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Motion of a rigid body in an unsteady non-uniform heavy fluid, an extension

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Difficulies with an earlier set of equations are explained by deriving the equations of motion of a rigid body moving in a perfect fluid that is itself accelerating and contains velocity gradients. The equations are derived on the assumption that changes in the fluid velocity over the vehicles length are small compared to the velocity of the stream in its neighbourhood. It is shown how the resultant perfect fluid equations can be augmented to include viscous forces and moments derived from other theoretical or experimental sources.

1. Introduction

Aircraft flight dynamics are usually formulated on the basis of the equations of motion of a body in-vacuo to which are added the various external forces and moments applied to the airframe. In particular the aerodynamic contributions are taken to be a function of the relative velocity between the vehicle and the air, with a correction added for acceleration effects in the form of $\dot{\alpha}$ or $\dot{w}$ derivatives. In the case of a heavy fluid i.e. one who's density is of the same order as that of the vehicle, such as is the case with submersibles, airships and parafoils, the acceleration effects are substantial and more complex formulations are in order.

The Lagrangian formulation of the equations of motion of a body immersed in a steady but heavy perfect fluid is outlined in Lamb (1932) and a resulting set of equations for the unsteady fluid case, is given in Lewis et al (1984) in a form suitable for the flight dynamics of underwater vehicles. These equations were successfully used to simulate the motion in a steady sea, of the SEAPUP remotely operated under water vehicle by Lewis (1985). They have also been used to model the motion of airships in a steady uniform atmosphere by Cook et al (1990) and were successfully applied to simulate the motion of the YEZ-2A airship by Nippress & Gomes (1990). Recently however the author has had some difficulty in applying these equations to the motion of other vehicles in steady or turbulent winds.

In principle the equations should be applicable to not only underwater vehicles and airships, but also to parafoils and aircraft. Two major problems of the equations in Lewis et al (1984) are that they do not reduce to the small perturbation equations that are used for aircraft in gusts, see Stengel (1966) and Mulder & van der Vaart (1993). Also as will be seen later, the fluids inertial velocity causes difficulty when they are applied to such things as dynamic wind tunnel models. This paper identifies the source of the problems and provides an alternative formulation that corrects the difficulties and so provides a general set of equations for the motion of a wide range of vehicles in an accelerating fluid. In addition it also extends the analysis further to the more general case of motion in fluids containing velocity gradients.

2. Original equations

The form of the equations given in Lewis et al (1984) is an expression of Newton's second law of motion in body axes,

$$\mathbf{M}\dot{\mathbf{x}} = \mathbf{F_d} + \mathbf{F_f} + \mathbf{A} + \mathbf{F} \tag{1}$$

where

$\mathbf{x} = \begin{bmatrix} u & v & w & p & q & r \end{bmatrix}^{\mathrm{T}}$ the inertial body axis velocities of the vehicle
$\mathbf{M}$ = the 6x6 mass matrix including added masses and inertias (2)
$\mathbf{F_d}$ = the dynamics vector arising from body fixed rotating axes
$\mathbf{F_f}$ = the vector of forces and moments due to the fluids inertial motion
$\mathbf{A}$ = the vector of the fluid dynamic forces and moments due to relative velocity
$\mathbf{F}$ = the vector of other external forces and moments

From, Lewis et al (1984) we define effective masses and inertia's as, (the bar refers to the displaced fluid, and conventional derivative terms are used to represent the added masses and inertia's),

$$\begin{aligned}
m_x &= m - X_{\dot u} & \overline{m}_x &= \overline{m} - X_{\dot u} \\
m_y &= m - Y_{\dot v} & \overline{m}_y &= \overline{m} - Y_{\dot v} \\
m_z &= m - Z_{\dot w} & \overline{m}_z &= \overline{m} - Z_{\dot w} \\
J_x &= I_x - L_{\dot p} & J_{yz} &= I_{yz} + M_{\dot r} = I_{zy} + N_{\dot q} \\
J_y &= I_y - M_{\dot q} & J_{zx} &= I_{zx} + N_{\dot p} = I_{xz} + L_{\dot r} \\
J_z &= I_z - N_{\dot r} & J_{xy} &= I_{xy} + L_{\dot q} = I_{yx} + M_{\dot p}
\end{aligned} \tag{3}$$

If the c.g. has body axis co-ordinates of $\left(a_x, a_y, a_z\right)$ then the mass matrix is,

$$\mathbf{M} = \begin{bmatrix}
m_x & 0 & 0 & -X_{\dot p} & a_z m - X_{\dot q} & -a_y m - X_{\dot r} \\
0 & m_y & 0 & -a_z m - Y_{\dot p} & -Y_{\dot q} & a_x m - Y_{\dot r} \\
0 & 0 & m_z & a_y m - Z_{\dot p} & a_x m - Z_{\dot q} & -Z_{\dot r} \\
-L_{\dot u} & -a_z m - L_{\dot v} & a_y m - L_{\dot w} & J_{xx} & -J_{xy} & -J_{zx} \\
a_z m - M_{\dot u} & -M_{\dot v} & -a_x m - M_{\dot w} & -J_{xy} & J_{yy} & -J_{yz} \\
-a_y m - N_{\dot u} & a_x m - N_{\dot v} & -N_{\dot w} & -J_{zx} & -J_{yz} & J_{zz}
\end{bmatrix} \tag{4}$$

and if the co-ordinates of the centre of buoyancy are $\left(b_x, b_y, b_z\right)$ then the dynamics vector is given by,

$$\mathbf{F_d} = \begin{bmatrix} -m_z wq + m_y rv + m[a_x(q^2+r^2) - a_y pq - a_z rp] \\ -m_x ur + m_z pw + m[-a_x pq + a_y(p^2+r^2) - a_z rq] \\ -m_y vp + m_x qu + m[-a_x rp - a_y rq + a_z(q^2+p^2)] \\ -(J_z - J_y)rq + J_{yz}(q^2 - r^2) + J_{zx}pq - J_{xy}pr + m[-a_y(vp-qu) + a_z(ur-pw)] \\ -(J_x - J_z)pr - J_{yz}pq + J_{zx}(r^2-p^2) + J_{xy}qr + m[a_x(vp-qu) - a_z(wq-rv)] \\ -(J_y - J_x)qp + J_{yz}pr - J_{zx}qr + J_{xy}(p^2-q^2) + m[-a_x(ur-pw) + a_y(wq-rv)] \end{bmatrix} \quad (5)$$

