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Working Paper WP8

The Linearised Small Perturbation Equations of Motion for an Airship

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July 1987

Revised January 1990

1. AXES AND ASSUMPTIONS

1.1 Generalised body axes are shown on fig. 1. The origin of the orthogonal axis is chosen arbitrarily on the centre line of the envelope and the ox axis is coincident with the centre line. The plane oxz defines the plane of symmetry of the airship.

1.2 The trimmed axes notation is shown in fig. 2. For the general case when the airship is in straight wings level flight. The steady state body attitude and speed are denoted Θ_0 and V_0 respectively. The buoyancy force B acts at the centre of buoyancy which has coordinates (b_x, b_y, b_z) and the weight W acts at the centre of gravity which has coordinates (a_x, a_y, a_z).

1.3 Initially the main assumptions are;

(i) steady rectilinear flight

(ii) the airship is "rigid", and hence aeroelastic effects are ignored.

(iii) disturbances from equilibrium are small.

(iv) constant mass.

(v) the body is symmetric and both the c.b. and c.g. lie in the plane of symmetry. (i.e. $a_y = b_y = 0$).

2. EQUATIONS OF MOTION

In addition to the usual aerodynamic terms the equations of motion of the airship will also include significant force and moment terms due to static buoyancy and inertial terms due to the displaced mass of air. A very complete mathematical model of a remotely operated underwater vehicle (ROV) is given in reference 1. Since the conditions applying are similar to those experienced by the airship then these equations may be used to describe its motion. However, since the airship has symmetry, well defined equilibrium conditions and operates in a less dense medium so the equations may be simplified accordingly.

The equations of motion may thus be written

$$M \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = F_d(u,v,w,p,q,r) + S(\dot{u}_g,\dot{v}_g,\dot{w}_g,u_g,v_g,w_g,p_g,q_g,r_g) + A\,(u,v,w,p,q,r) + G(\lambda_{13}, \lambda_{23}, \lambda_{33}) + C \text{ (control forces and moments)} + P \text{ (propulsion forces and moments)}$$

where M is the mass matrix and F_d, S, A, G and C are vectors defined below.

2.1 Mass Matrix M

The mass of air displaced by the airship in motion gives rise to virtual mass and inertia effects expressed as equivalent aerodynamic acceleration derivatives.

2.1.1. Components of mass

$$m_x = m - X_{\dot{u}}$$
$$m_y = m - Y_{\dot{v}}$$
$$m_z = m - Z_{\dot{w}}$$

where m is the mass of the airship and $X_{\dot{u}}$, $Y_{\dot{v}}$, $Z_{\dot{w}}$ are the virtual mass components.

2.1.2. Components of displaced air mass

$$\bar{m}_x = \bar{m} - X_{\dot{u}}$$
$$\bar{m}_y = \bar{m} - Y_{\dot{v}}$$
$$\bar{m}_z = \bar{m} - Z_{\dot{w}}$$

where m is the mass of air displaced by the airship.

2.1.3. Moments of inertia

$$J_x = I_x - L_{\dot{p}}$$
$$J_y = I_y - M_{\dot{q}}$$
$$J_z = I_z - N_{\dot{r}}$$

where I_x, I_y, I_z are the moments of inertia about the ox, oy, oz axes respectively and $L_{\dot{p}}$, $M_{\dot{q}}$, $N_{\dot{r}}$ are the virtual inertia components.

2.1.4. Products of inertia

$$J_{yz} = I_{yz} + M_{\dot{r}} \equiv I_{yz} + N_{\dot{q}}$$
$$J_{zx} = I_{zx} + N_{\dot{p}} \equiv I_{zx} + L_{\dot{r}}$$
$$J_{xy} = I_{xy} + L_{\dot{q}} \equiv I_{xy} + M_{\dot{p}}$$

where I_{yz}, I_{zx} and I_{xy} are the three products of inertia about the ox, oy, oz axes respectively.

Since the airship is symmetric about the oxz plane,

$$I_{yz} = I_{xy} = 0$$

and,

$$M_{\dot{r}} = N_{\dot{q}} = L_{\dot{q}} = M_{\dot{p}} = 0$$

for small disturbances.

