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FULLY DISCRETE ARBITRARY-ORDER SCHEMES
FOR A MODEL HYPERBOLIC CONSERVATION LAW

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Abstract

We investigate the fully discrete methodology and establish a formula from which two-level explicit fully discrete arbitrary-order (both in space and time) conservative numerical schemes for a model hyperbolic conservation law can be derived. To illustrate this approach fully discrete second, third and fourth order numerical schemes are presented.

1 Introduction

An important research subject in Computational Fluid Dynamics (CFD) is the development of high-order numerical schemes. One example of considerable interest is Acoustics, which needs long time evolution of weak flow features. For this kind of problems low-order methods will produce unacceptable dispersive and diffusive errors in a very short time. Another example concerns problems containing weak shocks in which the physical effects of diffusion and dispersion are important mechanisms. Low-order methods contain large numerical diffusion and dispersion and are thus totally inaccurate for simulating the propagation of weak shocks. In large computational problems low-order methods would require vast amounts of computer memory (possibly not available in current computers) in order to attain a satisfactory degree of accuracy. A high-order method would attain the same accuracy with coarser meshes requiring less sophisticated hardware.

There are two different techniques which can be used to construct high-order numerical schemes: semi-discrete and fully discrete methods. In the semi-discrete method [1] one divides the discretization process into two separate stages. In the first stage one discretizes in space only leaving the problem continuous in time; in the second stage one has sets of Ordinary Differential Equations (ODE) in time, which can be discretized appropriately. Often this technique is called **the method of lines.** The MUSCL approach introduced by Van Leer [2] can be utilised in conjunction with the method of lines. The most newly developed ENO schemes [3]-[5] belongs to this category. The main idea of the ENO scheme is that the spacial high-order approximations to the flux at a cell interface can be defined using high-order interpolation in space, and then the high-order temporal accuracy can be achieved by another discretization applying a high-order ODE solver. To our knowledge most high-order numerical schemes rely on the semi-discrete approach.

In this paper we investigate the fully discrete approach to obtain arbitrary -order (in space and time) numerical methods. The analysis is carried out in the context of a model hyperbolic conservation law. The resulting schemes are expressed in conservative form, as it is this form the one required for computing discontinuous

solutions to non-linear hyperbolic systems of conservation laws.

This paper is organized as follows: section 2 establishes a formula from which 2-level explicit fully discrete arbitrary-order non-conservative numerical schemes can be derived. Section 3 develops a method to transform the non-conservative schemes to conservative form. In section 4 we apply the technique to construct some high-order fully discrete conservative numerical schemes and give the stability condition for the schemes. Section 5 contains some numerical experiments and section 6 is conclusions.

2 Fully Discrete Formula

We consider the initial value problem (IVP) for one dimensional linear scalar hyperbolic partial differential equation (PDE), namely

$$u_t + au_x = 0 \quad -\infty < x < \infty, t \geq 0 \tag{1}$$
$$u(x,0) = u_0(x)$$

Here, $u(x,t)$ is the unknown function and a is a constant wave propagation speed.

We discretize the computational half plane by choosing a uniform mesh with a mesh width $h = \triangle x$ and a time step $k = \triangle t$, and define the computational grid $x_j = jh$, $t_n = nk$. We use U_j^n to denote the computed approximation to the exact solution $u(x_j, t_n)$ of equation (1).

In this section a fully discrete technique for the model equation is investigated. The fully discrete approach is based on Taylor series expansion in both space and time in a single stage.

Theorem 2.1 *The fully discrete formula from which a two-level fully discrete explicit m-th order accurate finite difference method can be derived for the model hy-*

perbolic equation, $u_t + a u_x = 0$, is defined as

$$U_j^{n+1} = \sum_{\alpha=1}^{p} B_{k_\alpha} U_{j+k_\alpha}^n \tag{2}$$

where α is the grid point number; p is the number of grid points used, $p = m + 1$; m is the order of accuracy; B_{k_α} are constant coefficients determined by

$$\begin{cases} B_{k_\alpha=0} = 1 - \sum_{\alpha=1, k_\alpha \neq 0}^{m} B_{k_\alpha} \\ \begin{bmatrix} B_{k_1} \\ B_{k_2} \\ \vdots \\ B_{k_m} \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & \dots & k_m \\ k_1^2 & k_2^2 & \dots & k_m^2 \\ \vdots & \vdots & \vdots & \vdots \\ k_1^m & k_2^m & \dots & k_m^m \end{bmatrix}^{-1} \begin{bmatrix} -c \\ c^2 \\ \vdots \\ (-c)^m \end{bmatrix} \\ (k_\alpha \neq 0) \end{cases} \tag{3}$$

where c is Courant number, $c = \frac{a\Delta t}{\Delta x}$.

