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ARBITRARY-ORDER HIGH RESOLUTION SCHEMES FOR MODEL HYPERBOLIC CONSERVATION LAWS

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Abstract

This report investigates the general theory and methodology of high resolution numerical schemes for one-dimensional hyperbolic conservation laws.

The Universal Formula from which 2-level explicit conservative arbitrary-order numerical methods can be derived is developed.

This report also explores the issue of linear stability. A new approach to linear stability analysis is presented.

The generalized formulation for TVD methods with stable region of $-1 \leq c \leq 1$ is proposed.

To demonstrate the theories, some third order and fourth order TVD methods are generated. [1]

[1]This report is a part of Ph.D research work supervised by Dr. E.F. Toro

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Chapter 1

INTRODUCTION

1.1 Background Review

The scientific study of numerical solution of hyperbolic conservation laws has been carried out for decades, and has been gathering momentum with each succeeding year.

Research into discontinuous nonlinear hyperbolic systems is difficult, not only by virtue of the complexity of the subject itself, but also because of the emergence of discontinuities or shocks with time evolution during wave propagation.

However difficult is it, a great deal of progress has been made over last fifteen years since shock-capturing methods were introduced. In recent years, there have been intensive efforts in developing Riemann-problem based shock-capturing schemes. Presently, a variety of such methods are available. Usually, these schemes are entitled 'High Resolution Methods', which have the following essential characteristics:

- They are at least of second-order accurate in smooth regions of the flow.
- They sharply resolve discontinuities without generating excessive smearing.
- They are free of spurious oscillationns in the computed solution.

The recent high resolution schemes include TVD (total variation diminishing) methods, TVB (total variation bounded) methods, and the most newly developed ENO (essentially non-oscillatory) methods. See [1-11].

Although these schemes, say ENO schemes, can achieve high order accuracy in both smooth regions and up to discontinuities, the highest order in practice is only second order in space and time. It is well known, even in the case of smooth initial data, that linear hyperbolic systems, within a short time, evolve an inevitable and unacceptable amplitude error caused by dissipation, and phase errors due to dispersion from the second order truncation error of these second order methods. Therefore, higher order methods for the approximation of the hyperbolic conservation laws are needed for both scientific research and manufacturing purposes.

There are different methods which can be used to construct numerical methods, but there is an absence of a general theory and universal formula to define the high order numerical methods, and, hence, reveal the intrinsic structure of the numerecal methods and the relationship between these methods. Here rises a question: does there exist a general theory and formula which governs the numerical methods?

As is well known, a numerical method is useless if this method will not converge to the differential equation. To prove convergence, there is a fundamental theorem for linear finite difference methods, which declares that for a consistent linear method stability is necessary and sufficient for convergence [12] [13]. Although linear stability is not a sufficient condition for guaranteeing nonlinear stability, it is still a necessary condition for achieving nonlinear stability. Therefore, linear stability analysis plays a significant role in the development of a numerical method.

There are several techniques which can be used to prove linear stability. But, it isby no means an easy task, especially for a method of more than second order accuracy. Normally, quite complicated algebraic triangular functions will be encountered, which are very difficult to analyse. Presently, there is no efficient and effective way to deal with the problem, and the popular method is experimental. This is time-consuming and tedious. Obviously, a simple and reliable method for proving linear stability is desired. Here rises another question: How can we derive such a method?

All we have discussed so far is linear systems. When extending these methods to nonlinear conservation laws we expect to meet two new problems.

1. The method might coverge to the wrong weak solution;
2. Nonlinear instability.

In order to overcome the first problem, a numerical method must take the conservation form. That is, to guarantee that a numerical method will converge to the true solution of the PDE, the numerical method valid for linear systems has to be transformed to a conservative method, i.e. given the form of numerical flux function.

Supporse we had got a high order numerical method which takes the form $U_j^{n+1} = \sum_{k=0}^{\infty} B_k U_{j+k}^n$, the question is: how can we transform it into numerical flux form?

To get rid of the second problem, Total Variation (TV) Stable Methods are needed. In terms of ENO's scheme, the idea of uniformly high order both for smooth and discontinuities is very attractive.

Analysis of the problem of nonlinear unstability, clearly, it is trigered by oscillations caused by discontinuities developed with time evolution. But, if we could find a method in which extra conditions were supplemented at the discontinuities, therefore, treated differently, then, the whole function would become a piecewise smooth function except the discontinuous points. Subsquently, all high order numerical methods could be used peacefully throughout as in the linear case.

Here rises the question: how can we supplement the extra conditions at discontinuities and how can we treat discontinuous points differently from smooth points?

Answering all the questions presented so far are the objectives of this project.

1.2 Objectives

The objectives of this project can be outlined as follows:

- Explore the general theory and universal formula defining the numerical methods in the form of $U_j^{n+1} = \sum_{k=0}^{\infty} B_k U_{j+k}^n$, for linear scalar advection equation.
- Investigate a general and simple approach for linear stability study.
- Find a way to transform a numerical method of the form: $U_j^{n+1} = \sum_{k=0}^{\infty} B_k U_{j+k}^n$ into the form: $U_j^{n+1} = U_j^n - \frac{k}{h}\left[F(U^n;j) - F(U^n;j-1)\right]$.
- Derive the general high order TVD or ENO methods for nonlinear hyperbolic conservation laws.
- Implement the high resolution methods to solve real problems.

Because two-level explicit numerical methods have obvious advantages over other methods, this project will concentrate on developing two-level explicit numerical methods.

The final goal of this project is to produce a general numerical method to deal with nonlinear hyperbolic conservation laws for real applications such as gas dynamics. This is an ambitious scheme in which we adopt a different approach from other schemes. The report presented here is only a part of a series of works on this scheme.

Because this is a large project, a good strategy is obviously needed.

1.3 The Strategy

First of all, we start with the linear scalar advection equation and generate the universal formula for two-level explicit arbitrary-order numerical methods.

Then, we investigate the issue of linear stability and develop a general approach to linear stability analysis.

After that, we extend these methods to nonlinear hyperbolic conservation laws, i.e. firstly ensure these methods are in the form of conservitive form, secondly, prove these methods are TVD methods, and further more, develop a uniformly high order accuracy method throughout both the smooth region and the shock points.

Finally, use these methods to solve real problems in gas dynamics and aeronautical flows.

Chapter 2

THE UNIVERSAL FORMULA DEFINING ARBITRARY-ORDER NUMERICAL METHODS

2.1 Introduction

In this chapter we are interested in finding a universal formula defining arbitrary-order numerical methods for the linear scalar advection equation and identifying the structure and relationship between these methods.

In one space dimension, the linear scalar advection equation takes the following form:

$$u_t + au_x = 0 \qquad -\infty < x < \infty, t \geq 0 \tag{2.1a}$$

$$u(x,0) = u_0(x) \tag{2.1b}$$

Here, u is the conserved variable and a is a constant wave propagation speed.

Historically, there are a variety of ways to define finite difference numerical methods, which include:

- Direct derivation by replacing the derivatives in 2.1a by relevant finite difference approximations.
- Interpolation methods using the grid points.
- The Taylor Series Expansion (TSE).

Although the direct derivation method is more natural than the Taylor Series Expansion method, and most of the explicit numerical methods used today are generated from direct finite difference approximations to the PDE, the TSE method has great advantages over the first two methods.

Firstly, any methods derived from TSE are automatically consistent with the PDE.

Secondly, the truncation error, therefore, the order of the method resulting from TSE is self evident.

Further more, any TSE numerical methods are stable under certain CFL conditions.

Finally, any high order finite difference methods can be achieved theoretically using the TSE method.

In spite of these advantages above, unfortunately, the TSE method has not gained the widespread applications as it should deserve. In the past, only one numerical method, i.e. the Lax-Wendroff method comes from the TSE.

The objective of this chapter is to investigate the Taylor Series Expansion method in detail and as a result find a method defining the general theory and universal formula for arbitrary-order explicit finite difference methods. In order to distinguish this method from others we define this method as the Truncation Error Vanish Method (TEV).

2.2 The Universal Formula Defining Arbitrary-Order Hyperbolic Numerical Methods

We discretize the computational plane by choosing a uniform mesh with a mesh width $h = \triangle x$ and a time step $k = \triangle t$, and define the computational grid $x_j = jh, \quad t_n = nk$. We use U_j^n to denote the computed approximation to the exact solution $u(x_j, t_n)$ of 2.1a.

THEOREM 1

The universal formula from which a two-level explicit arbitrary-order numerical method can be derived for the model hyperbolic equation, $u_t + au_x = 0$, is defined as

$$U_j^{n+1} = \sum_{\alpha=1}^{p} B_{k_\alpha} U_{j+k_\alpha}^n \tag{2.2}$$

where α is the grid point number; p is the number of grid points used, $P = m + 1$; m is the accurate order; B_{k_α} are constant coefficients which are determined by

$$B_0 = 1 - \sum_{\alpha=1, k_\alpha \neq 0}^{m} B_{k_\alpha} \tag{2.3}$$

$$\sum_{\alpha=1, k_\alpha \neq 0}^{m} B_{k_\alpha} k_\alpha^n = (-c)^n \quad (n = 1, 2, \ldots, m) \tag{2.4}$$

where c is Courant number, $c = \frac{a\triangle t}{\triangle x}$.

Note, in the notation of 2.2, k_α need not only be a integer, but could also a fraction.

PROOF

In order to prove Theorem 1, we first analyse the local truncation error of 2.2 by Taylor Series expansion of both sides of the equation, then set the truncation error equal to zero, namely, let the truncation error vanish. This is the essential idea of TEV method.

The truncation error of 2.2 can be written as:

$$\begin{aligned} E(x,t) &= u(x,t+\Delta t) - \sum_{\alpha=1}^{p} B_{k_\alpha} u(x + k_\alpha \triangle x, t) \\ &= u(x,t) + \sum_{n=1}^{m} \frac{\Delta t^n}{n!} u_{t^n} + O(\Delta t^{n+1}) \\ &\quad - \sum_{\alpha=1}^{p} B_{k_\alpha} \left[u(x,t) + \sum_{n=1}^{m} \frac{(k_\alpha \triangle x)^n}{n!} u_{x^n} \right] + O(\Delta x^{n+1}) \end{aligned} \qquad (2.5)$$

where m is the order of accuracy of the scheme, $1 \leq m < \infty$; $u_{t^2} = u_{tt}$, $u_{t^3} = u_{ttt}$, etc. and the same for u_{x^n}; $\Delta t^n = (\Delta t)^n$ for simplisity. The relationship between m and p obviously is:

$$p = m + 1 \qquad (2.6)$$

From scalar equation 2.1a, it is easy to get that:

$$u_t = -a u_x, \; u_{tt} = a^2 u_{xx}, \; u_{t^3} = -a^3 u_{x^3}, \; \ldots$$

Therefore,

$$u_{t^n} = (-a)^n u_{x^n} \qquad (2.7)$$

Substitution of 2.7 into 2.5:

$$\begin{aligned} E(x,t) &= u(x,t) - \sum_{\alpha=1}^{p} B_{k_\alpha} u(x,t) + \sum_{n=1}^{m} \frac{\Delta t^n}{n!} (-a)^n u_{x^n} \\ &\quad - \sum_{\alpha=1}^{p} B_{k_\alpha} \sum_{n=1}^{m} \frac{(k_\alpha \triangle x)^n}{n!} u_{x^n} + O(\Delta t^{m+1}) + O(\Delta x^{m+1}) \\ &= \left(1 - \sum_{\alpha=1}^{p} B_{k_\alpha}\right) u(x,t) + \sum_{n=1}^{m} \left[\frac{\Delta t^n}{n!} (-a)^n u_{x^n} \right. \\ &\quad \left. - \sum_{\alpha=1}^{p} B_{k_\alpha} \frac{(k_\alpha \triangle x)^n}{n!} u_{x^n} \right] + O(\Delta t^{m+1}) + O(\Delta x^{m+1}) \end{aligned} \qquad (2.8)$$

In order to achieve mth order accurate numerical method, the following equations must be satisfied:

$$1 - \sum_{\alpha=1}^{p} B_{k_\alpha} = 0 \tag{2.9a}$$

$$\frac{\Delta t^n}{n!}(-a)^n u_{x^n} - \sum_{\alpha=1}^{p} B_{k_\alpha} \frac{(k_\alpha \bigtriangleup x)^n}{n!} u_{x^n} = 0 \tag{2.9b}$$

$$(n = 1, 2, 3, \ldots m)$$

where: $a = \frac{c\Delta x}{\Delta t}$; c is the Courant number.

Simplify equation 2.9b, it becomes:

$$(-1)^n \frac{c^n}{n!} \bigtriangleup x^n - \sum_{\alpha=1}^{p} B_{k_\alpha} \frac{k_\alpha^n}{n!} \bigtriangleup x^n = 0$$

i.e.

$$\sum_{\alpha=1}^{p} k_\alpha^n B_{k_\alpha} = (-c)^n, (n = 1, 2, \cdots, m) \tag{2.10}$$

Therefore, equations 2.9a and 2.9b can be rewritten as:

$$\begin{cases} B_{k_\alpha=0} = 1 - \sum_{\alpha=1,k_\alpha\neq 0}^{m} B_{k_\alpha} \\ \sum_{\alpha=1,k_\alpha\neq 0}^{m} k_\alpha^n B_{k_\alpha} = (-c)^n \quad (n = 1, 2, \cdots, m) \end{cases} \tag{2.11}$$

This is equation 2.3 and 2.4, and Theorem 1 is proved. Equation 2.11 can be transformed into other forms:

$$\begin{cases} B_{k_\alpha=0} = 1 - \sum_{\alpha=1,k_\alpha\neq 0}^{m} B_{k_\alpha} \\ k_1 B_{k_1} + k_2 B_{k_2} + \cdots + k_m B_{k_m} = -c \\ k_1^2 B_{k_1} + k_2^2 B_{k_2} + \cdots + k_m^2 B_{k_m} = c^2 \\ \vdots \\ k_1^m B_{k_1} + k_2^m B_{k_2} + \cdots + k_m^m B_{k_m} = (-c)^m \\ (k_\alpha \neq 0) \end{cases} \tag{2.12}$$

and:

$$\begin{cases} B_{k_\alpha=0} = 1 - \sum_{\alpha=1,k_\alpha\neq 0}^{m} B_{k_\alpha} \\ \begin{bmatrix} B_{k_1} \\ B_{k_2} \\ \vdots \\ B_{k_m} \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & \ldots & k_m \\ k_1^2 & k_2^2 & \ldots & k_m^2 \\ \vdots & \vdots & \vdots & \vdots \\ k_1^m & k_2^m & \ldots & k_m^m \end{bmatrix}^{-1} \begin{bmatrix} -c \\ c^2 \\ \vdots \\ (-c)^m \end{bmatrix} \\ (k_\alpha \neq 0) \end{cases} \tag{2.13}$$

Assuming we want a m-th order numerical method, we know from 2.6 that we have to work out p coefficients $B_{k_\alpha}(\alpha = 1, 2, \cdots, p)$ in 2.2. As you can see, 2.11 or 2.12 or 2.13 have p equations, hence, these equations are closed. Using these equations all coefficients B_{k_α} wanted by 2.2 can be derived, so that arbitrary-order numerical methods for linear model equation 2.1a can be achieved.

We define Theorem 1 the Universal Formula Theorem of hyperbolic numerical methods.

2.3 Some Features of the Universal Formula

Some interesting features can be drawn by analysing the formula.

1. The Grid Points: By definition of 2.2, we know that k_α is not limited to the integers. This implies that the computational points for a numerical method can be both integer points (i.e. $j = \pm 1, \pm 2, \ldots, \pm\infty$), and fraction points (i.e.

$j = \pm\frac{1}{2}, \pm\frac{1}{3}, \pm 0.1, \ldots$). What this means is that a numerical method can be constructed in such a way that it is not mandatory to take all even divided points in a computational domain.

2. The Order of Accuracy: As described before, the order of accuracy of a numerical method depends on the number of nonlinear related grid points used, that is, the more the computational points are involved the higher the order gained. They are governed by 2.6, $m = p - 1$.

3. The Number of Numerical Methods Which Have the Same Order: Using the same number of grid points, but by using different points, different numerical methods can be obtained. Considering that the grid points can be both integer and fraction points, the number of numerical methods with the same order is infinite. If only considering the integer points, the number of numerical methods(N) which have the same order equal to the number of the points used.

$$N = P \tag{2.14}$$

For example, for four integer points schemes we can find four third order numerical methods.

In next section we will use some examples to demonstrate these features and also to show how to apply the universal formula to derive high order numerical methods.

2.4 Applications of the Universal Formula

2.4.1 Three Points Schemes

First, we consider the three points schemes. We will see that some familiar numerical methods can be found from the Universal Formula.

Frome equation 2.6, we know that the highest order m with three points schemes is

$$m = p - 1 = 3 - 1 = 2$$

Therefore,

$$\begin{cases} p = 3 \\ m = 2 \end{cases}$$

1. THREE POINTS SCHEMES FOR: $U_j^{n+1} = f(U_j^n, U_{j-1}^n, U_{j+1}^n)$

Here, $k_1 = 0,\ k_2 = -1,\ k_3 = 1$

So the numerical method takes the form

$$U_j^{n+1} = B_0 U_j^n + B_{-1} U_{j-1}^n + B_1 U_{j+1}^n \tag{2.15}$$

using equation 2.12, we get

$$\begin{cases} -B_{-1} + B_1 = -c \\ B_{-1} + B_1 = c^2 \\ B_0 = 1 - B_{-1} - B_1 \end{cases}$$

and

$$\begin{cases} B_0 = 1 - c^2 \\ B_{-1} = \frac{c}{2}(c+1) \\ B_1 = \frac{c}{2}(c-1) \end{cases} \tag{2.16}$$

substitution of 2.16 into 2.15

$$U_j^{n+1} = (1 - c^2)U_j^n + \frac{1}{2}(c^2 + c)U_{j-1}^n + \frac{1}{2}(c^2 - c)U_{j+1}^n \tag{2.17}$$

Note: here we get the famous Lax-Wendroff method.

2. THREE POINTS SCHEMES FOR: $U_j^{n+1} = f(U_j^n, U_{j-1}^n, U_{j-2}^n)$

Here, $k_1 = 0,\ k_2 = -1,\ k_3 = -2$

So the numerical method is

$$U_j^{n+1} = B_0 U_j^n + B_{-1} U_{j-1}^n + B_{-2} U_{j-2}^n \tag{2.18}$$

From 2.12

$$\begin{cases} -B_{-1} - 2B_{-2} = -c \\ B_{-1} + 4B_{-2} = c^2 \\ B_0 = 1 - B_{-1} - B_{-2} \end{cases}$$

Therefore

$$\begin{cases} B_0 = 1 + \frac{1}{2}c^2 - \frac{3}{2}c \\ B_{-1} = 2c - c^2 \\ B_{-2} = \frac{1}{2}c^2 - \frac{1}{2}c \end{cases} \tag{2.19}$$

Replace 2.19 into 2.18

$$U_j^{n+1} = (1 + \frac{1}{2}c^2 - \frac{3}{2}c)U_j^n + (2c - c^2)U_{j-1}^n + (\frac{1}{2}c^2 - \frac{1}{2}c)U_{j-2}^n \tag{2.20}$$

Note: this is the Beam-Warming method.

3. THREE POINTS SCHEMES FOR: $U_j^{n+1} = f(U_j^n, U_{j+1}^n, U_{j+2}^n)$

Following the same routine above, we can get another second order method. It is

$$U_j^{n+1} = (1 + \frac{1}{2}c^2 + \frac{3}{2}c)U_j^n - (c^2 + 2c)U_{j+1}^n + (\frac{1}{2}c^2 + \frac{1}{2}c)U_{j+2}^n \tag{2.21}$$

This is like Beam-Warming method for $c < 0$.

So far, we have got all three integer points numerical methods for three points schemes. Now, we will consider some numerical methods in which fraction points are involved for three points schemes.

4. THREE POINTS SCHEMES FOR: $U_j^{n+1} = f(U_j^n, U_{j-\frac{1}{2}}^n, U_{j-1}^n)$

Here, $k_1 = 0,\ k_2 = -\frac{1}{2},\ k_3 = -1$

So

$$U_j^{n+1} = B_0 U_j^n + B_{-\frac{1}{2}} U_{j-\frac{1}{2}}^n + B_{-1} U_{j-1}^n \tag{2.22}$$

and

$$\begin{cases} B_0 = 1 - B_{-\frac{1}{2}} - B_{-1} \\ -\frac{1}{2} B_{-\frac{1}{2}} - B_{-1} = -c \\ \frac{1}{4} B_{-\frac{1}{2}} + B_{-1} = c^2 \end{cases}$$

Therefore

$$\begin{cases} B_0 = 1 + 2c^2 - 3c \\ B_{-\frac{1}{2}} = 4(c - c^2) \\ B_{-1} = 2c^2 - c \end{cases} \tag{2.23}$$

Substitution of 2.23 into 2.22

$$U_j^{n+1} = (1 + 2c^2 - 3c)U_j^n + 4(c - c^2)U_{j-\frac{1}{2}}^n + (2c^2 - c)U_{j-1}^n \tag{2.24}$$

5. THREE POINTS SCHEMES FOR: $U_j^{n+1} = f(U_j^n, U_{j-\frac{1}{3}}^n, U_{j-1}^n)$

Similarly, we get the numerical method

$$U_j^{n+1} = (1 + 3c^2 - 4c)U_j^n + \frac{9}{2}(c - c^2)U_{j-\frac{1}{3}}^n + \frac{1}{2}(3c^2 - c)U_{j-1}^n \tag{2.25}$$

2.4.2 Four Points Schemes

From here on, we only consider the integer points schemes. Due to 2.14, we expect to get four third order numerical methods for four points schemes.