2.1.5. Mass matrix

$$\begin{bmatrix} m_x & 0 & 0 & -X_{\dot{p}} & ma_z - X_{\dot{q}} & -X_{\dot{r}} \\ 0 & m_y & 0 & -ma_z - Y_{\dot{p}} & -Y_{\dot{q}} & ma_x - y_{\dot{r}} \\ 0 & 0 & m_z & -Z_{\dot{p}} & -ma_x - Z_{\dot{q}} & -Z_{\dot{r}} \\ -L_{\dot{u}} & -ma_z - L_{\dot{v}} & -L_{\dot{w}} & J_x & 0 & -J_{zx} \\ ma_z - M_{\dot{u}} & -M_{\dot{v}} & -ma_x - M_{\dot{w}} & 0 & J_y & 0 \\ -N_{\dot{u}} & ma_x - N_{\dot{v}} & -N_{\dot{w}} & -J_{zx} & 0 & J_z \end{bmatrix}$$

noting that $a_y = 0$ (symmetry).

Assuming longitudinal-lateral motion coupling effects to be insignificant then the mass matrix may be simplified since,

$$X_{\dot{p}} = X_{\dot{r}} = Y_{\dot{q}} = Z_{\dot{p}} = Z_{\dot{r}} = L_{\dot{u}} = L_{\dot{w}} = M_{\dot{v}} = N_{\dot{u}} = N_{\dot{w}} = 0$$

and the matrix becomes

$$\begin{bmatrix} m_x & 0 & 0 & 0 & ma_z - X_{\dot{q}} & 0 \\ 0 & m_y & 0 & -ma_z - Y_{\dot{p}} & 0 & ma_x - Y_{\dot{r}} \\ 0 & 0 & m_z & 0 & -ma_x - Z_{\dot{q}} & 0 \\ 0 & -ma_z - L_{\dot{v}} & 0 & J_x & 0 & -J_{xz} \\ ma_z - M_{\dot{u}} & 0 & -ma_x - M_{\dot{w}} & 0 & J_y & 0 \\ 0 & ma_x - N_{\dot{v}} & 0 & -J_{xz} & 0 & J_z \end{bmatrix}$$

2.2 Dynamics Vector F_d

Since products and squares of small quantities may be ignored, the dynamics vector may be written,

$$F_d = \begin{bmatrix} -m_z\, wq + m_y\, rv \\ -m_x\, ur + m_z\, pw \\ -m_y\, vp + m_x\, qu \\ ma_z(ur - pw) \\ m[a_x(vp - qu) - a_z(wq - rv)] \\ -ma_x(ur - pw) \end{bmatrix}$$

Now

$$V_o = (U^2 + V^2 + W^2)^{\frac{1}{2}} \qquad \text{steady components of speed.}$$

And
$$u = u' + U$$
$$v = v' + V \qquad \text{and } V = o \text{ for straight flight.}$$
$$w = w' + W$$

Since u', v', w' are small then the dynamics vector may be rewritten,

$$F_d = \begin{bmatrix} -m_z\, qW + m_y\, \cancelto{0}{rV} \\ -m_x\, rU + m_z\, pW \\ -m_y\, \cancelto{0}{pV} + m_x\, qU \\ ma_z(rU - pW) \\ m[a_x(\cancelto{0}{pV} - qU) - a_z(qW - \cancelto{0}{rV})] \\ -ma_x(rU - pW) \end{bmatrix} = \begin{bmatrix} -m_z qW \\ -m_x rU + m_z pW \\ +m_x qU \\ ma_z(rU - pW) \\ -mq(a_x U + a_z W) \\ -ma_x(rU - pW) \end{bmatrix}$$

2.3 Atmospheric disturbances Vector S

Assuming a quasi-steady atmosphere with no gusts or turbulence then

$$S = 0$$

2.4 Aerodynamics Vector A

Assuming no longitudinal-lateral coupling, then in terms of aerodynamic derivatives

$$A = \begin{bmatrix} X_{a_o} + X_u u + X_w w + X_q q \\ Y_{a_o} + Y_v v + Y_p p + Y_r r \\ Z_{a_o} + Z_u u + Z_w w + Z_q q \\ L_{a_o} + L_v v + L_p p + L_r r \\ M_{a_o} + M_u u + M_w w + M_q q \\ N_{a_o} + N_v v + N_p p + N_r r \end{bmatrix}$$

where terms subscripted (a_o) are steady state values.