Proof

In order to prove the theorem, we first analyse the local truncation error of equation (2) by Taylor series expansion of both sides of equation (2). This can be written as:

$$\begin{aligned} E(x,t) &= u(x,t) + \sum_{n=1}^{m} \frac{\Delta t^n}{n!} u_{t^n} + O(\Delta t^{m+1}) \\ &\quad - \sum_{\alpha=1}^{p} B_{k_\alpha} \left[u(x,t) + \sum_{n=1}^{m} \frac{(k_\alpha \Delta x)^n}{n!} u_{x^n} \right] + O(\Delta x^{m+1}) \end{aligned} \tag{4}$$

where m is the order of accuracy of the scheme, $1 \leq m < \infty$; $u_{t^n} = \frac{\partial^n u}{\partial t^n}$, $u_{x^n} = \frac{\partial^n u}{\partial x^n}$; $\Delta t^n = (\Delta t)^n$. The relationship between m and p obviously is

$$p = m + 1 \tag{5}$$

For the scalar equation in (1), it is easy to obtain:

$$u_{t^n} = (-a)^n u_{x^n} \tag{6}$$

Substitution of equation (6) into equation (4) gives

$$\begin{aligned} E(x,t) &= \left(1 - \sum_{\alpha=1}^{p} B_{k_\alpha}\right) u(x,t) + \sum_{n=1}^{m} \left[\frac{\Delta t^n}{n!}(-a)^n u_{x^n} \right. \\ &\quad \left. - \sum_{\alpha=1}^{p} B_{k_\alpha} \frac{(k_\alpha \Delta x)^n}{n!} u_{x^n} \right] + O(\Delta t^{m+1}) + O(\Delta x^{m+1}) \end{aligned} \tag{7}$$

In order to achieve $m - th$ order of accuracy, it is sufficient to require that

$$1 - \sum_{\alpha=1}^{p} B_{k_\alpha} = 0 \tag{8a}$$

$$\frac{\Delta t^n}{n!}(-a)^n u_{x^n} - \sum_{\alpha=1}^{p} B_{k_\alpha}\frac{(k_\alpha \Delta x)^n}{n!} u_{x^n} = 0 \tag{8b}$$

$$(n = 1, 2, 3, \ldots m)$$

Simplifying equation (8b), equations (8a) and (8b) can be rewritten as

$$\begin{cases} B_{k_\alpha=0} = 1 - \sum_{\alpha=1, k_\alpha \neq 0}^{m} B_{k_\alpha} \\ \sum_{\alpha=1, k_\alpha \neq 0}^{m} k_\alpha^n B_{k_\alpha} = (-c)^n \quad (n = 1, 2, \cdots, m) \end{cases} \tag{9}$$

Equations (9) can be transformed into the alternative forms

$$\begin{cases} B_{k_\alpha=0} = 1 - \sum_{\alpha=1, k_\alpha \neq 0}^{m} B_{k_\alpha} \\ k_1 B_{k_1} + k_2 B_{k_2} + \cdots + k_m B_{k_m} = -c \\ k_1^2 B_{k_1} + k_2^2 B_{k_2} + \cdots + k_m^2 B_{k_m} = c^2 \\ \vdots \\ k_1^m B_{k_1} + k_2^m B_{k_2} + \cdots + k_m^m B_{k_m} = (-c)^m \\ (k_\alpha \neq 0) \end{cases} \tag{10}$$

or

$$\begin{cases} B_{k_\alpha=0} = 1 - \sum_{\alpha=1, k_\alpha \neq 0}^{m} B_{k_\alpha} \\ \begin{bmatrix} B_{k_1} \\ B_{k_2} \\ \vdots \\ B_{k_m} \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & \ldots & k_m \\ k_1^2 & k_2^2 & \ldots & k_m^2 \\ \vdots & \vdots & \vdots & \vdots \\ k_1^m & k_2^m & \ldots & k_m^m \end{bmatrix}^{-1} \begin{bmatrix} -c \\ c^2 \\ \vdots \\ (-c)^m \end{bmatrix} \\ (k_\alpha \neq 0) \end{cases} \tag{11}$$

which is the formula (3) and establishes the theorem.

The same result can be found in Roe [6] who derived it in a different manner and utilised it for different purposes to those of this paper. Our ultimate aim is to utilise this result to develop high-order schemes for CFD.

For an $m - th$ order numerical method, according to equation (5), p coefficients $B_{k_\alpha}(\alpha = 1, 2, \cdots, p)$ are needed in equation (2). Equations (9), or (10), or (11),

having p equations, are therefore closed so that arbitrary-order numerical methods for the linear scalar equation (1) can be obtained.

Some interesting observation can be made. The first one concerns the order of accuracy; this depends on the number of nonlinearly related grid points used, that is, the more grid points are involved the higher the order of accuracy achieved; see equation (5). A second aspect relates to the stencil of the scheme. Using the same number of grid points, but in different stencils, different numerical schemes can be obtained. If only integer points are considered, the number of numerical methods (N) which have the same order equal to the number of the points used, i.e. $N = p$. For example, for four-integer-point schemes we can find four third-order numerical methods.

In section 4 we will use some examples to illustrate these remarks and to show how to apply the formula to derive high-order numerical methods.

3   Conservative Schemes

Using theorem 2.1 we can construct 2-level fully discrete explicit arbitrary-order non-conservative numerical schemes. But these schemes are only suitable for linear systems or non-linear systems with smooth solutions. When extending these methods to nonlinear conservation laws we expect to meet two new problems. First the method might coverge to a wrong weak solution and second the method might suffer nonlinear instability.

In this section we reformulate the arbitrary-order finite difference schemes of the previous section in a conservative form. Following Leveque's notation [1] a method in conservation form reads

$$U_j^{n+1} = U_j^n - \frac{k}{h} \left[F(U^n; j) - F(U^n; j - 1) \right] \quad \forall \; j \tag{12}$$

where $F(U^n; j)$ is a numerical flux function which satisfies the consistency condition

$$F(\bar{u}, \bar{u}, \ldots, \bar{u}) = f(\bar{u}) \tag{13}$$

here $\bar{u}$ is constant.

