Cranfield UNIVERSITY

Comparison of the Time Vector Method and the State-Space Method (Eigenvector Analysis) for Aircraft Parameter Identification

Y.Baek

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Notations

c	Spring constant or stiffness	
f	Frequency	(Hz)
m	Mass	
k_c	Damping Coefficient	
q	Pitch rate perturbation	(deg/sec)
P	Period	(sec)
t_D	Relaxation time	(sec)
U_0	Trim speed along X-body axis	(m/sec)
$\mathbf{V}$	Eigenvector or modal matrix	
$\mathbf{v}_i$	Eigenvector	
w	Velocity perturbation along OZ axis	(m/sec)
$\mathbf{x}_0$	Initial condition	
$\mathbf{x}_i$	Zero-input response	
$\mathbf{x}_s$	Zero-state response	

Greek Symbols

ε_D	Damping angle	(deg)
δ	Logarithmic decrement	
$\Phi(t)$	State transient matrix (STM)	
Λ	Diagonal eigenvalue matrix	
λ_i	Eigenvalues	
θ	Pitch attitude perturbation	(deg)
ω	damped frequency	(rad/sec)
ω_n	Undamped Natural frequency	(rad/sec)
ζ	Damping ratio	
ζ_p	Damping ratio of phugoid mod	
σ	Product of ω_n and ζ	(rad/sec)

Aerodynamic Derivatives

M_η	Dimensional elevator power	(1/sec^2/rad)
M_q	Dimensional pitch damping derivative	(1/sec)
M_u^*	Pitching moment with forward speed	(1/m-sec)
M_w	Pitching moment due to vertical velocity	(1/m)

Z_η	Z-force due to elevator deflection	(1/sec^2/rad)
Z_q	Z-force due to pitch rate	(1/sec)
Z_u^*	Z-force due to forward speed	(1/m-sec)
Z_w	Z-force due to vertical velocity	(1/m-sec)
X_η	X-force due to elevator deflection	(1/sec^2/rad)
X_q	X-force due to pitch rate	(1/sec)
X_u^*	X-force due to forward speed	(1/m-sec)
X_w	X-force due to vertical velocity	(1/m-sec)

Abbreviations

EVM	Eigen Vector Method
STM	State Transient Matrix
TVM	Time Vector Method

1. Introduction

Many methods for aircraft parameter identification have been developed and are currently in use. Prior to the development of computer methods the graphical Time Vector Method (TVM) was introduced during 1950's (ref. 1,2). Basically the TVM is based on the mechanical vibration theory. When the aircraft responses have oscillatory modes they can be considered as vibrations even though the typical frequency of these may be higher than those of the mechanical vibrations. Because of the graphical approach, the TVM may produce inaccurate results. But it gives us thorough understanding of the physical system. Owing to the rapid development of computers and sophisticated parameter identification algorithms, the rather limited graphical method has fallen into disuse and the others have emerged to replace it. They are, typically, the output error methods, the equation error methods, the Kalman filter estimator, the maximum likelihood technique, and so on (ref. 3~6).

The damping angle, natural frequency, relative magnitude and phase angle of each dynamic mode are the basis of the TVM. In this respect the state-space models can provide the same information. In particular the eigenvalues and eigenvectors which involve all the response information of the system, have the same properties as the time vectors. So, the objective of this report is to show the relationship between the TVM and the state-space method of an analysis and hence to facilitate the state-space method (eigenvector analysis) for aircraft parameter identification.

2. Review of the Time Vector Method

2.1 One Degree of Freedom Oscillator without damping

First the simple spring-mass arrangement of Fig. 2.1a may be considered. When released from an initial deflection, x_0, it oscillates harmonically about its equilibrium position. As no energy is dissipated, the amplitude remains constant. The time history of the displacement is therefore simply a cosine-line, Fig. 2.1b.

A cosine-line can be considered as generated by a 'time vector' of fixed length rotating with constant angular velocity, ω_n, about the origin, Fig. 2.1c. During one period, P, of the oscillation the vector rotates through 2π radians, hence :

$$\omega_n = \frac{2\pi}{P} = 2\pi f \text{ (rad/sec)} \tag{2.1}$$

Thus ω_n is also a measure for the frequency, f (Hertz), with which the vector completes full circular motions.

From the circular motion of the time vector the cosine-wave is generated by plotting the projection of the vector on a fixed line, L, through the origin against the phase angle, i.e., against the instantaneous angular position, $\varepsilon = \omega t$, of the vector. This is indicated by the second scale, ωt, on the abscissa in Fig. 2.1b.

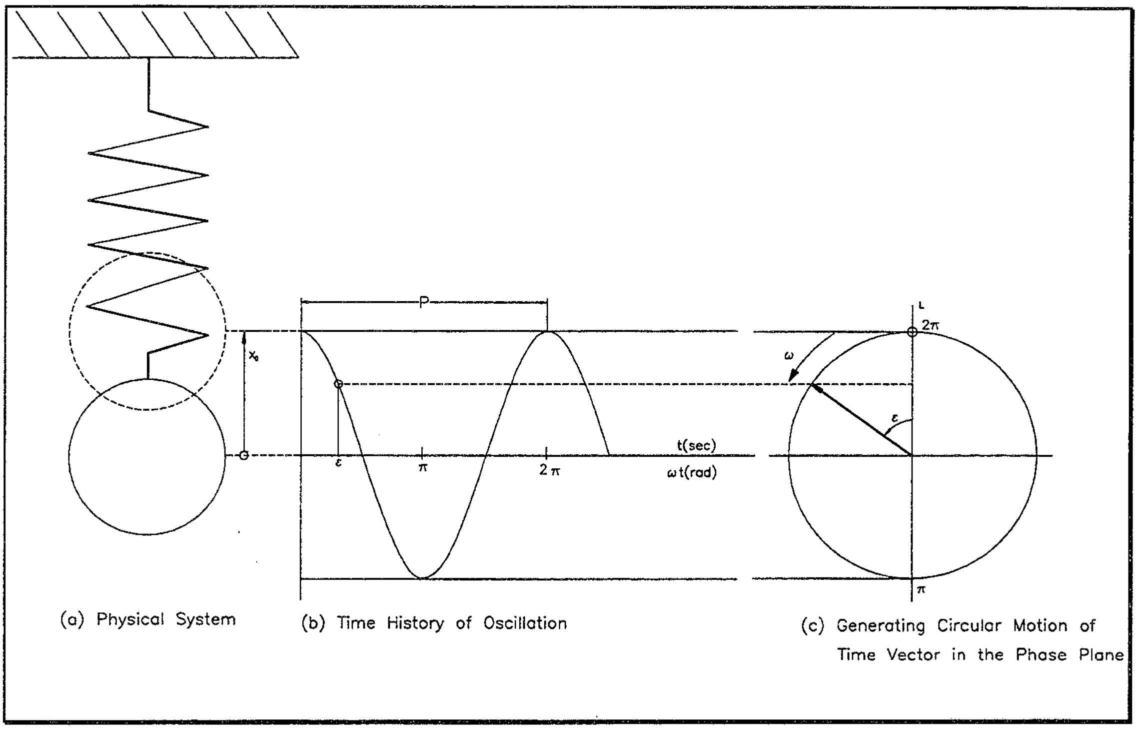


Figure 2.1 Undamped Oscillator with One Degree of Freedom

The velocity and acceleration of the oscillating mass are easily obtained by differentiating the cosine function with respect to time. They are shown in Fig. 2.2b. If the displacement is given by

$$x = x_0 \cos \omega t \tag{2.2}$$

the velocity is

$$\frac{dx}{dt} = \dot{x} = -x_0 \omega \sin \omega t = x_0 \omega \cos(\omega t + \frac{\pi}{2}) \tag{2.3}$$

and the acceleration

$$\frac{d^2 x}{dt^2} = \ddot{x} = -x_0 \omega^2 \cos \omega t = x_0 \omega^2 \cos(\omega t + \pi) \tag{2.4}$$

Thus when any derivative of a component of the motion is differentiated the new derivative multiplies its amplitude by ω and leads the preceding one by 90 deg.