2.5 Gravitational Force/Moment Vector G

$$G = \begin{bmatrix} \lambda_{13}(W-B) & \\ \lambda_{23}(W-B) & \\ \lambda_{33}(W-B) & \\ -\lambda_{23}(a_zW + b_zB) & \\ \lambda_{13}(a_zW + b_zB) & -\lambda_{33}(a_xW + b_xB) \\ \lambda_{23}(a_xW + b_xB) & \end{bmatrix}$$

where λ_{ij} derive from the standard direction cosine matrix,

$$\lambda_{13} = -\sin(\Theta+\Theta_o)$$

$$\lambda_{23} = \sin\phi\cos(\Theta+\Theta_o)$$

$$\lambda_{33} = \cos\phi\cos(\Theta+\Theta_o)$$

where Θ_o is the steady state pitch attitude of the airship and is not necessarily assumed to be a small angle. Expanding the above and assuming ψ,Θ,ϕ small then,

$$\lambda_{13} = -\Theta\cos\Theta_o - \sin\Theta_o$$

$$\lambda_{23} = \phi\cos\Theta_o$$

$$\lambda_{33} = \cos\Theta_o - \Theta\sin\Theta_o$$

whence,

$$G = \begin{bmatrix} -\Theta(mg - B)\cos\Theta_o - (mg - B)\sin\Theta_o \\ +\phi(mg - B)\cos\Theta_o \\ (mg - B)\cos\Theta_o - \Theta(mg - B)\sin\Theta_o \\ -\phi(a_zmg + b_zB)\cos\Theta_o \\ \Theta\{mg(a_x\sin\Theta_o - a_z\cos\Theta_o) + B(b_x\sin\Theta_o - b_z\cos\Theta_o)\} - mg(a_x\cos\Theta_o + a_z\sin\Theta_o) \\ \qquad - B(b_x\cos\Theta_o + b_z\sin\Theta_o) \\ \phi\cos\Theta_o(a_xmg + b_xB) \end{bmatrix}$$

2.6 Propulsion Forces and Moments

Referring to fig. 3 which shows the geometric arrangement of propulsion forces, the following is assumed;

(i) One turbo prop engine producing thrust in a direction parallel to ox in the plane of symmetry and acting at a distance c_z below the ox axis.

(ii) Two symmetrically displaced diesel engine driven ducted fan thrusters. Each thrust magnitude being independently controlled. The thrust vector direction can be varied synchronoulsy in a plane parallel to the oxz plane by rotation through the angle μ.

Whence,

$$P = \begin{bmatrix} X_{prop} \\ Y_{prop} \\ Z_{prop} \\ L_{prop} \\ M_{prop} \\ N_{prop} \end{bmatrix} = \begin{bmatrix} T_t + (T_{d_s} + T_{d_p}) \cos \mu \\ 0 \\ -(T_{d_s} + T_{d_p}) \sin \mu \\ (T_{d_p} - T_{d_s}) \sin \mu \; d_y \\ T_t \, c_z + (T_{d_s} + T_{d_p})(d_z \cos \mu - dx \sin \mu) \\ (T_{d_p} - T_{d_s}) \cos \mu \; d_y \end{bmatrix}$$

2.7 Control Forces and Moments

The aerodynamic controls comprise four independently actuated surfaces with the "vertical" surfaces acting like differential rudders and the "horizontal" surfaces acting like tailerons as shown on fig. 4. Positive control deflections are defined as follows;

Upper rudder δr_u, trailing edge to port
Lower rudder δr_ℓ, trailing edge to port
Port elevator δe_p, trailing edge down
Starboard elevator δe_s, trailing edge down

whence,

$$C = \begin{bmatrix} X_c \\ Y_c \\ Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix} = \begin{bmatrix} X_\delta \, (\delta r_u + \delta r_\ell + \delta e_p + \delta e_s) \\ Y_\delta \, (\delta r_u + \delta r_\ell) \\ Z_\delta \, (\delta e_p + \delta e_s) \\ L_\delta \, (\delta r_u - \delta r_\ell + \delta e_p - \delta e_s) \\ M_{\delta_e}(\delta e_p + \delta e_x) + M_{\delta_r}(\delta r_u - \delta r_\ell)^* \\ M_{\delta_r} (\delta r_u + \delta r_\ell) + N_{\delta_e} (\delta e_s - \delta e_p)^* \end{bmatrix}$$

*moment terms due to differential drag effects - assumed negligibly small for all practical purposes.