whilst the fluid motion vector is,

$$\mathbf{F_f} = \begin{bmatrix} \overline{m}_x \dot{u}_f + \overline{m}_z w_f q - \overline{m}_y r v_f \\ \overline{m}_y \dot{v}_f + \overline{m}_x u_f r - \overline{m}_z p w_f \\ \overline{m}_z \dot{w}_f + \overline{m}_y v_f p - \overline{m}_x q u_f \\ -L_{\dot{u}}\dot{u}_f - (b_z\overline{m} + L_{\dot{v}})\dot{v}_f + (b_y\overline{m} - L_{\dot{w}})\dot{w}_f + \overline{m}[b_y(v_f p - q u_f) - b_z(u_f r - p w_f)] \\ (b_z\overline{m} + M_{\dot{v}})\dot{u}_f - M_{\dot{v}}\dot{v}_f - (b_x\overline{m} - M_{\dot{w}})\dot{w}_f + \overline{m}[-b_x(v_f p - q u_f) + b_z(w_f q - r v_f)] \\ -(b_y\overline{m} + N_{\dot{u}})\dot{u}_f + (b_x\overline{m} - N_{\dot{v}})\dot{v}_f - M_{\dot{w}}\dot{w}_f + \overline{m}[b_x(u_f r - p w_f) - b_y(w_f q - r v_f)] \end{bmatrix} \quad (6)$$

In the derivation of the equations given in Lewis et al (1984) some added mass terms have been assumed to be zero (Lambs A', B' and C') and some perfect fluid terms have been moved to the right hand side of the equation and absorbed into the vector **A**. In addition of course as indicated above, the vehicle mass and inertia's have been combined with the added mass and inertia terms to give 'effective inertia's'.

2.1 Difficulties

Several difficulties arise with the above equations. The most obvious is if the fluid is unsteady, then the fluid motion vector $\mathbf{F_f}$ is a function of the fluid inertial velocity as well as its inertial acceleration and this is counter intuitive. This can be clearly seen by giving the body the mass and inertia properties of the fluid that it displaces. In that case the relative acceleration between the body and the fluid should be zero, but the above equations do not reduce to this. A less obvious difficulty arises with the mass matrix **M** if we attempt to apply the equations to a vehicle such as a partially constrained dynamic wind tunnel model. Then it would be expected that the forces and moments due to the vehicles inertia would depend upon its inertial acceleration whilst the forces and moments due to the fluid acceleration (the added mass and inertia terms) would depend upon the relative acceleration of the fluid and the vehicle and so we might rearrange the equations to the form,

$$\mathbf{M_I\dot{x}} + \mathbf{M_r\dot{x}_r} = -\mathbf{M_r\dot{x}_f} + \mathbf{F_d} + \mathbf{F_f} + \mathbf{A} + \mathbf{F} \quad (7)$$

where

$$\mathbf{x_r} = \mathbf{x} - \mathbf{x_f} = \text{ relative velocities of vehicle and fluid, and} \quad (8)$$

$$\mathbf{M}_i = \begin{bmatrix} m & 0 & 0 & 0 & ma_z & -ma_y \\ 0 & m & 0 & -ma_z & 0 & ma_x \\ 0 & 0 & m & ma_y & -ma_x & 0 \\ 0 & -ma_z & ma_y & I_{xx} & -I_{xy} & -I_{xz} \\ ma_z & 0 & -ma_x & -I_{xy} & I_{yy} & -I_{yz} \\ -ma_y & ma_x & 0 & -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \quad (9)$$

$$\mathbf{M}_r = \begin{bmatrix} -X_{\dot{u}} & -X_{\dot{v}} & -X_{\dot{w}} & -X_{\dot{p}} & -X_{\dot{q}} & -X_{\dot{r}} \\ -Y_{\dot{u}} & -Y_{\dot{v}} & -Y_{\dot{w}} & -Y_{\dot{p}} & -Y_{\dot{q}} & -Y_{\dot{r}} \\ -Z_{\dot{u}} & -Z_{\dot{v}} & -Z_{\dot{w}} & -Z_{\dot{p}} & -Z_{\dot{q}} & -Z_{\dot{r}} \\ -L_{\dot{u}} & -L_{\dot{v}} & -L_{\dot{w}} & -L_{\dot{p}} & -L_{\dot{q}} & -L_{\dot{r}} \\ -M_{\dot{u}} & -M_{\dot{v}} & -M_{\dot{w}} & -M_{\dot{p}} & -M_{\dot{q}} & -M_{\dot{r}} \\ -N_{\dot{u}} & -N_{\dot{v}} & -N_{\dot{w}} & -N_{\dot{p}} & -N_{\dot{q}} & -N_{\dot{r}} \end{bmatrix} \quad (10)$$

The $\mathbf{M}_r$ matrix above is full but in Lewis et al (1984) some elements are zero.

The above fluid motion and fluid acceleration vectors can be combined into a new fluid motion vector. This results in most of the acceleration terms cancelling but not all and the fluid motion vector remains a function of the inertial fluid velocity. The source of this is party due to the fact that in the derivation used in Lewis et al (1984) some but not all perfect fluid relative velocity terms have been absorbed into the vector **A**. As a result a full set of equations involving all the perfect fluid terms is required prior to any rearrangement such as that just given above.

3. Alternative Derivation

Lamb (1932) derives the general expression for the kinetic energy of a body moving in a steady fluid but does not give the general expressions for the forces and moments. Imlay (1961) carries out the necessary differentiation's of the full energy equation and presents the general expressions for the forces and moments. Lipscombe (1981) attributes to Burnett the addition of the bulk fluid motion terms to Lambs original analysis and uses a partially complete energy equation to derive the forces and moments. However the present author has been unable to trace Burnett's work and Lipscombe (1981) only covers the zero velocity gradient case and quotes but does not derive the required form of Lagrange's equations. Approximate expressions for the forces on a stationary rigid body in a perfect fluid with velocity gradients are given by Taylor (1928).

Due to the complexity of the real fluid case the approximate approach based upon the ideas of Lamb and Taylor is used as the starting point. The development of the equations is done in several stages. First the equations of motion of a rigid body in a non uniform unsteady perfect fluid are derived for the case in which the undisturbed fluid velocities do not change significantly over distances comparable to the dimensions of the vehicle. Secondly the perfect fluid forces and moments are identified and seperated into inertial and relative velocity effects.

Finally gust penetration effects are related to the velocity gradients so as to represent the variation over the vehicle of the undisturbed moving fluid velocities and these are combined with the perfect fluid terms and the viscous forces and moments, that are a function of relative velocity alone.