Lemma 3.1 *The conservative form of the scheme (2) is*

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[\frac{1}{c}F_j^n - \sum_{\alpha=1}^{p}\frac{1}{c}B_{k_\alpha}F_{j+k_\alpha}^n\right] \tag{14}$$

where the vector B_{k_α} is determined by equation (3).

Proof

By manipulating equation (2) we have

$$U_j^{n+1} = \sum_{\alpha=1}^{p}B_{k_\alpha}U_{j+k_\alpha}^n$$

$$= U_j^n - (1 - B_0)U_j^n + \sum_{\alpha=1,k_\alpha\neq0}^{p}B_{k_\alpha}U_{j+k_\alpha}^n$$

$$= U_n^n - \frac{k}{h}\left[\frac{1}{c}F_j^n - \sum_{\alpha=1}^{p}\frac{1}{c}B_{k_\alpha}F_{j+k_\alpha}^n\right]$$

which is equation (14).

From here we can derive the numerical flux by the following theorem.

Theorem 3.1 *Scheme (2) written in conservative form (12) has numerical flux*

$$\begin{cases} F(U^n;j) = \sum_{\alpha=2}^{p}B_\alpha F_{j+k_\alpha}^n \\ F(U^n;j-1) = \sum_{\alpha=1}^{p-1}B_{\alpha+1}F_{j+k_\alpha}^n \end{cases} \tag{15}$$

the coefficients B_α are defined by

$$\begin{cases} B_p = A_p \\ B_2 = -A_1 \\ B_\alpha - B_{\alpha+1} = A_\alpha \\ (\alpha = 2, 3, \ldots, p-1) \end{cases} \tag{16}$$

Proof

If we rewrite the conservative scheme (14) as

$$U_j^{n+1} = U_j^n - \frac{k}{h} \sum_{\alpha=1}^{p} A_\alpha F_{j+k_\alpha}^n \qquad (17)$$

where α is the grid point number, A_α are the coefficients and the p points are arranged as

$$k_1 < k_2 < k_3 < \ldots < k_p \qquad (18)$$

then

$$F(U^n; j) - F(U^n; j-1) = \sum_{\alpha=2}^{p} B_\alpha F_{j+k_\alpha}^n - \sum_{\alpha=1}^{p-1} B_{\alpha+1} F_{j+k_\alpha}^n$$

$$= B_p F_{j+k_p}^n - B_2 F_{j+k_1}^n + \sum_{\alpha=2}^{p-1} (B_\alpha - B_{\alpha+1}) F_{j+k_\alpha}^n$$

From equation (17) we have

$$B_p F_{j+k_p}^n - B_2 F_{j+k_1}^n + \sum_{\alpha=2}^{p-1} (B_\alpha - B_{\alpha+1}) F_{j+k_\alpha}^n = \sum_{\alpha=1}^{p} A_\alpha F_{j+k_\alpha}^n \qquad (19)$$

Comparing the coefficients on both sides of equation (19) we have the following

$$\begin{cases} B_p = A_p \\ B_2 = -A_1 \\ B_\alpha - B_{\alpha+1} = A_\alpha \\ (\alpha = 2, 3, \ldots, p-1) \end{cases}$$

which is equation (16) and the proof is complete.

4 Conservative High-order Numerical Schemes

In this section, we use some examples to demonstrate how to apply the method presented previously to derive high-order numerical schemes.

4.1 Fully Discrete Second-order Schemes

From equation (5) second-order schemes need at least three grid points. It will be seen that some familiar numerical schemes such as the Lax-Wendroff and Beam-Warming schemes can be derived using our approach.

4.1.1 Centered Scheme

Let us denote the 3-point centered scheme as $U_j^{n+1} = f(U_j^n, U_{j-1}^n, U_{j+1}^n)$.

Here, $k_1 = 0$, $k_2 = -1$, $k_3 = 1$ in equation (10), which gives

$$\begin{cases} -B_{-1} + B_1 = -c \\ B_{-1} + B_1 = c^2 \\ B_0 = 1 - B_{-1} - B_1 \end{cases}$$

i.e.

$$\begin{cases} B_0 = 1 - c^2 \\ B_{-1} = \frac{c}{2}(c+1) \\ B_1 = \frac{c}{2}(c-1) \end{cases} \tag{20}$$

Therefore the non-conservative numerical scheme is

$$U_j^{n+1} = (1 - c^2)U_j^n + \frac{1}{2}(c^2 + c)U_{j-1}^n + \frac{1}{2}(c^2 - c)U_{j+1}^n \tag{21}$$

which is the Lax-Wendroff (L-W) scheme [7].

The conservative form of L-W scheme according to equation (14), is

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[-\frac{1}{2}(1 + c)F_{j-1}^n + cF_j^n + \frac{1}{2}(1 - c)F_{j+1}^n \right] \tag{22}$$

Note that here, from equation (18), $k_1 = -1$, $k_2 = 0$, $k_3 = 1$.

From equation (17) we have

$$\begin{cases} A_1 = -\frac{1}{2}(1 + c) \\ A_2 = c \\ A_3 = \frac{1}{2}(1 - c) \end{cases}$$

and from equation (16) we have

$$\begin{cases} B_3 = A_3 = \frac{1}{2}(1 - c) \\ B_2 = -A_1 = \frac{1}{2}(1 + c) \\ B_2 - B_3 = A_2 = c \end{cases}$$

Therefore the numerical flux of the L-W scheme is

$$F^{L-W}(U^n; j) = \frac{1}{2}(1 + c)F_j^n + \frac{1}{2}(1 - c)F_{j+1}^n \tag{23}$$

Applying the stability analysis method introduced in [8], the amplification factor λ of the scheme is

$$\lambda = 1 - 2c^2 \tag{24}$$

For stability one requires $|\lambda| \leq 1$ which is satisfied if

$$|c| \leq 1 \tag{25}$$

4.1.2 Upwind-biased Scheme

We denote the scheme by $U_j^{n+1} = f(U_j^n, U_{j-1}^n, U_{j-2}^n)$ when we assume $a > 0$ in equation (1).