From 2.6, the highest order m with four points schemes is

$$\begin{cases} p = 4 \\ m = 3 \end{cases}$$

1. FOUR POINTS SCHEMES FOR: $U_j^{n+1} = f(U_j^n, U_{j-1}^n, U_{j+1}^n, U_{j+2}^n)$

Here, $k_1 = 0, \ k_2 = -1, \ k_3 = 1, \ and k_4 = 2$

From equation 2.12

$$\begin{cases} B_0 = 1 - B_{-1} - B_1 - B_2 \\ -B_{-1} + B_1 + 2B_2 = -c \\ B_{-1} + B_1 + 4B_2 = c^2 \\ -B_{-1} + B_1 + 8B_2 = -c^3 \end{cases}$$

and

$$\begin{cases} B_0 = 1 + \frac{1}{2}c - c^2 - \frac{1}{2}c^3 \\ B_{-1} = \frac{1}{6}c^3 + \frac{1}{2}c^2 + \frac{1}{3}c \\ B_1 = \frac{1}{2}c^3 + \frac{1}{2}c^2 - c \\ B_2 = \frac{1}{6}c - \frac{1}{6}c^3 \end{cases}$$

Therefore

$$\begin{aligned} U_j^{n+1} &= (1 + \frac{1}{2}c - c^2 - \frac{1}{2}c^3)U_j^n + (\frac{1}{6}c^3 + \frac{1}{2}c^2 + \frac{1}{3}c)U_{j-1}^n \\ &\quad + (\frac{1}{2}c^3 + \frac{1}{2}c^2 - c)U_{j+1}^n + (\frac{1}{6}c - \frac{1}{6}c^3)U_{j+2}^n \end{aligned} \tag{2.26}$$

Using the same routine we can acheive all other third order methods as follows:

2. FOUR POINTS SCHEMES FOR: $U_j^{n+1} = f(U_j^n, U_{j+1}^n, U_{j+2}^n, U_{j+3}^n)$

$$\begin{aligned} U_j^{n+1} &= (1+\frac{11}{6}c+c^2+\frac{1}{6}c^3)U_j^n - (\frac{1}{2}c^3+\frac{5}{2}c^2+3c)U_{j+1}^n \\ &+(\frac{1}{2}c^3+2c^2+\frac{3}{2}c)U_{j+2}^n - (\frac{1}{6}c^3+\frac{1}{2}c^2+\frac{1}{3}c)U_{j+3}^n \end{aligned} \quad (2.27)$$

3. FOUR POINTS SCHEMES FOR: $U_j^{n+1} = f(U_{j-2}^n, U_{j-1}^n, U_j^n, U_{j+1}^n)$

$$\begin{aligned} U_j^{n+1} &= (1+\frac{1}{2}c^3-c^2-\frac{1}{2}c)U_j^n + (\frac{1}{6}c^3-\frac{1}{6}c)U_{j-2}^n \\ &-(\frac{1}{6}c^3-\frac{1}{2}c^2+\frac{1}{3}c)U_{j+1}^n + (c+\frac{1}{2}c^2-\frac{1}{2}c^3)U_{j-1}^n \end{aligned} \quad (2.28)$$

4. FOUR POINTS SCHEMES FOR: $U_j^{n+1} = f(U_{j-3}^n, U_{j-2}^n, U_{j-1}^n, U_j^n)$

$$\begin{aligned} U_j^{n+1} &= (1-\frac{1}{6}c^3+c^2-\frac{11}{6}c)U_j^n + (\frac{1}{2}c^3-\frac{5}{2}c^2+3c)U_{j-1}^n \\ &-(\frac{1}{2}c^3-2c^2+\frac{3}{2}c)U_{j-2}^n + (\frac{1}{6}c^3-\frac{1}{2}c^2+\frac{1}{3}c)U_{j-3}^n \end{aligned} \quad (2.29)$$

2.4.3 Five Points Schemes

From equation 2.6, the highest order m we can get with five points schemes is

$$\begin{cases} p = 5 \\ m = 4 \end{cases}$$

1. FIVE POINTS SCHEMES FOR: $U_j^{n+1} = f(U_{j-2}^n, U_{j-1}^n, U_j^n, U_{j+1}^n, U_{j+2}^n)$

Here, $k_1 = -2,\ k_2 = -1,\ k_3 = 1,\ k_4 = 2,\ k_5 = 0$

From 2.12, we get

$$\begin{cases} B_0 = 1 - B_{-2} - B_{-1} - B_1 - B_2 \\ -2B_{-2} - B_{-1} + B_1 + 2B_2 = -c \\ 4B_{-2} + B_{-1} + B_1 + 4B_2 = c^2 \\ -8B_{-2} - B_{-1} + B_1 + 8B_2 = -c^3 \\ 16B_{-2} + B_{-1} + B_1 + 16B_2 = c^4 \end{cases}$$

and

$$\begin{cases} B_0 = 1 + \frac{1}{4}c^4 - \frac{5}{4}c^2 \\ B_{-2} = \frac{1}{2}(\frac{1}{12}c^4 + \frac{1}{6}c^3 - \frac{1}{12}c^2 - \frac{1}{6}c) \\ B_{-1} = \frac{1}{3}(2c + 2c^2 - \frac{1}{2}c^3 - \frac{1}{2}c^4) \\ B_1 = \frac{1}{6}(c^3 + 4c^2 - c^4 - 4c) \\ B_2 = \frac{1}{4}(\frac{1}{3}c - \frac{1}{6}c^2 - \frac{1}{3}c^3 + \frac{1}{6}c^4) \end{cases}$$

Therefore

$$\begin{aligned} U_j^{n+1} &= (1 + \frac{1}{4}c^4 - \frac{5}{4}c^2)U_j^n + (\frac{1}{24}c^4 + \frac{1}{12}c^3 - \frac{1}{24}c^2 - \frac{1}{12}c)U_{j-2}^n \\ &+(\frac{2}{3}c + \frac{2}{3}c^2 - \frac{1}{6}c^3 - \frac{1}{6}c^4)U_{j-1}^n + (\frac{1}{6}c^3 + \frac{2}{3}c^2 - \frac{1}{6}c^4 - \frac{2}{3}c)U_{j+1}^n \\ &+(\frac{1}{12}c - \frac{1}{24}c^2 - \frac{1}{12}c^3 + \frac{1}{24}c^4)U_{j+2}^n \end{aligned} \tag{2.30}$$

2. FIVE POINTS SCHEMES FOR: $U_j^{n+1} = f(U_{j-4}^n, U_{j-3}^n, U_{j-2}^n, U_{j-1}^n, U_j^n)$

$$\begin{aligned} U_j^{n+1} &= (1 - \frac{25}{12}c + \frac{35}{24}c^2 - \frac{5}{12}c^3 + \frac{1}{24}c^4)U_j^n + (\frac{1}{24}c^4 - \frac{1}{4}c^3 + \frac{11}{24}c^2 - \frac{1}{4}c)U_{j-4}^n \\ &+(\frac{4}{3}c - \frac{7}{3}c^2 + \frac{7}{6}c^3 - \frac{1}{6}c^4)U_{j-3}^n + (\frac{1}{4}c^4 - 2c^3 + \frac{19}{4}c^2 - 3c)U_{j-2}^n \\ &+(4c - \frac{13}{3}c^2 + \frac{3}{2}c^3 - \frac{1}{6}c^4)U_{j-1}^n \end{aligned} \tag{2.31}$$

3. FIVE POINTS SCHEMES FOR: $U_j^{n+1} = f(U_{j-3}^n, U_{j-2}^n, U_{j-1}^n, U_j^n, U_{j+1}^n)$

$$\begin{aligned} U_j^{n+1} &= (1-\frac{1}{6}c^4+\frac{5}{6}c^3-\frac{5}{6}c^2-\frac{5}{6}c)U_j^n+(\frac{1}{24}c^4-\frac{1}{12}c^3-\frac{1}{24}c^2+\frac{1}{12}c)U_{j-3}^n \\ &+(\frac{1}{2}c^3+\frac{1}{6}c^2-\frac{1}{6}c^4-\frac{1}{2}c)U_{j-2}^n+(\frac{1}{4}c^4-c^3+\frac{1}{4}c^2+\frac{3}{2}c)U_{j-1}^n \\ &+(\frac{1}{24}c^4-\frac{1}{4}c^3+\frac{11}{24}c^2-\frac{1}{4}c)U_{j+1}^n \end{aligned} \tag{2.32}$$

4. FIVE POINTS SCHEMES FOR: $U_j^{n+1} = f(U_{j-1}^n, U_j^n, U_{j+1}^n, U_{j+2}^n, U_{j+3}^n)$

$$\begin{aligned} U_j^{n+1} &= (1-\frac{1}{6}c^4-\frac{5}{6}c^3-\frac{5}{6}c^2+\frac{5}{6}c)U_j^n+(\frac{1}{24}c^4+\frac{1}{4}c^3+\frac{11}{24}c^2+\frac{1}{4}c)U_{j-1}^n \\ &+(\frac{1}{4}c^4+c^3+\frac{1}{4}c^2-\frac{3}{2}c)U_{j+1}^n-\frac{1}{6}(c^4+3c^3-c^2-3c)U_{j+2}^n \\ &+\frac{1}{24}(c^4+2c^3-c^2-2c)U_{j+3}^n \end{aligned} \tag{2.33}$$

5. FIVE POINTS SCHEMES FOR: $U_j^{n+1} = f(U_j^n, U_{j+1}^n, U_{j+2}^n, U_{j+3}^n, U_{j+4}^n)$

$$\begin{aligned} U_j^{n+1} &= (1+\frac{1}{24}c^4+\frac{5}{12}c^3+\frac{35}{24}c^2+\frac{25}{12}c)U_j^n-(\frac{1}{6}c^4+\frac{3}{2}c^3+\frac{13}{3}c^2+4c)U_{j+1}^n \\ &+(\frac{1}{4}c^4+2c^3+\frac{19}{4}c^2+3c)U_{j+2}^n-\frac{1}{6}(c^4+7c^3+14c^2+8c)U_{j+3}^n \\ &+\frac{1}{24}(c^4+6c^3+11c^2+6c)U_{j+4}^n \end{aligned} \tag{2.34}$$

So far, we have considered three, four and five points schemes. Actually, we can develop any high order numerical methods using more points according to the Universal Formula Theorem. To illustrate this the 20th order numerical method is presented in the appendix. When deriving higher order methods, equation 2.13 is recomanded by computing program.

Chapter 3

NEW APPROACH FOR LINEAR STABILITY ANALYSIS

3.1 Introduction

The Lax Equivalence Theorem says that for a consistent linear numerical method stability is necessary and sufficient for convergence to the true solution of the PDE.

As discussed in the last chapter, any numerical methods developed from Theorem 1 are automatically consistent with the PDE. Therefore, provided we can prove these methods are stable, then these methods are guaranteed to converge to the real solution of the Partial Differential Equation.

However, the difficulty lies in that it is not a simple task to theoretically prove and analyse the stability of a proposed numerical method, even for linear scalar advection problems.

At present, there are several techniques available to analyse stability. This includes:

- Discrete perturbation method.
- Maximal norm method.
- Fourier Series method.

In this chapter, we are going to further investigate the Fourier Series method and develop a new approach to the linear stability analysis.

3.2 New Approach for Linear Stability study

THEOREM 2

The stability condition of any two-level explicit linear numerical method,

$$U_j^{n+1} = \sum_{\alpha=1}^{p} B_{k_\alpha} U_{j+k_\alpha}^n$$

can be defined by

$$\lambda = 1 - 2 \sum_{\alpha=1, k_\alpha=\pm 1, \pm 3, \ldots}^{p} B_{k_\alpha} \tag{3.1}$$

where λ is the amplifier factor, $|\lambda| \leq 1$ and k_α are odd integer numbers.

PROOF

The Fourier Series method based on assuming that

$$U_j^n = A_L^n e^{iLj\Delta x} \tag{3.2}$$

where, A_L^n is the amplitude at time level n; L is the wave number in x-direction, $L = \frac{2\pi}{\tau}$; τ is the wavelength; i is the complex number, $i = \sqrt{-1}$.

Considering the general form of linear numerical methods

$$U_j^{n+1} = \sum_{\alpha=1}^{p} B_{k_\alpha} U_{j+k_\alpha}^n \tag{3.3}$$

From 3.2, we have

$$\begin{cases} U_j^{n+1} = A_L^{n+1} e^{iLj\Delta x} \\ U_{j+k_\alpha}^n = A_L^n e^{iL(j+k_\alpha)\Delta x} \end{cases} \tag{3.4}$$

Replace 3.4 into 3.3

$$A_L^{n+1} e^{iLj\Delta x} = \sum_{\alpha=1}^{p} B_{k_\alpha} A^n L e^{iL(j+k_\alpha)\Delta x}$$

Dividing both sides by $A_L^n e^{iLj\Delta x}$, we get amplifier factor λ:

$$\begin{aligned} \lambda(\theta) &= \frac{A_L^{n+1}}{A_L^n} \\ &= \sum_{\alpha=1}^{p} B_{k_\alpha} e^{iLk_\alpha \Delta x} \\ &= \sum_{\alpha=1}^{p} B_{k_\alpha} e^{ik_\alpha \theta} \end{aligned} \tag{3.5}$$

where, θ is the phase angle, $\theta = L \bigtriangleup x$.

The absolute value of the amplifier factor $|\lambda|$ is called amplifier coefficient. Obviously if $|\lambda| > 1$, the numerical method will not be stable, otherwise, it is stable.

We can rewrite 3.5 as:

$$\begin{aligned}
\lambda &= \sum_{\alpha=1}^{p} B_{k_\alpha} \cos k_\alpha\theta + i \sum_{\alpha=1}^{p} B_{k_\alpha} \sin k_\alpha\theta \\
&= \gamma_r + i\gamma_i
\end{aligned} \tag{3.6}$$

where,

$$\begin{aligned}
\gamma_r &= \sum_{\alpha=1}^{p} B_{k_\alpha} \cos k_\alpha\theta \\
\gamma_i &= \sum_{\alpha=1}^{p} B_{k_\alpha} \sin k_\alpha\theta
\end{aligned}$$

Therefore

$$|\lambda| = \sqrt{\gamma_r^2 + \gamma_i^2} \tag{3.7}$$

This is the normal approach of analysing the stability in practice using the Fourier method. But, as you can see, generally 3.7 is a very complicated triangular algebra, especially for high order numerical methods, say, over second order. For a method over second order, $|\lambda|$ is very difficult to work out, or even, impossible to manipulate using 3.7.

Here we are going to adopt a new approach.

Equation 3.5 can be rewritten as:

$$\begin{aligned}
\lambda &= \sum_{\alpha=1}^{p} B_{k_\alpha} e^{ik_\alpha\theta} \\
&= B_0 + \sum_{\alpha=1, k_\alpha\neq 0}^{p} B_{k_\alpha} e^{ik_\alpha\theta}
\end{aligned}$$

$$= 1 - \sum_{\alpha=1,k_\alpha\neq 0}^{p} B_{k_\alpha} + \sum_{\alpha=1,k_\alpha\neq 0}^{p} B_{k_\alpha} e^{ik_\alpha\theta}$$
$$= 1 - \sum_{\alpha=1,k_\alpha\neq 0}^{p} B_{k_\alpha}\left(1 - e^{ik_\alpha\theta}\right) \tag{3.8}$$

Since $B_0 = 1 - \sum_{\alpha=1,k_\alpha\neq 0}^{p} B_{k_\alpha}$ from 2.3.

Because the amplipier factor λ is a real number, to satisfy this if and only if the following condition is qualified:

$$k_\alpha\theta = n\pi$$

where n are all real integer numbers.

Consider the case in which the k_α are integer points,

$$k_\alpha = 0, \pm 1, \pm 2, \ldots, \pm\infty.$$

In this case, 3.8 simplely becomes

$$\lambda = 1 - \sum_{\alpha=1,k_\alpha\neq 0}^{p} B_{k_\alpha}\left(1 - e^{ik_\alpha\pi}\right)$$
$$= 1 - 2 \sum_{\alpha=1,k_\alpha=\pm 1,\pm 3,\ldots}^{p} B_{k_\alpha}$$

This implies 3.1, and proves the Theorem 2.

Next section, we will demonstrate with some examples how convenient to use Theorem 2 for linear stability study.

3.3 Stability Analysis for Three, Four and Five Points Schemes

In last chapter, using the Universal Formula Theorem, we produced numerical methods involving three, four, and five points. Now, we will deal with the stability of these methods according to Theorem 2 .

3.3.1 Three Points Schemes

1. Lax-Wendroff method. See equation 2.17.

$$U_j^{n+1} = (1-c^2)U_j^n + \frac{1}{2}(c^2+c)U_{j-1}^n + \frac{1}{2}(c^2-c)U_{j+1}^n$$

here,

$$\begin{cases} B_0 = 1-c^2 & k_\alpha = 0 \\ B_{-1} = \frac{1}{2}(c^2+c) & k_\alpha = -1 \\ B_1 = \frac{1}{2}(c^2-c) & k_\alpha = 1 \end{cases}$$

Substitution of these into 3.1, gives

$$\begin{aligned} \lambda &= 1-2(B_{-1}+B_1) \\ &= 1-2c^2 \end{aligned} \tag{3.9}$$

Equation 3.9 is the amplification function $\lambda(c)$ in terms of Courant number c. **Figure 3.1** illustrates this function.

As is clearly shown, when λ moves from -1 to 1, c is contained the region between -1 and 1 but not elsewhere. Therefore, the stable region of Lax-Wendroff method is:

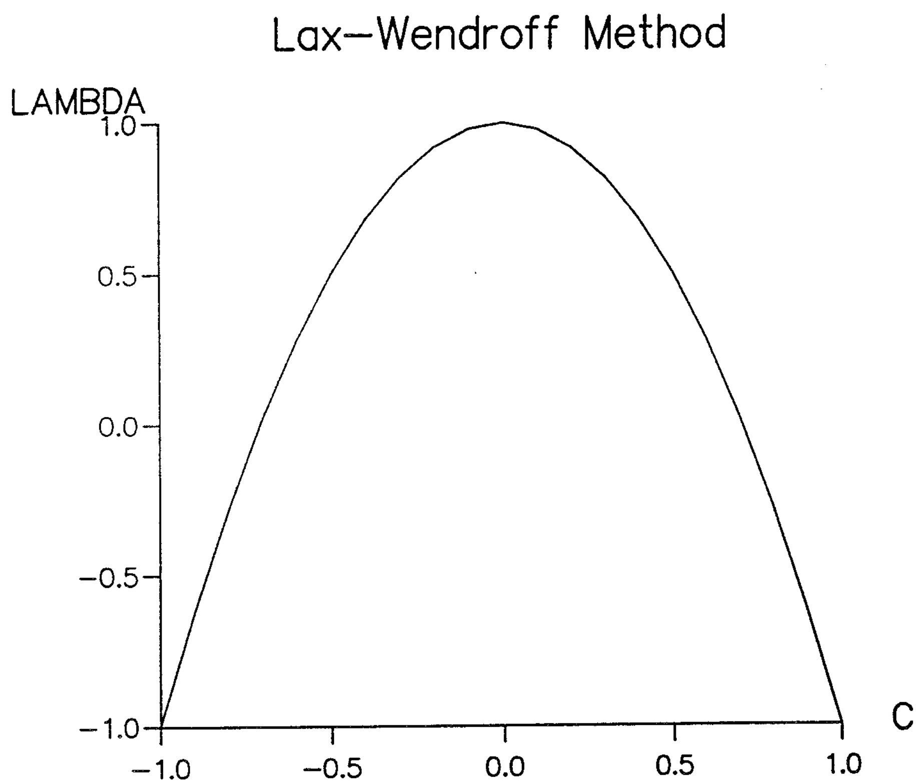

Figure 3.1: Stable Region for L-W Method

$$-1 \leq c \leq 1 \tag{3.10}$$

2. Beam-Warming method. See 2.20.

$$U_j^{n+1} = (1 + \frac{1}{2}c^2 - \frac{3}{2}c)U_j^n + (2c - c^2)U_{j-1}^n + \frac{1}{2}(c^2 - c)U_{j-2}^n$$

here,

$$\begin{cases} B_0 = 1 + \frac{1}{2}c^2 - \frac{3}{2}c & k_\alpha = 0 \\ B_{-1} = 2c - c^2 & k_\alpha = -1 \\ B_{-2} = \frac{1}{2}c^2 - \frac{1}{2}c & k_\alpha = -2 \end{cases}$$