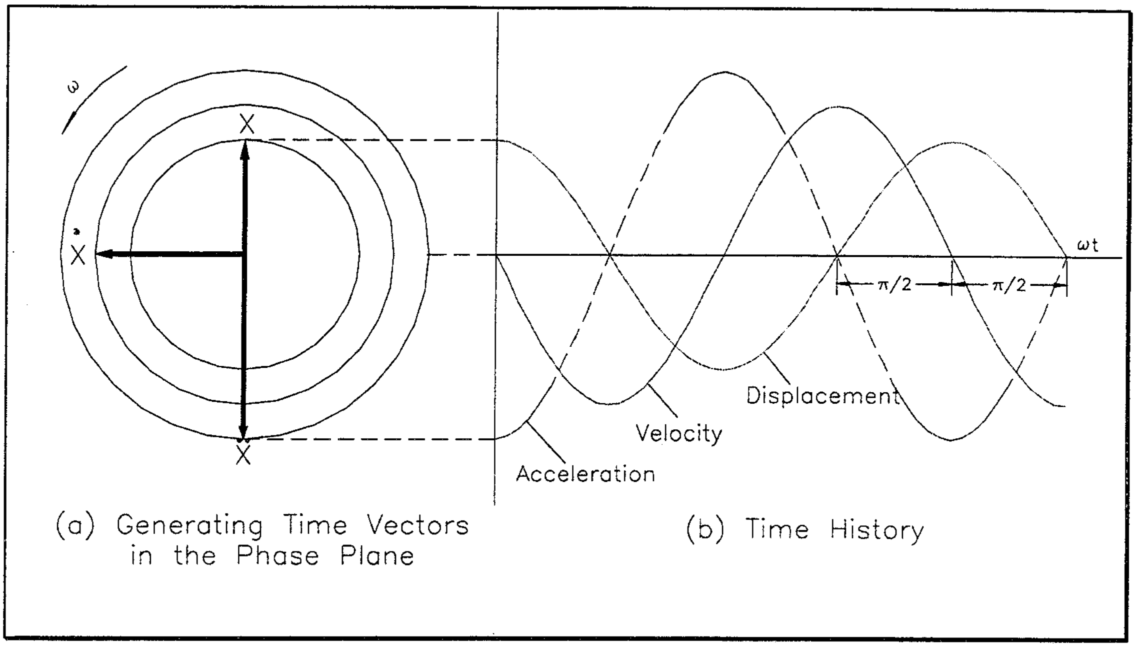


Figure 2.2 Undamped Oscillator, Motion Variables

The forces acting on the mass are spring force and the inertial resistance. Hence the equation of motion is

$$m\ddot{x} + cx = 0 \tag{2.5}$$

where *c* is the spring constant or stiffness. From equations (2.2) and (2.4) equation (2.5) can be written,

$$m\omega^2 x_0 = cx_0 \tag{2.6}$$

whence, the undamped natural frequency is given by,

$$\omega^2 = \frac{c}{m} \tag{2.7}$$

In order to avoid confusion it should be stressed that time vectors have no real physical vector character like force or speed themselves ; they are just graphical representations of oscillating quantities, whether scalar or vector, in the phase plane. The same mathematical rules, however, apply to time vectors as to real vectors. For example, two or more time vectors representing forces may be added geometrically to give one resultant time vector.

It is probably more useful to describe a vector by a single complex number which is then called the *complex amplitude*. The phase plane is then replaced by the complex plane. This is indeed advantageous because the process of projecting the time vector onto a line through the origin to obtain the actual displacement, etc., of the oscillator becomes equivalent to simply isolating the real part of the complex amplitude. This process is illustrated by Fig. 2.3.