Here, $k_1 = 0$, $k_2 = -1$, $k_3 = -2$ in equation (10), which gives

$$\begin{cases} -B_{-1} - 2B_{-2} = -c \\ B_{-1} + 4B_{-2} = c^2 \\ B_0 = 1 - B_{-1} - B_{-2} \end{cases}$$

$$\begin{cases} B_0 = 1 + \frac{1}{2}c^2 - \frac{3}{2}c \\ B_{-1} = 2c - c^2 \\ B_{-2} = \frac{1}{2}c^2 - \frac{1}{2}c \end{cases} \tag{26}$$

Therefore

$$U_j^{n+1} = (1 + \frac{1}{2}c^2 - \frac{3}{2}c)U_j^n + (2c - c^2)U_{j-1}^n + (\frac{1}{2}c^2 - \frac{1}{2}c)U_{j-2}^n \tag{27}$$

which is the Beam-Warming (B-W) scheme [9].

The conservative form of B-W scheme according to equation (14) is

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[\frac{1}{2}(3-c)F_j^n - (2-c)F_{j-1}^n - \frac{1}{2}(c-1)F_{j-2}^n\right] \tag{28}$$

where from equation (18), $k_1 = -2$, $k_2 = -1$, $k_3 = 0$.

From equation (17) we have

$$\begin{cases} A_1 = -\frac{1}{2}(1-c) \\ A_2 = c - 2 \\ A_3 = \frac{1}{2}(3-c) \end{cases}$$

and from equation (16) we have

$$\begin{cases} B_3 = A_3 = \frac{1}{2}(3-c) \\ B_2 = -A_1 = \frac{1}{2}(c-1) \\ B_2 - B_3 = A_2 = c - 2 \end{cases}$$

The numerical flux of B-W scheme is

$$F^{B-W}(U^n; j) = \frac{1}{2}(c-1)F_{j-1}^n - \frac{1}{2}(c-3)F_j^n \tag{29}$$

The amplification factor of the scheme is

$$\lambda = 2c^2 - 4c + 1 \tag{30}$$

For stability $|\lambda| \leq 1$ which is satisfied if

$$0 \leq c \leq 2 \tag{31}$$

4.2   Fully Discrete Third-order Schemes

From equation (5) third-order schemes use at least four grid points. Let us consider a scheme which is denoted as $U_j^{n+1} = f(U_j^n, U_{j-1}^n, U_{j+1}^n, U_{j+2}^n)$.

Here, $k_1 = 0$, $k_2 = -1$, $k_3 = 1$, and $k_4 = 2$. From equation (10) we have

$$\begin{cases} B_0 = 1 - B_{-1} - B_1 - B_2 \\ -B_{-1} + B_1 + 2B_2 = -c \\ B_{-1} + B_1 + 4B_2 = c^2 \\ -B_{-1} + B_1 + 8B_2 = -c^3 \end{cases}$$

or

$$\begin{cases} B_0 = 1 + \frac{1}{2}c - c^2 - \frac{1}{2}c^3 \\ B_{-1} = \frac{1}{6}c^3 + \frac{1}{2}c^2 + \frac{1}{3}c \\ B_1 = \frac{1}{2}c^3 + \frac{1}{2}c^2 - c \\ B_2 = \frac{1}{6}c - \frac{1}{6}c^3 \end{cases} \qquad (32)$$

Therefore the third-order (both in space and time) non-conservative numerical scheme is

$$\begin{aligned} U_j^{n+1} &= (1 + \frac{1}{2}c - c^2 - \frac{1}{2}c^3)U_j^n + (\frac{1}{6}c^3 + \frac{1}{2}c^2 + \frac{1}{3}c)U_{j-1}^n \\ &+ (\frac{1}{2}c^3 + \frac{1}{2}c^2 - c)U_{j+1}^n + (\frac{1}{6}c - \frac{1}{6}c^3)U_{j+2}^n \end{aligned} \qquad (33)$$

The conservative form of this scheme according to equation (12), is

$$\begin{aligned} U_j^{n+1} &= U_j^n - \frac{k}{h}\left[(\frac{1}{2}c^2 + c - \frac{1}{2})F_j^n - (\frac{1}{6}c^2 + \frac{1}{2}c + \frac{1}{3})F_{j-1}^n \right. \\ &\left. - (\frac{1}{2}c^2 + \frac{1}{2}c - 1)F_{j+1}^n - (\frac{1}{6} - \frac{1}{6}c^2)F_{j+2}^n \right] \end{aligned} \qquad (34)$$

where from equation (18), $k_1 = -1$, $k_2 = 0$, $k_3 = 1$, $k_4 = 2$ and from equation (17) we have

$$\begin{cases} A_1 = -(\frac{1}{6}c^2 + \frac{1}{2}c + \frac{1}{3}) \\ A_2 = \frac{1}{2}c^2 + c - \frac{1}{2} \\ A_3 = -(\frac{1}{2}c^2 + \frac{1}{2}c - 1) \\ A_4 = \frac{1}{6}c^2 - \frac{1}{6} \end{cases}$$