From 3.1

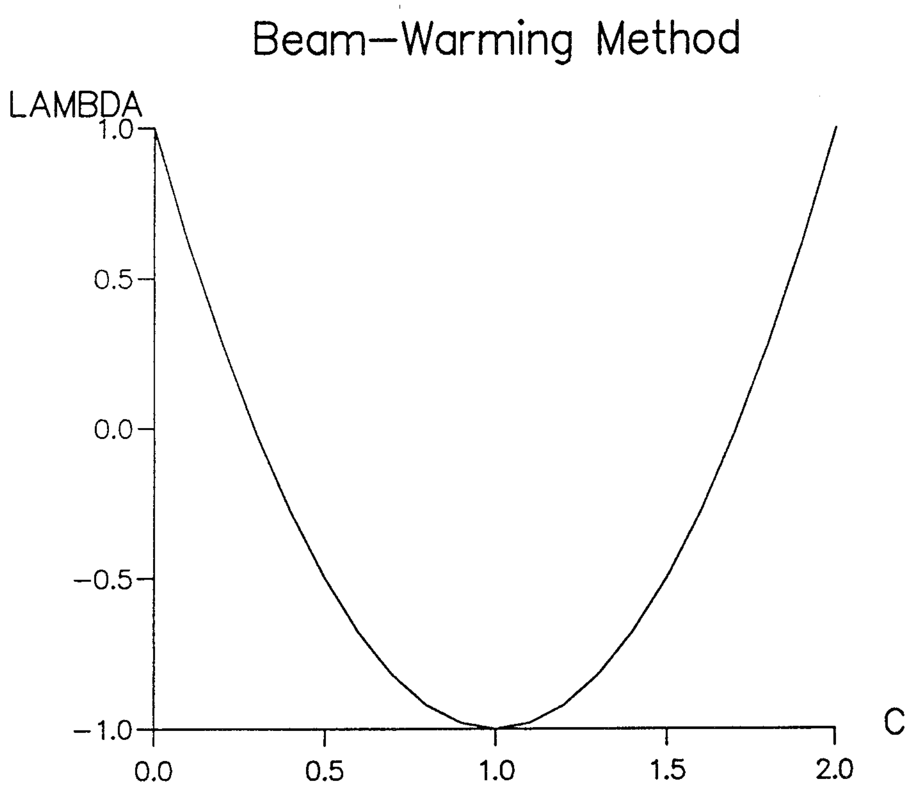

Figure 3.2: Stable Region for B-W Method

$$\begin{aligned} \lambda &= 1 - 2B_{-1} \\ &= 1 - 4c + 2c^2 \end{aligned} \tag{3.11}$$

From **Figure 3.2**, the stable region of this method is:

$$0 \leq c \leq 2 \tag{3.12}$$

3. Equation 2.21

$$U_j^{n+1} = (1 + \frac{1}{2}c^2 + \frac{3}{2}c)U_j^n - (c^2 + 2c)U_{j+1}^n + \frac{1}{2}(c^2 + c)U_{j+2}^n$$

and from 3.1

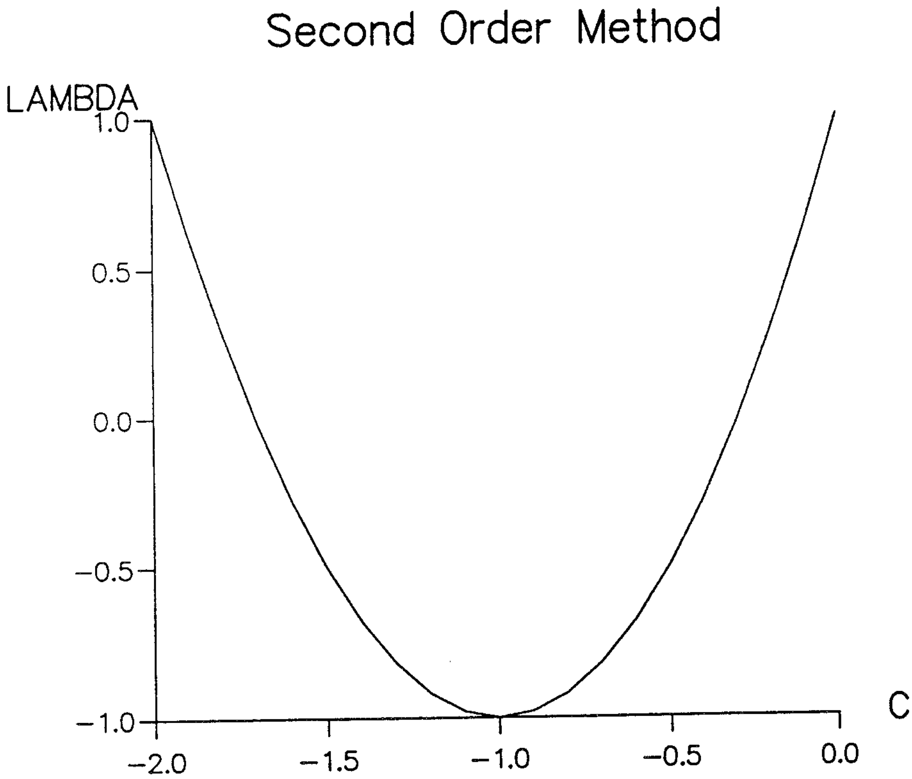

Figure 3.3: Stable Region for Equation 2.21

$$\lambda = 1 + 4c + 2c^2 \tag{3.13}$$

From **Figure 3.3**, the stable region of this method is:

$$-2 \leq c \leq 0 \tag{3.14}$$

3.3.2 Four Points Schemes

1. Equation 2.26

$$U_j^{n+1} = (1 + \frac{1}{2}c - c^2 - \frac{1}{2}c^3)U_j^n + (\frac{1}{6}c^3 + \frac{1}{2}c^2 + \frac{1}{3}c)U_{j-1}^n$$

$$+(\frac{1}{2}c^3+\frac{1}{2}c^2-c)U_{j+1}^n+(\frac{1}{6}c-\frac{1}{6}c^3)U_{j+2}^n$$

here,

$$\begin{cases} B_0 = 1+\frac{1}{2}c-c^2-\frac{1}{2}c^3 & k_\alpha = 0 \\ B_{-1} = \frac{1}{6}c^3+\frac{1}{2}c^2+\frac{1}{3}c & k_\alpha = -1 \\ B_1 = \frac{1}{2}c^3+\frac{1}{2}c^2-c & k_\alpha = 1 \\ B_2 = \frac{1}{6}c-\frac{1}{6}c^3 & k_\alpha = 2 \end{cases}$$

From 3.1

$$\lambda = 1-\frac{4}{3}c^3-2c^2+\frac{4}{3}c \tag{3.15}$$

Figure 3.4 shows that when λ moves from -1 to 1, there are multiple regions satisfying the condition of $|\lambda| \leq 1$. Namely, the CFL region is not unique. The three regions are:

$$-2 \leq c \leq -1.5 \tag{3.21a}$$

$$-1 \leq c \leq 0 \tag{3.21b}$$

$$0.5 \leq c \leq 1 \tag{3.21c}$$

Computational experiments tell us that only the region 3.21b is stable. Other regions are spurious.

Therefore, the stable region for this method is:

$$-1 \leq c \leq 0 \tag{3.22}$$

2. Equation 2.27

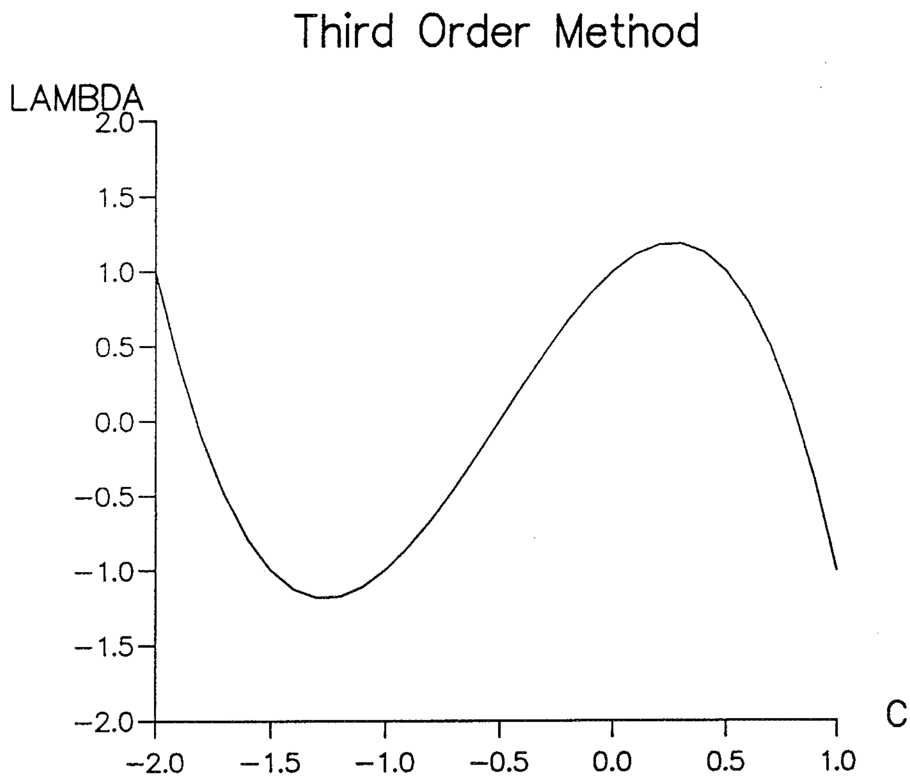

Figure 3.4: Stable Region for Equation 2.26

$$\begin{aligned} U_j^{n+1} &= (1+\frac{11}{6}c+c^2+\frac{1}{6}c^3)U_j^n-(\frac{1}{2}c^3+\frac{5}{2}c^2+3c)U_{j+1}^n \\ &+(\frac{1}{2}c^3+2c^2+\frac{3}{2}c)U_{j+2}^n-(\frac{1}{6}c^3+\frac{1}{2}c^2+\frac{1}{3}c)U_{j+3}^n \end{aligned}$$

From 3.1

$$\lambda = 1+\frac{4}{3}c^3+6c^2+\frac{20}{3}c \tag{3.23}$$

The stable region for this method is: (see **Figure 3.5**)

$$-2 \le c \le -1 \tag{3.24}$$

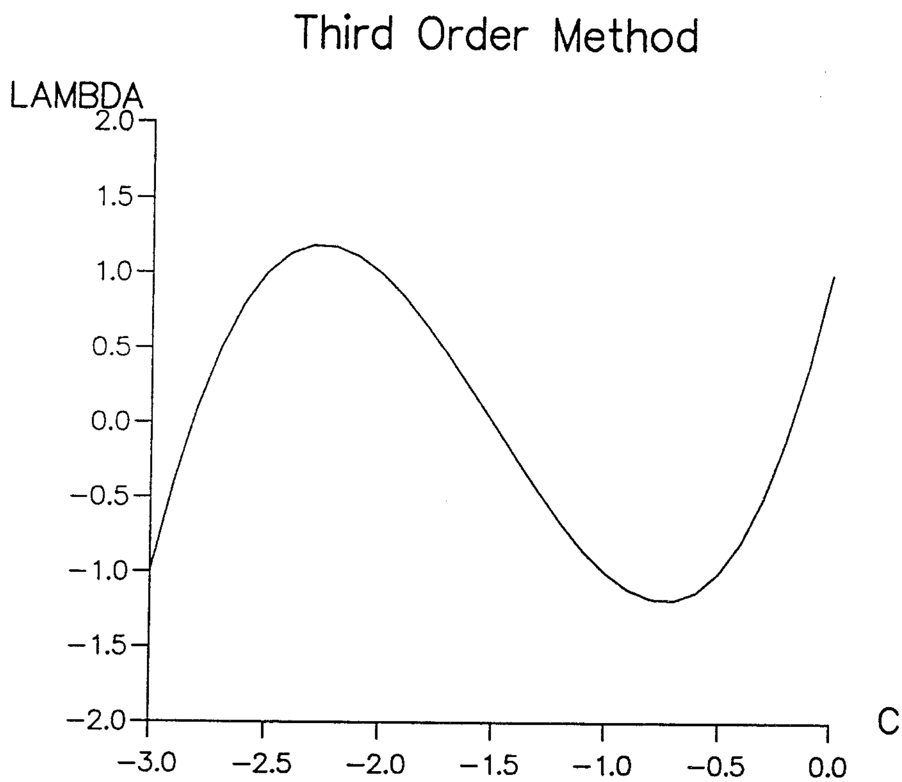

Figure 3.5: Stable Region for Equation 2.27

3. Equation 2.28

$$\begin{aligned} U_j^{n+1} &= (1+\frac{1}{2}c^3 - c^2 - \frac{1}{2}c)U_j^n - (\frac{1}{6}c^3 - \frac{1}{2}c^2 + \frac{1}{3}c)U_{j+1}^n \\ &\quad +(c+\frac{1}{2}c^2 - \frac{1}{2}c^3)U_{j-1}^n + (\frac{1}{6}c^3 - \frac{1}{6}c)U_{j-2}^n \end{aligned}$$

$$\lambda = 1 + \frac{4}{3}c^3 - 2c^2 - \frac{4}{3}c \tag{3.25}$$

The stable region for this method is: (see **Figure 3.6**)

$$0 \le c \le 1 \tag{3.26}$$

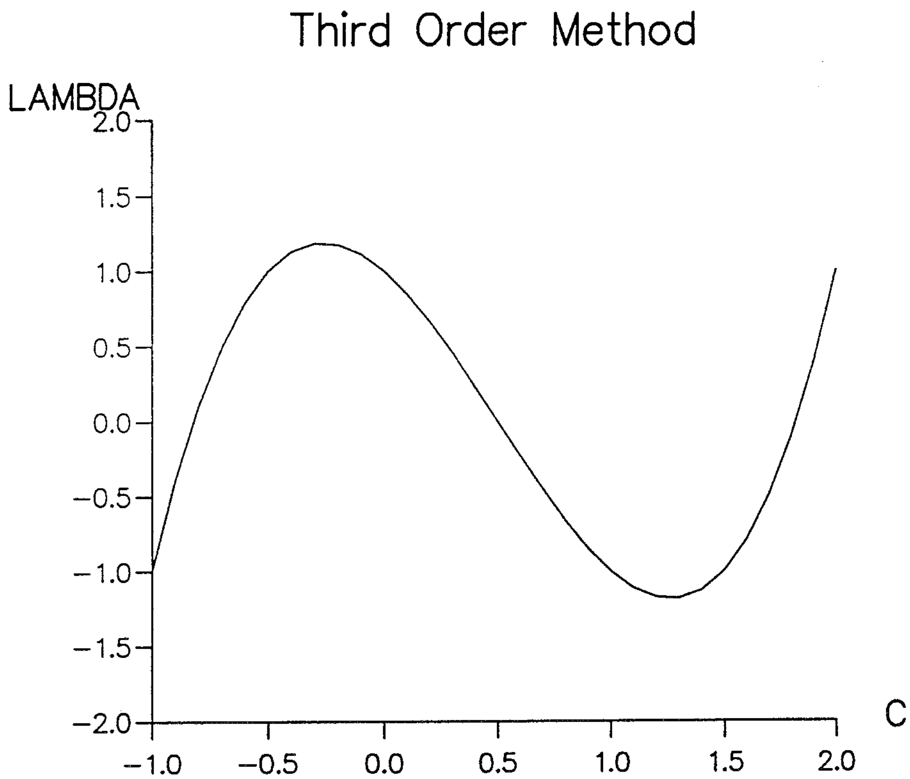

Figure 3.6: Stable Region for Equation 2.28

4. Equation 2.29

$$\begin{aligned} U_j^{n+1} &= (1 - \frac{1}{6}c^3 + c^2 - \frac{11}{6}c)U_j^n + (\frac{1}{2}c^3 - \frac{5}{2}c^2 + 3c)U_{j-1}^n \\ &\quad -(\frac{1}{2}c^3 - 2c^2 + \frac{3}{2}c)U_{j-2}^n + (\frac{1}{6}c^3 - \frac{1}{2}c^2 + \frac{1}{3}c)U_{j-3}^n \end{aligned}$$

$$\lambda = 1 - \frac{4}{3}c^3 + 6c^2 - \frac{20}{3}c \qquad (3.27)$$

The stable region for this method is: (see **Figure 3.7**)

$$1 \leq c \leq 2 \qquad (3.28)$$

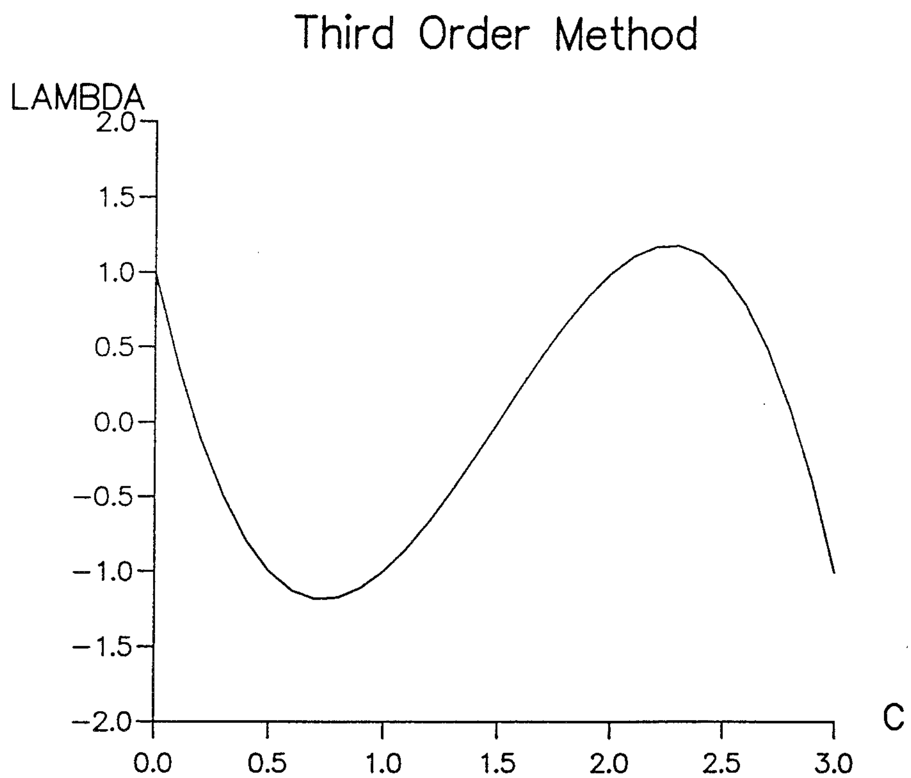

Figure 3.7: Stable Region for Equation 2.29

3.3.3 Five Points Schemes

1. Equation 2.30

$$\begin{aligned} U_j^{n+1} &= (1+\frac{1}{4}c^4-\frac{5}{4}c^2)U_j^n+(\frac{1}{24}c^4+\frac{1}{12}v^3-\frac{1}{24}c^2-\frac{1}{12}c)U_{j-2}^n \\ &+(\frac{2}{3}c+\frac{2}{3}c^2-\frac{1}{6}c^3-\frac{1}{6}c^4)U_{j-1}^n+(\frac{1}{6}c^3+\frac{2}{3}c^2-\frac{1}{6}c^4-\frac{2}{3}c)U_{j+1}^n \\ &+(\frac{1}{12}c-\frac{1}{24}c^2-\frac{1}{12}c^3+\frac{1}{24}c^4)U_{j+2}^n \end{aligned}$$

$$\lambda = 1-\frac{8}{3}c^2+\frac{2}{3}c^4 \qquad (3.29)$$

The stable region for this method is: (see **Figure 3.8**)

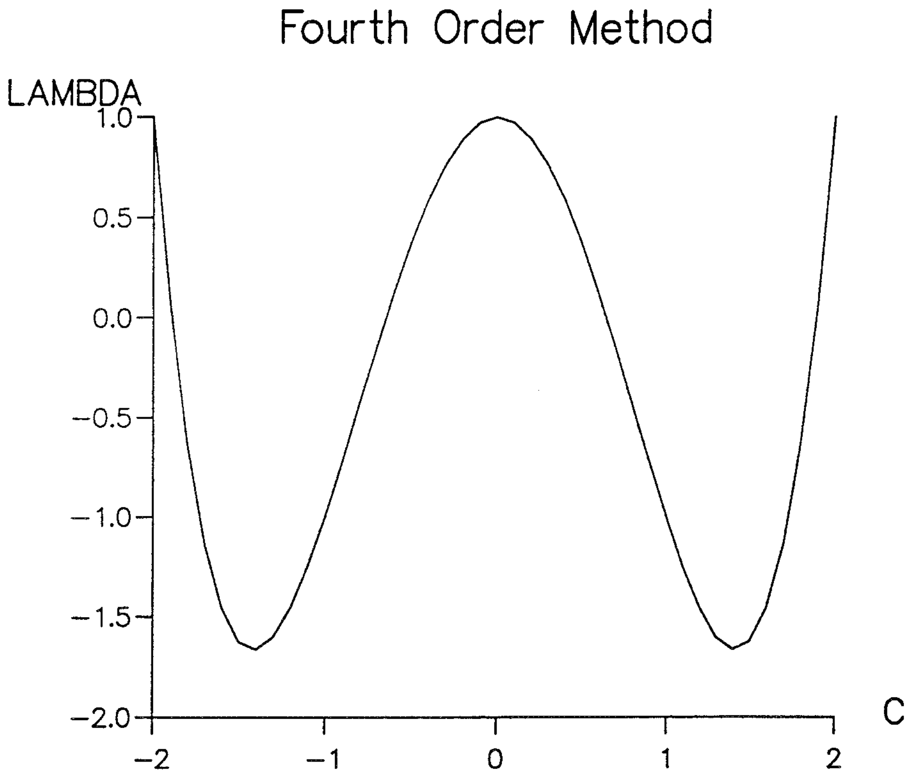

Figure 3.8: Stable Region for Equation 2.30

$$\begin{aligned} -2 &\leq c \leq -1.73 \\ -1 &\leq c \leq 1 \\ 1.73 &\leq c \leq 2 \end{aligned} \tag{3.30}$$

