Cranfield

College of Aeronautics Report No. 9210
....1992



# ARBITRARY-ORDER NUMERICAL SCHEMES FOR MODEL PARABOLIC EQUATION

Jian Shi

College of Aeronautics
Cranfield Institute of Technology
Cranfield, Bedford MK43 0AL. England

Cranfield

College of Aeronautics Report No. 9210
....1992



# ARBITRARY-ORDER NUMERICAL SCHEMES FOR MODEL PARABOLIC EQUATION

Jian Shi

College of Aeronautics
Cranfield Institute of Technology
Cranfield, Bedford MK43 0AL. England

ISBN 1 871564493

£ 8.00

Cranfield
COA REPORT No. 9210
SEPTEMBER, 1992

# ARBITRARY-ORDER NUMERICAL SCHEMES FOR MODEL PARABOLIC EQUATION

JIAN SHI

DEPARTMENT OF AEROSPACE SCIENCE
COLLEGE OF AERONAUTICS
CRANFIELD INSTITUTE OF TECHNOLOGY
CRANFIELD, BEDS., MK43 OAL

# Abstract

This report investigates the general theory and methodology of high order numerical schemes for one-dimensional model parabolic equation.

The Universal Formula from which a 2-level explicit arbitrary-order numerical methods for diffusion equation can be derived is developed. Using the Universal Formula some high order numerical methods are constructed.

Some important features of numerical methods are revealed through the construction of high order numerical methods and stability analysis.

Subject to the limitation of diffusion number, d, being positive, only the method that satisfies positive stable region is relevant.

# Acknowledgements

I would like to express my sincere thanks to Dr. E.F. Toro for his help during this work.

Special thanks go to S. Billett for his corrections and comments of the report, and to all those who helped me in this work.

# Contents

# List of Figures

# Chapter 1

# INTRODUCTION

## 1.1 Background Review

The model parabolic Partial Defferential Equations is the one-dimensional initial-value scalar diffusion equation:

$$\begin{aligned} u_t &= \nu u_{xx} \qquad\qquad\qquad\qquad\qquad\qquad (1.1) \\ u(x,0) &= u_0(x) \quad (-\infty < x < \infty,\ t > 0) \end{aligned}$$

where $\nu$ is a viscous coefficient.

Because equation 1.1 can describe heat transfer, diffusive phenomena, or viscous fluid flow, 1.1 is called the heat transfer equation or diffusion equation. In the heat transfer case the function $u$ gives the temperature at time $t$ and location $x$.

The problems presented by solving equation 1.1 numerically are not as severe as that of hyperbolic type. As is well known the convergence conditions are fairly tough for hyperbolic equations, but not as strict for schemes of the diffusion equation. Because of the smoothing inherent in 1.1 the discontinuities or shocks will not be formed during a computing process, and it is reasonable to believe that non-smooth initial functions should not seriously influence the convergence of the finite difference solutions to the PDE. Indeed a numerical method, as long as being consistent and linearly stable, will converge to the true solution of the PDE.

To analyse linear stability there is a stability Theorem 2 in [6]. Therefore, the main task of solving 1.1 numerically is how to construct good high order numerical methods.

Since the 2-level explicit numerical methods have obvious advantages over other methods, this report will study 2-level explicit numerical methods exclusively.

## 1.2 Objectives

So far, there is an absence of theory and formulae to define high order 2-level explicit numerical methods for the diffusion equation. Therefore, the objectives of this report is to

- develop a general theory and a universal formula defining 2-level explicit arbitrary-order finite difference methods for the model diffusion equation using TEV method introduced in [6].
- use the universal formula to construct some high order diffusion numerical schemes.
- analyse linear stabilities and find stable regions for these high order numerical schemes.
- discuss constructive features of these high order numerical schemes.

# Chapter 2

# THE UNIVERSAL FORMULA DEFINING NUMERICAL METHODS FOR MODEL PARABOLIC EQUATION

## 2.1 Introduction

At present the simplest 2-level explicit numerical method for 1.1 is obtained by directly replacing the derivatives of 1.1 by forward-time central-space finite difference scheme

$$U_j^{n+1} = (1-2d)U_j^n + d\left(U_{j-1}^n + U_{j+1}^n\right) \tag{2.1}$$

However this scheme is only first order accuracy in time and second order in space.