According to equation (16) we have

$$\begin{cases} B_4 = A_4 = \frac{1}{6}c^2 - \frac{1}{6} \\ B_2 = -A_1 = \frac{1}{6}c^2 + \frac{1}{2}c + \frac{1}{3} \\ B_2 - B_3 = A_2 = \frac{1}{2}c^2 + c - \frac{1}{2} \\ B_3 - B_4 = A_3 = 1 - \frac{1}{2}c - \frac{1}{2}c^2 \end{cases} \qquad (35)$$

Hence

$$B_3 = \frac{5}{6} - \frac{1}{3}c^2 - \frac{1}{2}c \qquad (36)$$

and the numerical flux of the scheme is

$$F^{4-P}(U^n; j) = (\frac{1}{6}c^2 + \frac{1}{2}c + \frac{1}{3})F_j^n + (\frac{5}{6} - \frac{1}{3}c^2 - \frac{1}{2}c)F_{j+1}^n + (\frac{1}{6}c^2 - \frac{1}{6})F_{j+2}^n \qquad (37)$$

Applying a stability analysis we have the condition

$$-1 \leq c \leq 0 \tag{38}$$

which implies that $a < 0$ in equation (1).

For positive speed a in (1) the corresponding four point scheme has flux

$$F^{4-P}(U^n; j) = (\frac{5}{6} - \frac{1}{3}c^2 + \frac{1}{2}c)F_j^n + (\frac{1}{3} + \frac{1}{6}c^2 - \frac{1}{2}c)F_{j+1}^n + \frac{1}{6}(c^2 - 1)F_{j-1}^n \tag{39}$$

The stability condition for this scheme is

$$0 \leq c \leq 1 \tag{40}$$

By unifying the two previous schemes we obtain a five-point third-order method which can accommodates arbitrary wave speeds and has flux

$$F(U^n; j) = \frac{1}{2}\left(F_j^n + F_{j+1}^n\right) - \frac{|a|}{2}\Delta U_{j+\frac{1}{2}} + |a|\left(\frac{1}{3} - \frac{|c|}{2} + \frac{c^2}{6}\right)\Delta U_{j+\frac{1}{2}}$$

$$+\frac{|a|}{6}\left(1 - c^2\right)\Delta U_{j+L+\frac{1}{2}} \tag{41}$$

where

$$\Delta U_{j+L+\frac{1}{2}} = U_{j+L+1} - U_{j+L} \tag{42}$$

$$\begin{cases} L = -1 & if \ a > 0 \\ L = 1 & if \ a < 0 \end{cases} \tag{43}$$

The stability condition of the method now is:

$$|c| \leq 1 \tag{44}$$

4.3 Fully Discrete Fourth-order Centered Scheme

From equation (5) the fourth-order scheme needs at least five grid points. We denote the fourth-order centered scheme as $U_j^{n+1} = f(U_{j-2}^n, U_{j-1}^n, U_j^n, U_{j+1}^n, U_{j+2}^n)$.

By repeating the same procedure as before, the fourth-order centered scheme has flux

$$F(U^n; j) = \frac{1}{2}\left(F_j^n + F_{j+1}^n\right) - \frac{|a|}{2}\Delta U_{j+\frac{1}{2}} + a\left(\frac{1}{12} + \frac{c}{24} - \frac{c^2}{12} - \frac{c^3}{24}\right)\Delta U_{j-\frac{1}{2}}$$
$$+ a\left(\frac{1}{2}\,sgn(a) - \frac{7}{12}c + \frac{c^3}{12}\right)\Delta U_{j+\frac{1}{2}} - a\left(\frac{c^3}{24} - \frac{c^2}{12} - \frac{c}{24} + \frac{1}{12}\right)\Delta U_{j+\frac{3}{2}} \quad (45)$$

The stability condition of the scheme is:

$$|c| \leq 1 \quad (46)$$

For the reader's benefit all other third and fourth-order schemes are included in the appendix to this paper.

5 Numerical Experiments

In this section we use some numerical experiments to demonstrate the performance of the fully discrete high-order numerical schemes. To this end we select a smooth initial condition

$$u(x, 0) = sin\frac{\pi}{2}x \quad (47)$$

We are interested in evolving the solution for long times. The chosen computational domain is therefore large and varies according to the evolution time. We thus select a fixed mesh width $\Delta x = 0.1$ and a Courant number coefficient 0.7.

Figure 1 shows a comparison between the numerical solution obtained by the Lax-Wendroff method (boxes) and the exact solution (solid line) after 1000 time steps. The dispersive errors of the method are evident and result in a trailing numerical solution. Clearly second-order methods can be very inaccurate in modelling long time behaviour.

Figure 2 shows a comparison between the numerical solution obtained by the third-order method and the exact solution after 6000 time steps. Although the solution looks more acceptable than the Lax-Wendroff solution evolved for only 1000 time

steps, the numerical diffusion of the thir-order scheme produces the inaccuracy observed.

Figure 3 shows a comparison between the numerical solution obtained by the fourth-order method and the exact solution after 20000 time steps. The numerical solution still looks accurate at the shown time. This indicates that the accuracy of numerical solutions is improved dramatically by changing from second to fourth order methods.

Figure 4 shows a comparison between the numerical solution obtained by the 20th-order centered method and the exact solution after 50000 time steps. As expected the numerical solution looks very accurate.