2. Equation 2.31

$$\begin{aligned} U_j^{n+1} &= (1 - \frac{25}{12}c + \frac{35}{24}c^2 - \frac{5}{12}c^3 + \frac{1}{24}c^4)U_j^n + (\frac{1}{24}c^4 - \frac{1}{4}c^3 + \frac{11}{24}c^2 - \frac{1}{4}c)U_{j-4}^n \\ &+(\frac{4}{3}c - \frac{7}{3}c^2 + \frac{7}{6}c^3 - \frac{1}{6}c^4)U_{j-3}^n + (\frac{1}{4}c^4 - 2c^3 + \frac{19}{4}c^2 - 3c)U_{j-2}^n \\ &+(4c - \frac{13}{3}c^2 + \frac{3}{2}c^3 - \frac{1}{6}c^4)U_{j-1}^n \end{aligned}$$

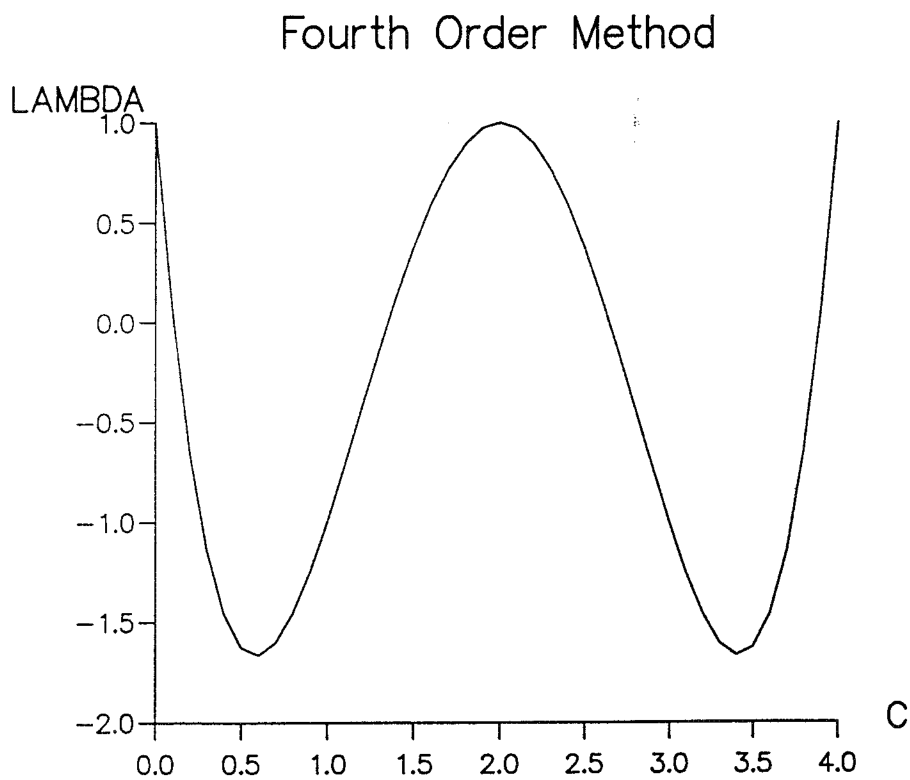

Figure 3.9: Stable Region for Equation 2.31

$$\lambda = 1 + \frac{2}{3}c^4 - \frac{16}{3}c^3 + \frac{40}{3}c^2 - \frac{32}{3}c \tag{3.31}$$

The stable region for this method is: (see **Figure 3.9**)

$$\begin{aligned} 1 &\leq c \leq 3 \\ 3.73 &\leq c \leq 4 \end{aligned} \tag{3.32}$$

3. Equation 2.32

$$U_j^{n+1} = \left(1 - \frac{1}{6}c^4 + \frac{5}{6}c^3 - \frac{5}{6}c^2 - \frac{5}{6}c\right) U_j^n + \left(\frac{1}{24}c^4 - \frac{1}{12}c^3 - \frac{1}{24}c^2 + \frac{1}{12}c\right) U_{j-3}^n$$

Figure 3.10: Stable Region for Equation 2.32

$$+\left(-\frac{1}{6}c^4+\frac{1}{2}c^3+\frac{1}{6}c^2-\frac{1}{2}c\right)U_{j-2}^n+\left(\frac{1}{4}c^4-c^3+\frac{1}{4}c^2+\frac{3}{2}c\right)U_{j-1}^n$$
$$+\left(\frac{1}{24}c^4-\frac{1}{4}c^3+\frac{11}{24}c^2-\frac{1}{4}c\right)U_{j+1}^n$$

$$\lambda = 1-\frac{2}{3}c^4+\frac{8}{3}c^3-\frac{4}{3}c^2-\frac{8}{3}c \tag{3.33}$$

The stable region for this method is: (see **Figure 3.10**)

$$\begin{aligned} -1 &\leq c \leq -0.73 \\ 0 &\leq c \leq 2 \\ 2.73 &\leq c \leq 3 \end{aligned} \tag{3.34}$$

4. Equation 2.33

$$\begin{aligned}
U_j^{n+1} &= \left(1-\frac{1}{6}c^4-\frac{5}{6}c^3-\frac{5}{6}c^2+\frac{5}{6}c\right)U_j^n+\left(\frac{1}{24}c^4+\frac{1}{4}c^3+\frac{11}{24}c^2+\frac{1}{4}c\right)U_{j-1}^n \\
&+\left(\frac{1}{4}c^4+c^3+\frac{1}{4}c^2-\frac{3}{2}c\right)U_{j+1}^n-\frac{1}{6}\left(c^4+2c^3-c^2-3c\right)U_{j+2}^n \\
&+\frac{1}{24}\left(c^4+2c^3-c^2-2c\right)U_{j+3}^n
\end{aligned}$$

$$\lambda = 1-\frac{2}{3}c^4-\frac{8}{3}c^3-\frac{4}{3}c^2+\frac{8}{3}c \qquad (3.35)$$

The stable region for this method is: (see **Figure 3.11**)

$$\begin{aligned}
-3 &\leq c \leq -2.73 \\
-2 &\leq c \leq 0 \\
0.73 &\leq c \leq 1
\end{aligned} \qquad (3.36)$$

5. Equation 2.34

$$\begin{aligned}
U_j^{n+1} &= \left(1+\frac{1}{24}c^4+\frac{5}{12}c^3+\frac{35}{24}c^2+\frac{25}{12}c\right)U_j^n-\left(\frac{1}{6}c^4+\frac{3}{2}c^3+\frac{13}{3}c^2+4c\right)U_{j+1}^n \\
&+\left(\frac{1}{4}c^4+2c^3+\frac{19}{4}c^2+3c\right)U_{j+2}^n-\frac{1}{6}\left(c^4+7c^3+14c^2+8c\right)U_{j+3}^n \\
&+\frac{1}{24}\left(c^4+6c^3+11c^2+6c\right)U_{j+4}^n
\end{aligned}$$

$$\lambda = 1+\frac{2}{3}c^4+\frac{16}{3}c^3+\frac{40}{3}c^2+\frac{32}{3}c \qquad (3.37)$$

The stable region for this method is: (see **Figure 3.12**)

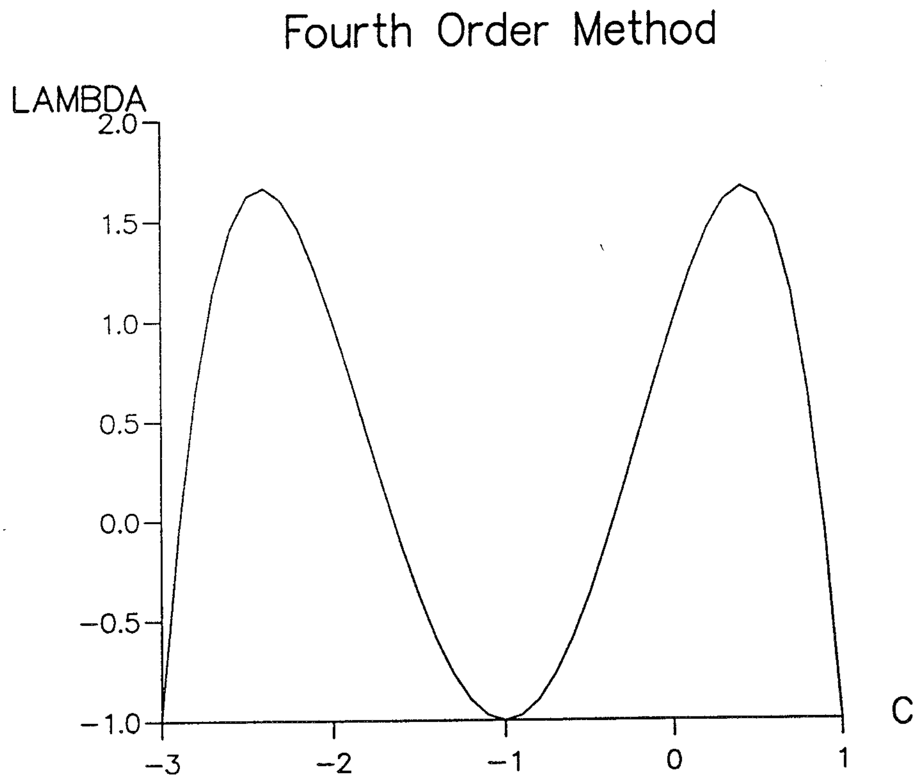

Figure 3.11: Stable Region for Equation 2.33

$$
\begin{aligned}
-4 &\leq c \leq -3.73 \\
-3 &\leq c \leq -1
\end{aligned}
\tag{3.38}
$$

3.3.4 Some Computational Results

Figure 3.13 to **Figure 3.15** show some computational results. As you can see from **Figure 3.13**, the second order method has an obvious error with time marching after only 1000 steps, and the third order method, **Figure 3.14**, begins a distinctive error at 6000 time steps, but the fourth order method, **Figure 3.15**, still looks very good after 20000 steps. This indicates that the accuracy of computational results enhances dramatically by moving from second to fourth order.

Figure 3.12: Stable Region for Equation 2.34

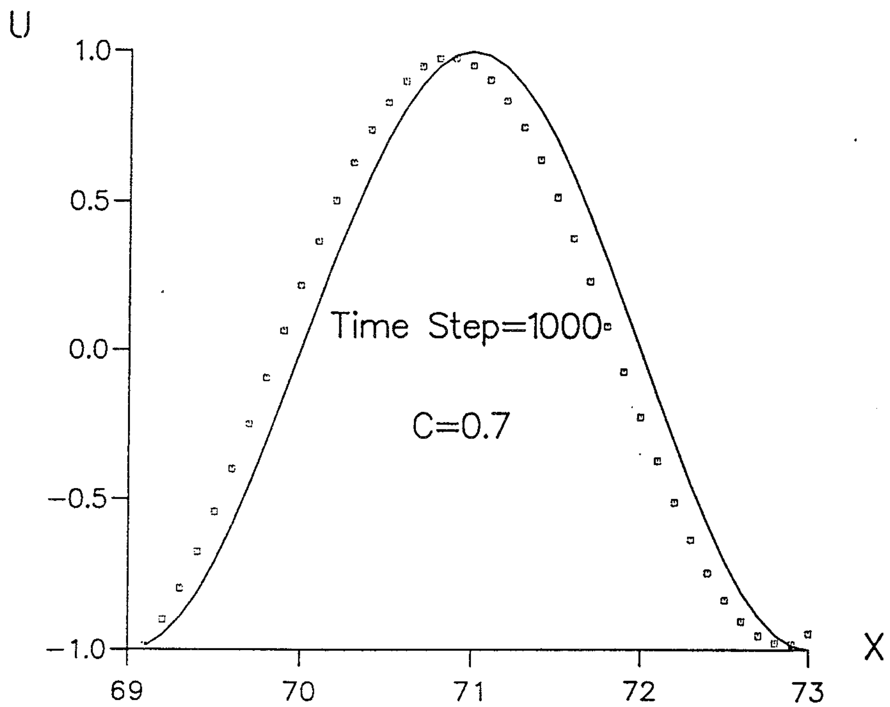

Figure 3.13: Computational Result of Lax-Wendroff Method

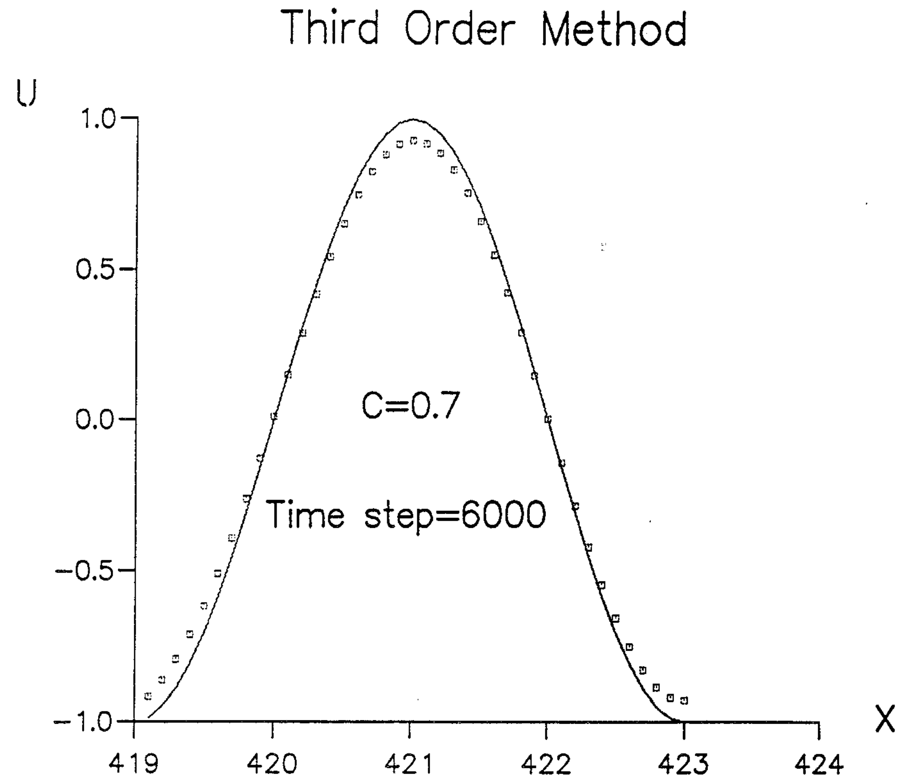

Figure 3.14: Computational result of Equation 2.28

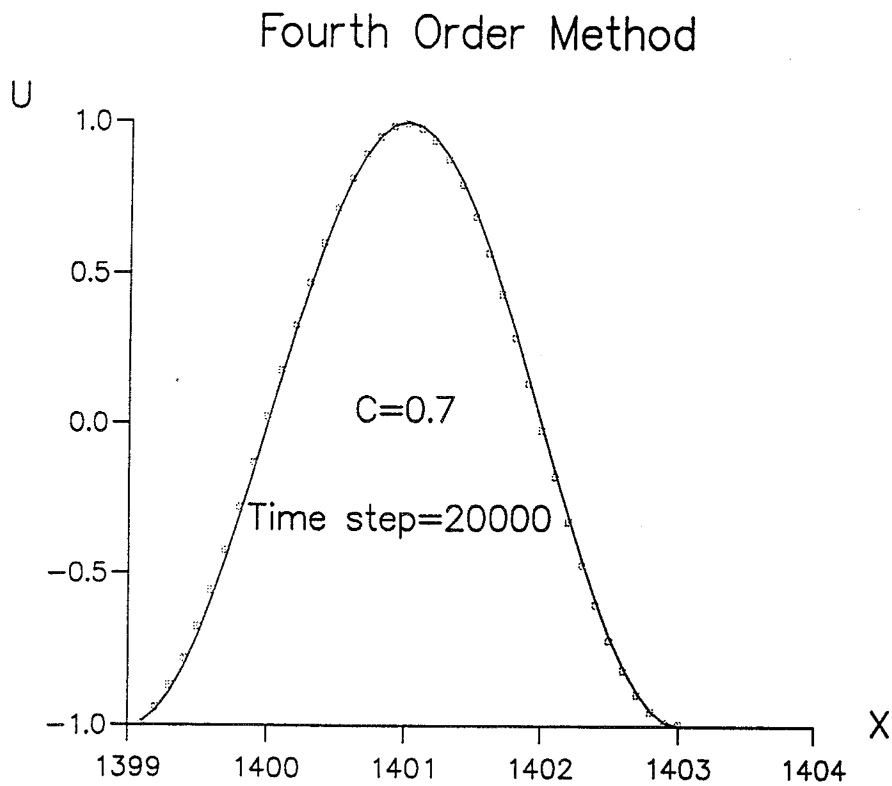

Figure 3.15: Computational Result of Equation 2.30

Chapter 4

CONSERVATIVE FORM OF THE UNIVERSAL FORMULA

4.1 Introduction

So far, we have created the Universal Formula (UF) in chapter 2, which can construct 2-level explicit arbitrary-order numerical methods. We have also developed a new method to deal with the linear stability problem in chapter 3. From now on, we will shift our interest to nonlinear problems, that is, extend the linear Universal Formula to nonlinear hyperbolic conservation laws and define the conservative form of the Universal Formula.

In this chapter, we intend to find a general way to transform the Universal Formula in the form of

$$U_j^{n+1} = \sum_{\alpha=1}^{p} B_{k_\alpha} U_{j+k_\alpha}^n \tag{4.1}$$

into conservative form in terms of numerical fluxes,

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[F(U^n; j) - F(U^n; j-1)\right] \tag{4.2}$$

so that these methods are guaranteed convergence to the true weak solution of the PDE when implementing these methods to nonlinear problems.

4.2 The Conservation form of the Universal Formula(UF)

4.2.1 Derivation of Conservation form of UF

THEOREM 3

The conservative form of the Universal Formula can be defined as

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[\frac{1}{c_0}F_j^n - \sum_{\alpha=1}^{p}\frac{1}{c_{k_\alpha}}B_{k_\alpha}F_{j+k_\alpha}^n\right] \tag{4.3}$$

here, $c_{k_\alpha} = f'(U_{j+k_\alpha}^n)\frac{k}{h}$; $F_{j+k_\alpha}^n = f(U_{j+k_\alpha}^n)$. For linear case, $F_{j+k_\alpha}^n = aU_{j+k_\alpha}^n$.

PROOF

Manipulating equation 4.1, we have

$$\begin{aligned} U_j^{n+1} &= \sum_{\alpha=1}^{p} B_{k_\alpha}U_{j+k_\alpha}^n \\ &= B_0U_j^n + \sum_{\alpha=1, k_\alpha\neq 0}^{p} B_{k_\alpha}U_{j+k_\alpha}^n \\ &= U_j^n - (1-B_0)U_j^n + \sum_{\alpha=1, k_\alpha\neq 0}^{p} B_{k_\alpha}U_{j+k_\alpha}^n \end{aligned}$$

$$= U_j^n - \frac{k}{h}\left[\frac{1}{c_0}(1 - B_0)F_j^n - \sum_{\alpha=1,k_\alpha\neq 0}^{p} \frac{1}{c_{k_\alpha}} B_{k_\alpha} F_{j+k_\alpha}^n\right]$$

$$= U_n^n - \frac{k}{h}\left[\frac{1}{c_0}F_j^n - \sum_{\alpha=1}^{p} \frac{1}{c_{k_\alpha}} B_{k_\alpha} F_{j+k_\alpha}^n\right]$$

This is the equation 4.3, and the proof is completed.

The equation 4.3 is called the Conservative Universal Formula.

4.2.2 Numerical Flux of the UF

The Conservative Universal Formula can be rewritten as

$$U_j^{n+1} = U_j^n - \frac{k}{h}\sum_{\alpha=1}^{p} A_\alpha F_{j+k_\alpha}^n \tag{4.4}$$

here α is the grid point number; A_α are the coefficients.

THEOREM 4

Assuming the numerical flux takes the following form:

$$F(U^n; j) = \sum_{\alpha=2}^{p} B_\alpha F_{j+k_\alpha}^n \tag{4.5a}$$

$$F(U^n; j-1) = \sum_{\alpha=1}^{p-1} B_{\alpha+1} F_{j+k_\alpha}^n \tag{4.5b}$$

the coefficients B_α are defined by

$$\begin{cases} B_p = A_p \\ B_2 = -A_1 \\ B_\alpha - B_{\alpha+1} = A_\alpha \\ (\alpha = 2, 3, \ldots, p-1) \end{cases} \tag{4.6}$$

PROOF

Consider P points schemes such that the p points are arranged as the follows:

$$k_1 < k_2 < k_3 < \ldots < k_p \tag{4.7}$$

Then,

$$\begin{aligned} F(U^n;j) - F(U^n;j-1) &= \sum_{\alpha=2}^{p} B_\alpha F^n_{j+k_\alpha} - \sum_{\alpha=1}^{p-1} B_{\alpha+1} F^n_{j+k_\alpha} \\ &= B_p F^n_{j+k_p} - B_2 F^n_{j+k_1} \\ &\quad + \sum_{\alpha=2}^{p-1} (B_\alpha - B_{\alpha+1}) F^n_{j+k_\alpha} \end{aligned}$$

From 4.4, we get

$$B_p F^n_{j+k_p} - B_2 F^n_{j+k_1} + \sum_{\alpha=2}^{p-1} (B_\alpha - B_{\alpha+1}) F^n_{j+k_\alpha} = \sum_{\alpha=1}^{p} A_\alpha F^n_{j+k_\alpha} \tag{4.8}$$

Comparing the coefficients of both sides of 4.8, we have the following equations:

$$\begin{cases} B_p = A_p \\ B_2 = -A_1 \\ B_\alpha - B_{\alpha+1} = A_\alpha \\ (\alpha = 2, 3, \ldots, p-1) \end{cases}$$

This is 4.6, and the proof is done.