What we will consider in this chapter is to use Truncation Error Vanish Method (TEV), see[6], to derive the universal formula defining 2-level explicit arbitrary-order numerical methods for the linear model parabolic equation 1.1, to develop some high order 2-level explicit finite difference schemes, and to analyse structures and features of the high order diffusion numerical methods.

## 2.2 The Universal Formula for Model Parabolic Numerical Schemes

We discretize the computational plane by choosing a uniform mesh with a mesh width $h = \Delta x$ and a time step $k = \Delta t$, and define the computational gride $x_j = jh,\ t_n = nk$. We use $U_j^n$ to denote the computed approximation to the exact solution $u(x_j, t_n)$ of 1.1.

**THEOREM**

The universal formula from which a 2-level explicit arbitrary-order numerical methods can be derived for the model parabolic equation, $u_t - \nu u_{xx} = 0$, is defined as

$$U_j^{n+1} = \sum_{\alpha=0}^{p} B_{k_\alpha} U_{j+k_\alpha}^n \tag{2.2}$$

where $\alpha$ is the grid point number; $p$ is the number of grid points used, $p = 2m+1$; $m$ is the accurate order in time; $B_{k_\alpha}$ are constant coefficients which can be determined by

$$B_0 = 1 - \sum_{\alpha=1,k_\alpha\neq 0}^{2m} B_{k_\alpha} \tag{2.3a}$$

$$\sum_{\alpha=1,k_\alpha\neq 0}^{2m} B_{k_\alpha} k_\alpha^n = 0 \qquad (n = 1, 3, \ldots, 2m-1) \tag{2.3b}$$

$$\sum_{\alpha=1,k_\alpha\neq 0}^{2m} B_{k_\alpha} k_\alpha^n = \frac{n!}{(\frac{n}{2})!} d^{\frac{n}{2}} \qquad (n = 2, 4, \ldots, 2m) \tag{2.3c}$$

where d is a diffusion number, $d = \frac{\nu \Delta t}{\Delta x^2}$.

**PROOF**

The local truncation error of 2.2 can be written as:

$$\begin{aligned} E(x,t) &= u(x,t+\Delta t) - \sum_{\alpha=1}^{p} B_{k_\alpha} u(x + k_\alpha \Delta x, t) \\ &= u(x,t) + \sum_{n=1}^{m} \frac{(\Delta t)^n}{n!} u_{t^n} + O((\Delta t)^{n+1}) \\ & \sum_{\alpha=1}^{p} B_{k_\alpha} \left[ u(x,t) + \sum_{n=1}^{m} \frac{(k_\alpha \Delta x)^n}{n!} u_{x^n} \right] + O((\Delta x)^{n+1}) \end{aligned} \tag{2.4}$$

where $u_{t^2} = u_{tt}$, $u_{t^3} = u_{ttt}$, etc. and the same for $u_{x^n}$.

From 1.1 it is easy to get

$$u_{t^n} = \nu^n u_{x^{2n}} \tag{2.5}$$

and here,

$$\nu = \frac{d(\Delta x)^2}{\Delta t} \tag{2.6}$$

Substitution of 2.5 into 2.4:

$$\begin{aligned} E(x,t) &= u(x,t) - \sum_{\alpha=1}^{p} B_{k_\alpha} u(x,t) + \sum_{n=1}^{m} \frac{(\Delta t)^n}{n!} \nu^n u_{x^{2n}} \\ & - \sum_{\alpha=1}^{p} B_{k_\alpha} \sum_{n=1}^{m} \frac{(k_\alpha \Delta x)^n}{n!} u_{x^n} + O\left((\Delta t)^{m+1}, (\Delta x)^{2m+1}\right) \\ &= \left(1 - \sum_{\alpha=1}^{p} B_{k_\alpha}\right) u(x,t) + \sum_{n=1}^{m} \left[ \frac{(\Delta t)^n}{n!} \nu^n u_{x^{2n}} \right. \end{aligned}$$

$$- \sum_{\alpha=1}^{p} B_{k_\alpha} \frac{(k_\alpha \Delta x)^n}{n!} u_{x^n} \Bigg] + O\left((\Delta t)^{m+1}, (\Delta x)^{2m+1}\right) \tag{2.7}$$