3.1 Perfect Fluid Equations

Consider a rigid body moving in a perfect fluid that is circulating in a doubly connected region, with the region itself accelerating. We shall consider a set of axes fixed in the vehicles body plus a non accelerating set of earth axes *(N,E,D)*. We invoke Taylors (1928) approximation that changes in the circulating fluid velocities over the length of the vehicle are small compared to the velocity of the stream in its neighbourhood. We also define the following quantities evaluated in vehicle body axes,

$\mathbf{x} = (u,v,w,p,q,r)^T$	the bodies velocity relative to the earth axes.
$\mathbf{x_f} = (u_f,v_f,w_f,0,0,0)^T$	the regions velocity relative to the earth axes.
$\mathbf{x_c} = (u_c,v_c,w_c,0,0,0)$	the steady circulating velocity relative to the region, that would exist if the body were absent.

We also have therefore,

$\mathbf{x_f} = (\dot{N}_f,\dot{E}_f,\dot{D}_f,0,0,0)$ in earth axes,

and we define the relative velocity as, $\mathbf{x_r} = \mathbf{x} - \mathbf{x_f} - \mathbf{x_c}$ (11)

also M_f = the mass of the fluid in the multiply connected region.

We can then write the Lagrangian of the system including the bulk translation of the doubly connected fluid, by considering the total of the kinetic energies of the following items,

1. The energy of the circulating fluid in the absence of the vehicle, $= 2K_0$
2. The energy of the fluid mass in the cyclic space due to its bulk translation $= M_f \mathbf{x_f}^T \mathbf{x_f}$.
3. The additional energy when the fluid in the space to be occupied by the body is given the same speed as the body $= \mathbf{x_r^b}^T (\mathbf{M_r^b} + \overline{\mathbf{M}}_\mathbf{i}^b) \mathbf{x_r^b}$
4. The actual energy of the body minus the energy of the fluid it replaces, $= \mathbf{x^g}^T \mathbf{M_i^g} \mathbf{x^g} - \mathbf{x^b}^T \overline{\mathbf{M}}_\mathbf{i}^b \mathbf{x^b}$

The superscripts b and g refer to items derived with the axis origin at the centre of buoyancy and the centre of gravity respectively and the **M**'s are the appropriate 6x6 mass matrices.

In equation form this is,

$$2T = 2K_0 + M_f \mathbf{x}_\mathbf{f}{}^T \mathbf{x}_\mathbf{f} + \mathbf{x}_\mathbf{r}^{b^T}\left(\mathbf{M}_\mathbf{r}^b + \overline{\mathbf{M}}_\mathbf{i}^b\right)\mathbf{x}_\mathbf{r}^b + \mathbf{x}^{g^T}\mathbf{M}_\mathbf{i}^g\mathbf{x}^g - \mathbf{x}^{b^T}\overline{\mathbf{M}}_\mathbf{i}^b\mathbf{x}^b \tag{12}$$

We can relate the velocity vector of the vehicles centre of gravity and centre of buoyancy to the velocity vector of the body axis origin by,

$$\mathbf{x}^g = \begin{bmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\mathbf{x} \qquad \mathbf{x}^b = \begin{bmatrix} \mathbf{I} & \mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\mathbf{x} \tag{13}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & a_z & -a_y \\ -a_z & 0 & a_x \\ a_y & -a_x & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & b_z & -b_y \\ -b_z & 0 & b_x \\ b_y & -b_x & 0 \end{bmatrix} \tag{14}$$

The elements of the added mass matrix change with any change in the position of the axis origin. By considering the kinetic energy of the fluid as constant it can be shown that the added mass matrix for an arbitary axis origin is given by,

$$\mathbf{M}_\mathbf{r} = \begin{bmatrix} \mathbf{I} & -\mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\mathbf{M}_\mathbf{r}^b\begin{bmatrix} \mathbf{I} & \mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{15}$$

Failure to bear this in mind can lead to serious errors as will be shown later in section 3.2.

Now the relative velocity between the fluid and the vehicle, at the vehicles centre of buoyancy is given by,

$$\mathbf{x}_\mathbf{r}^b = \mathbf{x}^b - \mathbf{x}_\mathbf{f} - \mathbf{x}_\mathbf{c}^b \tag{16}$$

Due to the velocity gradients in the circulating fluid, the circulating velocity that would exist at the centre of buoyancy if the vehicle were absent, is different from that which would exist at the body axis origin. As a result we can write,

$$\mathbf{x}_\mathbf{c}^b = \mathbf{x}_\mathbf{c} + \begin{bmatrix} \Phi^T \\ \mathbf{0} \end{bmatrix}\mathbf{b} \tag{17}$$

where $\mathbf{b} = \begin{pmatrix} b_x & b_y & b_z \end{pmatrix}$ and,

$$\Phi = \begin{bmatrix} \frac{\partial u_c}{\partial x} & \frac{\partial v_c}{\partial x} & \frac{\partial w_c}{\partial x} \\ \frac{\partial u_c}{\partial y} & \frac{\partial v_c}{\partial y} & \frac{\partial w_c}{\partial y} \\ \frac{\partial u_c}{\partial z} & \frac{\partial v_c}{\partial z} & \frac{\partial w_c}{\partial z} \end{bmatrix} \tag{18}$$

Substituting gives,

$$\mathbf{x}_r^b = \begin{bmatrix} \mathbf{I} & \mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \left(\mathbf{x}_r + \begin{bmatrix} \Phi^T \\ \mathbf{0} \end{bmatrix} \mathbf{b} \right) \tag{19}$$

The Lagrangian can now be written as,

$$2T = 2K_0 + M_f \mathbf{x}_f^T \mathbf{x}_f + \left(\mathbf{x}_r + \begin{bmatrix} \Phi^T \\ 0 \end{bmatrix} \mathbf{b} \right)^T (\mathbf{M}_r + \overline{\mathbf{M}}_1) \left(\mathbf{x}_r + \begin{bmatrix} \Phi^T \\ 0 \end{bmatrix} \mathbf{b} \right) + \mathbf{x}^T (\mathbf{M}_1 - \overline{\mathbf{M}}_1) \mathbf{x} \tag{20}$$

where all the items are referred to the body axis origin. The mass matrices $\mathbf{M}_r \& \mathbf{M}_1$ are as previously used and

$$\overline{\mathbf{M}}_1 = \begin{bmatrix} \overline{m} & 0 & 0 & 0 & \overline{m}b_z & -\overline{m}b_y \\ 0 & \overline{m} & 0 & -\overline{m}b_z & 0 & \overline{m}b_x \\ 0 & 0 & \overline{m} & \overline{m}b_y & -\overline{m}b_x & 0 \\ 0 & -\overline{m}b_z & \overline{m}b_y & 0 & 0 & 0 \\ \overline{m}b_z & 0 & -\overline{m}b_x & 0 & 0 & 0 \\ -\overline{m}b_y & \overline{m}b_x & 0 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

The vehicle equations of motion with the extra terms for the fluids motion can be derived using Lagrange's equations. However the above Lagrangian is not in terms of generalised co-ordinates and their velocities and so the conventional form of Lagrange's equations cannot be used.