Late on, we will use some examples to demonstrate how to apply Theorem 4 to obtain the numerical flux.

4.3 Consistency of the Conservative Form of UF

The conservative method is consistent with the original PDE, provided the numerical flux function F reduces to the physical flux f in the case of constant flow.

If $u(x,t) = \bar{u}$, we expect

$$F(\bar{u}, \bar{u}, \ldots, \bar{u}) = f(\bar{u}) \tag{4.9}$$

where, $\bar{u}$ is constant.

This is equivalent to

$$\begin{aligned} F(U^n; j) - F(U^n; j-1) &= F(\bar{u}, \bar{u}, \ldots, \bar{u}) - F(\bar{u}, \bar{u}, \ldots, \bar{u}) \\ &= f(\bar{u}) - f(\bar{u}) \\ &= 0 \end{aligned} \tag{4.10}$$

From 4.3, we have

$$F(U^n; j) - F(U^n; j-1) = \frac{1}{c} F_j^n - \sum_{\alpha=1}^{P} \frac{1}{c} B_{k_\alpha} F_{j+k_\alpha}^n \tag{4.11}$$

If we can prove 4.11 equals 0, then, the conservative universal formula is consistent with the real PDE.

Assuming $U_{j+k_\alpha}^n = \bar{u} \;\; \forall\alpha$, so that $F_{j+k_\alpha}^n = f(\bar{u}) \;\; \forall\alpha$, 4.11 becomes

$$F(U^n; j) - F(U^n; j-1) = \frac{1}{c}\left(1 - \sum_{\alpha=1}^{p} B_{k_\alpha}\right) f(\bar{u}) \tag{4.12}$$

Recall, $1 - \sum_{\alpha=1}^{p} B_{k_\alpha} = 0$ therefore, 4.11 is 0. This proves that our conservation form of universal formula is consistent method.

4.4 Conservation Form of Three, Four and Five Point Schemes

In chapter 2 we have derived three point, four point and five point numerical methods. In this section, we will transform them into conservative form according to Theorem 3 and Theorem 4.

4.4.1 Three Points Schemes

1. $U_j^{n+1} = f(U_{j-1}^n, U_j^n, U_{j+1}^n)$ **(L-W method)**

According to 4.3, we can rewrite the L-W scheme for the linear equation as

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[-\frac{1}{2}(1+c)F_{j-1}^n + cF_j^n + \frac{1}{2}(1-c)F_{j+1}^n\right] \tag{4.13}$$

Hence, according to 4.7, $k_1 = -1, \ k_2 = 0, \ k_3 = 1$

and from 4.4,

$$\begin{cases} A_1 = -\frac{1}{2}(1+c) \\ A_2 = c \\ A_3 = \frac{1}{2}(1-c) \end{cases}$$

From 4.6, we get

$$\begin{cases} B_3 = A_3 = \frac{1}{2}(1-c) \\ B_2 = -A_1 = \frac{1}{2}(1+c) \\ B_2 - B_3 = A_2 = c \end{cases}$$

Therefore, the numerical flux is

$$F^{L-W}(U^n;j) = \frac{1}{2}(1+c)F_j^n + \frac{1}{2}(1-c)F_{j+1}^n \tag{4.14}$$

2. $U_j^{n+1} = f(U_{j-2}^n, U_{j-1}^n, U_j^n)$ **(B-W method)**

From 4.3 the B-W scheme can be write as

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[-\frac{1}{2}(c-1)F_{j-2}^n - (2-c)F_{j-1}^n - \frac{1}{2}(c-3)F_j^n\right] \tag{4.15}$$

Here, $k_1 = -2,\ k_2 = -1,\ k_3 = 0$, and

$$\begin{cases} B_3 = A_3 = -\frac{1}{2}(c-3) \\ B_2 = -A_1 = \frac{1}{2}(c-1) \\ B_2 - B_3 = A_2 = c-2 \end{cases}$$

So the numerical flux is

$$F^{B-W}(U^n;j) = \frac{1}{2}(c-1)F_{j-1}^n - \frac{1}{2}(c-3)F_j^n \tag{4.16}$$

3. $U_j^{n+1} = f(U_j^n, U_{j+1}^n, U_{j+2}^n)$**. See 2.21**

As before we get

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[-\frac{1}{2}(c+3)F_j^n + (2+c)F_{j+1}^n - \frac{1}{2}(1+c)F_{j+2}^n\right] \tag{4.17}$$

and, the numerical flux is

$$F^{3-P}(U^n;j) = \frac{1}{2}(3+c)F_{j+1}^n - \frac{1}{2}(1+c)F_{j+2}^n \tag{4.18}$$

4.4.2 Four Points Schemes

1. $U_j^{n+1} = f(U_{j-3}^n, U_{j-2}^n, U_{j-1}^n, U_j^n)$. **See 2.29**

From 4.3 we get

$$\begin{aligned} U_j^{n+1} &= U_j^n - \frac{k}{h}\Big[-(\frac{1}{6}c^2 - \frac{1}{2}c + \frac{1}{3})F_{j-3}^n + (\frac{1}{2}c^2 - 2c + \frac{3}{2})F_{j-2}^n \\ &\quad -(\frac{1}{2}c^2 - \frac{5}{2}c + 3)F_{j-1}^n + (\frac{1}{6}c^2 - c + \frac{11}{6})F_j^n\Big] \end{aligned} \tag{4.19}$$

here, $k_1 = -3,\ k_2 = -2,\ k_3 = -1,\ k_4 = 0$

and,

$$\begin{cases} A_1 = -(\frac{1}{6}c^2 - \frac{1}{2}c + \frac{1}{3}) \\ A_2 = \frac{1}{2}c^2 - 2c + \frac{3}{2} \\ A_3 = -(\frac{1}{2}c^2 - \frac{5}{2}c + 3) \\ A_4 = \frac{1}{6}c^2 - c + \frac{11}{6} \end{cases}$$

According to 4.6

$$\begin{cases} B_4 = A_4 = \frac{1}{6}c^2 - c + \frac{11}{6} \\ B_2 = -A_1 = \frac{1}{6}c^2 - \frac{1}{2}c + \frac{1}{3} \\ B_2 - B_3 = A_2 = \frac{1}{2}c^2 - 2c + \frac{3}{2} \\ B_3 - B_4 = A_3 = \frac{5}{2}c - \frac{1}{2}c^2 - 3 \end{cases}$$

Hence,

$$B_3 = \frac{3}{2}c - \frac{1}{3}c^2 - \frac{7}{6}$$

and, the numerical flux is

$$F^{4-P}(U^n;j) = (\frac{1}{6}c^2 - \frac{1}{2}c + \frac{1}{3})F^n_{j-2} + (\frac{3}{2}c - \frac{1}{3}c^2 - \frac{7}{6})F^n_{j-1} + (\frac{1}{6}c^2 - c + \frac{11}{6})F^n_j \tag{4.20}$$

2. $U^{n+1}_j = f(U^n_{j-2}, U^n_{j-1}, U^n_j, U^n_{j+1})$**. See 2.28.**

Doing the same routine we get

$$U^{n+1}_j = U^n_j - \frac{k}{h}\left[-(\frac{1}{6}c^2 - \frac{1}{6})F^n_{j-2} - (1 + \frac{1}{2}c - \frac{1}{2}c^2)F^n_{j-1} - (\frac{1}{2}c^2 - c - \frac{1}{2})F^n_j + (\frac{1}{6}c^2 - \frac{1}{2}c + \frac{1}{3})F^n_{j+1}\right] \tag{4.21}$$

and, the numerical flux is

$$F^{4-P}(U^n;j) = (\frac{1}{6}c^2 - \frac{1}{6})F^n_{j-1} + (\frac{1}{2}c - \frac{1}{3}c^2 + \frac{5}{6})F^n_j + (\frac{1}{6}c^2 - \frac{1}{2}c + \frac{1}{3})F^n_{j+1} \tag{4.22}$$

3. $U^{n+1}_j = f(U^n_{j-1}, U^n_j, U^n_{j+1}, U^n_{j+2})$**. See 2.26.**

The conservation form of this method is

$$U^{n+1}_j = U^n_j - \frac{k}{h}\left[(\frac{1}{2}c^2 + c - \frac{1}{2})F^n_j - (\frac{1}{6}c^2 + \frac{1}{2}c + \frac{1}{3})F^n_{j-1} - (\frac{1}{2}c^2 + \frac{1}{2}c - 1)F^n_{j+1} - \frac{1}{6}(1 - c^2)F^n_{j+2}\right] \tag{4.23}$$

The numerical flux is

$$F^{4-P}(U^n;j) = (\frac{1}{6}c^2+\frac{1}{2}c+\frac{1}{3})F_j^n+(\frac{5}{6}-\frac{1}{3}c^2-\frac{1}{2}c)F_{j+1}^n + \frac{1}{6}(c^2-1)F_{j+2}^n \quad (4.24)$$

4. $U_j^{n+1} = f(U_j^n, U_{j+1}^n, U_{j+2}^n, U_{j+3}^n)$. **See 2.27.**

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[-(\frac{11}{6}+c+\frac{1}{6}c^2)F_j^n+(\frac{1}{2}c^2+\frac{5}{2}c+3)F_{j+1}^n -(\frac{1}{2}c^2+2c+\frac{3}{2})F_{j+2}^n+(\frac{1}{6}c^2+\frac{1}{2}c+\frac{1}{3})F_{j+3}^n\right] \quad (4.25)$$

The numerical flux is

$$F^{4-P}(U^n;j) = (\frac{11}{6}+c+\frac{1}{6}c^2)F_{j+1}^n-(\frac{1}{3}c^2+\frac{3}{2}c+\frac{1}{6})F_{j+2}^n +(\frac{1}{6}c^2+\frac{1}{2}c+\frac{1}{3})F_{j+3}^n \quad (4.26)$$

4.4.3 Five Points schemes

1. $U_j^{n+1} = f(U_{j-2}^n, U_{j-1}^n, U_j^n, U_{j+1}^n, U_{j+2}^n)$. **See 2.30.**

From 4.3

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[(\frac{5}{4}c-\frac{1}{4}c^3)F_j^n-(\frac{1}{24}c^3+\frac{1}{12}c^2-\frac{1}{24}c-\frac{1}{12})F_{j-2}^n -(\frac{2}{3}+\frac{2}{3}c-\frac{1}{6}c^2-\frac{1}{6}c^3)F_{j-1}^n-(\frac{1}{6}c^2+\frac{2}{3}c-\frac{1}{6}c^3-\frac{2}{3})F_{j+1}^n -(\frac{1}{12}-\frac{1}{24}c-\frac{1}{12}c^2+\frac{1}{24}c^3)F_{j+2}^n\right] \quad (4.27)$$

here, $k_1 = -2,\ k_2 = -1,\ k_3 = 0,\ k_4 = 1,\ k_5 = 2$, and,

$$\begin{cases} A_1 = -(\frac{1}{24}c^3 + \frac{1}{12}c^2 - \frac{1}{24}c - \frac{1}{12}) \\ A_2 = -(\frac{2}{3} + \frac{2}{3}c - \frac{1}{6}c^2 - \frac{1}{6}c^3) \\ A_3 = \frac{5}{4}c - \frac{1}{4}c^3 \\ A_4 = -(\frac{1}{6}c^2 + \frac{2}{3}c - \frac{1}{6}c^3 - \frac{2}{3} \\ A_5 = -(\frac{1}{12} - \frac{1}{24}c - \frac{1}{12}c^2 + \frac{1}{24}c^3) \end{cases}$$

From 4.6, we have

$$\begin{cases} B_5 = A_5 = -(\frac{1}{12} - \frac{1}{24}c - \frac{1}{12}c^2 + \frac{1}{24}c^3) \\ B_2 = -A_1 = \frac{1}{24}c^3 + \frac{1}{12}c^2 - \frac{1}{24}c - \frac{1}{12} \\ B_2 - B_3 = A_2 = -(\frac{2}{3} + \frac{2}{3}c - \frac{1}{6}c^2 - \frac{1}{6}c^3) \\ B_3 - B_4 = A_3 = \frac{5}{4}c - \frac{1}{4}c^3 \\ B_4 - B_5 = A_4 = -(\frac{1}{6}c^2 + \frac{2}{3}c - \frac{1}{6}c^3 - \frac{2}{3}) \end{cases}$$

Work out B_3 and B_4

$$\begin{cases} B_3 = \frac{7}{12} + \frac{5}{8}c - \frac{1}{12}c^2 - \frac{1}{8}c^3 \\ B_4 = \frac{1}{8}c^3 - \frac{1}{12}c^2 - \frac{5}{8}c + \frac{7}{12} \end{cases}$$

we get the numerical flux of this method:

$$\begin{aligned} F^{5-P}(U^n;j) &= (\frac{1}{24}c^3 + \frac{1}{12}c^2 - \frac{1}{24}c - \frac{1}{12})F^n_{j-1} + (\frac{7}{12} + \frac{5}{8}c - \frac{1}{12}c^2 - \frac{1}{8}c^3)F^n_j \\ &\quad +(\frac{1}{8}c^3 - \frac{1}{12}c^2 - \frac{5}{8}c + \frac{7}{12})F^n_{j+1} - (\frac{1}{24}c^3 - \frac{1}{12}c^2 \\ &\quad -\frac{1}{24}c + \frac{1}{12})F^n_{j+2} \end{aligned} \tag{4.28}$$

2. $U^{n+1}_j = f(U^n_{j-4}, U^n_{j-3}, U^n_{j-2}, U^n_{j-1}, U^n_j)$. **See 2.31.**

Going the same routine as above we get

$$\begin{aligned} U_j^{n+1} &= U_j^n - \frac{k}{h}\Big[(\frac{25}{12} - \frac{35}{24}c + \frac{5}{12}c^2 - \frac{1}{24}c^3)F_j^n - (\frac{1}{24}c^3 - \frac{1}{4}c^2 + \frac{11}{24}c - \frac{1}{4})F_{j-4}^n \\ & -(\frac{4}{3} - \frac{7}{3}c + \frac{7}{6}c^2 - \frac{1}{6}c^3)F_{j-3}^n - (\frac{1}{4}c^3 - 2c^2 + \frac{19}{4}c - 3)F_{j-2}^n \\ & -(4 - \frac{13}{3}c + \frac{3}{2}c^2 - \frac{1}{6}c^3)F_{j-1}^n\Big] \end{aligned} \tag{4.29}$$

and the numerical flux of this method is

$$\begin{aligned} F^{5-P}(U^n;j) &= (\frac{1}{24}c^3 - \frac{1}{4}c^2 + \frac{11}{24}c - \frac{1}{4})F_{j-3}^n + (\frac{13}{12} - \frac{15}{8}c + \frac{11}{12}c^2 - \frac{1}{8}c^3)F_{j-2}^n \\ & +(\frac{1}{8}c^3 - \frac{13}{12}c^2 + \frac{23}{8}c - \frac{23}{12})F_{j-1}^n \\ & +(\frac{25}{12} - \frac{35}{24}c + \frac{5}{12}c^2 - \frac{1}{24}c^3)F_j^n \end{aligned} \tag{4.30}$$

3. $U_j^{n+1} = f(U_{j-3}^n, U_{j-2}^n, U_{j-1}^n, U_j^n, U_{j+1}^n)$. **See 2.32.**

$$\begin{aligned} U_j^{n+1} &= U_j^n - \frac{k}{h}\Big[(\frac{1}{6}c^3 - \frac{5}{6}c^2 + \frac{5}{6}c + \frac{5}{6})F_j^n - (\frac{1}{24}c^3 - \frac{1}{12}c^2 - \frac{1}{24}c + \frac{1}{12})F_{j-3}^n \\ & +(\frac{1}{6}c^3 - \frac{1}{2}c^2 - \frac{1}{6}c + \frac{1}{2})F_{j-2}^n - (\frac{1}{4}c^3 - c^2 + \frac{1}{4}c + \frac{3}{2})F_{j-1}^n \\ & -(\frac{1}{24}c^3 - \frac{1}{4}c^2 + \frac{11}{24}c - \frac{1}{4})F_{j+1}^n\Big] \end{aligned} \tag{4.31}$$

The numerical flux is defined as

$$\begin{aligned} F^{5-P}(U^n;j) &= (\frac{1}{24}c^3 - \frac{1}{12}c^2 - \frac{1}{24}c + \frac{1}{12})F_{j-2}^n - (\frac{1}{8}c^3 - \frac{5}{12}c^2 - \frac{1}{8}c + \frac{5}{12})F_{j-1}^n \\ & +(\frac{1}{8}c^3 - \frac{7}{12}c^2 + \frac{3}{8}c + \frac{13}{12})F_j^n \\ & -(\frac{1}{24}c^3 - \frac{1}{4}c^2 + \frac{11}{24}c - \frac{1}{4})F_{j+1}^n \end{aligned} \tag{4.32}$$

4. $U_j^{n+1} = f(U_{j-1}^n, U_j^n, U_{j+1}^n, U_{j+2}^n, U_{j+3}^n)$. **See 2.33.**

$$\begin{aligned} U_j^{n+1} &= U_j^n - \frac{k}{h}\Big[(\frac{1}{6}c^3 + \frac{5}{6}c^2 + \frac{5}{6}c - \frac{5}{6})F_j^n - (\frac{1}{24}c^3 + \frac{1}{4}c^2 + \frac{11}{24}c + \frac{1}{4})F_{j-1}^n \\ &\quad -(\frac{1}{4}c^3 + c^2 + \frac{1}{4}c - \frac{3}{2})F_{j+1}^n + \frac{1}{6}(c^3 + 3c^2 - c - 3)F_{j+2}^n \\ &\quad -\frac{1}{24}(c^3 + 2c^2 - c - 2)F_{j+3}^n\Big] \end{aligned} \tag{4.33}$$

The numerical flux is

$$\begin{aligned} F^{5-P}(U^n;j) &= (\frac{1}{24}c^3 + \frac{1}{4}c^2 + \frac{11}{24}c + \frac{1}{4})F_j^n - (\frac{1}{8}c^3 + \frac{7}{12}c^2 + \frac{3}{8}c - \frac{13}{12})F_{j+1}^n \\ &\quad +(\frac{1}{8}c^3 + \frac{5}{12}c^2 - \frac{1}{8}c - \frac{5}{12})F_{j+2}^n \\ &\quad -\frac{1}{24}(c^3 + 2c^2 - c - 2)F_{j+3}^n \end{aligned} \tag{4.34}$$

5. $U_j^{n+1} = f(U_j^n, U_{j+1}^n, U_{j+2}^n, U_{j+3}^n, U_{j+4}^n)$. **See 2.34.**

$$\begin{aligned} U_j^{n+1} &= U_j^n - \frac{k}{h}\Big[-(\frac{1}{24}c^3 + \frac{5}{12}c^2 + \frac{35}{24}c + \frac{25}{12})F_j^n + (\frac{1}{6}c^3 + \frac{3}{2}c^2 + \frac{13}{3}c + 4)F_{j+1}^n \\ &\quad -(\frac{1}{4}c^3 + 2c^2 + \frac{19}{4}c + 3)F_{j+2}^n + \frac{1}{6}(c^3 + 7c^2 + 14c + 8)F_{j+3}^n \\ &\quad -\frac{1}{24}(c^3 + 6c^2 + 11c + 6)F_{j+4}^n\Big] \end{aligned} \tag{4.35}$$

The numerical flux is

$$\begin{aligned} F^{5-P}(U^n;j) &= (\frac{1}{24}c^3 + \frac{5}{12}c^2 + \frac{35}{24}c + \frac{25}{12})F_{j+1}^n - (\frac{1}{8}c^3 + \frac{13}{12}c^2 + \frac{23}{8}c + \frac{23}{12})F_{j+2}^n \\ &\quad +(\frac{1}{8}c^3 + \frac{11}{12}c^2 + \frac{15}{8}c + \frac{13}{12})F_{j+3}^n \\ &\quad -\frac{1}{24}(c^3 + 6c^2 + 11c + 6)F_{j+4}^n \end{aligned} \tag{4.36}$$

Chapter 5

GENERALIZED FORMULATION FOR CENTRAL HYPERBOLIC TVD SCHEMES

5.1 Introduction

When we apply high order conservative methods to deal with nonliear hyperbolic problems, we run into a dilemma. Accompanying these high order methods are spurious oscillations invoked by numerical dispersion. In most cases, the oscillations are trigered by discontinuities. As is well known, nonlinear hyperbolic systems may produce discontinuities (shocks, contact discontinuities) even if the initial condition is a smooth function. This phenomena is characteristic of nonlinear hyperbolic convervation laws.

Recently, there have been intense efforts made towards developing a theory and method to resolve this problem. TVD theory and method has been proved very useful in fighting spurious oscillations, although at discontinuities schemes have to reduce to first order accuracy.

In this chapter, we will further investigate the TVD methods and define TVD regions for some high order numerical methods.