To summarize the situation Figure 5 shows the comparison of the Lax-Wendroff method (crosses), the fourth-order central scheme (boxes) and the exact solution (solid line) after 10000 time steps. The effects of dispersion and diffusion have made the solution of the Lax-Wendroff method meaningless. This justifies the necessity for higher-order numerical schemes for problems which involve long time evolution.

6   Conclusions and Discussions

An approach for constructing two-level explicit fully discrete arbitrary-order conservative numerical methods for one dimensional scalar hyperbolic equation has been presented. To illustrate the technique fully discrete second, third and fourth-order conservative numerical schemes are given. Numerical experiments indicate that the second-order methods are not accurate for problems requiring long time evolution, and a dramatic improvement of the numerical solution is seen when the accuracy changes from second to fourth order.

In this paper we have laid the fundations for developing high-order methods that are applicable to non-linear systems of hyperbolic conservation laws. There are two major problems to be addressed. One is the problem of monotonicity of the schemes near high gradients (e.g. shocks) and the other is the extension of the approach to

non-linear systems. These issues are the subjects of current research.

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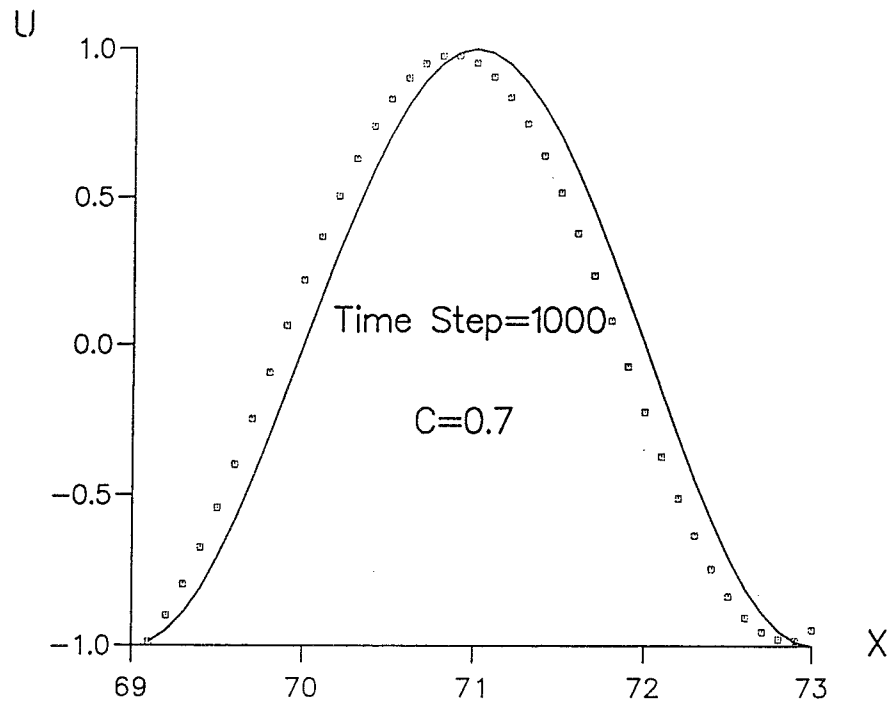
Figure 1: Numerical Solution by the Lax-Wendroff Method (symbol) and the Exact Solution (line)
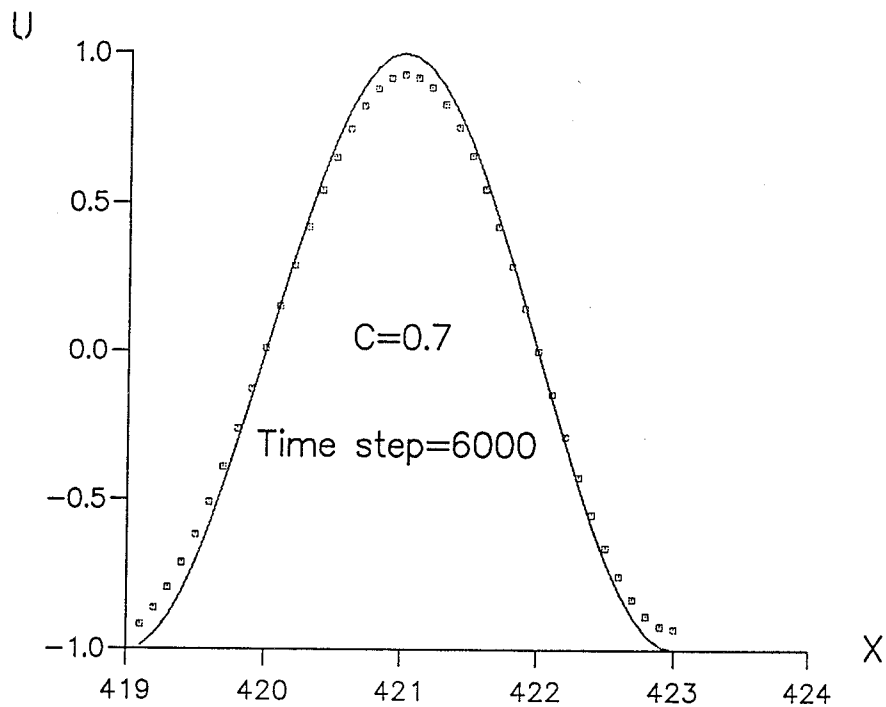