This problem belongs to a class that can be solved by the use of 'quasi co-ordinates' and this general class of problems is described by Whittaker (1937) and Meirovitch (1970).

The true co-ordinates of the current problem are taken as

$$\mathbf{q} = (N, E, D, \phi, \theta, \psi, N_f, E_f, D_f, N_c, E_c, D_c) \tag{22}$$

whilst the rates of change of the quasi co-ordinates are

$$\omega = (u, v, w, p, q, r, u_f, v_f, w_f, u_c, v_c, w_c)\cdot \tag{23}$$

Lagrange's equations in matrix form are,

$$\dot{\mathbf{T}}_{\dot{\mathbf{q}}} - \mathbf{T}_{\mathbf{q}} = \mathbf{Q} \tag{24}$$

where

$$\overline{\mathbf{T}}_{\mathbf{q}} = \begin{bmatrix} \frac{\partial \overline{T}}{\partial N} & \frac{\partial \overline{T}}{\partial E} & \frac{\partial \overline{T}}{\partial D} & \frac{\partial \overline{T}}{\partial \phi} & \frac{\partial \overline{T}}{\partial \theta} & \frac{\partial \overline{T}}{\partial \psi} & \frac{\partial \overline{T}}{\partial N_f} & \frac{\partial \overline{T}}{\partial E_f} & \frac{\partial \overline{T}}{\partial D_f} & \frac{\partial \overline{T}}{\partial N_c} & \frac{\partial \overline{T}}{\partial E_c} & \frac{\partial \overline{T}}{\partial D_c} \end{bmatrix}^T \tag{25}$$

and the Lagrangian $\overline{T} = \overline{T}(\dot{\mathbf{q}} \quad \mathbf{q})$ is expressed in terms of the generalised co-ordinates and their velocities and Q represents the generalised forces. To convert this to quasi co-ordinates first express the rates of change of the quasi co-ordinates as linear combinations of the rates of change of the true co-ordinates,

$$\omega = [\alpha]^T \dot{\mathbf{q}} \tag{26}$$

then the inverse relation is

$$\dot{\mathbf{q}} = [\beta]\omega \tag{27}$$

For the present problem we have,

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{N} \\ \dot{E} \\ \dot{D} \end{bmatrix} \tag{28}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & \sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{29}$$

$$\begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{N}_f \\ \dot{E}_f \\ \dot{D}_f \end{bmatrix} \tag{30}$$

Plus a similar equation to (30) for $[u_c \quad v_c \quad w_c]^T$.
The above equations define the $[\alpha]$ matrix.

$$[\alpha]^T = \begin{bmatrix} \mathbf{BE} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{BE} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{BE} \end{bmatrix} \tag{31}$$

and the 3x3 matrices **BE** & **R** are defined above.

Pre multiply Lagrange's equation by $[\beta]^T$ to give,

$$[\beta]^T\left[\dot{\overline{\mathbf{T}}}_{\dot{\mathbf{q}}} - \overline{\mathbf{T}}_{\mathbf{q}}\right] = [\beta]^T\mathbf{Q} = \Pi \tag{32}$$

and we can write,

$$\overline{\mathbf{T}}_{\dot{\mathbf{q}}} = \omega\left[\frac{\partial}{\partial \dot{\mathbf{q}}}\right]^T \mathbf{T}_\omega = [\alpha]^T \mathbf{T}_\omega \tag{33}$$

where $\mathbf{T} = \mathbf{T}(\omega, \mathbf{q})$ is the Lagrangian expressed solely in terms of the generalised co-ordinates and the quasi-co-ordinate velocities. Therefore

$$[\beta]^T\left[[\alpha]\dot{\mathbf{T}}_\omega + [\dot{\alpha}]\mathbf{T}_\omega - \dot{\mathbf{T}}_{\mathbf{q}}\right] = \Pi \tag{34}$$

but

$$\overline{\mathbf{T}}_{\mathbf{q}} = \mathbf{T}_{\mathbf{q}} + \omega\left[\frac{\partial}{\partial \mathbf{q}}\right]^T \mathbf{T}_\omega \tag{35}$$

and writing

$$[\beta]^T \mathbf{T}_{\mathbf{q}} = \mathbf{T}_\pi \tag{36}$$

gives

$$\dot{\mathbf{T}}_\omega + [\beta]^T\left([\dot{\alpha}] - \omega\left[\frac{\partial}{\partial \mathbf{q}}\right]\right)\mathbf{T}_\omega - \mathbf{T}_\pi = \Pi \tag{37}$$

After considerable algebra the indicated transformations yield the Lagrangian equations of motion.

$$\dot{\mathbf{T}}_{\mathbf{x}} + (\mathbf{P} + \mathbf{W})\mathbf{T}_{\mathbf{x}} + \mathbf{W}_{\mathbf{f}}\mathbf{T}_{\mathbf{f}} + \mathbf{W}_{\mathbf{c}}\mathbf{T}_{\mathbf{c}} - \mathbf{T}_{\mathbf{s}} = \mathbf{F} \tag{38}$$

where,

$$\mathbf{P} = \begin{bmatrix} 0 & -r & q & 0 & 0 & 0 \\ r & 0 & -p & 0 & 0 & 0 \\ -q & p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -r & q \\ 0 & 0 & 0 & r & 0 & -p \\ 0 & 0 & 0 & -q & p & 0 \end{bmatrix} \quad \mathbf{W} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -w & v & 0 & 0 & 0 \\ w & 0 & -u & 0 & 0 & 0 \\ -v & u & 0 & 0 & 0 & 0 \end{bmatrix} \tag{39}$$