5.2 Oscillation Free Criteria

There is a local maximum criteria, see[15], which states that if

$$\begin{aligned} 0 &\leq \frac{U_j^{n+1} - U_j^n}{U_{j-1}^n - U_j^n} \leq 1, \quad for \;\; a > 0 \\ 0 &\leq \frac{U_j^{n+1} - U_j^n}{U_j^n - U_{j+1}^n} \leq 1, \quad for \;\; a < 0 \end{aligned} \tag{5.1}$$

then, U_j^{n+1} at the new time level is bounded by the data. However, 5.1 only valid for a CFL number in the region $-1 \leq c \leq 1$.

In linear cases, the value of U_j^{n+1} is determined by the CFL number. For example, if the stable region of a numerical method is $1 \leq c \leq 2$, then the value of U_j^{n+1} at the next time level will lie between U_{j-1}^n and U_{j-2}^n, depending on the CFL number c.

Therefore, the general form for local maximum criteria should have the following form:

LOCAL MAXIMUM CRITERIA

$$\begin{aligned} 0 &\leq \frac{U_j^{n+1} - U_{j-k}^n}{U_{j-(k+1)}^n - U_{j-k}^n} \leq 1 \quad k \leq c \leq k+1 \\ 0 &\leq \frac{U_j^{n+1} - U_{j-k}^n}{U_{j-k}^n - U_{j+(k+1)}^n} \leq 1 \quad -(k+1) \leq c \leq -k \end{aligned} \tag{5.2}$$

here, k is a positive integer, i.e. $k = 0, 1, 2, \ldots, \infty$.

When developing a TVD numerical method, the linear stable region of this method must be taken into account as we will see in the next section.

5.3 TVD Region for Arbitrary-Order Numerical Methods

There are several techniques in use to curb the oscillations. We will follow the flux-limiter method here.

5.3.1 Generalized Formulation Defining TVD Region for Methods with CFL Number: $0 \leq c \leq 1$

THEOREM 5

Any numerical flux of hyperbolic numerical methods can be written as the following form:

$$\begin{aligned} F(U^n;j) &= a\Big[U_j^n + \left(D_0 \triangle U_{j+\frac{1}{2}} + D_{-1} \triangle U_{j-\frac{1}{2}}\right)\phi_j \\ &+ \sum_{k=-\infty, k\neq -1,0}^{\infty} D_k \triangle U_{j+k+\frac{1}{2}}\ \phi_{j+k}\Big] \end{aligned} \tag{5.3}$$

where ϕ_j are flux limiters, D_k are coefficients, and

$$\begin{aligned} \Delta U_{j-\frac{1}{2}} &= U_j^n - U_{j-1}^n \\ \Delta U_{j+\frac{1}{2}} &= U_{j+1}^n - U_j^n \\ &\quad etc. \end{aligned}$$

For a method with CFL number: $0 \le c \le 1$, the flux limiters are determined by

$$
\begin{aligned}
\phi_j &\le \frac{(1-c)\theta_j}{c(D_{-1}\theta_j + D_0 - D_1)} \\
\phi_{j-1} &\le \frac{1-c}{c(D_0 + D_{-1}\theta_{j-1})} \\
\phi_{j-2} &\le \frac{-(1-c)}{c\theta_{j-1}D_{-2}} \\
\phi_{j+k} &\le \frac{-(1-c)\theta_j\theta_{j+1}\ldots\theta_{j+k}}{c(D_k - D_{k+1})} \qquad (k = 1, 2, \ldots, \infty) \\
\phi_{j+k} &\le \frac{-(1-c)}{c(D_k - D_{k+1})\theta_{j-1}\theta_{j-2}\ldots\theta_{j+k+1}} \quad (k = -3, -4, \ldots, -\infty)
\end{aligned}
\tag{5.4}
$$

where $\theta_j = \frac{\Delta U_{j-\frac{1}{2}}}{\Delta U_{j+\frac{1}{2}}}$.

PROOF

From 5.3, the numerical method is

$$
\begin{aligned}
U_j^{n+1} &= U_j^n - c\Big[U_j^n + \Big(D_0 \triangle U_{j+\frac{1}{2}} + D_{-1} \triangle U_{j-\frac{1}{2}}\Big)\phi_j \\
&\quad + \sum_{k=-\infty, k\neq -1,0}^{\infty} D_k \triangle U_{j+k+\frac{1}{2}}\ \phi_{j+k} \\
&\quad - U_{j-1}^n - \Big(D_0 \triangle U_{j-\frac{1}{2}} + D_{-1} \triangle U_{j-\frac{3}{2}}\Big)\phi_{j-1} \\
&\quad - \sum_{k=-\infty, k\neq -1,0}^{\infty} D_k \triangle U_{j+k-\frac{1}{2}}\ \phi_{j+k-1}\Big] \\
&= U_j^n - c\Big[\Delta U_{j-\frac{1}{2}} + D_0 \triangle U_{j+\frac{1}{2}}\ \phi_j + D_{-1} \triangle U_{j-\frac{1}{2}}\ \phi_j \\
&\quad - D_0 \triangle U_{j-\frac{1}{2}}\ \phi_{j-1} \\
&\quad - D_{-1} \triangle U_{j-\frac{3}{2}}\ \phi_{j-1} - D_1 \triangle U_{j+\frac{1}{2}}\ \phi_j + D_{-2} \triangle U_{j-\frac{3}{2}}\ \phi_{j-2} \\
&\quad + \sum_{k=-\infty, k\neq -2,-1,0}^{\infty} (D_k - D_{k+1}) \triangle U_{j+k+\frac{1}{2}}\ \phi_{j+k}\Big]
\end{aligned}
\tag{5.5}
$$

Modifying 5.5, we get

$$
\begin{aligned}
\frac{U_j^{n+1} - U_j^n}{-\triangle U_{j-\frac{1}{2}}} &= c\Bigg[(1 + D_{-1}\phi_j - D_0\phi_{j-1}) + (D_0 - D_1)\frac{\phi_j}{\theta_j} - D_{-1}\phi_{j-1}\theta_{j-1} \\
&\quad + D_{-2}\phi_{j-2}\theta_{j-1} + \sum_{k=-\infty, k\neq -2,-1,0}^{\infty} (D_k - D_{k+1})\frac{\Delta U_{j+k+\frac{1}{2}}}{\Delta U_{j-\frac{1}{2}}}\phi_{j+k}\Bigg] \\
&= c\Bigg[1 + \left(D_{-1} + (D_0 - D_1)\frac{1}{\theta_j}\right)\phi_j - (D_0 + D_{-1}\theta_{j-1})\phi_{j-1} \\
&\quad + D_{-2}\theta_{j-1}\phi_{j-2} + \sum_{k=1}^{\infty}(D_k - D_{k+1})\frac{\phi_{j+k}}{\theta_j\theta_{j+1}\dots\theta_{j+k}} \\
&\quad + \sum_{k=-\infty}^{-3}(D_k - D_{k+1})\theta_{j-1}\theta_{j-2}\dots\theta_{j+k+1}\phi_{j+k}\Bigg]
\end{aligned}
\tag{5.6}
$$

where:

$$
\begin{aligned}
\phi_j &= \phi(\theta_j) \\
\theta_j &= \frac{\Delta U_{j-\frac{1}{2}}}{\Delta U_{j+\frac{1}{2}}}
\end{aligned}
\tag{5.7}
$$

For oscillation free solutions we require:

$$0 \leq \frac{U_j^{n+1} - U_j^n}{-\triangle U_{j-\frac{1}{2}}} \leq 1$$

and assuming:

$$\phi(\theta) = 0 \quad for \;\; \theta \leq 0 \tag{5.8}$$

we get

$$\left(D_{-1} + (D_0 - D_1)\frac{1}{\theta_j}\right)\phi_j \leq \frac{1-c}{c}$$
$$(D_0 + D_{-1}\theta_{j-1})\phi_{j-1} \leq \frac{1-c}{c}$$
$$-D_{-2}\theta_{j-1}\phi_{j-2} \leq \frac{1-c}{c}$$
$$-(D_k - D_{k+1})\frac{\phi_{j+k}}{\theta_j\theta_{j+1}\ldots\theta_{j+k}} \leq \frac{1-c}{c} \qquad (k = 1, 2, \ldots, \infty)$$
$$-(D_k - D_{k+1})\theta_{j-1}\theta_{j-2}\ldots\theta_{j+k+1}\phi_{j+k} \leq \frac{1-c}{c} \qquad (k = -3, -4, \ldots, -\infty)$$

Therefore,

$$\phi_j \leq \frac{(1-c)\theta_j}{c(D_{-1}\theta_j + D_0 - D_1)}$$
$$\phi_{j-1} \leq \frac{1-c}{c(D_0 + D_{-1}\theta_{j-1})}$$
$$\phi_{j-2} \leq \frac{-(1-c)}{c\theta_{j-1}D_{-2}}$$
$$\phi_{j+k} \leq \frac{-(1-c)\theta_j\theta_{j+1}\ldots\theta_{j+k}}{c(D_k - D_{k+1})} \qquad (k = 1, 2, \ldots, \infty)$$
$$\phi_{j+k} \leq \frac{-(1-c)}{c(D_k - D_{k+1})\theta_{j-1}\theta_{j-2}\ldots\theta_{j+k+1}} \qquad (k = -3, -4, \ldots, -\infty)$$

This is 5.4, and the proof is completed.

5.3.2 Generalized Formulation Defining TVD Region for Method with CFL Number $-1 \leq c \leq 0$

THEOREM 6

The numerical flux of any hyperbolic numerical method can be written as follows

$$\begin{aligned} F(U^n;j) &= a\Big[U^n_{j+1} + \Big(D_0 \triangle U_{j+\frac{1}{2}} + D_1 \triangle U_{j+\frac{3}{2}}\Big)\,\phi_j \\ &\quad + \sum_{k=-\infty,k\neq 1,0}^{\infty} D_k \triangle U_{j+k+\frac{1}{2}}\ \phi_{j+k}\Big] \end{aligned} \tag{5.9}$$

where, ϕ_j are flux limiters. D_k are coefficients.

$$\begin{aligned} &\Delta U_{j+\frac{3}{2}} = U^n_{j+2} - U^n_{j+1} \\ &\Delta U_{j+\frac{1}{2}} = U^n_{j+1} - U^n_j \\ &etc. \end{aligned}$$

For a method with CFL number: $-1 \le c \le 0$, the flux limiters are determined by

$$\begin{aligned} &\phi_j \le \frac{1+c}{c(D_0 + D_1\theta_j)} \\ &\phi_{j-1} \le \frac{(1+c)\theta_{j-1}}{c(D_1\theta_{j-1} + D_0 - D_{-1})} \\ &\phi_{j+1} \le -\frac{1+c}{cD_2\theta_j} \\ &\phi_{j+k} \le \frac{1+c}{c(D_k - D_{k+1})\theta_j\theta_{j+1}\dots\theta_{j+k-1}} \quad (k = 2,3,\dots,\infty) \\ &\phi_{j+k} \le \frac{(1+c)\theta_{j-1}\theta_{j-2}\dots\theta_{j+k}}{c(D_k - D_{k+1})} \quad (k = -2,-3,\dots,-\infty) \end{aligned} \tag{5.10}$$

PROOF

From 5.9, the numerical method is

$$U^{n+1}_j = U^n_j + c\Big[U^n_{j+1} + \Big(D_0 \triangle U_{j+\frac{1}{2}} + D_1 \triangle U_{j+\frac{3}{2}}\Big)\,\phi_j$$

$$
\begin{aligned}
& + \sum_{k=-\infty, k\neq 1,0}^{\infty} D_k \triangle U_{j+k+\frac{1}{2}} \phi_{j+k} \\
& -U_j^n - \left(D_0 \triangle U_{j-\frac{1}{2}} + D_1 \triangle U_{j+\frac{1}{2}}\right) \phi_{j-1} \\
& - \left.\sum_{k=-\infty, k\neq 1,0}^{\infty} D_k \triangle U_{j+k-\frac{1}{2}} \phi_{j+k-1}\right] \\
= & \; U_j^n + c\left[(1 + D_0\phi_j - D_1\phi_{j-1}) \triangle U_{j+\frac{1}{2}} + (D_{-1}\phi_{j-1} - D_0\phi_{j-1}) \triangle U_{j-\frac{1}{2}}\right. \\
& + D_1\phi_j \triangle U_{j+\frac{3}{2}} - D_2\phi_{j+1} \triangle U_{j+\frac{3}{2}} \\
& + \left.\sum_{k=-\infty, k\neq -1,0,1}^{\infty} (D_k - D_{k+1}) \triangle U_{j+k+\frac{1}{2}} \phi_{j+k}\right]
\end{aligned} \tag{5.11}
$$

Modifying 5.11 we get

$$
\begin{aligned}
\frac{U_j^{n+1} - U_j^n}{-\triangle U_{j+\frac{1}{2}}} = & -c\left[(1 + D_0\phi_j - D_1\phi_{j-1}) + (D_{-1} - D_0)\frac{\phi_{j-1}}{\theta_{j-1}} + D_1\phi_j\theta_j\right. \\
& \left. -D_2\phi_{j+1}\theta_j + \sum_{k=-\infty, k\neq -1,0,1}^{\infty} (D_k - D_{k+1})\frac{\Delta U_{j+k+\frac{1}{2}}}{\Delta U_{j+\frac{1}{2}}}\phi_{j+k}\right] \\
= & \; c\left[-1 - (D_0 + D_1\theta_j)\phi_j - \left((D_{-1} - D_0)\frac{1}{\theta_{j-1}} - D_1\right)\phi_{j-1}\right. \\
& + D_2\theta_{j+1}\phi_{j+1} - \sum_{k=2}^{\infty}(D_k - D_{k+1})\theta_j\theta_{j+1}\ldots\theta_{j+k-1}\phi_{j+k} \\
& \left. - \sum_{k=-\infty}^{-2}(D_k - D_{k+1})\frac{\phi_{j+k}}{\theta_{j-1}\theta_{j-2}\ldots\theta_{j+k}}\right]
\end{aligned} \tag{5.12}
$$

where:

$$
\begin{aligned}
\phi_j &= \phi(\theta_j) \\
\theta_j &= \frac{\Delta U_{j+\frac{3}{2}}}{\Delta U_{j+\frac{1}{2}}}
\end{aligned} \tag{5.13}
$$

For oscillation free solutions we require:

$$0 \le \frac{U_j^{n+1} - U_j^n}{- \triangle U_{j+\frac{1}{2}}} \le 1$$

and assuming:

$$\phi(\theta) = 0 \quad for \quad \theta \le 0 \tag{5.14}$$

we get

$$\begin{aligned}
&\left(D_1 - (D_{-1} - D_0)\frac{1}{\theta_{j-1}}\right)\phi_{j-1} \le \frac{1+c}{c} \\
&(D_0 + D_1\theta_j)\phi_j \le \frac{1+c}{c} \\
&-D_2\theta_j\phi_{j+1} \le \frac{1+c}{c} \\
&(D_k - D_{k+1})\theta_j\theta_{j+1}\dots\theta_{j+k-1}\phi_{j+k} \le \frac{1+c}{c} \quad (k = 2, 3, \dots, \infty) \\
&(D_k - D_{k+1})\frac{\phi_{j+k}}{\theta_{j-1}\theta_{j-2}\dots\theta_{j+k}} \le \frac{1+c}{c} \qquad (k = -2, -3, \dots, -\infty)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\phi_j \le \frac{1+c}{c(D_0 + D_1\theta_j)} \\
&\phi_{j-1} \le \frac{(1+c)\theta_{j-1}}{c(D_1\theta_{j-1} + D_0 - D_{-1})} \\
&\phi_{j+1} \le -\frac{1+c}{cD_2\theta_j} \\
&\phi_{j+k} \le \frac{1+c}{c(D_k - D_{k+1})\theta_j\theta_{j+1}\dots\theta_{j+k-1}} \quad (k = 2, 3, \dots, \infty) \\
&\phi_{j+k} \le \frac{(1+c)\theta_{j-1}\theta_{j-2}\dots\theta_{j+k}}{c(D_k - D_{k+1})} \qquad (k = -2, -3, \dots, -\infty)
\end{aligned}$$

This implies 5.10, and proves Theorem 6.

5.3.3 TVD Region for Third Order Methods

As applications of Theorem 5 and Theorem 6, in this section we are going to find the flux limiters for equations 4.22 and 4.24, and then analysing TVD regions for these methods.

1. TVD method for 4.21

4.21 has a stable region of $0 \leq c \leq 1$. The numerical flux of this method can be written as follows for linear advection case: See 4.22.

$$F^{4-P}(U^n;j) = a\left[U_j^n + (\frac{1}{3} - \frac{c}{2} + \frac{c^2}{6}) \triangle U_{j+\frac{1}{2}} + \frac{1}{6}(1-c^2) \triangle U_{j-\frac{1}{2}}\right] \tag{5.15}$$

here,

$$\begin{aligned} \triangle U_{j-\frac{1}{2}} &= U_j^n - U_{j-1}^n \\ \triangle U_{j+\frac{1}{2}} &= U_{j+1}^n - U_j^n \end{aligned}$$

The first term on the right side of the equation is the first order upwind method, the rest terms are the high order corrections which increase the accuracy but also give rise to the spurious oscillations.

According to 5.3, the limiter form of 5.15 is

$$F^{4-P}(U^n;j) = a\left[U_j^n + \left((\frac{1}{3} - \frac{c}{2} + \frac{c^2}{6}) \triangle U_{j+\frac{1}{2}} + \frac{1}{6}(1-c^2) \triangle U_{j-\frac{1}{2}}\right) \phi_j\right] \tag{5.16}$$

here:

$$
\begin{aligned}
D_0 &= \frac{1}{3} - \frac{c}{2} + \frac{c^2}{6} \\
D_{-1} &= \frac{1}{6}(1 - c^2)
\end{aligned}
$$

From 5.4 we have

$$
\phi_j \leq \frac{6\theta_j}{c[\theta_j(1+c) + 2 - c]} \tag{5.17}
$$

$$
\phi_{j-1} \leq \frac{6}{c[2 - c + (1+c)\theta_{j-1}]} \tag{5.18}
$$

here,

$$
\theta_j = \frac{\Delta U_{j-\frac{1}{2}}}{\Delta U_{j+\frac{1}{2}}} \tag{5.19}
$$

As you can see, the flux limiters are functions of θ and Courant number c, this means that different c has different TVD region. For $c = 1$, 5.17 and 5.18 become

$$
\phi_j \leq \frac{6\theta_j}{1 + 2\theta j} \tag{5.20}
$$

$$
\phi_j \leq \frac{6}{1 + 2\theta_j} \tag{5.21}
$$

Figure 5.1 shows the TVD regions for $c = 0.7$ and $c = 1$. As you can see, the TVD region will enlarge with the value of c decreasing.

Figure 5.2 shows the computational result of this method with time step of 20000.

The CFL number used is 0.7, and the TVD limiters adopted in the computation are as follows:

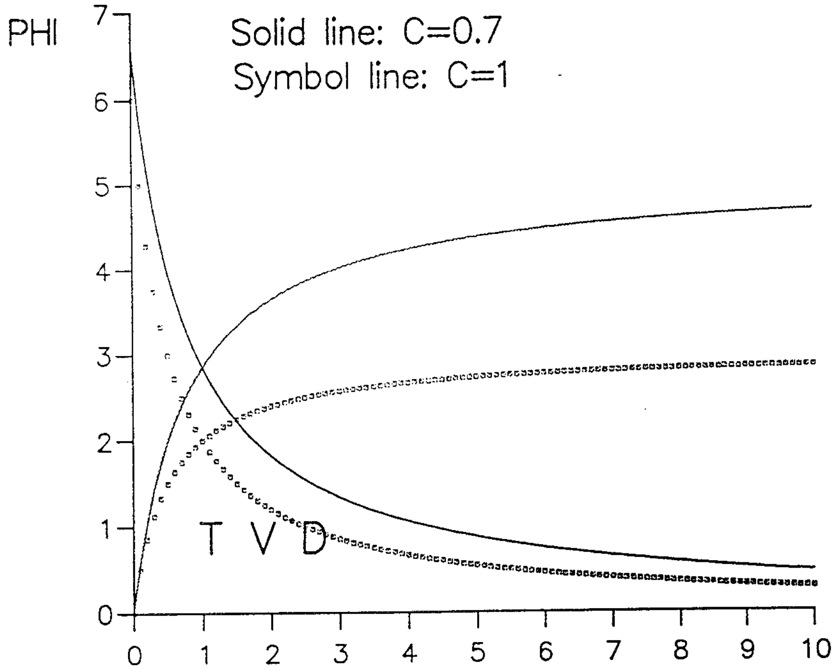

Figure 5.1: TVD Region for Third Order Method

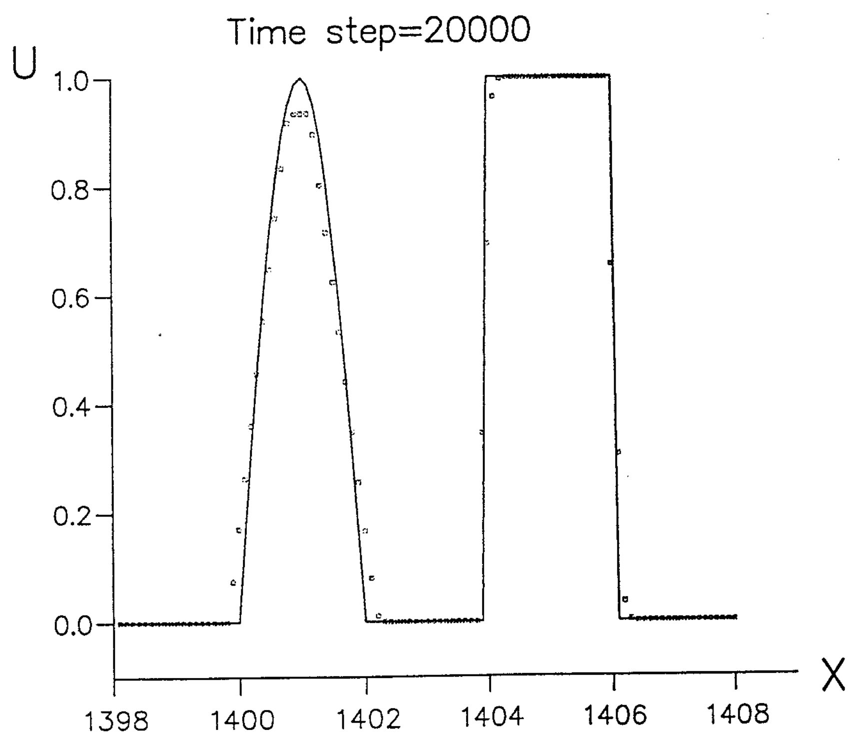

Figure 5.2: Computational Result of Third Order TVD Method

$$\begin{aligned} \phi_j &= \frac{6\theta_j}{0.7(1.7\theta_j + 1.3)} & 0 \le \theta_j < 0.6 \\ \phi_j &= 1 & 0.6 \le \theta_j < 1.8 \\ \phi_j &= \frac{6}{0.7(1.3 + 1.7\theta_j)} & \theta_j \ge 1.8 \end{aligned} \tag{5.22}$$

2. TVD method for 4.23

This method hase stable region of $-1 \le c \le 0$.