3. THE DECOUPLED LONGITUDINAL EQUATIONS OF MOTION

$$m_x\dot{u} + (ma_z - X_{\dot{q}})\dot{q} = -m_z qW + X_{a_o} + X_u u + X_w w + X_q q - \Theta(mg - B)\cos\Theta_o - (mg - B)\sin\Theta_o + T_t + (T_{d_s} + T_{d_p})\cos\mu + X_c$$

$$m_z\dot{w} - (ma_x + Z_{\dot{q}})\dot{q} = m_x qU + Z_{a_o} + Z_u u + Z_w w + Z_q q + (mg - B)\cos\Theta_o - \Theta(mg - B)\sin\Theta_o - (T_{d_s} + T_{d_p})\sin\mu + Z_c$$

$$(ma_z - M_{\dot{u}})\dot{u} - (ma_x + M_{\dot{w}})\dot{w} + J_y\dot{q} = -mg(a_x U + a_z W) + M_{a_o} + M_u u + M_w w + M_q q + \Theta\{mg(a_x\sin\Theta_o - a_z\cos\Theta_o) + B(b_x\sin\Theta_o - b_z\cos\Theta_o)\} - mg(a_x\cos\Theta_o + a_z\sin\Theta_o) - B(b_x\cos\Theta_o + b_z\sin\Theta_o) + T_t c_z + (T_{d_s} + T_{d_p})(d_z\cos\mu - d_x\sin\mu) + M_c$$

3.1 Steady state trim conditions

When the small perturbation variables are zero, trimmed equilibrium prevails and the equation may be written,

$$Xa_o - (mg - B)\sin\Theta_o + T_t + (T_{d_s} + T_{d_p})\cos\mu + X_{c_o} = 0$$

$$Za_o + (mg - B)\cos\Theta_o - (T_{d_s} + T_{d_p})\sin\mu + Z_{c_o} = 0$$

$$Ma_o - mg(a_x\cos\Theta_o + a_z\sin\Theta_o) - (b_x\cos\Theta_o + b_z\sin\Theta_o) + T_t C_z + (T_{d_s} + T_{d_p})(d_z\cos\mu - d_x\sin\mu) + M_{c_o} = 0$$

where X_{c_o}, Z_{c_o}, M_{c_o} represent the trimmed control forces and moments.

Removing these terms from the decoupled equations leaves the small perturbation equations with respect to the steady flight condition.

3.2 The longitudinal small perturbation equations

$$m_x\dot{u} + (ma_z - X_{\dot{q}})\dot{q} = -m_z qW + X_u u + X_w w + X_q q - \Theta(mg - B)\cos\Theta_o + \Delta X_c$$

$$M_z\dot{w} - (ma_x + Z_{\dot{q}})\dot{q} = m_x qU + Z_u u + Z_w w + Z_q q - \Theta(mg - B)\sin\Theta_o + \Delta Z_c$$

$$(ma_z - M_{\dot{u}})\dot{u} - (ma_x + M_{\dot{w}})\dot{w} + J_y\dot{q} = -mq(a_xU + a_zW) + M_uu + M_ww + M_qq$$
$$+\Theta\{mg(a_x\sin\Theta_o - a_z\cos\Theta_o) + B(b_x\sin\Theta_o - b_z\cos\Theta_o)\}$$
$$+ \Delta M_c$$

where ΔX_c, ΔZ_c and ΔM_c are the control force and moment perturbations about their trim values.

The above assumes the thrust propulsion configuration to be fixed at the steady state values.

Rearranging,

$$m_x\dot{u} - X_uu - X_ww + (ma_z - X_{\dot{q}})\dot{q} - (X_q - m_zW)q + \Theta(mg - B)\cos\Theta_o = \Delta X_c$$

$$-Z_uu + m_z\dot{w} - Z_ww - (ma_x + Z_{\dot{q}})\dot{q} - (Z_q + m_xU)q + \Theta(mg - B)\sin\Theta_o = \Delta Z_c$$

$$(ma_z - M_{\dot{u}})\dot{u} - M_uu - (ma_x + M_{\dot{w}})\dot{w} - M_ww + J_y\dot{q} - (M_q - ma_xU + ma_zW)q$$
$$-\Theta[mg(a_x\sin\Theta_o - a_z\cos\Theta_o) + B(b_x\sin\Theta_o - b_z\cos\Theta_o)] = \Delta M_c$$