$$\mathbf{W}_f = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -w_f & v_f & 0 & 0 & 0 \\ w_f & 0 & -u_f & 0 & 0 & 0 \\ -v_f & u_f & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{W}_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -w_c & v_c & 0 & 0 & 0 \\ w_c & 0 & -u_c & 0 & 0 & 0 \\ -v_c & u_c & 0 & 0 & 0 & 0 \end{bmatrix} \tag{40}$$

$$\mathbf{T}_f = \begin{bmatrix} \frac{\partial T}{\partial u_f} & \frac{\partial T}{\partial v_f} & \frac{\partial T}{\partial w_f} & 0 & 0 & 0 \end{bmatrix}^T \qquad \mathbf{T}_c = \begin{bmatrix} \frac{\partial T}{\partial u_c} & \frac{\partial T}{\partial v_c} & \frac{\partial T}{\partial w_c} & 0 & 0 & 0 \end{bmatrix}^T \tag{41}$$

$$\mathbf{T}_x = \begin{bmatrix} \frac{\partial T}{\partial u} & \frac{\partial T}{\partial v} & \frac{\partial T}{\partial w} & \frac{\partial T}{\partial p} & \frac{\partial T}{\partial q} & \frac{\partial T}{\partial r} \end{bmatrix}^T \qquad \mathbf{T}_s = \begin{bmatrix} \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} & \frac{\partial T}{\partial \xi} & \frac{\partial T}{\partial \eta} & \frac{\partial T}{\partial \zeta} \end{bmatrix}^T \tag{42}$$

and $(x \quad y \quad z \quad \xi \quad \eta \quad \zeta)$ are linear and angular deflections about the body axes.

These equations are similar to Lamb's (1932) but with fluid motion and velocity gradient terms included. They consist of six equations whilst the problem has twelve degrees of freedom. The remaining six equations are the equations of motion of the doubly connected region. For the present problem the motion of the region is regarded as prescribed and hence the equations are not required.

The equations of motion (38) can be written as,

$$\dot{\mathbf{T}}_x = \mathbf{F} - (\mathbf{P} + \mathbf{W})\mathbf{T}_x - \mathbf{W}_f\mathbf{T}_f - \mathbf{W}_c\mathbf{T}_c - \mathbf{T}_s \tag{43}$$

differentiating the Lagrangian and remembering $\mathbf{x}_r = \mathbf{x} - \mathbf{x}_f - \mathbf{x}_c$ gives,

$$\begin{aligned} \mathbf{T}_x &= (\mathbf{M}_r + \overline{\mathbf{M}}_i)\left(\mathbf{x} - \mathbf{x}_f - \mathbf{x}_c + \begin{bmatrix} \mathbf{\Phi}^T \\ \mathbf{0} \end{bmatrix}\mathbf{b}\right) + (\mathbf{M}_i - \overline{\mathbf{M}}_i)\mathbf{x} \\ \dot{\mathbf{T}}_x &= (\mathbf{M}_r + \overline{\mathbf{M}}_i)(\dot{\mathbf{x}} - \dot{\mathbf{x}}_f) + (\mathbf{M}_i - \overline{\mathbf{M}}_i)\dot{\mathbf{x}} \\ \mathbf{T}_f &= M_f\mathbf{x}_f - \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}(\mathbf{M}_r + \overline{\mathbf{M}}_i)\left(\mathbf{x} - \mathbf{x}_f - \mathbf{x}_c + \begin{bmatrix} \mathbf{\Phi}^T \\ \mathbf{0} \end{bmatrix}\mathbf{b}\right) \\ \mathbf{T}_c &= -\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}(\mathbf{M}_r + \overline{\mathbf{M}}_i)\left(\mathbf{x} - \mathbf{x}_f - \mathbf{x}_c + \begin{bmatrix} \mathbf{\Phi}^T \\ \mathbf{0} \end{bmatrix}\mathbf{b}\right) \end{aligned} \tag{44}$$

Substituting in equation (43) gives

$$\begin{aligned}\dot{\mathbf{T}}_x &= (\mathbf{M}_r + \mathbf{M}_i)\dot{\mathbf{x}} - (\mathbf{M}_r + \overline{\mathbf{M}}_i)\dot{\mathbf{x}}_f \\ &= \mathbf{F} - (\mathbf{P}+\mathbf{W})(\mathbf{M}_r + \overline{\mathbf{M}}_i)\left(\mathbf{x}_r + \begin{bmatrix}\Phi^T \\ \mathbf{0}\end{bmatrix}\mathbf{b}\right) - (\mathbf{P}+\mathbf{W})(\mathbf{M}_i + \overline{\mathbf{M}}_i)\mathbf{x} \\ &+ \mathbf{W}_f(\mathbf{M}_r + \overline{\mathbf{M}}_i)\left(\mathbf{x}_r + \begin{bmatrix}\Phi^T \\ \mathbf{0}\end{bmatrix}\mathbf{b}\right) + \mathbf{W}_c(\mathbf{M}_r + \overline{\mathbf{M}}_i)\left(\mathbf{x}_r + \begin{bmatrix}\Phi^T \\ \mathbf{0}\end{bmatrix}\mathbf{b}\right) + \mathbf{T}_s\end{aligned} \tag{45}$$

or writing $\mathbf{W}_r = \mathbf{W} - \mathbf{W}_f - \mathbf{W}_c$ gives the equations of motion as,

$$(\mathbf{M}_r + \mathbf{M}_i)\dot{\mathbf{x}} = \mathbf{F} - (\mathbf{P}+\mathbf{W})(\mathbf{M}_i - \overline{\mathbf{M}}_i)\mathbf{x} + (\mathbf{M}_r + \overline{\mathbf{M}}_i)\dot{\mathbf{x}}_f - (\mathbf{P}+\mathbf{W}_r)(\mathbf{M}_r + \overline{\mathbf{M}}_i)\left(\mathbf{x}_r + \begin{bmatrix}\Phi^T \\ \mathbf{0}\end{bmatrix}\mathbf{b}\right) + \mathbf{T}_s \tag{46}$$