The numerical flux of this method takes the following form in the linear case. See 4.24

$$F^{4-P}(U^n; j) = a\left[U^n_{j+1} - (\frac{1}{3} + \frac{c}{2} + \frac{c^2}{6}) \bigtriangleup U_{j+\frac{1}{2}} + \frac{1}{6}(1-c^2) \bigtriangleup U_{j+\frac{3}{2}}\right] \tag{5.23}$$

Adding flux limiters, it becomes

$$F^{4-P}(U^n; j) = a\left[U^n_{j+1} - \left((\frac{1}{3} + \frac{c}{2} + \frac{c^2}{6}) \bigtriangleup U_{j+\frac{1}{2}} + \frac{1}{6}(1-c^2) \bigtriangleup U_{j+\frac{3}{2}}\right)\phi_j\right] \tag{5.24}$$

here,

$$\begin{aligned} D_0 &= -(\frac{1}{3} + \frac{c}{2} + \frac{c^2}{6}) \\ D_1 &= -\frac{1}{6}(1-c^2) \end{aligned}$$

From 5.10, the flux limiters for this method are

$$\phi_{j-1} \le \frac{-6\theta_{j-1}}{c[\theta_{j-1}(1-c) + 2 + c]} \tag{5.25}$$

$$\phi_j \le \frac{-6}{c[2 + c + (1-c)\theta_j]} \tag{5.26}$$

here,

$$\theta_j = \frac{\Delta U_{j+\frac{3}{2}}}{\Delta U_{j+\frac{1}{2}}} \tag{5.27}$$

5.3.4 TVD Region for Fourth Order Method

Consider the fourth order method 4.27 which is stable in the region of $-1 \leq c \leq 1$. The numerical flux of this method is, (see 4.28)

$$F^{5-P}(U^n;j) = a\left[U_j^n + D_{-1} \triangle U_{j-\frac{1}{2}} + D_0 \triangle U_{j+\frac{1}{2}} + D_1 \triangle U_{j+\frac{3}{2}}\right] \tag{5.28}$$

here,

$$\begin{aligned} D_{-1} &= \frac{1}{12} + \frac{c}{24} - \frac{c^2}{12} - \frac{c^3}{24} \\ D_0 &= \frac{1}{2} - \frac{7}{12}c + \frac{c^3}{12} \\ D_1 &= -(\frac{c^3}{24} - \frac{c^2}{12} - \frac{c}{24} + \frac{1}{12}) \end{aligned}$$

From 5.3, the flux limiter form of this method is

$$F^{5-P}(U^n;j) = a\left[U_j^n + \left(D_{-1} \triangle U_{j-\frac{1}{2}} + D_0 \triangle U_{j+\frac{1}{2}}\right)\phi_j + D_1 \triangle U_{j+\frac{3}{2}}\phi_{j+1}\right] \tag{5.29}$$

using 5.4 and doing some manipulations, we finally get the following limiters

$$\phi_j \leq \frac{24\theta_j}{c\left[(2+c)(1+c)\theta_j + 2(7+c) - 3c(1+c)\right]} \tag{5.30}$$

$$\phi_{j-1} \leq \frac{24}{c\left[(2+c)(1+c)\theta_{j-1}+2(1+c)(6-c^2)-2c^2\right]} \quad (5.31)$$

$$\phi_{j+1} \leq \frac{\theta_{j+1}\theta_j}{c(2-c)(1+c)} \quad (5.32)$$

here,

$$\begin{aligned} \theta_j &= \frac{\Delta U_{j-\frac{1}{2}}}{\Delta U_{j+\frac{1}{2}}} \qquad a>0 \\ \theta_j &= \frac{\Delta U_{j+\frac{3}{2}}}{\Delta U_{j+\frac{1}{2}}} \qquad a<0 \end{aligned} \quad (5.33)$$

and,

$$c = |c| \quad (5.34)$$

Figure 5.3 shows the TVD region for $c = 1$ and $c = 0.7$.

Figure 5.4 shows the computational result. The CFL number used is 0.7. The TVD limiters are:

$$\begin{aligned} \phi_j &= \frac{24\theta_j}{3.465\theta_j + 8.281} \qquad 0 < \theta_j \leq 0.3 \\ \phi_j &= 1 \qquad 0.3 < \theta_j \leq 2 \\ \phi_j &= \frac{24}{3.465\theta_j + 12.4278} \qquad \theta_j > 2 \end{aligned} \quad (5.35)$$

and

$$\begin{aligned} \phi_{j+1} &= 0 \qquad \theta_{j+1}\theta_j \leq 0 \\ \phi_{j+1} &= \frac{\theta_{j+1}\theta_j}{1.547} \qquad 0 < \theta_{j+1}\theta_j \leq 1 \\ \phi_{j+1} &= 1 \qquad \theta_{j+1}\theta_j > 1 \end{aligned} \quad (5.36)$$

here,

$$\theta_{j+1}\theta_j = \frac{\Delta U_{j-\frac{1}{2}}}{\Delta U_{j+\frac{3}{2}}} \tag{5.37}$$

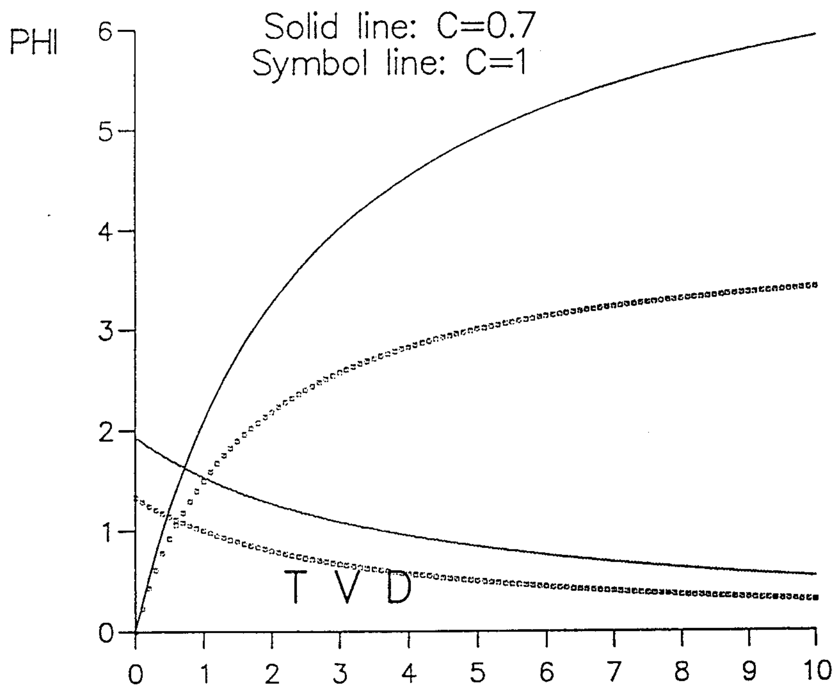

Figure 5.3: TVD Region for Fourth Order Method

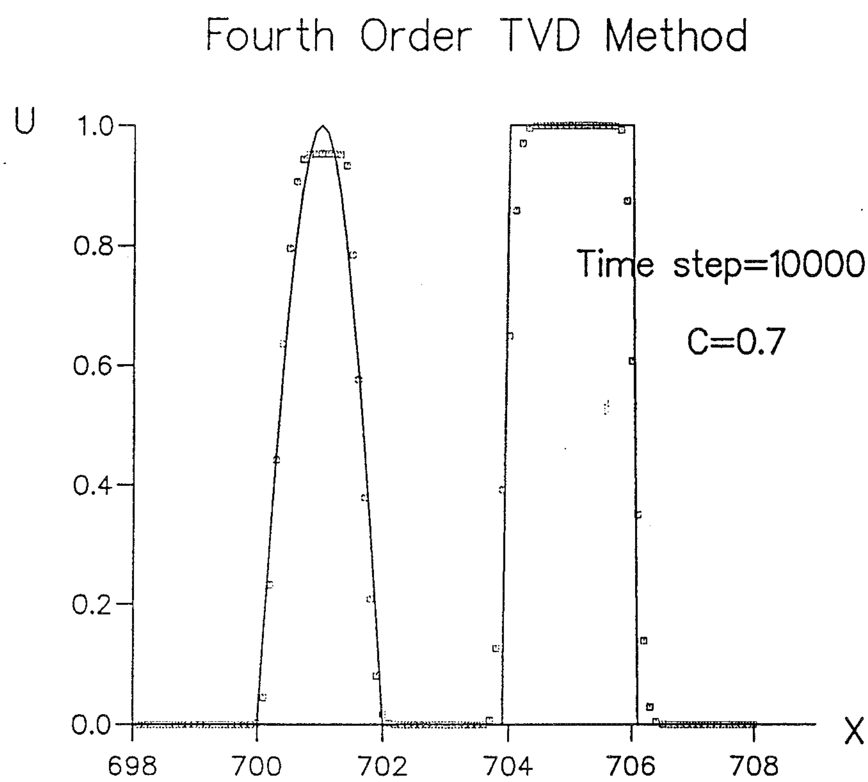

Figure 5.4: Computational Result of Fourth Order TVD Method

Chapter 6

CONCLUSIONS

Based on the findings in this report, the following conclusions can be drawn:

A series of theorems constructing 2-level explicit arbitrary-order high resolution methods for one-dimensional model hyperbolic equation are established.

Theorem 1 defines the law from which 2-level explicit arbitrary-order numerical methods for the model hyperbolic equation can be derived.

Theorem 2 gives the rule of dealing with the problem of linear stability analysis.

Theorems 3 and 4 show a way of developing 2-level explicit conservative arbitrary-order numerical methods and numerical flux.

Theorems 5 and 6 produce TVD methods for central numerical schemes with stable region of $-1 \leq c \leq 1$.

In order to demonstrate how to implement these theorems, as examples, some third order and fourth order high resolution schemes are presented. Computational results of these schemes show some previously unseen distinguishing features of high resolution methods.

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Appendix A

The 20th Order Numerical Method

The 20th order method takes the following form:

$$\begin{aligned} U_j^{n+1} &= B_0 U_j^n + B_{-10} U_{j-10}^n + B_{-9} U_{j-9}^n + B_{-8} U_{j-8}^n + B_{-7} U_{j-7}^n \\ &+ B_{-6} U_{j-6}^n + B_{-5} U_{j-5}^n + B_{-4} U_{j-4}^n + B_{-3} U_{j-3}^n + B_{-2} U_{j-2}^n \\ &+ B_{-1} U_{j-1}^n + B_1 U_{j+1}^n + B_2 U_{j+2}^n + B_3 U_{j+3}^n + B_4 U_{j+4}^n \\ &+ B_5 U_{j+5}^n + B_6 U_{j+6}^n + B_7 U_{j+7}^n + B_8 U_{j+8}^n + B_9 U_{j+9}^n \\ &+ B_{10} U_{j+10}^n \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} B_{-10} &= -5.41254412746970 11E-07\ C - 5.4125442193536856E-08\ C^2 \\ &+ 8.334060771409744E-07\ C^3 + 8.3340608286850486E-08\ C^4 \\ &- 3.488244178100931E-07\ C^5 - 3.4882441891155766E-08\ C^6 \\ &+ 6.2018699671233378E-08\ C^7 + 6.2018697781630719E-09\ C^8 \end{aligned}$$

$$- 5.6212428632890577E-09\ C^{9} - 5.62124286952818E-10\ C^{10}$$
$$+ 2.8337919393891736E-10\ C^{11} + 2.8337919414809486E-11\ C^{12}$$
$$- 8.2180457465338512E-12\ C^{13} - 8.2180457507619695E-13\ C^{14}$$
$$+ 1.354185244170744E-13\ C^{15} + 1.3541852446737389E-14\ C^{16}$$
$$- 1.171440522636276E-15\ C^{17} - 1.1714405229588017E-16\ C^{18}$$
$$+ 4.1103176232824669E-18\ C^{19} + 4.1103176241380932E-19\ C^{20}$$

$$B_{-9} = 1.2027875799663232E-05\ C + 1.3364306711344411E-06\ C^{2}$$
$$- 1.8491921508798121E-05\ C^{3} - 2.0546579573255938E-06\ C^{4}$$
$$+ 7.7085597515460868E-06\ C^{5} + 8.5650664136481394E-07\ C^{6}$$
$$- 1.3605424588926793E-06\ C^{7} - 1.5117138455282181E-07\ C^{8}$$
$$+ 1.219016208481073E-07\ C^{9} + 1.3544624551725358E-08\ C^{10}$$
$$- 6.0415222193403872E-09\ C^{11} - 6.7128024702854551E-10\ C^{12}$$
$$+ 1.7100969895060985E-10\ C^{13} + 1.9001077669921326E-11\ C^{14}$$
$$- 2.7243020794222938E-12\ C^{15} - 3.0270023115029691E-13\ C^{16}$$
$$+ 2.2491658034449778E-14\ C^{17} + 2.4990731155980741E-15\ C^{18}$$
$$-7.3985717218632879E-17\ C^{19} - 8.2206352482552116E-18\ C^{20}$$

$$B_{-8} = -1.2854792258442681E-04\ C - 1.6068490601938150E-05\ C^{2}$$
$$+ 1.9721086119309995E-04\ C^{3} + 2.4651357766736245E-05\ C^{4}$$
$$- 8.1743718940279278E-05\ C^{5} - 1.0217964890385482E-05\ C^{6}$$
$$+ 1.428065355387014E-05\ C^{7} + 1.7850816965332447E-06\ C^{8}$$
$$- 1.2592043798505395E-06\ C^{9} - 1.5740054761137869E-07\ C^{10}$$
$$+ 6.0977941920976703E-08\ C^{11} + 7.622242744484152E-09\ C^{12}$$
$$- 1.6720311077765168E-09\ C^{13} - 2.0900388856030767E-10\ C^{14}$$
$$+ 2.5554272136247108E-11\ C^{15} + 3.194284018082011E-12\ C^{16}$$

$$
\begin{aligned}
&- 2.005506174699382E-13\ C^{17} \ - \ 2.5068827190492179E-14\ C^{18} \\
&+ 6.2476827872381071E-16\ C^{19} \ + \ 7.8096034858409144E-17\ C^{20}
\end{aligned}
$$

$$
\begin{aligned}
B_{-7} \ = \ & 8.8147147022441235E-04\ C \ + \ 1.2592449720362899E-04\ C^{2} \\
&- 1.3480868260251767E-03\ C^{3} \ - \ 1.9258383301538196E-04\ C^{4} \\
&+ 5.5414611748462071E-04\ C^{5} \ + \ 7.9163731210419160E-05\ C^{6} \\
&- 9.5373632669074515E-05\ C^{7} \ - \ 1.3624804681203933E-05\ C^{8} \\
&+ 8.218213675544677E-06\ C^{9} \ + \ 1.1740305258767458E-06\ C^{10} \\
&- 3.8533056741743935E-07\ C^{11} \ - \ 5.5047223943406495E-08\ C^{12} \\
&+ 1.0134817956397296E-08\ C^{13} \ + \ 1.4478311371637456E-09\ C^{14} \\
&- 1.4754246524325741E-10\ C^{15} \ - \ 2.1077495041098964E-11\ C^{16} \\
&+ 1.1020912436507576E-12\ C^{17} \ + \ 1.5744160627642396E-13\ C^{18} \\
&- 3.2800334632538536E-15\ C^{19} \ - \ 4.6857620914379306E-16\ C^{20}
\end{aligned}
$$

$$
\begin{aligned}
B_{-6} \ = \ & -4.3706293678705118E-03\ C \ - \ 7.2843823472016191E-04\ C^{2} \\
&+ 6.6520539913242586E-03\ C^{3} \ + \ 1.1086756682913929E-03\ C^{4} \\
&- 2.6992754360052182E-03\ C^{5} \ - \ 4.4987923993807736E-04\ C^{6} \\
&+ 4.5398869824749103E-04\ C^{7} \ + \ 7.566478310195485E-05\ C^{8} \\
&- 3.7788748938689926E-05\ C^{9} \ - \ 6.2981248265185162E-06\ C^{10} \\
&+ 1.6925148814571187E-06\ C^{11} \ + \ 2.8208581368916746E-07\ C^{12} \\
&- 4.2229287043591243E-08\ C^{13} \ - \ 7.0382145095313115E-09\ C^{14} \\
&+ 5.8407602736300839E-10\ C^{15} \ + \ 9.7346004587333441E-11\ C^{16} \\
&- 4.1700939722246099E-12\ C^{17} \ - \ 6.9501566220910104E-13\ C^{18} \\
&+ 1.1948693330177068E-14\ C^{19} \ + \ 1.9914488888179928E-15\ C^{20}
\end{aligned}
$$

$$
B_{-5} \ = \ 1.6783216790265943E-02\ C \ + \ 3.3566433739730434E-03\ C^{2}
$$

$$
\begin{aligned}
& - 2.5338759119334336E-02\ C^3 \ - \ 5.0677518335719013E-03\ C^4 \\
& + 1.0061219734637984E-02\ C^5 \ + \ 2.012243948862976E-03\ C^6 \\
& - 1.6287905249829739E-03\ C^7 \ - \ 3.2575810519266637E-04\ C^8 \\
& + 1.283826360655336E-04\ C^9 \ + \ 2.5676527224143184E-05\ C^{10} \\
& - 5.3947515886538981E-06\ C^{11} \ - \ 1.0789503180983456E-06\ C^{12} \\
& + 1.2690531937984211E-07\ C^{13} \ + \ 2.5381063883369351E-08\ C^{14} \\
& - 1.6710964543258815E-09\ C^{15} \ - \ 3.3421929095318834E-10\ C^{16} \\
& + 1.1470745596579926E-11\ C^{17} \ + \ 2.2941491198815431E-12\ C^{18} \\
& - 3.1863182212735747E-14\ C^{19} \ - \ 6.3726364440528092E-15\ C^{20}
\end{aligned}
$$

$$
\begin{aligned}
B_{-4} \ = \ & -5.2447552382367639E-02\ C \ - \ 1.3111888163451239E-02\ C^2 \\
& + 7.8003552315233024E-02\ C^3 \ + \ 1.9500888102217392E-02\ C^4 \\
& - 2.9733434540143964E-02\ C^5 \ - \ 7.4333586397198753E-03\ C^6 \\
& + 4.4892831985170432E-03\ C^7 \ + \ 1.1223208001032861E-03\ C^8 \\
& - 3.24214353407299E-04\ C^9 \ - \ 8.1053588378368919E-05\ C^{10} \\
& + 1.2643031133976332E-05\ C^{11} \ + \ 3.1607577843734094E-06\ C^{12} \\
& - 2.8073362574115141E-07\ C^{13} \ - \ 7.0183406452931007E-08\ C^{14} \\
& + 3.539489732800221E-09\ C^{15} \ + \ 8.8487243340955198E-10\ C^{16} \\
& - 2.3515028472187363E-11\ C^{17} \ - \ 5.8787571193929876E-12\ C^{18} \\
& + 6.3726364423175025E-14\ C^{19} \ + \ 1.5931591109380331E-14\ C^{20}
\end{aligned}
$$

$$
\begin{aligned}
B_{-3} \ = \ & 0.13986013993322489\ C \ + \ 4.6620046671199018E-02\ C^2 \\
& - 0.2012107160430074\ C^3 - \ 6.7070238726304382E-02\ C^4 \\
& + 6.9933004598079787E-02\ C^5 \ + \ 2.3311001541969804E-02\ C^6 \\
& - 9.156660441459236E-03\ C^7 \ - \ 3.05222014810262E-03\ C^8 \\
& + 5.9537875973539355E-04\ C^9 \ + \ 1.9845958663193471E-04\ C^{10}
\end{aligned}
$$