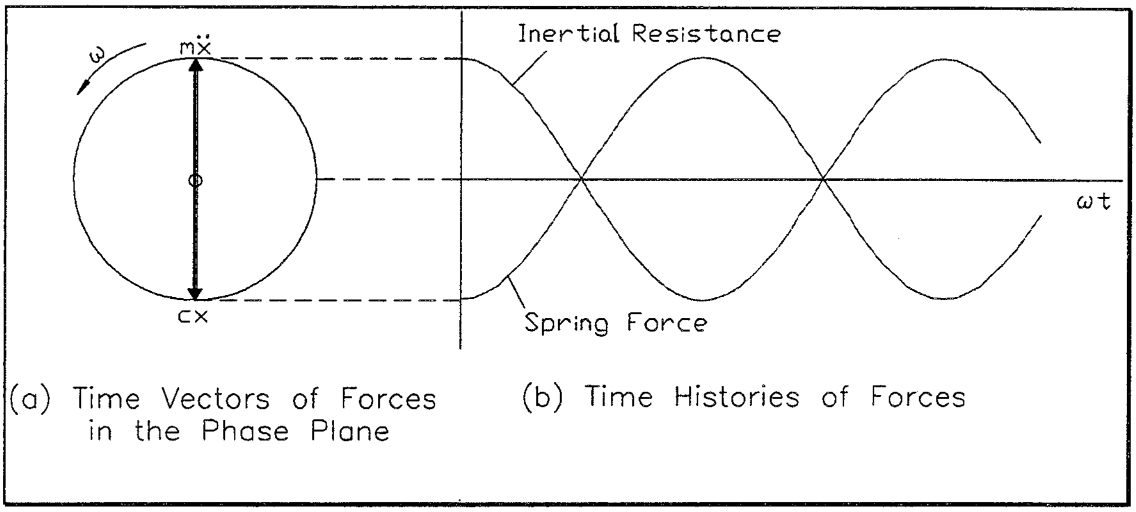


Figure 2.3 Forces Acting in the Oscillator

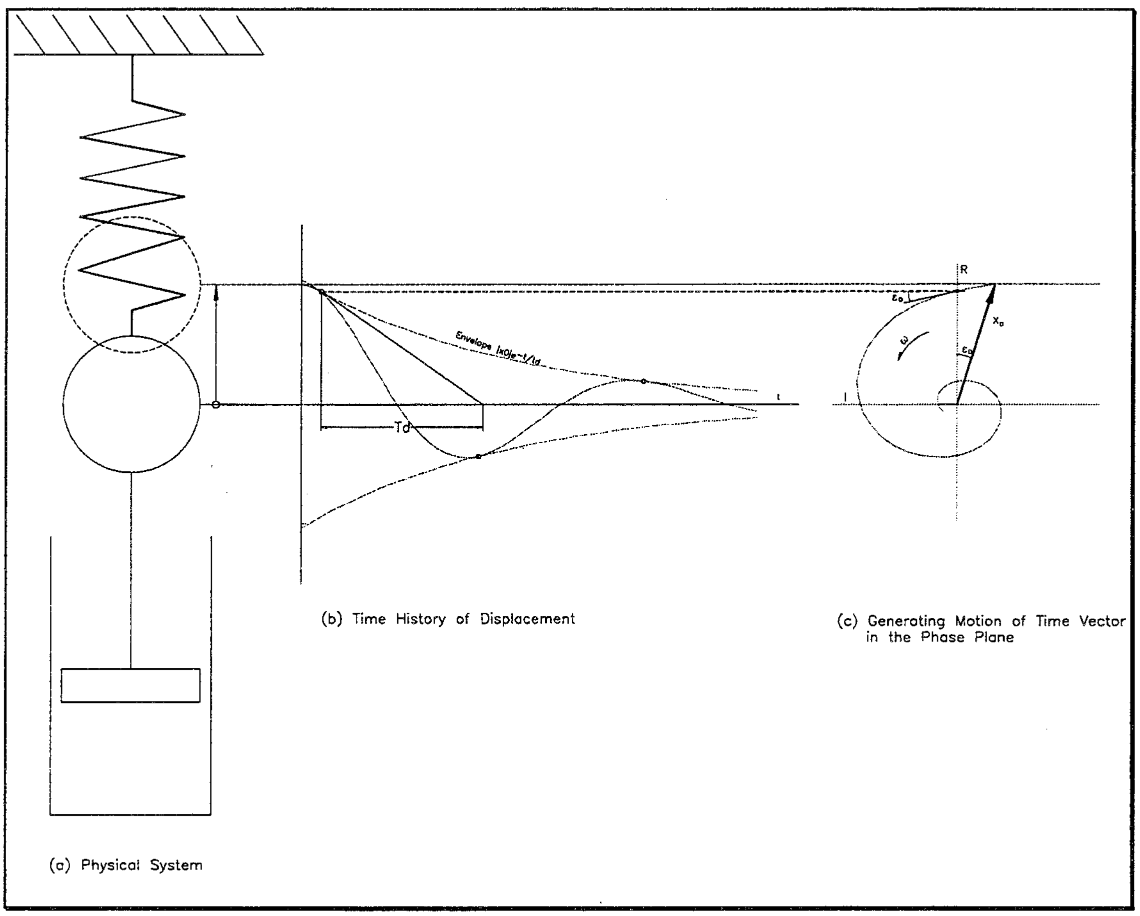


Figure 2.4 Damped Oscillator with One Degree of Freedom

2.2 One Degree of Freedom Oscillator with Damping

2.2.1 Equation of Motion

Next the oscillator with spring, mass and damper in Fig. 2.4a will be considered. The damper is assumed to add purely viscous friction so that $F_{(\dot{x})} = k\dot{x}$. Hence the differential equation describing unforced motion of this oscillator is ,

$$m\ddot{x} + k\dot{x} + cx = 0 \tag{2.8}$$

and the general solution is assumed to be,

$$\begin{aligned} x &= \exp(\lambda t) \\ \dot{x} &= \lambda \exp(\lambda t) \\ \ddot{x} &= \lambda^2 \exp(\lambda t) \end{aligned} \tag{2.9}$$

where λ is constant. Substituting equation (2.9) into equation (2.8) yields

$$(\lambda^2 + \frac{k}{m}\lambda + \frac{c}{m})\exp(\lambda t) = 0 \tag{2.10}$$

where the quadratic equation is the so-called *characteristic equation* and the solutions of this equation are,

$$\lambda_{1,2} = -\frac{k}{2m} \pm \sqrt{\left(\frac{k}{2m}\right)^2 - \frac{c}{m}} \tag{2.11}$$

Whence, the general solution of equation (2.8) may be written

$$x = A\exp(\lambda_1 t) + B\exp(\lambda_2 t) \tag{2.12}$$

In order to examine the properties of the general solution, substitute equation (2.11) into equation (2.12) to obtain,

$$x = \exp\left(-\frac{k}{2m}t\right)\left\{A\exp\left(\sqrt{\left(\frac{k}{2m}\right)^2 - \frac{c}{m}}t\right) + B\exp\left(-\sqrt{\left(\frac{k}{2m}\right)^2 - \frac{c}{m}}t\right)\right\} \tag{2.13}$$

In the general solution the decay factor is,

$$\exp\left(-\frac{k}{2m}t\right)=\exp\left(-\frac{t}{t_D}\right) \tag{2.14}$$

in which t_D is the damping time during which the amplitude decrease to $1/e = 0.368$ of its original value. It is sometimes called the *relaxation time*.

When the system is critically damped, which means the value of the square root in equation (2.13) is zero, the damping coefficient is,

$$k_c = 2m\sqrt{\frac{c}{m}} = 2m\omega_n \tag{2.15}$$

The fixed amplitude ratio of two consecutive peaks in response, i.e., a full period apart, is usually given by its Naperian logarithm and is called the *logarithmic decrement*, δ. This is connected with damping time by the simple relation :

$$\delta = \frac{P}{t_D} = \frac{2\pi}{\omega t_D} \tag{2.16}$$

2.2.2 The Damping Angle

When making the geometrical projection of the amplitude time vector it is seen that this projection reaches its maximum numerical value not when the vector lies along the positive real axis but, slightly earlier, as indicated in Fig. 2.4c. The phase angle for this position may be called the damping angle, ε_D. This angle is the same as that between the perpendicular to the radius vector and the tangent to the spiral trajectory in the phase plane which is constant. The angle is zero for no damping and it increases with damping, making the spiral trajectory converge more rapidly. It becomes negative for negative damping when the spiral is divergent.

In the engineering applications the damping is often expressed by its ratio to the critical damping,

$$\zeta = \frac{k}{k_c} = \frac{k}{2m\omega_n} = \frac{1}{t_D\omega_n} \tag{2.17}$$

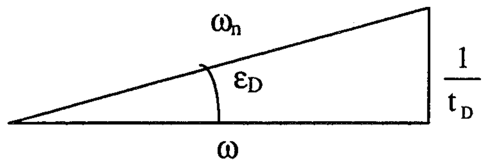
where ζ is the damping ratio.

As described earlier the TVM is connected with the oscillating system, which is underdamped . Now the circular frequency or *damped natural frequency* ,ω, defined as

$$\omega = \sqrt{\frac{c}{m}-\left(\frac{k}{2m}\right)^2} = \sqrt{\omega_n^{\,2}-\left(\frac{1}{t_D}\right)^2} = \omega_n\sqrt{1-\zeta^2} \tag{2.18}$$

It is evident that the addition of viscous friction reduces the circular frequency. Furthermore it is seen that this reduction is independent of the sign of t_D, i.e., the frequency is always reduced whether the oscillation is positively damped or divergent. The time history of such a damped oscillation is shown in Fig. 2.4b. The envelope is the time function of the decay factor and the damping time t_D appears as the sub-tangent to the envelope at any point.

Also the change of circular frequency with viscous friction can be expressed in terms of the damping angle as follows. Equation (2.18) can be represented by the relationship of the sides of a right angle triangle as shown below,



Using this diagram the circular frequency can be redefined and ε_D can be expressed in terms of damping time and undamped circular frequency,

$$\omega = \omega_n \cos\varepsilon_D$$
$$\varepsilon_D = \sin^{-1}\left(\frac{1}{t_D\omega_n}\right) = \sin^{-1}\zeta \qquad (2.19)$$