Note here, the order of the truncation error in equation 2.7 is $m+1$ in time and $2m+1$ in space because $\Delta t \sim \Delta x^2$, see 2.6. Obviously the relationship between m and p is:

$$p = 2m + 1 \tag{2.8}$$

In order to achieve an mth order accurate numerical method in time, the following equations must be satisfied:

$$1 - \sum_{\alpha=1}^{p} B_{k_\alpha} = 0 \tag{2.9}$$

$$\frac{(\Delta t)^n}{n!} \nu^n u_{x^{2n}} - \sum_{\alpha=1}^{p} B_{k_\alpha} \frac{(k_\alpha \Delta x)^n}{n!} u_{x^n} = 0 \tag{2.10}$$

$$(n = 1, 2, 3, \ldots m)$$

Replacing 2.6 into 2.10 and simplifying it we get

$$\frac{(d(\Delta x)^2)^n}{n!} u_{x^{2n}} - \sum_{\alpha=1}^{p} \frac{(\Delta x)^n}{n!} B_{k_\alpha} k_\alpha^n u_{x^n} = 0 \tag{2.11}$$

Incorporating left hand side of 2.11 in terms of n and reorganizing it we finally get the following equations in order to obtain a numerical method of an mth order of accuracy in time.

$$\begin{cases} B_0 = 1 - \sum_{\alpha=1, k_\alpha \neq 0}^{2m} B_{k_\alpha} & \\ \sum_{\alpha=1, k_\alpha \neq 0}^{2m} B_{k_\alpha} k_\alpha^n = 0 & (n = 1, 3, \ldots, 2m-1) \\ \sum_{\alpha=1, k_\alpha \neq 0}^{2m} B_{k_\alpha} k_\alpha^n = \frac{n!}{(\frac{n}{2})!} d^{\frac{n}{2}} & (n = 2, 4, \ldots, 2m) \end{cases} \tag{2.12}$$

This is equations 2.3a, 2.3b, and 2.3c and Theorem 1 is proved.

2.12 can be written in other forms:

$$\begin{cases} B_{k_\alpha=0} = 1 - \sum_{\alpha=1,k_\alpha\neq 0}^{m} B_{k_\alpha} \\ k_1 B_{k_1} + k_2 B_{k_2} + \cdots + k_{2m} B_{k_{2m}} = 0 \\ k_1^2 B_{k_1} + k_2^2 B_{k_2} + \cdots + k_{2m}^2 B_{k_{2m}} = 2d \\ \vdots \\ k_1^n B_{k_1} + k_2^n B_{k_2} + \cdots + k_{2m}^n B_{k_{2m}} = \begin{cases} 0 & if\ n\ is\ an\ odd\ number \\ \frac{n!}{(\frac{n}{2})!} d^{\frac{n}{2}} & if\ n\ is\ an\ even\ number \end{cases} \\ \vdots \\ k_1^{2m} B_{k_1} + k_2^{2m} B_{k_2} + \cdots + k_{2m}^{2m} B_{k_{2m}} = \frac{(2m)!}{m!} d^m \\ (k_\alpha \neq 0) \end{cases} \tag{2.13}$$

and:

$$\begin{cases} B_{k_\alpha=0} = 1 - \sum_{\alpha=1,k_\alpha\neq 0}^{m} B_{k_\alpha} \\ \begin{bmatrix} B_{k_1} \\ B_{k_2} \\ \vdots \\ \vdots \\ \vdots \\ B_{k_{2m}} \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & \dots & k_{2m} \\ k_1^2 & k_2^2 & \dots & k_{2m}^2 \\ \vdots & \vdots & \vdots & \vdots \\ k_1^n & k_2^n & \dots & k_{2m}^n \\ \vdots & \vdots & \vdots & \vdots \\ k_1^{2m} & k_2^{2m} & \dots & k_{2m}^{2m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2d \\ \vdots \\ y \\ \vdots \\ \frac{(2m)!}{m!} d^m \end{bmatrix} \\ (k_\alpha \neq 0) \end{cases} \tag{2.14}$$

here

$$y = \begin{cases} 0 & if\ n\ is\ an\ odd\ number \\ \frac{n!}{(\frac{n}{2})!} d^{\frac{n}{2}} & if\ n\ is\ an\ even\ number \end{cases}$$