$$- 2.1597279728755198E-05\ C^{11}\ -\ 7.1990932446989605E-06\ C^{12}$$
$$+\ 4.560971098645676E-07\ C^{13}\ +\ 1.5203236999083242E-07\ C^{14}$$
$$-\ 5.5501202564010796E-09\ C^{15}\ -\ 1.8500400858978121E-09\ C^{16}$$
$$+\ 3.5941669532496461E-11\ C^{17}\ +\ 1.19805565135244E-11\ C^{18}$$
$$-\ 9.5589546628469194E-14\ C^{19}\ -\ 3.1863182216990149E-14\ C^{20}$$

$$\begin{aligned} B_{-2} \;=\; & -0.3409090907740962\ C\ -\ 0.1704545455456840\ C^2 \\ & +\ 0.4431026355289627\ C^3\ +\ 0.2215513178433360\ C^4 \\ & -\ 0.1141806087503913\ C^5\ -\ 5.709030439213017E-02\ C^6 \\ & +\ 1.2714396381982809E-02\ C^7\ +\ 6.3571981927759303E-03\ C^8 \\ & -\ 7.5256550066543344E-04\ C^9\ -\ 3.762827504347923E-04\ C^{10} \\ & +\ 2.5750408250353993E-05\ C^{11}\ +\ 1.2875204128610629E-05\ C^{12} \\ & -\ 5.233979019443609E-07\ C^{13}\ -\ 2.6169895104208433E-07\ C^{14} \\ & +\ 6.205243213563923E-09\ C^{15}\ +\ 3.1026216076239978E-09\ C^{16} \\ & -\ 3.9454585366658115E-11\ C^{17}\ -\ 1.9727292688812399E-11\ C^{18} \\ & +\ 1.0355534216861282E-13\ C^{19}\ +\ 5.1777671099087964E-14\ C^{20} \end{aligned}$$

$$\begin{aligned} B_{-1} \;=\; & 0.9090909089340871\ C\ +\ 0.9090909091587303\ C^2 \\ & -\ 0.499788846365348\ C^3\ -\ 0.4997888465311628\ C^4 \\ & +\ 0.1000945339323041\ C^5\ +\ 0.1000945339715666\ C^6 \\ & -\ 9.9309289012765676E-03\ C^7\ -\ 9.9309289056588156E-03\ C^8 \\ & +\ 5.52176686112127E-04\ C^9\ +\ 5.5217668637241141E-04\ C^{10} \\ & -\ 1.8201402847760972E-05\ C^{11}\ -\ 1.8201402856741592E-05\ C^{12} \\ & +\ 3.6126371888276745E-07\ C^{13}\ +\ 3.6126371906854591E-07\ C^{14} \\ & -\ 4.2155308676830449E-09\ C^{15}\ -\ 4.2155308699409659E-09\ C^{16} \\ & +\ 3.6510167586153395E-11\ C^{17}\ +\ 2.6510167600918578E-11\ C^{18} \\ & -\ 6.9036894753918626E-14\ C^{19}\ -\ 6.903689479377323E-14\ C^{20} \end{aligned}$$

$$
\begin{aligned}
B_1 \;=\; & -0.9090909089982441\ C \;+\; 0.9090909090107639\ C^2 \\
& + 0.4997888467038099\ C^3 \;-\; 0.4997888464761192\ C^4 \\
& - 0.1000945340215663\ C^5 \;+\; 0.1000945339597436\ C^6 \\
& + 9.9309289117927961E-03\ C^7 \;-\; 9.9309289044145661E-03\ C^8 \\
& - 5.5217668675993216E-04\ C^9 \;+\; 5.5217668630038223E-04\ C^{10} \\
& + 1.8201402870649327E-05\ C^{11} \;-\; 1.8201402854301842E-05\ C^{12} \\
& - 3.612637193622773E-07\ C^{13} \;+ 3.6126371901919222E-07\ C^{14} \\
& + 4.2155308735369533E-09\ C^{15} \;-\; 4.2155308693570004E-09\ C^{16} \\
& - 2.6510167624410223E-11\ C^{17} \;+\; 2.6510167597207196E-11\ C^{18} \\
& + 6.9036894856845310E-14\ C^{19} \;-\; 6.9036894784028248E-14\ C^{20}
\end{aligned}
$$

$$
\begin{aligned}
B_2 \;=\; & 0.3409090911590382\ C \;-\; 0.1704545454095551\ C^2 \\
& - 0.4431026357790551\ C^3 \;+\; 0.2215513177642915\ C^4 \\
& + 0.1141806088171811\ C^5 \;-\; 5.7090304375038911E-02\ C^6 \\
& - 1.2714396389902835E-02\ C^7 \;+\; 6.3571981909751434E-03\ C^8 \\
& + 7.525655011554771E-04\ C^9 \;-\; 3.7628275033047855E-04\ C^{10} \\
& - 2.5750408267719344E-05\ C^{11} \;+\; 1.2875204125074939\ C^{12} \\
& + 5.2339790230882308E-07\ C^{13} \;-\; 2.6169895097051142E-07\ C^{14} \\
& - 6.2052432180178555E-09\ C^{15} \;+\; 3.1026216067765952E-09\ C^{16} \\
& + 3.9454585395780889E-11\ C^{17} \;-\; 1.9727292683423773E-11\ C^{18} \\
& - 1.0355534224698324E-13\ C^{19} \;+\; 5.1777671084932366E-14\ C^{20}
\end{aligned}
$$

$$
\begin{aligned}
B_3 \;=\; & -0.1398601398680919\ C \;+\; 4.6620046550289472E-02\ C^2 \\
& + 0.2012107161932965\ C^3 \;-\; 6.707023865762228E-02\ C^4 \\
& - 6.9933004638802393E-02\ C^5 \;+\; 2.3311001527102117E-02\ C^6
\end{aligned}
$$

$$+\ 9.1566604463335817E-03\ C^7\ -\ 3.0522201465355389E-03\ C^8$$
$$-\ 5.9537876003893763E-04\ C^9\ +\ 1.9845958654107830E-04\ C^{10}$$
$$+\ 2.1597279739557117E-05\ C^{11}\ -\ 7.1990932416161747E-06\ C^{12}$$
$$-\ 4.5609711009187323E-07\ C^{13}\ +\ 1.5203236992836355E-07\ C^{14}$$
$$+\ 5.5501202591831552E-09\ C^{15}\ -\ 1.8500400851574985E-09\ C^{16}$$
$$-\ 3.5941669550703171E-11\ C^{17}\ +\ 1.1980556508912785E-11\ C^{18}$$
$$+\ 9.5589546677486802E-14\ C^{19}\ -\ 3.1863182204603885E-14\ C^{20}$$

$$B_4 = 5.2447552542760158E-02\ C\ -\ 1.3111888087354322E-02\ C^2$$
$$-\ 7.8003552387288705E-02\ C^3\ +\ 1.950088805984191E-02\ C^4$$
$$+\ 2.9733434559988135E-02\ C^5\ -\ 7.4333586305919156E-03\ C^6$$
$$-\ 4.4892832009135923E-03\ C^7\ +\ 1.1223207991417077E-03\ C^8$$
$$+\ 3.242143535572775E-04\ C^9\ -\ 8.1053588322611519E-05\ C^{10}$$
$$-\ 1.2643031139325643E-05\ C^{11}\ +\ 3.1607577824812954E-06\ C^{12}$$
$$+\ 2.8073362585376238E-07\ C^{13}\ -\ 7.0183406414589029E-08\ C^{14}$$
$$-\ 3.5394897341775433E-09\ C^{15}\ +\ 8.8487243295521279E-10\ C^{16}$$
$$+\ 2.3515028481189938E-11\ C^{17}\ -\ 5.8787571165019582E-12\ C^{18}$$
$$-\ 6.3726364447379082E-14\ C^{19}\ +\ 1.5931591101781878E-14\ C^{20}$$

$$B_5 = -1.6783216790265943E-02\ C\ +\ 3.3566433397294302E-03\ C^2$$
$$+\ 2.5338759146446066E-02\ C^3\ -\ 5.0677518143916545E-03\ C^4$$
$$-\ 1.0061219742240127E-02\ C^5\ +\ 2.012243944745893E-03\ C^6$$
$$+\ 1.6287905259103595E-03\ C^7\ -\ 3.2575810475890017E-04\ C^8$$
$$-\ 1.2838263612394298E-04\ C^9\ +\ 2.5676527198963036E-05\ C^{10}$$
$$+\ 5.3947515907457231E-06\ C^{11}\ -\ 1.1789503172426883E-06\ C^{12}$$
$$-\ 1.269053194239874E-07\ C^{13}\ +\ 2.5381063866006513E-08\ C^{14}$$

$$+\ 1.6710964548665981E-09\ C^{15}\ -\ 3.342192907471848E-10\ C^{16}$$
$$-\ 1.1470745600117163E-11\ C^{17}\ +\ 2.2941491185692477E-12\ C^{18}$$
$$+\ 3.1863182222250577E-14\ C^{19}\ -\ 6.3726364406004012E-15\ C^{20}$$

$$\begin{aligned} B_6 \;=\;& 4.37062937989995E-03\ C\ -\ 7.2843822351700458E-04\ C^2 \\ & -\ 6.6520539990204087E-03\ C^3\ +\ 1.1086756619139275E-03\ C^4 \\ & +\ 2.6992754381966501E-03\ C^5\ -\ 4.4987923857653351E-04\ C^6 \\ & -\ 4.5398869851632942E-04\ C^7\ +\ 7.5664782958773701E-05\ C^8 \\ & +\ 3.77887489556616E-05\ C^9\ -\ 6.2981248182117565E-06\ C^{10} \\ & -\ 1.692514882065163E-06\ C^{11}\ +\ 2.8208581340692872E-07\ C^{12} \\ & +\ 4.22292870564138E-08\ C^{13}\ -\ 7.0382145038040535E-09\ C^{14} \\ & -\ 5.8407602751987296E-10\ C^{15}\ +\ 9.7346004519377128E-11\ C^{16} \\ & +\ 4.1700939732493913E-12\ C^{17}\ -\ 6.9501566177616943E-13\ C^{18} \\ & -\ 1.1948693332930086E-14\ C^{19}\ +\ 1.0014488876789506E-15\ C^{20} \end{aligned}$$

$$\begin{aligned} B_7 \;=\;& -8.8147147022441235E-04\ C\ +\ 1.2592449419145932E-04\ C^2 \\ & +\ 1.348086827653146E-03\ C^3\ -\ 1.9258383147415187E-04\ C^4 \\ & -\ 5.5414611795211459E-04\ C^5\ +\ 7.916373088834124E-05\ C^6 \\ & +\ 9.5373632726828108E-05\ C^7\ -\ 1.3624804646819964E-05\ C^8 \\ & -\ 8.2182136792058245E-06\ C^9\ +\ 1.1740305238792094E-06\ C^{10} \\ & +\ 3.8533056754888451E-07\ C^{11}\ -\ 5.5047223875425271E-08\ C^{12} \\ & -\ 1.0134817959171436E-08\ C^{13}\ +\ 1.4478311357821706E-09\ C^{14} \\ & +\ 1.4754246527719211E-10\ C^{15}\ -\ 2.1077495024685307E-11\ C^{16} \\ & -\ 1.1020912438723225E-12\ C^{17}\ +\ 1.5744160617175219E-13\ C^{18} \\ & +\ 3.2800334638485865E-15\ C^{19}\ -\ 4.6857620886819089E-16\ C^{20} \end{aligned}$$

$$\begin{aligned}
B_8 \;=\; & 1.2854792233381353E-04\ C \;-\; 1.6068490020642247E-05\ C^2 \\
& -\ 1.9721086142894054E-04\ C^3 \;+\; 2.4651357511000552E-05\ C^4 \\
& +\ 8.174371900901958E-05\ C^5 \;-\; 1.0217964836322332E-05\ C^6 \\
& -\ 1.4280653562422396E-05\ C^7 \;+\; 1.78508169084522E-06\ C^8 \\
& +\ 1.2592043803954657E-06\ C^9 \;-\; 1.5740054728028971E-07\ C^{10} \\
& -\ 6.0977941940620429E-08\ C^{11} \;+\; 7.6222427331900794E-09\ C^{12} \\
& +\ 1.6720311081924454E-09\ C^{13} \;-\; 2.0900388833025625E-10\ C^{14} \\
& -\ 2.5554272141348674E-11\ C^{15} \;+\; 3.1942840153432374E-12\ C^{16} \\
& +\ 2.0055061750332314E-13\ C^{17} \;-\; 2.5068827172994957E-14\ C^{18} \\
& -\ 6.2476827881359847E-16\ C^{19} \;+\; 7.8096034812267166E-17\ C^{20}
\end{aligned}$$

$$\begin{aligned}
B_9 \;=\; & -1.2027875830989893E-05\ C \;+\; 1.3364306203536201E-06\ C^2 \\
& +\ 1.849192153014139E-05\ C^3 \;-\; 2.0546579310704337E-06\ C^4 \\
& -\ 7.708559757771679E-06\ C^5 \;+\; 8.565066358162458E-07\ C^6 \\
& +\ 1.3605424596677215E-06\ C^7 \;-\; 1.5117138396853592E-07\ C^8 \\
& -\ 1.2190162089731396E-07\ C^9 \;+\; 1.3544624517689925E-08\ C^{10} \\
& +\ 6.0415222211036277E-09\ C^{11} \;-\; 6.7128024586735774E-10\ C^{12} \\
& -\ 1.710096989876961E-10\ C^{13} \;+\; 1.9001077646274995E-11\ C^{14} \\
& +\ 2.7243020798742975E-12\ C^{15} \;-\; 3.0270023086891944E-13\ C^{16} \\
& -\ 2.2491658037391251E-14\ C^{17} \;+\; 2.4990731138014241E-15\ C^{18} \\
& +\ 7.3985717226506696E-17\ C^{19} \;-\; 8.220635243519829E-18\ C^{20}
\end{aligned}$$

$$\begin{aligned}
B_{10} \;=\; & 5.4125441176801199E-07\ C \;-\; 5.4125439871243214E-08\ C^2 \\
& -\ 8.3340607804876036E-07\ C^3 \;+\; 8.3340607014844473E-08\ C^4 \\
& +\ 3.4882441807603583E-07\ C^5 \;-\; 3.4882441623128323E-08\ C^6
\end{aligned}$$

$$
\begin{aligned}
& - 6.2018699704524168E-08\ C^7 + 6.2018699499478518E-09\ C^8 \\
& + 5.6212428654146464E-09\ C^9 - 5.6212428530759774E-10\ C^{10} \\
& - 2.8337919401552493E-10\ C^{11} + 2.833791935858306E-11\ C^{12} \\
& + 8.2180457481542722E-12\ C^{13} - 8.2180457392895429E-13\ C^{14} \\
& - 1.3541852443693002E-13\ C^{15} + 1.2541852433059194E-14\ C^{16} \\
& + 1.1714405227661243E-15\ C^{17} - 1.1714405220838428E-16\ C^{18} \\
& - 4.110317623631571E-18\ C^{19} + 4.1103176218282869E-19\ C^{20}
\end{aligned}
$$

$$
\begin{aligned}
B_0 \ = \ & 1 + 9.643184E-09\ C - 1.549768\ C^2 - 4.8558619E-09\ C^3 \\
& + 0.6598717\ C^4 + 2.192701E-10\ C^5 - 0.121028\ C^6 \\
& + 3.0618016E-10\ C^7 + 1.1531415E-02\ C^8 + 1.6614871E-11 C^9 \\
& - 6.2741595E-04\ C^{10} - 4.3139212E-13\ C^{11} + 2.0418935E-05 C^{12} \\
& + 2.4922193E-14\ C^{13} - 4.0202718E-07\ C^{14} - 1.3142628E-16 C^{15} \\
& + 4.6662447E-09\ C^{16} + 3.0843404E-19\ C^{17} - 2.9237123E-11 C^{18} \\
& - 7.3362479E-22\ C^{19} + 7.5940576E-14\ C^{20}
\end{aligned}
$$

where, C is the CFL number.

The stable function of this method is

$$
\begin{aligned}
\lambda \ = \ & 1 - 2\,(-2.1820901E-08\ C + 1.91839\ C^2 + 2.2592523E-08 C^3 \\
& - 1.144243\ C^4 - 6.8021899E-09\ C^5 + 0.2509956\ C^6 \\
& + 1.1990592E-09\ C^7 - 2.6645366E-02\ C^8 + 4.4792958E-11 C^9 \\
& + 1.5550008E-03\ C^{10} + 2.3386265E-13\ C^{11} - 5.3070333E-05 C^{12} \\
& - 1.5452447E-14\ C^{13} + 1.0802879E-06\ C^{14} + 4.9565642E-16 C^{15} \\
& - 1.2842341E-08\ C^{16} - 4.3901846E-18\ C^{17} + 8.1889627E-11 C^{18} \\
& + 4.1853998E-21\ C^{19} - 2.1549903E-13\ C^{20}) \qquad \text{(A.2)}
\end{aligned}
$$

The stable regions of this method are: (see **Figure A.1**)

$$\begin{aligned} -4 &\leq C \leq -3.75 \\ -3 &\leq C \leq -2.58 \\ -2 &\leq C \leq -1.3 \\ -1 &\leq C \leq 1 \\ 1.3 &\leq C \leq 2 \\ 2.58 &\leq C \leq 3 \\ 3.75 &\leq C \leq 4 \end{aligned} \tag{A.3}$$

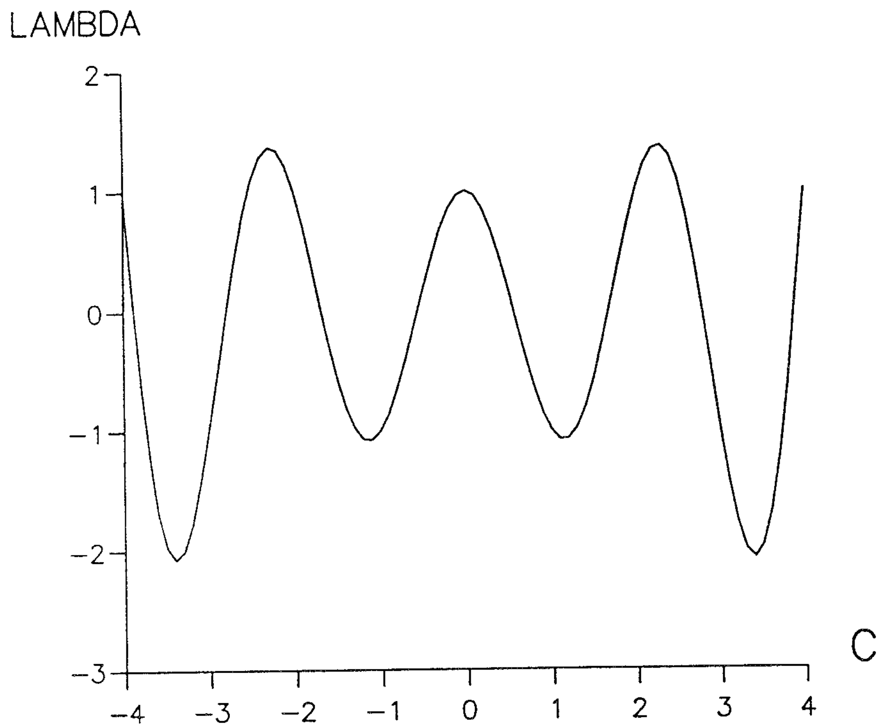

Figure A.1: Stable Region of the 20th Method

Figure A.2 showes the computational result. The initial date of the calculation is absolutely smooth. As you can see that even thought the time evolves fifty-thousants steps there is hardly any error.

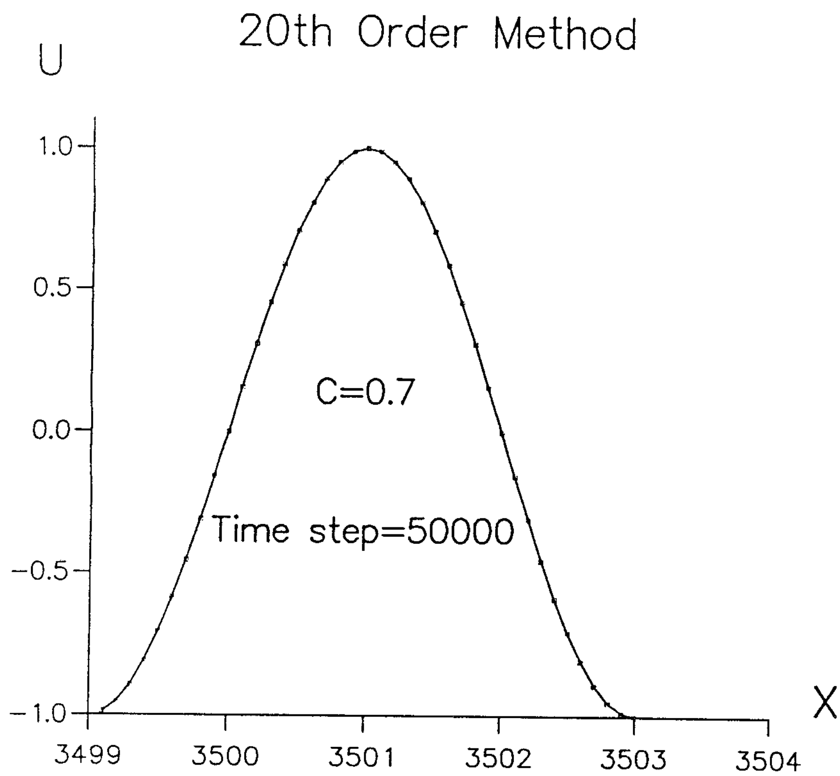

Figure A.2: Computational Result of the 20th Numerical Method