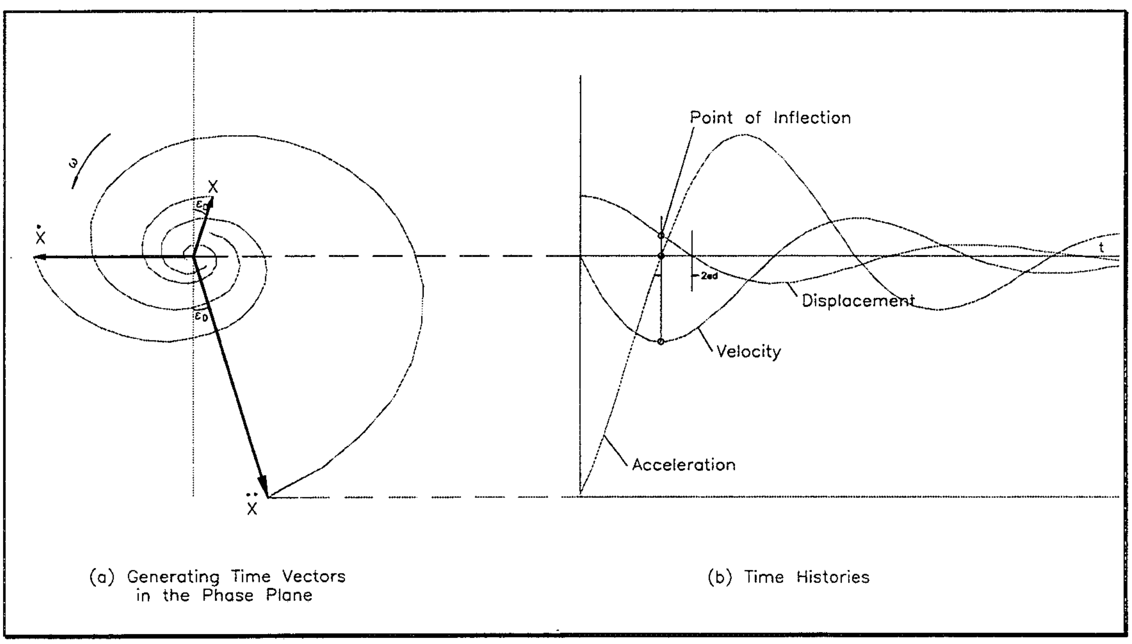


Figure 2.5 Damped Oscillator, Motion Variables

2.2.3 Velocity and Acceleration

As stated above the displacement vector is offset from the vertical axis in the phase plane by ε_D, i.e., in the position indicated in Fig. 2.4c. At this instant, t = 0, there must be zero projection of the velocity vector because the velocity of the physical system is zero. This means that the vector must fall along the imaginary axis, Fig 2.5a. Thus the phase angle between the time vector of displacement and that of velocity is 90 deg + ε_D.

After rotating the diagram into the position of maximum projection for the velocity vector the same reasoning can be applied to show that the phase angle between the acceleration and the velocity time vectors is again 90 deg + ε_D. Hence the acceleration vector not in counter-phase to the displacement vector but leads by an additional angle of $2\varepsilon_D$, Fig. 2.5a.

This difference is also apparent in the time history of the variables of motion, Fig 2.5b, where evidently zero acceleration must coincide with the point of inflection of the displacement curve. This is $2\varepsilon_D$ ahead of zero displacement.

The amplitude of velocity and acceleration may be obtained by analytic differentiation of equation (2.13). Substituting equation (2.14) and (2.18) into equation (2.13) yields

$$x = \exp\left(-\frac{t}{t_D}\right)\{A\exp(i\omega t) + B\exp(-i\omega t)\}$$

$$= \exp\left(-\frac{t}{t_D}\right)\{(A+B)\cos\omega t\_i(A-B)\sin\omega t\} \tag{2.20}$$

where $\exp(\pm i\omega t) = \cos\omega t \pm i \sin\omega t$. Using the initial condition of zero velocity, equation (2.20) can be expressed as follows.

$$x = x_0 \exp\left(-\frac{t}{t_D}\right)\{\cos\omega t + \tan\varepsilon_D \sin\omega t\}$$

$$= x_0 \frac{\omega_n}{\omega}\exp\left(-\frac{t}{t_D}\right)\cos(\omega t - \varepsilon_D) \tag{2.21}$$

and the velocity is,

$$\begin{aligned}\dot{x} &= x_0 \frac{\omega_n}{\omega} \exp\left(-\frac{t}{t_D}\right)\left\{\left(-\frac{1}{t_D}\right)\cos(\omega t - \varepsilon_D) - \omega \sin(\omega t - \varepsilon_D)\right\} \\ &= -x_0 \frac{\omega_n^{\ 2}}{\omega} \exp\left(-\frac{t}{t_D}\right)\sin \omega t \\ &= x_0 \frac{\omega_n^{\ 2}}{\omega} \exp\left(-\frac{t}{t_D}\right)\cos\left(\omega t + \frac{\pi}{2}\right)\end{aligned} \tag{2.22}$$

Similarly,

$$\begin{aligned}\ddot{x} &= -x_0 \frac{\omega_n^{\ 2}}{\omega} \exp\left(-\frac{t}{t_D}\right)\left\{\left(-\frac{1}{t_D}\right)\sin \omega t + \omega \cos \omega t\right\} \\ &= -x_0 \frac{\omega_n^{\ 3}}{\omega} \exp\left(-\frac{t}{t_D}\right)\cos(\omega t + \varepsilon_D) \\ &= x_0 \frac{\omega_n^{\ 3}}{\omega} \exp\left(-\frac{t}{t_D}\right)\cos(\omega t + \pi + \varepsilon_D)\end{aligned} \tag{2.23}$$

It can be summarized that when any derivative of a component of motion is differentiated the new derivative multiplies its amplitude by ω_n and leads it by 90 deg + ε_D.

2.2.4 Isosceles Triangle of the Forces Acting on the Oscillator

Having determined the relative phase angles of the motion variables the corresponding time vectors of the forces acting on the damped oscillator, can be plotted. The spring and the inertial forces are not in counter phase and together with the damping force they must form a closed polygon in order that at every instant the projection of all three vectors may add up to zero. From the phase relationship, Fig. 2.6a and 2.7a, it can be seen that this polygon must be an isosceles triangle, Fig 2.7b.

The isosceles triangle of forces is most important for the application of the TVM. It will be seen by reference to Fig. 2.7 that the vertex angle of this triangle is $2\varepsilon_D$, showing the degree of damping. The ratio of the height of the triangle to the sides is equal to $\cos\varepsilon_D$ and can therefore be used as a measure of the frequency of the damped motion , ω, as compared with that of the undamped motion, ω_n. A notable consequence of the isosceles triangle is the fact that the modulus of the spring force vector is always equal to that of the inertial force vector.