The remaining term to be considered is $\mathbf{T}_s$. This is the change in energy resulting from quasi-coordinate displacements i.e. displacements along and around the body axes. The corresponding forces and moments arise from the velocity gradients in the circulating fluid. The only component of the Lagrangian that contributes is,

$$\left(\mathbf{x}_r + \begin{bmatrix}\Phi^T \\ \mathbf{0}\end{bmatrix}\mathbf{b}\right)^T (\mathbf{M}_r + \overline{\mathbf{M}}_i)\left(\mathbf{x}_r + \begin{bmatrix}\Phi^T \\ \mathbf{0}\end{bmatrix}\mathbf{b}\right) \tag{47}$$

and we have from equation (19)

$$\left(\mathbf{x}_r + \begin{bmatrix}\Phi^T \\ \mathbf{0}\end{bmatrix}\mathbf{b}\right) = \begin{bmatrix}\mathbf{I} & -\mathbf{B} \\ \mathbf{0} & \mathbf{I}\end{bmatrix}\mathbf{x}_r^b \tag{48}$$

as a result the expression can be written as,

$$\mathbf{x}_r^{bT}\begin{bmatrix}\mathbf{I} & \mathbf{0} \\ \mathbf{B} & \mathbf{I}\end{bmatrix}(\mathbf{M}_r + \overline{\mathbf{M}}_i)\begin{bmatrix}\mathbf{I} & -\mathbf{B} \\ \mathbf{0} & \mathbf{I}\end{bmatrix}\mathbf{x}_r^b \tag{49}$$

or

$$(\mathbf{x}^b - \mathbf{x}_f - \mathbf{x}_c^b)^T\begin{bmatrix}\mathbf{I} & \mathbf{0} \\ \mathbf{B} & \mathbf{I}\end{bmatrix}(\mathbf{M}_r + \overline{\mathbf{M}}_i)\begin{bmatrix}\mathbf{I} & -\mathbf{B} \\ \mathbf{0} & \mathbf{I}\end{bmatrix}(\mathbf{x}^b - \mathbf{x}_f - \mathbf{x}_c^b) \tag{50}$$

Differentiating this with respect to the quasi-co-ordinates gives,

$$\mathbf{T}_s = -\begin{bmatrix}\Phi & \mathbf{0} \\ \mathbf{B}\Phi & \mathbf{0}\end{bmatrix}(\mathbf{M}_r + \overline{\mathbf{M}}_i)\left(\mathbf{x}_r + \begin{bmatrix}\Phi^T \\ \mathbf{0}\end{bmatrix}\mathbf{b}\right) \tag{51}$$

as a result the final form of the equations of motion becomes,

$$
\begin{aligned}
(\mathbf{M}_r + \mathbf{M}_i)\dot{\mathbf{x}} &= \mathbf{F} - (\mathbf{P} + \mathbf{W})(\mathbf{M}_i - \overline{\mathbf{M}}_i)\mathbf{x} + (\mathbf{M}_r + \overline{\mathbf{M}}_i)\dot{\mathbf{x}}_f \\
&-(\mathbf{P} + \mathbf{W}_r)(\mathbf{M}_r + \overline{\mathbf{M}}_i)\left(\mathbf{x}_r + \begin{bmatrix}\Phi^T \\ \mathbf{0}\end{bmatrix}\mathbf{b}\right) \\
&-\begin{bmatrix}\boldsymbol{\Phi} & \mathbf{0} \\ \mathbf{B}\boldsymbol{\Phi} & \mathbf{0}\end{bmatrix}(\mathbf{M}_r + \overline{\mathbf{M}}_i)\left(\mathbf{x}_r + \begin{bmatrix}\Phi^T \\ \mathbf{0}\end{bmatrix}\mathbf{b}\right)
\end{aligned}
\qquad (52)
$$

It is instructive to compare these equations with the original equations (1,2,3,4,5,6). Firstly the mass matrix $(\mathbf{M}_r + \mathbf{M}_i)$ is the same as before but the dynamics vector $-(\mathbf{P} + \mathbf{W})(\mathbf{M}_i - \overline{\mathbf{M}}_i)\mathbf{x}$ now involves the difference between the vehicle's mass and that of the displaced fluid and so is zero if the two are the same. Secondly the fluid motion terms $(\mathbf{M}_r + \overline{\mathbf{M}}_i)\dot{\mathbf{x}}_f$ do not now depend upon the fluids inertial velocity, they are only a function of the fluids acceleration and the sum of the displaced fluid mass and its added mass. Finally the remaining two terms are the perfect fluid forces and moments that are a function of the relative velocity and the velocity gradients alone.

3.2 Gust Penetration Perfect Fluid Effects

The term $(\mathbf{P} + \mathbf{W}_r)(\mathbf{M}_r + \overline{\mathbf{M}}_i)\mathbf{x}_r$ in the equation (52) is a function of the relative velocities only and could therefore be absorbed into the vector **A** as the perfect fluid component of the aerodynamic forces and moments due to the relative velocity between the fluid and the vehicle, this also applies to the $\boldsymbol{\Phi}$ terms. In many cases the elements of the vector **A** will be derived empirically from wind tunnel or tank facilities and will include some or all of the perfect fluid effects.

The perfect fluid relative velocity terms that may be absorbed into the vector **A** are,

$$
-(\mathbf{P}+\mathbf{W}_r)(\mathbf{M}_r+\overline{\mathbf{M}}_i)(\mathbf{x}_r+\begin{bmatrix}\Phi^T \\ \mathbf{0}\end{bmatrix}\mathbf{b}) =
$$

