ma_x \\ rm & 0 & -pm & -pma_y & rma_z + pma_x & -ma_y \\ -qm & pm & 0 & -pma_z & -qma_z & qma_y + pma_x \\ -rma_z - qma_y & -wm + qma_x & vm - rma_x & wma_z + vma_y - rI_{xy} + qI_{xz} & -vma_x - rI_{yy} + qI_{yz} & -wma_x - rI_{yz} + qI_{zz} \\ wm + pma_y & -rma_z - pma_x & -um + rma_y & -uma_y + rI_{xx} - pI_{xz} & wma_z + uma_x + rI_{xy} - pI_{yz} & -wma_y - rI_{xz} - pI_{zz} \\ -vm + pma_z & um + qma_z & -qma_y + pma_x & -uma_z - qI_{xx} + pI_{xy} & vma_z - qI_{xy} + pI_{yy} & vma_y + uma_x - qI_{xz} + pI_{yz} \end{bmatrix}$$

$$\begin{bmatrix} 0 & -r & q & 0 & 0 & 0 \\ r & 0 & -p & 0 & 0 & 0 \\ -q & p & 0 & 0 & 0 & 0 \\ 0 & -w & v & 0 & -r & q \\ w & 0 & -u & r & 0 & -p \\ -v & u & 0 & -q & p & 0 \end{bmatrix} \times \begin{bmatrix} \overline{m} & 0 & 0 \\ 0 & \overline{m} & 0 \\ 0 & 0 & \overline{m} \\ 0 & -\overline{m}b_z & \overline{m}b_y \\ \overline{m}b_z & 0 & \overline{m}b_x \\ \overline{m}b_y & \overline{m}b_x & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & -r\overline{m} & q\overline{m} \\ r\overline{m} & 0 & -p\overline{m} \\ -q\overline{m} & p\overline{m} & 0 \\ -r\overline{m}b_z - q\overline{m}b_y & -w\overline{m} + q\overline{m}b_x & v\overline{m} - r\overline{m}b_x \\ w\overline{m} + p\overline{m}b_y & -r\overline{m}b_z - p\overline{m}b_x & -u\overline{m} + r\overline{m}b_y \\ -v\overline{m} + p\overline{m}b_z & u\overline{m} + q\overline{m}b_z & -q\overline{m}b_y + p\overline{m}b_x \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -w_s & v_s \\ w_s & 0 & -u_s \\ -v_s & u_s & 0 \end{bmatrix} \times \left\{ \begin{bmatrix} X_{\dot{u}} - \overline{m} & X_{\dot{v}} & X_{\dot{w}} & X_{\dot{p}} & -\overline{m}b_z - X_{\dot{q}} & \overline{m}b_y + X_{\dot{r}} \\ Y_{\dot{u}} & Y_{\dot{v}} - \overline{m} & Y_{\dot{w}} & \overline{m}b_z + Y_{\dot{p}} & Y_{\dot{q}} & -\overline{m}b_x - Y_{\dot{r}} \\ Z_{\dot{u}} & Z_{\dot{v}} & Z_{\dot{w}} - \overline{m} & -\overline{m}b_y - Z_{\dot{p}} & \overline{m}b_x + Z_{\dot{q}} & Z_{\dot{r}} \end{bmatrix} \times \begin{bmatrix} u - u_s \\ v - v_s \\ w - w_s \\ p \\ q \\ r \end{bmatrix} + M_s \begin{bmatrix} u_s \\ v_s \\ w_s \end{bmatrix} \right\} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -w_s Y_{\dot{u}} + v_s Z_{\dot{u}} & -(Y_{\dot{v}} - \overline{m})w_s + v_s Z_{\dot{v}} & (Z_{\dot{w}} - \overline{m})v_s - w_s Y_{\dot{w}} & -(\overline{m}b_y + Z_{\dot{p}})v_s - (\overline{m}b_z + Y_{\dot{p}})w_s & (\overline{m}b_x + Z_{\dot{q}})v_s - w_s Y_{\dot{p}} & (\overline{m}b_x + Y_{\dot{r}})w_s + v_s Z_{\dot{r}} \\ (X_{\dot{u}} - \overline{m})w_s - u_s Z_{\dot{u}} & w_s X_{\dot{v}} - u_s Z_{\dot{v}} & -(Z_{\dot{w}} - \overline{m})u_s + w_s X_{\dot{w}} & (\overline{m}b_y + Z_{\dot{p}})u_s + w_s X_{\dot{p}} & -(\overline{m}b_x + Z_{\dot{q}})u_s - (\overline{m}b_z + X_{\dot{q}})w_s & (\overline{m}b_y + X_{\dot{r}})w_s - u_s Z_{\dot{r}} \\ -(X_{\dot{u}} - \overline{m})v_s + u_s Y_{\dot{u}} & (Y_{\dot{v}} - \overline{m})u_s - v_s X_{\dot{v}} & -v_s X_{\dot{w}} + u_s Y_{\dot{w}} & (\overline{m}b_z + Y_{\dot{p}})u_s - v_s X_{\dot{p}} & (\overline{m}b_z + X_{\dot{q}})v_s + u_s Y_{\dot{q}} & -(\overline{m}b_x + Y_{\dot{r}})u_s - (\overline{m}b_y + X_{\dot{r}})v_s \end{bmatrix} \times \begin{bmatrix} u - u_s \\ v - v_s \\ w - w_s \\ p \\ q \\ r \end{bmatrix}$$