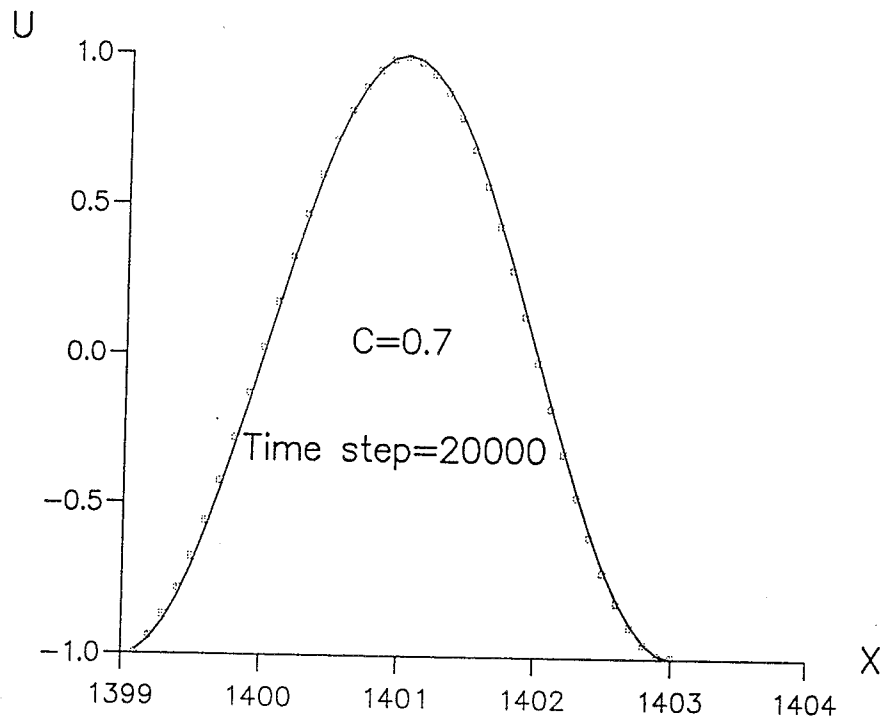Figure 2: Numerical Solution by the 3rd-order Method (symbol) and the Exact Solution (line)

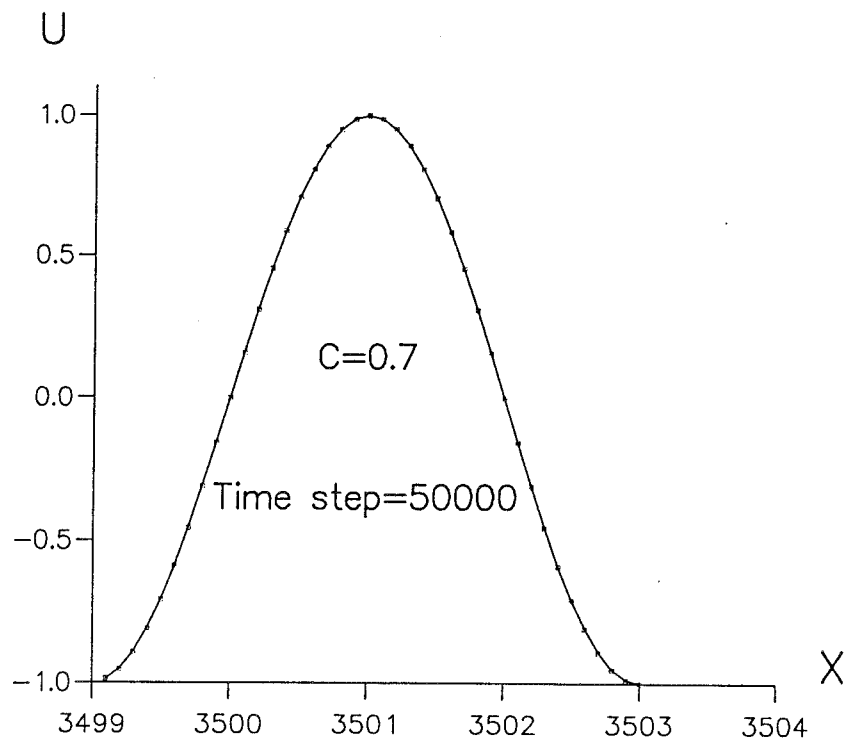Figure 3: Numerical Solution by the 4th-order Method (symbol) and the Exact Solution (line)

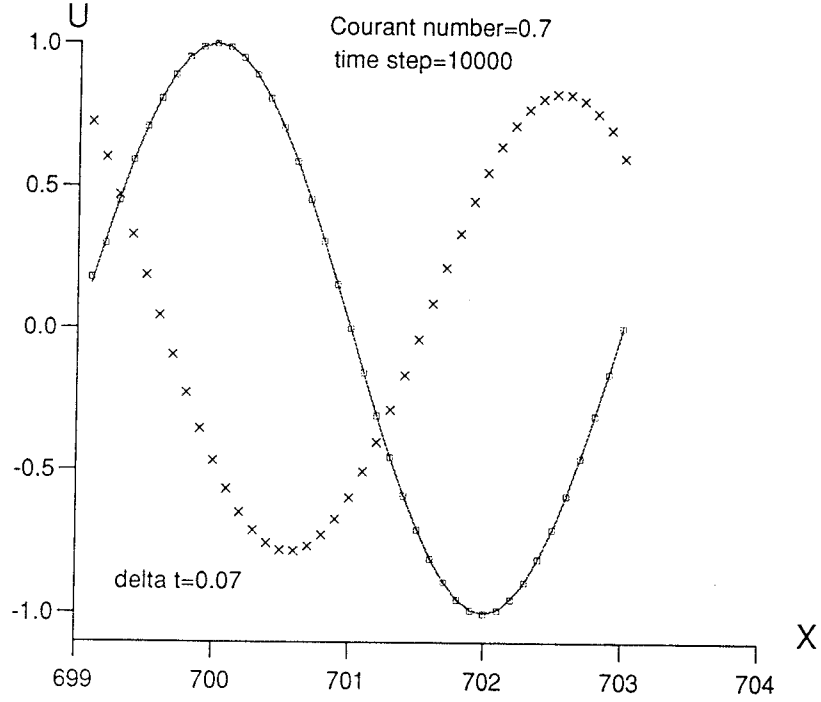Figure 4: Numerical Solution by the 20th-order Method (symbol) and the Exact Solution (line)