4. THE DECOUPLED LATERAL EQUATIONS OF MOTION

$$m_y\dot{v} - (ma_z - Y_{\dot{p}})\dot{p} + (ma_x - Y_{\dot{r}})\dot{r} = - m_xrU + m_zpW + Ya_o + Y_vv + Y_pp$$
$$+ Y_rr + \phi(mg \quad B) + Y_c$$

$$-(ma_z + L_{\dot{v}})\dot{v} + J_x\dot{p} - J_{xz}\dot{r} = ma_z(rU - pW) + La_o + L_vv + L_pp + L_rr$$
$$-\phi(a_zmg + b_zB) + (T_{d_p} - T_{d_s})d_y\sin\mu + L_c$$

$$(ma_x - N_{\dot{v}})\dot{v} - J_{xz}\dot{p} + J_z\dot{r} = -ma_x(rU - pW) + Na_o + N_vv + N_pp + N_rr$$
$$+ \phi\cos\Theta_o(a_xmg + b_xB) + (T_{d_p} - T_{d_s})d_y\cos\mu + N_c$$

4.1 Steady state trim conditions

When the small perturbation variables become zero the above equations reduce to,

$$Ya_o + Y_{c_o} = 0$$

$$La_o + (T_{d_p} - T_{d_s})d_y \sin\mu + L_{c_o} = 0$$

$$Na_o + (T_{d_p} - T_{d_s})d_y \cos\mu + N_{c_o} = 0$$

whence, the small perturbation equation with respect to trimmed steady state follow;

$$m_y\dot{v} - (ma_z - Y_{\dot{p}})\dot{p} + (ma_x - Y_{\dot{r}})\dot{r} = -m_x rU + m_z pW + Y_v v + Y_p p + Y_r r$$
$$+ \phi(mg + B) + \Delta Y_c$$

$$-(ma_z + L_{\dot{v}})\dot{v} + J_x\dot{p} - J_{xz}\dot{r} = ma_z(rU - pW) + L_v v + L_p p + L_r r$$
$$- \phi(a_z mg + b_z B) + \Delta L_c$$

$$(ma_x - N_{\dot{v}})\dot{v} - J_{xz}\dot{p} + J_z\dot{r} = -ma_x(rU - pW) + N_v v + N_p p + N_r r$$
$$+ \phi \mathrm{Cos}\Theta_o(a_x mg + b_x B) + \Delta N_c$$

where ΔY_c, ΔL_c and ΔN_c are the control force and moment perturbations about their steady state trim values.

Thrust and propulsion components are assumed constant and trimmed out. i.e. steady state aerodynamic and control forces and moments balance out the steady state propulsion effects.

Rearranging,

$$m_y\dot{v} - Y_v v - (ma_z - Y_{\dot{p}})\dot{p} - (Y_p + m_z W)p - \phi(mg + B) + (ma_x - Y_{\dot{r}})\dot{r}$$
$$-(Y_r - m_x U)r = \Delta Y_c$$

$$-(ma_z + L_{\dot{v}})\dot{v} - L_v v + J_x\dot{p} - (L_p - ma_z W)p + \phi(a_z mg + b_z B)$$
$$-J_{xz}\dot{r} - (L_r + ma_z U)r = \Delta L_c$$

$$(ma_x - N_{\dot{v}})\dot{v} - N_v v - J_{xz}\dot{p} - (N_p + ma_x W)p - \phi \mathrm{Cos}\Theta_o(a_x mg + b_x B)$$
$$+ J_z\dot{r} - (N_r - ma_x U)r = \Delta N_c$$

5. SIMPLIFICATIONS

Further simplifications can be made to the equations if the following additional assumptions are made;