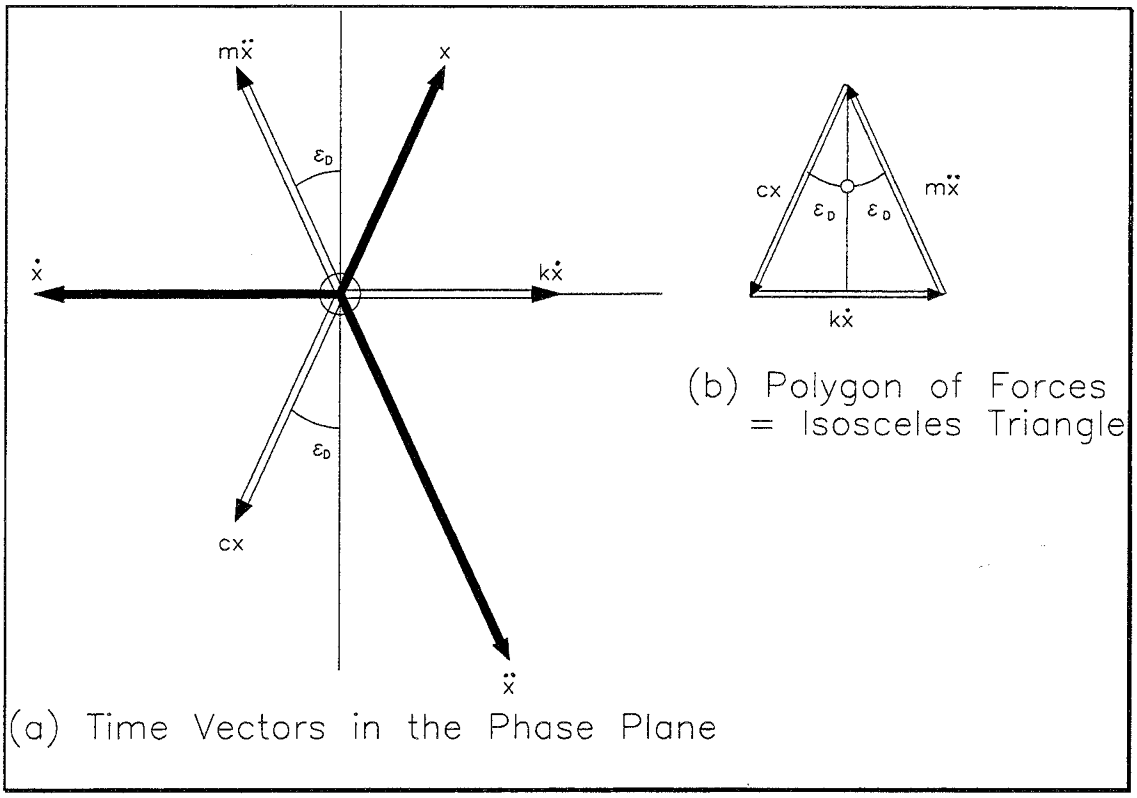


Figure 2.6 Forces Acting on the Damped Oscillator

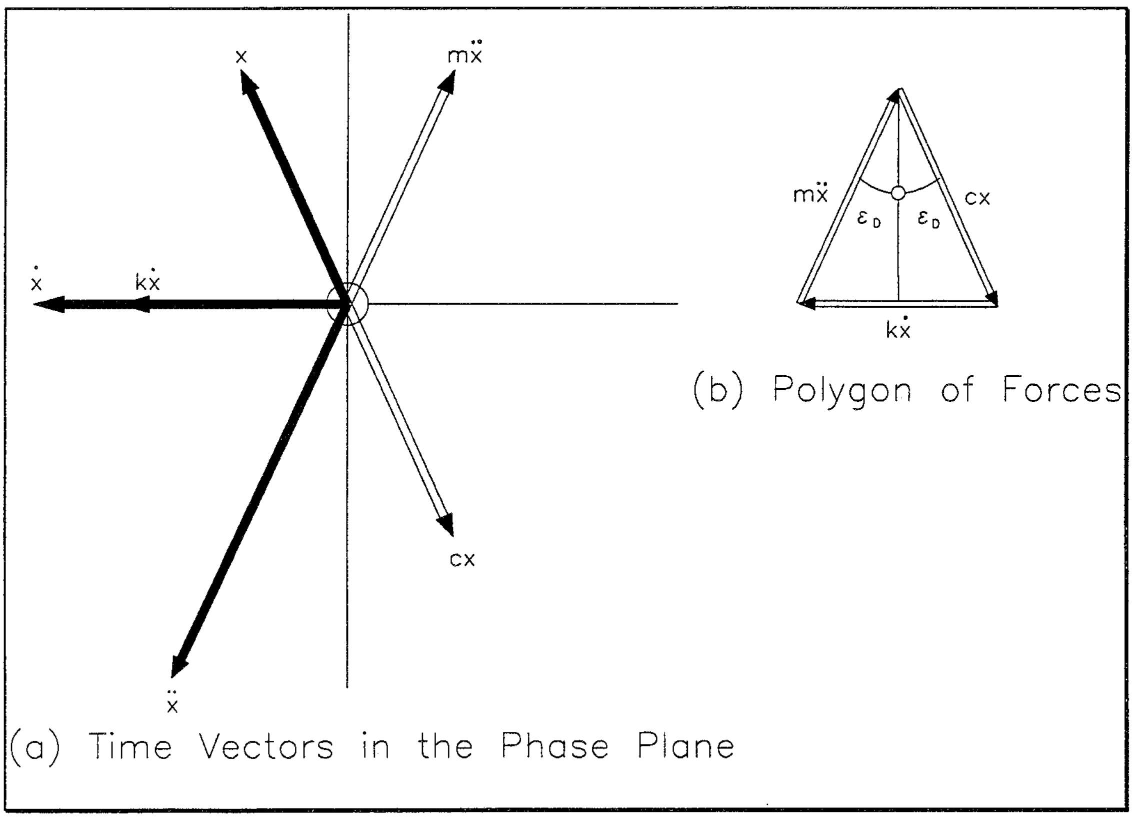


Figure 2.7 Forces Acting on the Oscillator with Negative Damping

2.2.5 Vector Notation

The notation of the basic equations for the time vector representation may be chosen similar to that of the differential equations; it must, however, give amplitude and phase of each term explicitly. How this can be achieved may best be demonstrated by means of the one degree of freedom oscillator described by equation (2.8), repeated below. The results are also correct for systems with more than one degree of freedom as long as only one particular oscillatory mode of motion is considered.

$$m\ddot{x}+k\dot{x}+cx=0 \tag{2.8}$$

With equation (2.20), (2.21) and (2.22) this becomes

$$\left\{mx_0\omega_n^{\;2}\exp\left(-\frac{t}{t_D}\right)\cos(\omega t+\pi+\varepsilon_D)\right\}+\left\{kx_0\omega_n\exp\left(-\frac{t}{t_D}\right)\cos(\omega t+\frac{\pi}{2})\right\}$$

$$+\left\{cx_0\exp\left(-\frac{t}{t_D}\right)\cos(\omega t-\varepsilon_D)\right\}=0 \tag{2.24}$$

or, equivalently,

$$\text{inertial force + damping force + spring force = 0} \tag{2.25}$$

The decay factor $\exp\left(-\frac{t}{t_D}\right)$ can be omitted if the time vector system is to be considered at a given instant, e.g., t = 0. The cosine expressions give the phase of each term and the remainder is the amplitude of each time vector. Thus the desired separate identification of both amplitude and phase can be achieved.

When writing the amplitude and phase components it will be sufficient, and indeed more convenient for the subsequent graphical treatment, to indicate the phase only by the motion variables, e.g., $\bar{x}, \bar{\dot{x}}$ and $\bar{\ddot{x}}$, to which the vector term refers together with the appropriate sign, to indicate whether the force or moment acts in phase with, or in counterphase to, the variable. The phase of the variable itself is then derived graphically. The following notation is therefore used,

Moduli :	$mx_0\omega_n^2$	$kx_0\omega_n$	cx_0	
Phases :	$\ddot{x}$	$\dot{x}$	$\bar{x}$	= 0

The zero on the right-hand side is to be understood as postulating the closure of the polygon made up by the time vectors listed on the left.

3. State-Space Method

So far the graphical method has been considered in order to investigate the properties of a simple oscillatory system. Since the computational mechanism is based on the use of matrix algebra it is most conveniently handled by a digital computer. In this chapter the general analysis of the equivalent state-space model and its eigenstructure is considered.

3.1 State-Space Equations

Consider a linear time invariant (LTI) system, then the matrix state and output equation can be expressed,

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned} \tag{3.1}$$

The variables **x**(t), **u**(t), and **y**(t) are column vectors, and **A, B, C**, and **D** are matrices having constant elements.