Figure 5: Comparision between the Lax-Wendroff Solution (crosses) and the 4th-order Method Solution (boxes)

APPENDIX

A Fully Discrete Third-order and Fourth-order Schemes

1. Seven-point Third-order Scheme

By unifying the pair of one-side four-point schemes

$$\begin{cases} U_j^{n+1} = f(U_{j-3}^n, U_{j-2}^n, U_{j-1}^n, U_j^n) \\ U_j^{n+1} = f(U_{j+3}^n, U_{j+2}^n, U_{j+1}^n, U_j^n) \end{cases}$$

the numerical flux of the scheme can be written as

$$\begin{aligned} F(U^n; j) = {} & \frac{1}{2}\left(F_j^n + F_{j+1}^n\right) - \frac{|a|}{2}\Delta U_{j+\frac{1}{2}} + |a|\left(\frac{5}{6} + \frac{c^2}{6} - |c|\right)\Delta U_{j+k_1+\frac{1}{2}} \\ & - |a|\left(\frac{1}{3} - \frac{|c|}{2} + \frac{c^2}{6}\right)\Delta U_{j+k_2+\frac{1}{2}} \end{aligned} \tag{48}$$

where

$$\begin{cases} k_1 = -1, & k_2 = -2 \quad if \quad c > 0 \\ k_1 = 1, & k_2 = 2 \quad\;\; if \quad c < 0 \end{cases} \tag{49}$$

The stability condition of the scheme is:

$$1 \leq |c| \leq 2 \tag{50}$$

2. Seven-point Fourth-order Scheme

Unifying the following pair of five-point schemes

$$\begin{cases} U_j^{n+1} = f(U_{j-3}^n, U_{j-2}^n, U_{j-1}^n, U_j^n, U_{j+1}^n) \\ U_j^{n+1} = f(U_{j+3}^n, U_{j+2}^n, U_{j+1}^n, U_j^n, U_{j-1}^n) \end{cases}$$

the numerical flux of the scheme

$$F(U^n; j) = \frac{1}{2}(F_j^n + F_{j+1}^n) - \frac{|a|}{2}\Delta U_{j+\frac{1}{2}} + |a| \left(\frac{1}{4} - \frac{11}{24}|c| + \frac{1}{4}c^2 - \frac{1}{24}|c|^3\right)\Delta U_{j+\frac{1}{2}}$$

$$+|a| \left(\frac{1}{12}|c|^3 - \frac{1}{3}c^2 - \frac{|c|}{12} + \frac{1}{3}\right)\Delta U_{j+k_1+\frac{1}{2}} - |a| \left(\frac{1}{24}|c|^3 - \frac{1}{12}c^2 - \frac{1}{24}|c| + \frac{1}{12}\right)\Delta U_{j+k_2+\frac{1}{2}} \tag{51}$$

where

$$\begin{cases} k_1 = -1, & k_2 = -2 \quad if \quad c > 0 \\ k_1 = 1, & k_2 = 2 \quad\;\; if \quad c < 0 \end{cases} \tag{52}$$

The stability condition of the scheme is:

$$|c| \leq 2 \tag{53}$$

3. Nine-point Fourth-order Scheme

From the pair of five-point one-side upwind schemes

$$\begin{cases} U_j^{n+1} = f(U_{j-4}^n, U_{j-3}^n, U_{j-2}^n, U_{j-1}^n, U_j^n) \\ U_j^{n+1} = f(U_{j+4}^n, U_{j+3}^n, U_{j+2}^n, U_{j+1}^n, U_j^n) \end{cases}$$

we have the numerical flux of the scheme as

$$F(U^n; j) = \frac{1}{2}(F_j^n + F_{j+1}^n) - \frac{|a|}{2}\Delta U_{j+\frac{1}{2}} + |a| \left(\frac{13}{12} - \frac{35}{24}|c| + \frac{5}{12}c^2 - \frac{1}{24}|c|^3\right)\Delta U_{j+k_1+\frac{1}{2}}$$

$$+|a| \left(\frac{|c|^3}{12} - \frac{2}{3}c^2 + \frac{17}{12}|c| - \frac{5}{6}\right)\Delta U_{j+k_2+\frac{1}{2}} + |a| \left(\frac{1}{4} - \frac{11}{24}|c| + \frac{1}{4}c^2 - \frac{1}{24}|c|^3\right)\Delta U_{j+k_3+\frac{1}{2}} \tag{54}$$

where

$$\begin{cases} k_1 = -1, \ k_2 = -2, \ k_3 = -3 & if \ c > 0 \\ k_1 = 1, \ k_2 = 2, \ k_3 = 3 & if \ c < 0 \end{cases} \tag{55}$$

The stability condition of the scheme is:

$$1 \leq |c| \leq 3 \tag{56}$$