## 2.3 Applications of the Universal Formula

In this section we will use some examples to demonstrate how to use the Universal Formula to derive high order numerical methods. We will also pursue stability analysis for these methods using stability Theorem 2 in [6]. Later on we will see that because of the limitation that d must be positive, some numerical methods are physically meaningless.

### 2.3.1 Three Point Schemes

From 2.8 the accurate order of three point schemes is $m = \frac{p-1}{2} = 1$ in time and second in space.

1. $U_j^{n+1} = f(U_{j-2}^n, U_{j-1}^n, U_j^n)$

Like hyperbolic schemes we still call this scheme the "upwind scheme", here $k_1 = -2$, $k_2 = -1$, and $k_3 = 0$.

So the numerical method takes the form:

$$U_j^{n+1} = B_0 U_j^n + B_{-2} U_{j-2}^n + B_{-1} U_{j-1}^n \tag{2.15}$$

From equation 2.13, we have

$$\begin{cases} -2B_{-2} - B_{-1} = 0 \\ 4B_{-2} + B_{-1} = 2d \\ B_0 = 1 - B_{-2} - B_{-1} \end{cases}$$

i.e.

$$\begin{cases} B_0 = 1 + d \\ B_{-1} = -2d \\ B_{-2} = d \end{cases} \tag{2.16}$$

replacing 2.16 into 2.15

$$U_j^{n+1} = (1+d)U_j^n - 2dU_{j-1}^n + dU_{j-2}^n \tag{2.17}$$

According to the stability Theorem 2 [6], the amplification function for this method is:

$$\lambda = 1 + 4d$$

therefore, the stable region is $|\lambda| \leq 1$, i.e.

$$-\frac{1}{2} \leq d \leq 0 \tag{2.18}$$

However, it is physically meaningless for diffusion number d being negtive. Obviously, upwind scheme 2.17 is physically not right. Actually, later on you will see that all upwind and downwind numerical schemes are physically violating for model parabolic equation.

**2.** $U_j^{n+1} = f(U_{j-1}^n, U_j^n, U_{j+1}^n)$

This is central scheme. Here $k_1 = -1,\ k_2 = 0$, and $k_3 = 1$.

So the numerical method is:

$$U_j^{n+1} = B_0 U_j^n + B_{-1} U_{j-1}^n + B_1 U_{j+1}^n \tag{2.19}$$

From 2.13

$$\begin{cases} -B_{-1} + B_1 = 0 \\ B_{-1} + B_1 = 2d \\ B_0 = 1 - B_{-1} - B_1 \end{cases}$$

Therefore

$$\begin{cases} B_0 = 1 - 2d \\ B_{-1} = d \\ B_1 = d \end{cases} \quad (2.20)$$

here $B_{-1} = B_1$. Later on we will see that this is a common feature for all numerical schemes. It states that mirror points of numerical schemes, such as points $j-1$ and $j+1$ in this scheme, have same coefficient values.

Substitution of 2.20 into 2.19

$$U_j^{n+1} = (1-2d)U_j^n + d(U_{j-1}^n + U_{j+1}^n) \quad (2.21)$$

The amplification function of 2.21 is

$$\lambda = 1 - 4d$$

Hence the stable region of this method is

$$0 \le d \le \frac{1}{2} \quad (2.22)$$

The order of accuracy of this scheme is first order in time and second order in space, i.e. order (1,2).