$$
\left\{\begin{matrix}
-rY_{\dot u}+qZ_{\dot u} & -rY_{\dot v}+qZ_{\dot v} & -rY_{\dot w}+qZ_{\dot w} & -rY_{\dot p}+qZ_{\dot p} & -rY_{\dot q}+qZ_{\dot q} & -rY_{\dot r}+qZ_{\dot r} \\
rX_{\dot u}-pZ_{\dot u} & rX_{\dot v}-pZ_{\dot v} & rX_{\dot w}-pZ_{\dot w} & rX_{\dot p}-pZ_{\dot p} & rX_{\dot q}-pZ_{\dot q} & rX_{\dot r}-pZ_{\dot r} \\
-qX_{\dot u}+pY_{\dot u} & -qX_{\dot v}+pY_{\dot v} & -qX_{\dot w}+pY_{\dot w} & -qX_{\dot p}+pY_{\dot p} & -qX_{\dot q}+pY_{\dot q} & -qX_{\dot r}+pY_{\dot r} \\
-w_rY_{\dot u}+v_rZ_{\dot u}-rM_{\dot u}+qN_{\dot u} & -w_rY_{\dot v}+v_rZ_{\dot v}-rM_{\dot v}+qN_{\dot v} & -w_rY_{\dot w}+v_rZ_{\dot w}-rM_{\dot w}+qN_{\dot w} & -w_rY_{\dot p}+v_rZ_{\dot p}-rM_{\dot p}+qN_{\dot p} & -w_rY_{\dot q}+v_rZ_{\dot q}-rM_{\dot q}+qN_{\dot q} & -w_rY_{\dot r}+v_rZ_{\dot r}-rM_{\dot r}+qN_{\dot r} \\
w_rX_{\dot u}-u_rZ_{\dot u}+rL_{\dot u}-pN_{\dot u} & w_rX_{\dot v}-u_rZ_{\dot v}+rL_{\dot v}-pN_{\dot v} & w_rX_{\dot w}-u_rZ_{\dot w}+rL_{\dot w}-pN_{\dot w} & w_rX_{\dot p}-u_rZ_{\dot p}+rL_{\dot p}-pN_{\dot p} & w_rX_{\dot q}-u_rZ_{\dot q}+rL_{\dot q}-pN_{\dot q} & w_rX_{\dot r}-u_rZ_{\dot r}+rL_{\dot r}-pN_{\dot r} \\
-v_rX_{\dot u}+u_rY_{\dot u}-qL_{\dot u}+pM_{\dot u} & -v_rX_{\dot v}+u_rY_{\dot v}-qL_{\dot v}+pM_{\dot v} & -v_rX_{\dot w}+u_rY_{\dot w}-qL_{\dot w}+pM_{\dot w} & -v_rX_{\dot p}+u_rY_{\dot p}-qL_{\dot p}+pM_{\dot p} & -v_rX_{\dot q}+u_rY_{\dot q}-qL_{\dot q}+pM_{\dot q} & -v_rX_{\dot r}+u_rY_{\dot r}-qL_{\dot r}+pM_{\dot r}
\end{matrix}\right\}(\mathbf{x}_r+\begin{bmatrix}\Phi^T \\ \mathbf{0}\end{bmatrix}\mathbf{b}) \qquad (53)
$$

$$
-\overline{m}\begin{bmatrix} 0 & -r & q & rb_z+qb_y & -qb_x & -rb_x \\ r & 0 & -p & -pb_y & rb_z+pb_x & -rb_y \\ -q & p & 0 & -pb_z & -qb_z & qb_y+pb_x \\ -rb_z-qb_y & 0 & 0 & w_rb_z+v_rb_y & 0 & 0 \\ 0 & -rb_z-pb_x & 0 & 0 & w_rb_z+u_rb_x & 0_y \\ 0 & 0 & -qb_y+pb_x & 0 & 0 & v_rb_y+u_rb_x \end{bmatrix}(\mathbf{x}_r+\begin{bmatrix}\Phi^T \\ \mathbf{0}\end{bmatrix}\mathbf{b})
$$

Depending upon how the elements of the **A** vector are determined, some or all of the above terms may already be included and care should be exercised that terms are neither omitted nor 'double accounted' for. The risk of double accounting can be seen by considering an axisymmetric body in a turn with $w = p = q = 0$. Then the perfect fluid yawing moment is from Equation(53),

$$N = -v_r X_{\dot{u}} u_r + u_r Y_{\dot{v}} v_r \tag{54}$$

Now if the vehicle is sidesliping at an angle β,

$$\sin\beta = \frac{v_r}{V_{tot}} \qquad \cos\beta = \frac{u_r}{V_{tot}} \tag{55}$$

Hence,

$$N = \frac{V_{tot}^2}{2}(Y_{\dot{v}} - X_{\dot{u}})\sin 2\beta \tag{56}$$

i.e. this is the classic perfect fluid moment given by Monk (1924), and this may be already included in the vector **A**, depending upon the source of the data. For example Lingard (1981) derives a set of equations for the longitudinal motion of a parafoil and rightly notes that previous investigators have erroneously left the terms like Equation (54) in the equations of motion whilst at the same time using wind tunnel data for the pitching moment, thereby double accounting. He therefore removes the terms from his equations but without correcting the rest of the added mass matrix for axes not at the centre of buoyancy (as described earlier in Section 3.1). As a result Lingards equations predict a motion that is dependant upon the chosen axis position, clearly a physically incorrect result.

In the same sideslipping case there is also a sideforce given by equation (53) as,

$$Y = u_r r(\overline{m} - X_{\dot{u}}) - v_r r X_{\dot{v}} \tag{57}$$

and any empirical data used in **A** to calculate the side force, such as whirling arm results, may or may not have been adjusted for this centrifugal force term.

With the axes at the centre of buoyancy and with all the above terms absorbed the equations of motion (52) become,

$$(\mathbf{M}_r+\mathbf{M}_i)\dot{\mathbf{x}} = -(\mathbf{P}+\mathbf{W})(\mathbf{M}_i-\overline{\mathbf{M}}_i)\mathbf{x}+(\mathbf{M}_r+\overline{\mathbf{M}}_i)\dot{\mathbf{x}}_f-\begin{bmatrix}\Phi & 0\\ 0 & 0\end{bmatrix}(\mathbf{M}_r+\overline{\mathbf{M}}_i)\mathbf{x}_r+\mathbf{A}+\mathbf{F}$$

or

$$\mathbf{M}\dot{\mathbf{x}} = \mathbf{F}_d+\mathbf{F}_f+\mathbf{F}_l+\mathbf{A}+\mathbf{F}$$

where

$$\mathbf{M} = (\mathbf{M}_r+\mathbf{M}_i)$$

$$\mathbf{F}_f = (\mathbf{M}_r+\overline{\mathbf{M}}_i)\dot{\mathbf{x}}_f$$

$$\mathbf{F}_l = -\begin{bmatrix}\Phi & 0\\ 0 & 0\end{bmatrix}(\mathbf{M}_r+\overline{\mathbf{M}}_i)\mathbf{x}_r$$

$$\mathbf{F}_d = -(\mathbf{P}+\mathbf{W})(\mathbf{M}_i-\overline{\mathbf{M}}_i)\mathbf{x} = \tag{58}$$

$$\begin{bmatrix}
(m-\overline{m})(rv-qw)+m[a_x(q^2+r^2)-a_y pq-a_z rp]\\
(m-\overline{m})(pw-ru)+m[a_y(p^2+r^2)-a_x pq-a_z rq]\\
(m-\overline{m})(qu-pv)+m[a_z(q^2+p^2)-a_y rq-a_x rp]\\
-(I_{zz}-I_{yy})rq+I_{yz}(q^2-r^2)+I_{zx}pq-I_{xy}pr+m[-a_y(vp-qu)+a_z(ur-pw)]\\
-(I_{xx}-I_{zz})pr-I_{yz}pq+I_{zx}(r^2-p^2)+I_{xy}qr+m[a_x(vp-qu)-a_z(wq-rv)]\\
-(I_{yy}-I_{xx})qp+I_{yz}pr-I_{zx}qr+I_{xy}(p^2-q^2)+m[-a_x(ur-pw)+a_y(wq-rv)]
\end{bmatrix}$$

These are more acceptable from a physical point of view since the fluid motion terms now only depend upon the fluids inertial acceleration. Giving the body the same mass and inertia's as the displaced fluid results in the relative acceleration becoming zero as would be expected. If the mass of displaced fluid is negligible they revert to the conventional aircraft equations of motion.