3.1.1 Homogeneous Solution (State Transition Matrix : STM)

The homogeneous state equation, with the input **u**(t)=0, is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \tag{3.2}$$

where **A** is a constant $n \times n$ matrix and **x** is an $n \times 1$ column vector. The general solution of the equation (3.2) is given by

$$\mathbf{x}(t) = \exp[\mathbf{A}(t - t_0)]\mathbf{x}(t_0) \tag{3.3}$$

The analogous exponential function of a square matrix **A** in equation (3.3) with $t_0 = 0$ using the infinite series becomes

$$e^{\mathbf{A}t} = \exp(\mathbf{A}t) = \mathbf{I} + \frac{\mathbf{A}t}{1!} + \frac{(\mathbf{A}t)^2}{2!} + \frac{(\mathbf{A}t)^3}{3!} + \cdots + \frac{(\mathbf{A}t)^k}{k!} + \cdots \tag{3.4}$$

Thus exp(**A**t) is a square matrix of the same order as **A**. It is common to call this the *state transition matrix* (STM) or the *fundamental matrix* of the system and to denote by

$$\Phi(t) = e^{\mathbf{A}t} = \exp(\mathbf{A}t) \tag{3.5}$$

The STM is descriptive of the unforced or natural response. Hence the solution of the equation (3.2) at $t_0 = 0$ will be

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) \tag{3.6}$$

3.1.2 Eigenvalues and Eigenvectors

Consider a system of equations represented by equation (3.2). One case for which a solution of this equation exists is if $\mathbf{x}$ and $\dot{\mathbf{x}}$ have the same direction in the state space but differ only in magnitude by a scalar proportionality factor λ. The solution must therefore have the form $\dot{\mathbf{x}}(t) = \lambda\mathbf{x}(t)$. Inserting this into equation (3.2) and rearranging terms yields

$$[\ \lambda\mathbf{I} - \mathbf{A}\ ]\mathbf{x}(t) = 0 \tag{3.7}$$

This equation has a nontrivial solution only if $\mathbf{x}$ is not zero. Therefore the determinant of the coefficients of $\mathbf{x}$ must be zero and,

$$\Delta(\lambda) \equiv |\lambda\mathbf{I} - \mathbf{A}| = 0 \tag{3.8}$$

The resulting polynomial equation (3.8) is called the *characteristic equation* and the roots λ_i of the characteristic equation are called *eigenvalues* of $\mathbf{A}$. An eigenvalue λ_i and its corresponding non-zero *eigenvector* $\mathbf{v}_i$ are such that

$$\mathbf{A}\mathbf{v}_i = \lambda_i\,\mathbf{v}_i \tag{3.9}$$

Therefore the equation (3.8) may be written,

$$[\ \lambda_i\mathbf{I} - \mathbf{A}\ ]\ \mathbf{v}_i = 0 \tag{3.10}$$

Since $\mathbf{v}_i \neq 0$ then $[\ \lambda_i\mathbf{I} - \mathbf{A}\ ]$ is singular. The eigenvectors $\mathbf{v}_i$ are always linearly independent hence the eigenvalues λ_i are distinct. When the eigenvalue is complex its corresponding eigenvector is also complex and the complex conjugate λ_i^* corresponds with the complex conjugate $\mathbf{v}_i^*$.

The *eigenvector* or *modal matrix* comprises all of the eigenvectors and is defined

$$\mathbf{V} = [\ \mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3 \cdots \mathbf{v}_m\ ] \tag{3.11}$$

Substituting equation (3.11) into (3.9) gives

$$\mathbf{AV} = \mathbf{V}\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \mathbf{0} \\ & & \ddots & \\ & \mathbf{0} & & \ddots & \\ & & & & \lambda_n \end{bmatrix} \equiv \mathbf{V}\Lambda \qquad (3.12)$$

where Λ is the *diagonal eigenvalue matrix.* Thus

$$\mathbf{V}^{-1}\mathbf{AV} = \Lambda \qquad (3.13)$$

and A is said to be similar to the diagonal eigenvalue matrix Λ.

Eigenvectors may be determined as follows. Now ,by definition,

$$[\lambda_i \mathbf{I} - \mathbf{A}]^{-1} = \frac{\mathrm{Adj}[\lambda_i \mathbf{I} - \mathbf{A}]}{|\lambda_i \mathbf{I} - \mathbf{A}|} \qquad (3.14)$$

and since, for any eigenvalue λ_i , $|\lambda_i\mathbf{I} - \mathbf{A}| = 0$, equation (3.14) may be rearranged and written,

$$[\lambda_i \mathbf{I} - \mathbf{A}]\mathrm{Adj}[\lambda_i \mathbf{I} - \mathbf{A}] = |\lambda_i \mathbf{I} - \mathbf{A}|\mathbf{I} = \mathbf{0} \qquad (3.15)$$

Comparing equation (3.15) with equation (3.10) the eigenvector $\mathbf{v}_i$ corresponding to the eigenvalue λ_i is defined

$$\mathbf{v}_i = \mathrm{Adj}[\lambda_i \mathbf{I} - \mathbf{A}] \qquad (3.16)$$

Any non-zero column of the adjoint matrix is an eigenvector and if there is more than one column they differ only by a constant factor, Eigenvectors are therefore unique in direction only and not in magnitude. However, the dynamic characteristics of a system determines the unique relationship between each of its eigenvectors.

3.1.3 Complete Solution of the State Equation

When an input $\mathbf{u}(t)$ is present, the complete solution for $\mathbf{x}(t)$ is obtained from equation (3.1). The derivatives of the product of two matrices are given by

$$\frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}(t)] = e^{-\mathbf{A}t}\left[\dot{\mathbf{x}}(t) - \mathbf{Ax}(t)\right] \qquad (3.17)$$

Utilising equation (3.1), in the right side of the equation (3.17) gives,

$$\frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}\mathbf{Bu}(t) \tag{3.18}$$

Integrating this equation between 0 and t gives,

$$e^{-At}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-A\tau}\mathbf{Bu}(\tau)d\tau \tag{3.19}$$

Multiplying by e^{At} and rearranging terms produces,

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}\mathbf{Bu}(\tau)d\tau \equiv \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{Bu}(\tau)d\tau \tag{3.20}$$

Using the STM and generalising for initial conditions at $t = t_0$ gives the solution to the state-variable equation with an input $\mathbf{u}(t)$ as,

$$\mathbf{x}(t) = \mathbf{\Phi}(t-t_0)\mathbf{x}(t_0) + \int_{t_0}^t \mathbf{\Phi}(t-\tau)\mathbf{Bu}(\tau)d\tau \qquad t > t_0 \tag{3.21}$$

This equation is called the *state transition equation*; i.e., it describes the change of state relative to the initial conditions $\mathbf{x}(t_0)$ and the input $\mathbf{u}(t)$. It may be more convenient to change the variables in the integral of equation (3.21) by letting $\beta = t - \tau$, which gives

$$\mathbf{x}(t) = \mathbf{\Phi}(\beta)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(\beta)\mathbf{Bu}(t-\beta)dt \tag{3.22}$$

All equations comprises two parts, and can be written in general form,

$$\mathbf{x}(t) = \mathbf{x}_i(t) + \mathbf{x}_s(t) \tag{3.23}$$

where $\mathbf{x}_i(t)$ is called the *zero-input response*, that is, $\mathbf{u}(t)=0$, and $\mathbf{x}_s(t)$ is called *zero-state response*, that is $\mathbf{x}(t_0)=0$.