**3.** $U_j^{n+1} = f(U_j^n, U_{j+1}^n, U_{j+2}^n)$

This is the three points downwind scheme. Undertaking the same procedure as obove the numerical method becomes:

$$U_j^{n+1} = (1+d)U_j^n - 2dU_{j+1}^n + dU_{j+2}^n \quad (2.23)$$

Like in the upwind scheme presented in equation 2.17, for stability, d must be nagtive. Clearly this scheme is physically not correct.

### 2.3.2 Four Point Schemes

**1.** $U_j^{n+1} = f(U_{j-3}^n, U_{j-2}^n, U_{j-1}^n, U_j^n)$

This is upwind scheme. Here $k_1 = -3,\ k_2 = -2,\ k_3 = -1$, and $k_4 = 0$.

So

$$U_j^{n+1} = B_0 U_j^n + B_{-1} U_{j-1}^n + B_{-2} U_{j-2}^n + B_{-3} U_{j-3}^n$$

From 2.13 we get

$$\begin{cases} -B_{-1} - 2B_{-2} - 3B_{-3} = 0 \\ B_{-1} + 4B_{-2} + 9B_{-3} = 2d \\ -B_{-1} - 8B_{-2} - 27B_{-3} = 0 \\ B_0 = 1 - B_{-1} - B_{-2} - B_{-3} \end{cases}$$

i.e.

$$\begin{cases} B_0 = 1 + 2d \\ B_{-1} = -5d \\ B_{-2} = 4d \\ B_{-3} = -d \end{cases}$$

The amplification function is

$$\lambda = 1 + 12d$$

and the stable region is

$$-\frac{1}{6} \le d \le 0$$

Again we proved the upwind schemes are physically unrealistic.

**2.** $U_j^{n+1} = f(U_{j-2}^n, U_{j-1}^n, U_j^n, U_{j+1}^n)$

Here $k_1 = -2,\ k_2 = -1,\ k_3 = 1$, and $k_4 = 0$.

So the numerical method takes the form:

$$U_j^{n+1} = B_0 U_j^n + B_1 U_{j+1}^n + B_{-1} U_{j-1}^n + B_{-2} U_{j-2}^n \tag{2.24}$$

From 2.13 and doing some manipulation we get

$$\begin{cases} B_0 = 1 - 2d \\ B_1 = d \\ B_{-1} = d \\ B_{-2} = 0 \end{cases} \tag{2.25}$$

Replacing 2.25 into 2.24

$$U_j^{n+1} = (1-2d)U_j^n + d(U_{j+1}^n + U_{j-1}^n)$$

This implies that since $B_{-2}$ is equal to 0 the four point scheme (-2,-1,0,1) has became the same as three point scheme (-1,0,1), and can not increase the order of accuracy.

**3.** $U_j^{n+1} = f(U_{j-1}^n, U_j^n, U_{j+1}^n, U_{j+2}^n)$

This is the mirror scheme of 2.24. Here $k_1 = 2,\ k_2 = 1,\ k_3 = -1$, and $k_4 = 0$.

So

$$U_j^{n+1} = B_0 U_j^n + B_1 U_{j+1}^n + B_{-1} U_{j-1}^n + B_2 U_{j+2}^n \tag{2.26}$$

and

$$\begin{cases} B_0 = 1 - 2d \\ B_1 = d \\ B_{-1} = d \\ B_2 = 0 \end{cases} \tag{2.27}$$

Like in the scheme 2.24 this scheme reduces to three point scheme (-1,0,1) as well.

**4.** $U_j^{n+1} = f(U_j^n, U_{j+1}^n, U_{j+2}^n, U_{j+3}^n)$

The numerical method is:

$$U_j^{n+1} = (1+2d)U_j^n - 5dU_{j+1}^n + 4dU_{j+2}^n - dU_{j+3}^n \qquad (2.28)$$

The stable region of the method is

$$-\frac{1}{6} \leq d \leq 0$$

As you can see that the stable region of this method is the same as the upwind scheme, hence this method is physically meaningless too.

We can conclude therefore, that four point schemes can not improve the order of accuracy of parabolic numerical schemes.

### 2.3.3 Five Point Schemes

Because upwind and downwind schemes are physically meaningless for model diffusion equation, from now on we will dispence these schemes.