The terms involving Φ represent the perfect fluid effects of velocity gradients. If we put the axes at the centre of buoyancy then the moments due to the gradients are zero and the forces are,

$$\Phi\begin{bmatrix}
u_r(\overline{m}-X_{\dot u})-v_rX_{\dot v}-w_rX_{\dot w}-pX_{\dot p}-qX_{\dot q}-rX_{\dot r}\\
-u_rY_{\dot u}+v_r(\overline{m}-Y_{\dot v})-w_rY_{\dot w}-pY_{\dot p}-qY_{\dot q}-rY_{\dot r}\\
-u_rZ_{\dot u}-v_rZ_{\dot v}+w_r(\overline{m}-Z_{\dot w})-pZ_{\dot p}-qZ_{\dot q}-rZ_{\dot r}
\end{bmatrix} \tag{59}$$

If we consider a stationary axisymmetric body with no rotation the x-force component becomes,

$$X = \left\{u_r(\overline{m}-X_{\dot u})\frac{\partial u}{\partial x}-v_rX_{\dot v}\frac{\partial v}{\partial x}-w_rX_{\dot w}\frac{\partial w}{\partial x}\right\} \tag{60}$$

and this is the same result as obtained by Lamb (1932) and Taylor (1928).

3.3 Gust Penetration Viscous Effects

If the vehicle is moving in a steady but non-uniform stream there will be time varying relative velocity components due to the bodies translation through the fluid. The perfect fluid effects of these gradients are already included in the equations as shown above but in many situations e.g. turbulence, the velocity field will not be irrotational. In this case the matrix of gradients Φ can be split, Prandtl (1934) into symmetric and anti-symmetric parts, the former representing the irrotational strain rates who's perfect fluid effects are given above, and the later representing vorticity. The effects of such vortical velocity gradients can be treated as 'effective rotation rates' since they produce velocity distributions similar to those due to rotation,

$$\begin{bmatrix} p_f & q_f & r_f \end{bmatrix}^T = \begin{bmatrix} \frac{\partial w_c}{\partial y} - \frac{\partial v_c}{\partial z} & \frac{\partial u_c}{\partial z} - \frac{\partial w_c}{\partial x} & \frac{\partial v_c}{\partial x} - \frac{\partial u_c}{\partial y} \end{bmatrix}^T = \frac{1}{2}\left(\Phi - \Phi^{\mathrm{T}}\right) \tag{61}$$

These 'effective' rotation rates can then be carefully applied to the vehicles rotary derivatives. For example in an aircraft the vertical dimensions are small compared to the tail arm and the span, and so the rotary derivatives are mainly due to changes in the incidence of the tailplane and the wing tips i.e. the result of x and y gradients, as a result the variation with z should be neglected, so that,

$$\begin{bmatrix} p_f & q_f & r_f \end{bmatrix}^T = \begin{bmatrix} \frac{\partial w_c}{\partial y}, & -\frac{\partial w_c}{\partial x}, & \frac{\partial v_c}{\partial x} - \frac{\partial u_c}{\partial y} \end{bmatrix}^T \tag{62}$$

This is the linear field approximation used by Etkin (1980).

If the vehicle has a velocity U (along the x-axis) relative to the fluid,

$$\begin{bmatrix} p_f & q_f & r_f \end{bmatrix}^T = \begin{bmatrix} \frac{\partial w_c}{\partial y}, & -\frac{\dot{w}^*_c}{U}, & \frac{\dot{v}^*_c}{U} - \frac{\partial u_c}{\partial y} \end{bmatrix}^T \tag{63}$$

Where the starred terms are the apparent rate of change due to the motion of the turbulence field past the vehicle i.e. NOT the inertial accelerations of the undisturbed fluid. The above expression is added to the vehicle rotation rates so as to give the effective relative angular velocity for use in the aerodynamic calculations of **A** (it being solely a function of the relative velocities). This avoids one of the problems with the formulation used in Lewis et al (1984), it had fluid velocity terms in the fluid motion vector and it was not clear how the 'effective' rotation rates could be included in the equations.

The distinction between the real and apparent fluid accelerations is important since in some derivations Mulder & van der Vaart (1993), the use of the frozen turbulence approximation leads to them being lumped together so that $Z_{\dot{w}_f} = Z_{\dot{w}} - Z_q$. In the case of a moving sea or non frozen turbulence this is not correct and the above treatment keeps them distinct.

One important point is that the linear field approximation must break down at some point as the wave length of the disturbance reduces, this applies to both the perfect fluid effects and the vortical effects. As a result more extensive gust or sea state models could be included at this point depending upon the application.

The acceleration dependant terms or 'added masses' arise from the work done in accelerating the perfect fluid, in a real fluid additional acceleration effects come into play such as the increase in vorticity and its convection past tailplanes etc. It is assumed that all such effects are added into the corresponding perfect fluid added mass terms. This is a reasonable assumption for streamlined vehicles supported by buoyancy, but is less exact for vehicles with substantial lift such as aircraft. The unsteady aspects of lift generation and the nature of the $\dot{\alpha}$ 'derivatives' is discussed by Etkin (1971) and Hancock (1995) and whilst the above assumption is not strictly correct, it is still a working approximation for many aircraft situations.

4. Conclusions

A new formulation of the equations of motion of a rigid body in an unsteady heavy fluid is given. This avoids the problems with an earlier set of equations used for submersibles and airships with regard to the unsteady case. The source of error in a previous parafoil model is also revealed. In addition it accounts for velocity gradients in a way that links both irrotational and rotational effects. It clearly separates out the inertia's, the added masses and the relative velocity effects and the small perturbation equations revert to those that are normally used for both buoyant and lifting vehicles. As a result the formulation provides a common set of equations of motion for describing the motion of underwater vehicles, airships, parafoils and aircraft in an unsteady heavy fluid.

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