3.2 One Degree of Freedom Oscillator with Damping

3.2.1 Equation of Motion

In order to compare the state space method with the time vector method the same equation of motion for a mass-spring-damper system is used. Again, writing,

$$m\ddot{x} + k\dot{x} + cx = 0 \quad (3.24)$$

The matrix state equation for this system may be realized by writing $z_1 = x$ and $z_2 = \dot{z}_1$. Using these relationships the equation (3.24) may be written in matrix form,

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (3.25)$$

or

$$\dot{\mathbf{z}} = \mathbf{Az} \quad (3.26)$$

3.2.2 The Eigenvalues

As described in paragraph 3.1.2 the characteristic equation may be written,

$$\lambda^2 + \frac{k}{m}\lambda + \frac{c}{m} = 0 \quad (3.27)$$

For a second order system the characteristic equation ,in general, is expressed in terms of the undamped natural frequency ,ω_n and damping ratio ,ζ. Hence equation (3.27) may be expressed by,

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n{}^2 = 0 \quad (3.28)$$

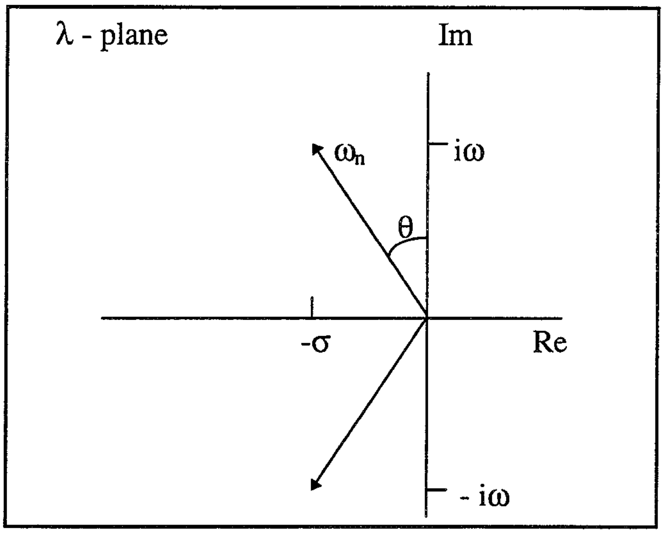
Whence, from the equation (3.27) and (3.28),

$$\omega_n{}^2 = \frac{c}{m}$$
$$\zeta = \frac{k}{2m\omega_n} = \frac{\sigma}{\omega_n} \quad (3.29)$$

Now, the roots or eigenvalues of equation (3.28) for $0 \le \zeta \le 1$ are,

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} = -\sigma \pm i\omega \quad (3.30)$$

where ω is damped frequency of oscillation. The diagram below clearly illustrates the relationships between damping ratio, undamped natural frequency and damped frequency.



From the diagram the angle θ between the eigenvalue and the imaginary axis can be defined as

$$\theta = \sin^{-1}\left(\frac{\sigma}{\omega_n}\right) = \sin^{-1}\zeta \tag{3.31}$$

3.2.3 The Solution of the Equation of Motion

Using the results of the complete solution of the state equation in paragraph 3.1.3 the solution of the equation of motion (3.26) will be just $\mathbf{z}(t) = \mathbf{z}_i(t)$ as input is zero in this case. In order to evaluate the state transient matrix the Cayley-Hamilton theorem is used (ref. 7) and the STM can be expressed by this theorem,

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \exp(\mathbf{A}t) = \sum_{k=0}^{n-1} \alpha_k(t)\mathbf{A}^k \tag{3.32}$$

where α_0 and α_1 are expressed below using equation (3.25), (3.28), and (3.30), diagram shown above and the Euler's equation(ref. 7),

$$\begin{aligned} \alpha_0 &= \frac{1}{2}\left[\left(e^{\lambda_2 t} + e^{\lambda_1 t}\right) - i\tan\theta\left(e^{\lambda_2 t} - e^{\lambda_1 t}\right)\right] = \exp\left(-\zeta\omega_n t\right)\left[\cos\omega t + \tan\theta\sin\omega t\right] \\ \alpha_1 &= i\frac{\left(e^{\lambda_2 t} - e^{\lambda_1 t}\right)}{2\omega} = -\frac{1}{\omega}\exp\left(-\zeta\omega_n t\right)\sin\omega t \end{aligned} \tag{3.33}$$

Substituting equation (3.33) into (3.32) assuming zero initial velocity yields,

$$x(t) = z_1(t) = x_0 \exp(-\zeta\omega_n t)\{\cos\omega t + \tan\theta \sin\omega t\}$$
$$= x_0 \frac{\omega_n}{\omega} \exp(-\zeta\omega_n t)\cos(\omega t - \theta) \quad (3.34)$$

and,

$$\dot{x} = \dot{z}_1 = x_0 \frac{\omega_n^2}{\omega} \exp(-\zeta\omega_n t)\sin(\omega t) = x_0 \frac{\omega_n^2}{\omega} \exp(-\zeta\omega_n t)\cos\left(\omega t + \frac{\pi}{2}\right) \quad (3.35)$$

4. Comparison of the Time Vector Method with the State-Space Method

4.1 Similarity

It has been shown above, as expected, both methods provide the same information although the notations are different. For convenience the analysis described earlier is summarized in table 4.1 to emphasize the obvious similarities.

	Time Vector Method	State-Space Method
EOM	$m\ddot{x} + k\dot{x} + cx = 0$	$m\ddot{x} + k\dot{x} + cx = 0$ $\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$
Characteristic Equation	$\lambda^2 + \frac{k}{m}\lambda + \frac{c}{m} = 0$	$\lambda^2 + \frac{k}{m}\lambda + \frac{c}{m} = 0$ $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$
Damping ratio	$\zeta = \frac{k}{k_c} = \frac{k}{2m\omega_n} = \frac{1}{t_D\omega_n}$	$\zeta = \frac{k}{2m\omega_n} = \frac{\sigma}{\omega_n}$
Damping Angle	$\varepsilon_D = \sin^{-1}\left(\frac{1}{t_D\omega_n}\right) = \sin^{-1}\zeta$	$\theta = \sin^{-1}\left(\frac{\sigma}{\omega_n}\right) = \sin^{-1}\zeta$
Undamped Natural Frequency	$\omega_n^2 = \frac{c}{m}$	$\omega_n^2 = \frac{c}{m}$
Solution	$x = x_0 \frac{\omega_n}{\omega} \exp\left(-\frac{t}{t_D}\right)\cos(\omega t - \varepsilon_D)$	$x = x_0 \frac{\omega_n}{\omega} \exp(-\zeta\omega_n t)\cos(\omega t - \theta)$

Table 4.1 Similarities of the time vector method and the state-space method

4.2 Comparison of the Procedures of Both Methods

An outline summary of the procedures for parameter identification by both methods is shown in table 4.2. However, the TVM is established as described earlier whereas, the state-space or, eigenvector method is merely suggested.

Step	Time Vector Method	State-Space Method
Logarithmic Decrement(δ)	Determine the positive peak values and plot their natural logarithm versus the time at which they occur	
Damped frequency	Calculate the period, P , then $\omega = \frac{2\pi}{P}$	
Phase Angle	$\phi = \frac{\text{TimeDifference}}{P} * 360(\text{deg})$	
Vector Magnitude	From the instant of the time calculate the relative magnitude of each mode	
Damping Time	$t_D = \frac{P}{\delta}$	
Damping Angle	$\varepsilon_D = \tan^{-1}\left(\frac{1}{t_D \omega_n}\right)$	
Damping Ratio		$\zeta = \sin \varepsilon_D$
Eigenvalues and Undamped Natural frequency	$\omega_n = \frac{\omega}{\cos \varepsilon_D}$	$\lambda_{1,2} = \zeta\omega_n \pm i\omega$ $\omega_n = \frac{\omega}{\sqrt{1-\zeta^2}}$
Diagonal Matrix		$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$
Eigenvector and Time Vector	Using the Phase angle and the relative magnitude construct time vector	Construct eigenvector matrix **V** from response data
Time Vector Polygon and Reconstruction of the State-Space Model	Using the Time vector and Equation of motion draw the time vector polygon and extract the parameters	From the Eigenvector and Diagonal matrix reconstruct the state-space model $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$

Table 4.2 Comparison of parameter identification procedures

4.3 Case Study of Aircraft Model in the Short Period Mode

For the case study the longitudinal dynamics the McDornnell Douglas Phantom F-4C aircraft, given in ref. 8 was chosen. In order to carry out the procedures described in section 4.2, the time responses shown in figure 4.1 for the TVM were obtained using state-space equation 4.1.