From 2.8 we know that the accuracy of five point schemes is second order in time and fourth order in space.

**1.** $U_j^{n+1} = f(U_{j-3}^n, U_{j-2}^n, U_{j-1}^n, U_j^n, U_{j+1}^n)$

The numerical method is:

$$\begin{aligned} U_j^{n+1} &= \left(1 - \frac{5}{3}d - 2d^2\right) U_j^n + \left(\frac{11}{12}d + \frac{1}{2}d^2\right) U_{j+1}^n + \left(\frac{1}{2}d + 3d^2\right) U_{j-1}^n \\ &\quad + \left(\frac{1}{3}d - 2d^2\right) U_{j-2}^n + \left(\frac{1}{2}d^2 - \frac{1}{12}d\right) U_{j-3}^n \end{aligned} \qquad (2.29)$$

The amplification function is:

$$\lambda = 1 - 2\left(\frac{4}{3}d + 4d^2\right)$$

The stable region is $|\lambda| \leq 1$, i.e.

$$0 \leq d \leq 0.36 \tag{2.30}$$

**2.** $U_j^{n+1} = f(U_{j-1}^n, U_j^n, U_{j+1}^n, U_{j+2}^n, U_{j+3}^n)$

This is the mirror scheme of 2.29. The numerical method is:

$$\begin{aligned} U_j^{n+1} &= \left(1 - \frac{5}{3}d - 2d^2\right)U_j^n + \left(\frac{11}{12}d + \frac{1}{2}d^2\right)U_{j-1}^n + \left(\frac{1}{2}d + 3d^2\right)U_{j+1}^n \\ &+ \left(\frac{1}{3}d - 2d^2\right)U_{j+2}^n + \left(\frac{1}{2}d^2 - \frac{1}{12}d\right)U_{j+3}^n \end{aligned} \tag{2.31}$$

The amplification function is:

$$\lambda = 1 - 2\left(\frac{4}{3}d + 4d^2\right)$$

and the stable region is $|\lambda| \leq 1$, i.e.

$$0 \leq d \leq 0.36 \tag{2.32}$$

Note this scheme has a same stable region as 2.29.

**3.** $U_j^{n+1} = f(U_{j-2}^n, U_{j-1}^n, U_j^n, U_{j+1}^n, U_{j+2}^n)$

The scheme is:

$$
\begin{aligned}
U_j^{n+1} &= \left(1 + 3d^2 - \frac{5}{2}d\right) U_j^n + \left(\frac{4}{3}d - 2d^2\right) \left(U_{j-1}^n + U_{j+1}^n\right) \\
&\quad + \left(\frac{1}{2}d^2 - \frac{1}{12}d\right) \left(U_{j-2}^n + U_{j+2}^n\right)
\end{aligned}
\tag{2.33}
$$

The amplification function is:

$$
\lambda = 1 - 2\left(\frac{8}{3}d - 4d^2\right)
$$

and the stable region is:

$$
0 \le d \le \frac{2}{3} \tag{2.34}
$$

### 2.3.4 Seven Point Central Scheme

From the observation we know that all central schemes are stable, moreover, they have the largest stable regions. Hence we only consider to construct high order central numerical schemes from now on.

From 2.8 we know that the accuracy with seven point schemes is third order in time and sixth order in space.

$$
\begin{aligned}
U_j^{n+1} &= \left(1 - \frac{10}{3}d^3 + \frac{14}{3}d^2 - \frac{49}{18}d\right) U_j^n + \left(\frac{15}{6}d^3 - \frac{13}{4}d^2 + \frac{3}{2}d\right) \left(U_{j-1}^n + U_{j+1}^n\right) \\
&\quad + \left(d^2 - d^3 - \frac{3}{20}d\right) \left(U_{j-2}^n + U_{j+2}^n\right) \\
&\quad + \left(\frac{1}{6}d^3 - \frac{1}{12}d^2 + \frac{1}{90}d\right) \left(U_{j-3}^n + U_{j+3}^n\right)
\end{aligned}
\tag{2.35}
$$

The amplification function of this scheme is:

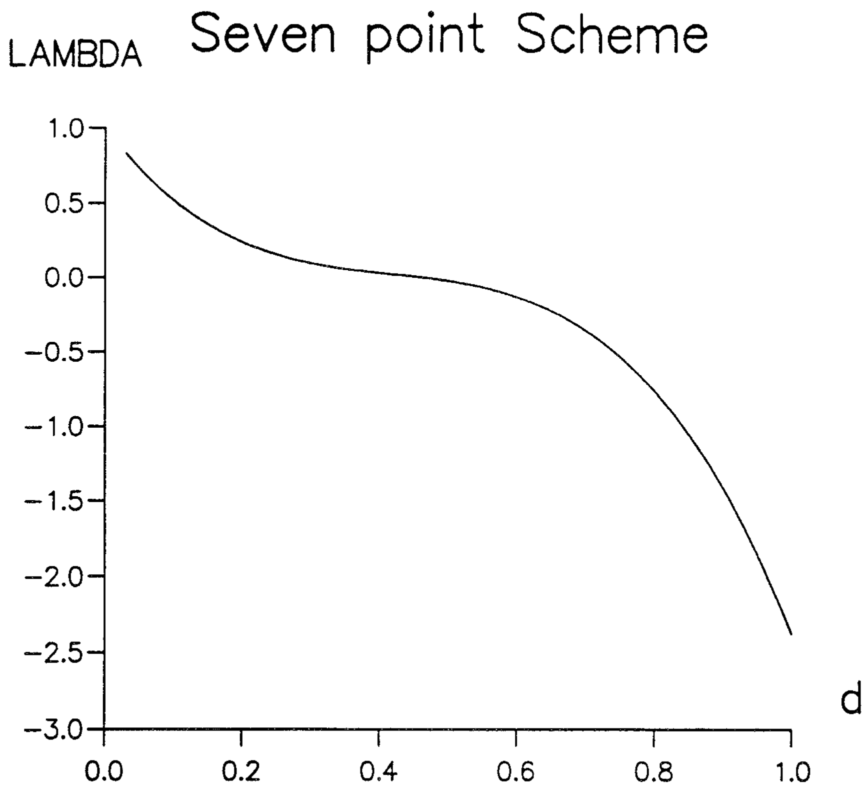


Figure 2.1: Stable Region for 7-point Central Scheme

$$\lambda = 1 - \frac{32}{3}d^3 + \frac{40}{3}d^2 - \frac{272}{45}d$$

and the stable region is $|\lambda| \leq 1$, see **Figure 2.1**, this is

$$0 \leq d \leq 0.85 \tag{2.36}$$

### 2.3.5 Nine Point Central Scheme

The accuracy of this scheme is 4th order in time and 8th order in space.

$$U_j^{n+1} = (1 - 2.847222d + 5.6875008d^2 - 6.24999984d^3 + 2.9166648d^4)U_j^n$$

$$+(1.6d - 4.066666656d^2 + 4.83333324d^3 - 2.33333184d^4)(U_{j-1}^n + U_{j+1}^n)$$
$$+(1.408333332d^2 - 0.2d - 2.16666667d^3 + 1.16666667d^4)(U_{j-2}^n + U_{j+2}^n)$$
$$+(0.025396824d - 0.2d^2 + 0.5d^3 - 0.33333216d^4)(U_{j-3}^n + U_{j+3}^n)$$
$$+(0.01453324d^2 - 0.00178572d - 0.04166664d^3 + 0.04166568d^4)$$
$$(U_{j-4}^n + U_{j+4}^n) \quad (2.37)$$

The amplification function is:

$$\lambda = 1 - 2(3.250794d - 8.533333d^2 + 10.66666692d^3 - 5.33333304d^4)$$

The stable region, see **Figure 2.2**, is:

$$0 \leq d \leq 1 \quad (2.38)$$

## 2.4 Important Features of Observation

From the observation of the numerical schemes in the last section, we can summarise some of the important features of parabolic numerical schemes as follows:

1. For arbitrary point schemes fully upwind and downwind schemes are physically violating.
2. For arbitrary point schemes the central point scheme has the largest stable region.
3. For central point schemes, the stable region increases proportion to the size of the stencil.
4. For mirror numerical schemes the coefficient values, $B_{k_\alpha}$, of mirror points are identical.