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u^* & X_w & -(W_0+X_q) & -g\cos\Theta_0 \\ Z_u^* & Z_w & (U_0+Z_q) & -g\sin\Theta_0 \\ M_u^* & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} X_\eta \\ Z_\eta \\ M_\eta \\ 0 \end{bmatrix} \eta \qquad (4.1)$$

where body axes are assumed. The corresponding numerical state equation is,

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.0677 & 0.0107 & 0 & -32.2 \\ 0.0226 & -2.11 & 1215 & 0 \\ 0.0033 & -0.0473 & -1.986 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} -2.2 \\ -250.18 \\ -60.917 \\ 0 \end{bmatrix} \eta \qquad (4.2)$$

By analysing the time responses in figure 4.1a and 4.1b following results were obtained for use in the TVM analysis of the longitudinal short period mode,

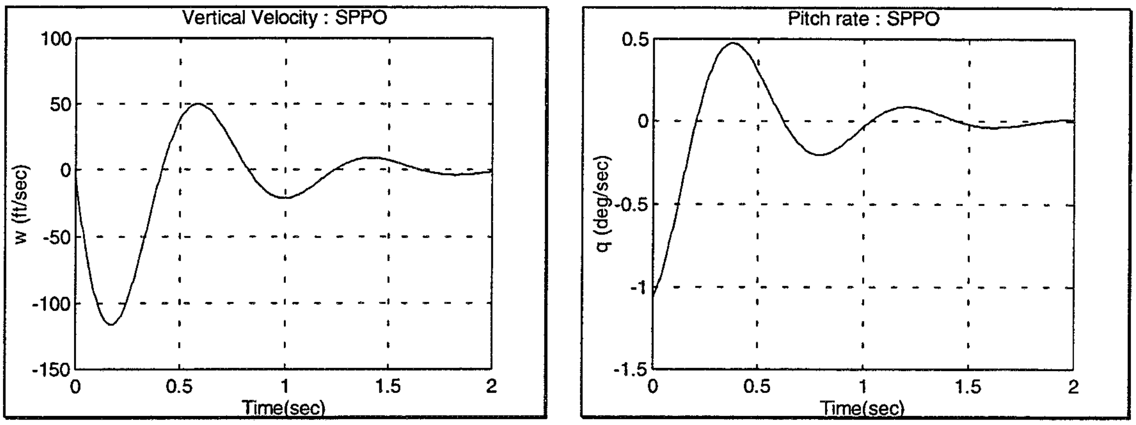
Period	P = 0.83 sec
Logarithmic Decrement	δ = 1.6931
Damped Frequency	ω = 7.5701 rad/sec
Phase Angle	ϕ = 91.08 deg
Relative Vector Magnitude	w = 158.941, q = 1
Damping Time	t_D = 0.4902 sec
Damping Angle	ε_D = 15.08 deg

The simplified vector equations that correspond to the equation 4.1 are

Moduli :	$\omega_n w$	$\omega_n w$	$U_0 q$	
Phases :	$+\dot{w}$	$-\bar{w}$	$-\bar{q}$	= 0

Moduli :	$\omega_n q$	$M_w w$	$M_q q$	
Phases :	$+\dot{q}$	$-\bar{w}$	$-\bar{q}$	= 0

The corresponding vector polygons are shown in figure 4.2.



(a) Vertical Velocity : w (b) Pitch Rate : q

Figure 4.1 Time Histories in Short Period Mode

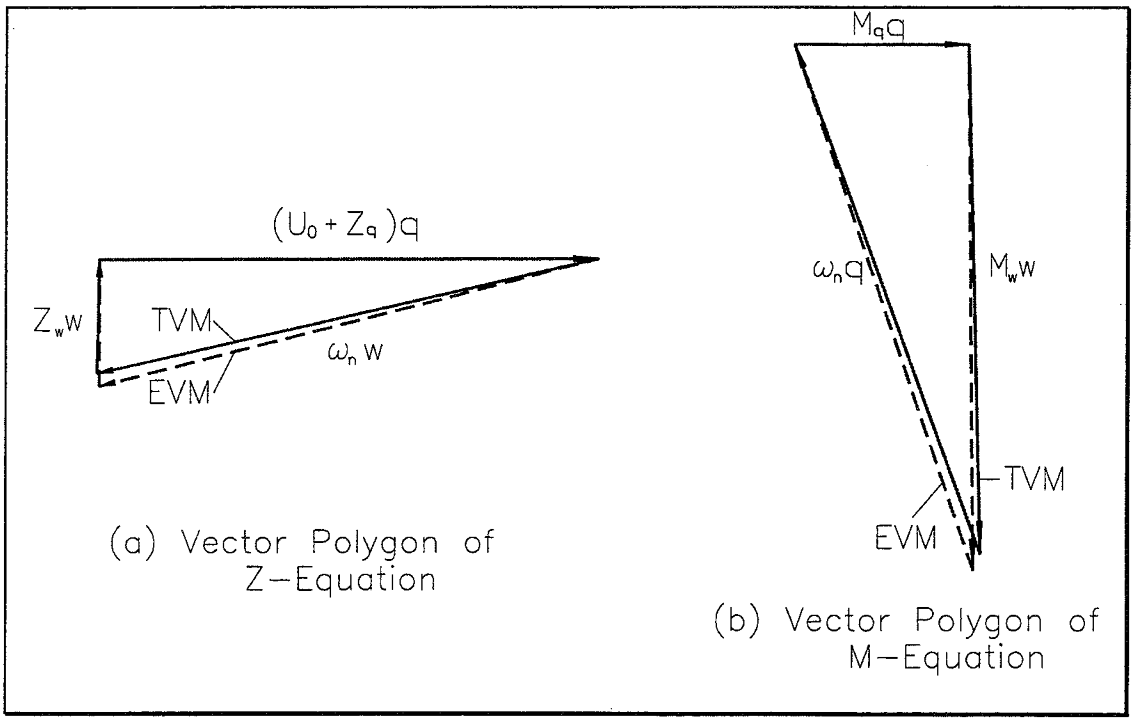


Figure 4.2 Vector Poligons in Short Period Mode

In order to compare the time vector method to the state-space method eigenvectors were obtained from the state matrix **A** and, in the same manner as for the TVM, are also plotted in the Fig. 4.2. By analysing all polygons, time vectors, eigenvectors, phase differences and relative magnitude of each vector the estimated derivatives were obtained in the table 4.3 and 4.4 together with the actual values obtained from the state-space method using the computational tools provided in PC-MATLAB.

	Phase Difference (deg)		Relative Magnitude	
	Time vector	Eigenvector	Time vector	Eigenvector
w	91.08	89.55	158.941	160.361
q	0	0	1	1

Table 4.3 Comparison of phase angle and relative magnitude

	Time Vector Method	State-Space method	Actual Value
ω_n	7.9434	7.9067	7.8492
Z_w	-1.9214	-2.11	-2.11
M_w	-0.0112	-0.0118	-0.0473
M_q	-2.0671	-2.0094	-1.9859

Table 4.4 Comparison of values of the parameters

It has been shown that, as might be expected, the results using the eigenvectors from state-space method give a better approximation than those obtained by TVM. Also, it is clear that the eigenvectors include all information about phase angle and relative magnitude which are extracted using the time vector method (ref. 9,10).

5. Conclusion

In this report the thorough study of the time vector method was made and compared with the state-space method for analysing linear oscillating motion. It was concluded that ;

- The analytical characteristics of the state-space method are exactly parallel to those of the time vector method

- The eigenvectors contain the same information about phase angle and relative magnitude as derived by the time vector method

- Hence the time vector method, the limited manual graphical approach for the aircraft parameter identification, can be replaced by a state-space method in order to capitalize in computational tools. If reconstruction of eigenvalues and eigenvectors from recorded flight data can be made satisfactorily, then it should be possible to estimate the state matrix.

- The next step in this study will be to design computer algorithms for the reconstruction of eigenvalues and eigenvectors from the flight simulation data.

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