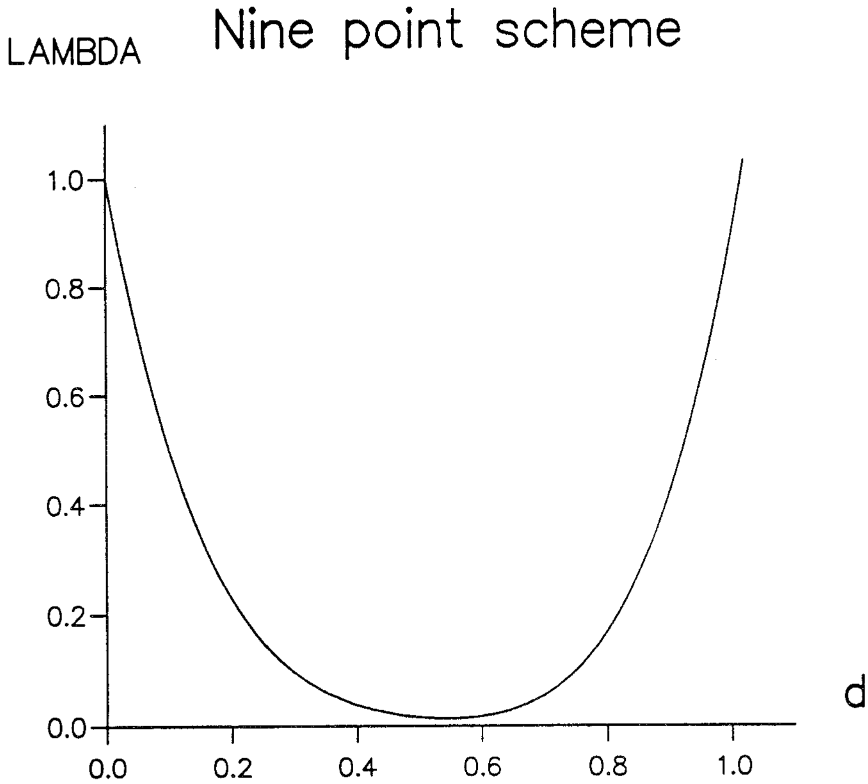


Figure 2.2: Stable Region for 9-point Central Scheme

5. Mirror numerical schemes have the same stable region.

Generalizing these features above we can conclude that the ideal 2-level explicit numerical methods for model parabolic equation are central point numerical schemes.

# Chapter 3

# CONCLUSIONS

In this report we developed a theory and a universal formula which can be used to construct arbitrary-order 2-level explicit numerical methods for the model parabolic equation.

However, because of the limitation of the diffusion number, some numerical schemes are physically violating. In order to obtain a physically correct and numerically stable method we have to select a numerical method which possesses a positive stable region in terms of diffusion number d.

The findings in this report indicate that the ideal numerical schemes for the model parabolic equation are central point schemes which have the largest stable region.

# Bibliography


[1] **ROE, P.L.**, Numerical Algorithms for the Linear Wave Equation, Technical Report 81047, Cranfield Institute of Technology.

[2] **RICHTMYER, R.D. and MORTON, K.W.**, Difference Methods for Initial-value Problems, Wiley-Interscience, 1967.

[3] **STRIKWERDA, J.C.**, Finite Difference Schemes and Partial Differential Equations, Wadsworth and Brooks/cole, 1989.

[4] **LEVEQUE, R.J.**, Numerical Methods for Conservation Laws, Birkhauser, 1990.

[5] **TORO, E.F.**, Shock-capturing Methods, Leture Note in Computational Fluid Dynamics, College of Aeronautics, Cranfield Institute of Technology, 1992.

[6] **SHI, J.**, Arbitrary-order High Resolution Schemes for Model Hyperbolic Conservation Laws, COA Report No. 9209, College of Aeronautics, Cranfield Institute of Technology, 1992.