(i) Neutral buoyancy $mg = B$

(ii) Level flight and a wind axis system,

$W = \Theta_o = 0$ and $U = V_o$

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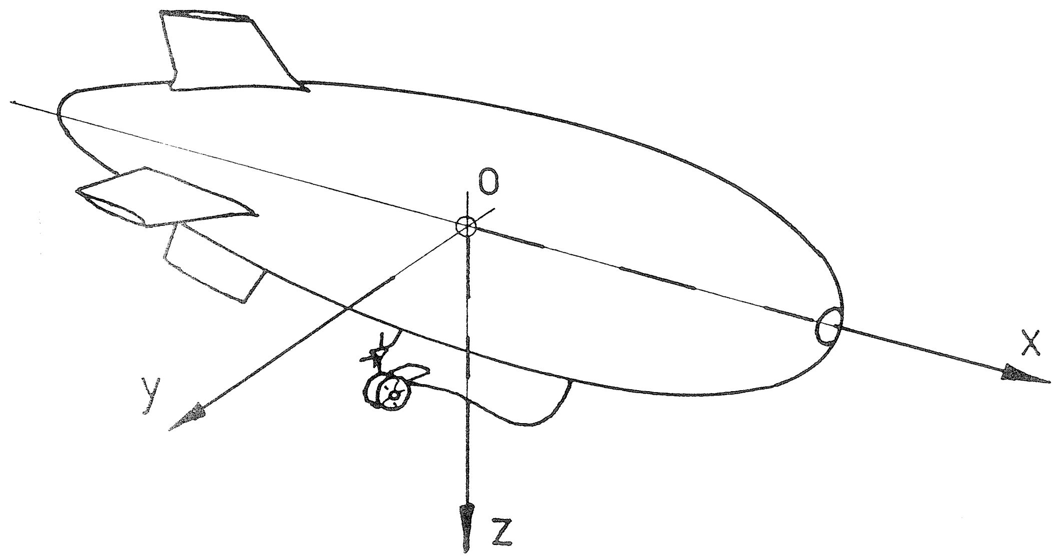

FIG. 1 GENERALISED BODY AXES

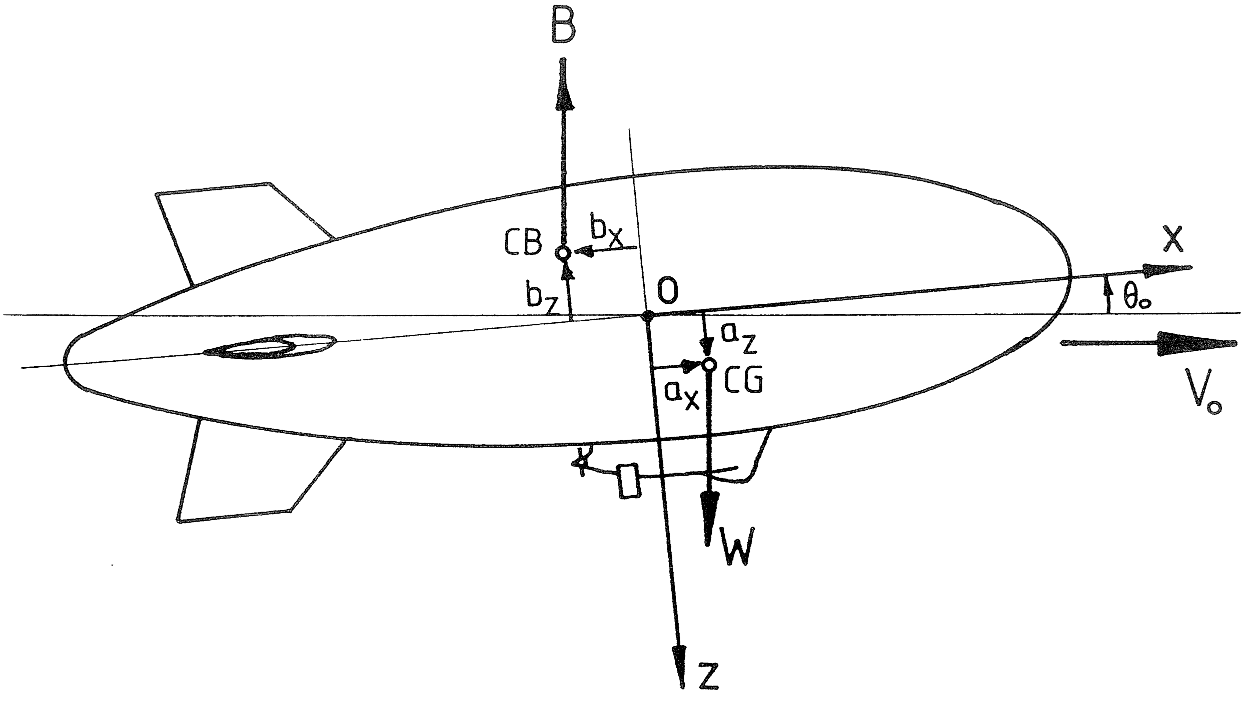

FIG. 2 TRIMMED AXES NOTATION

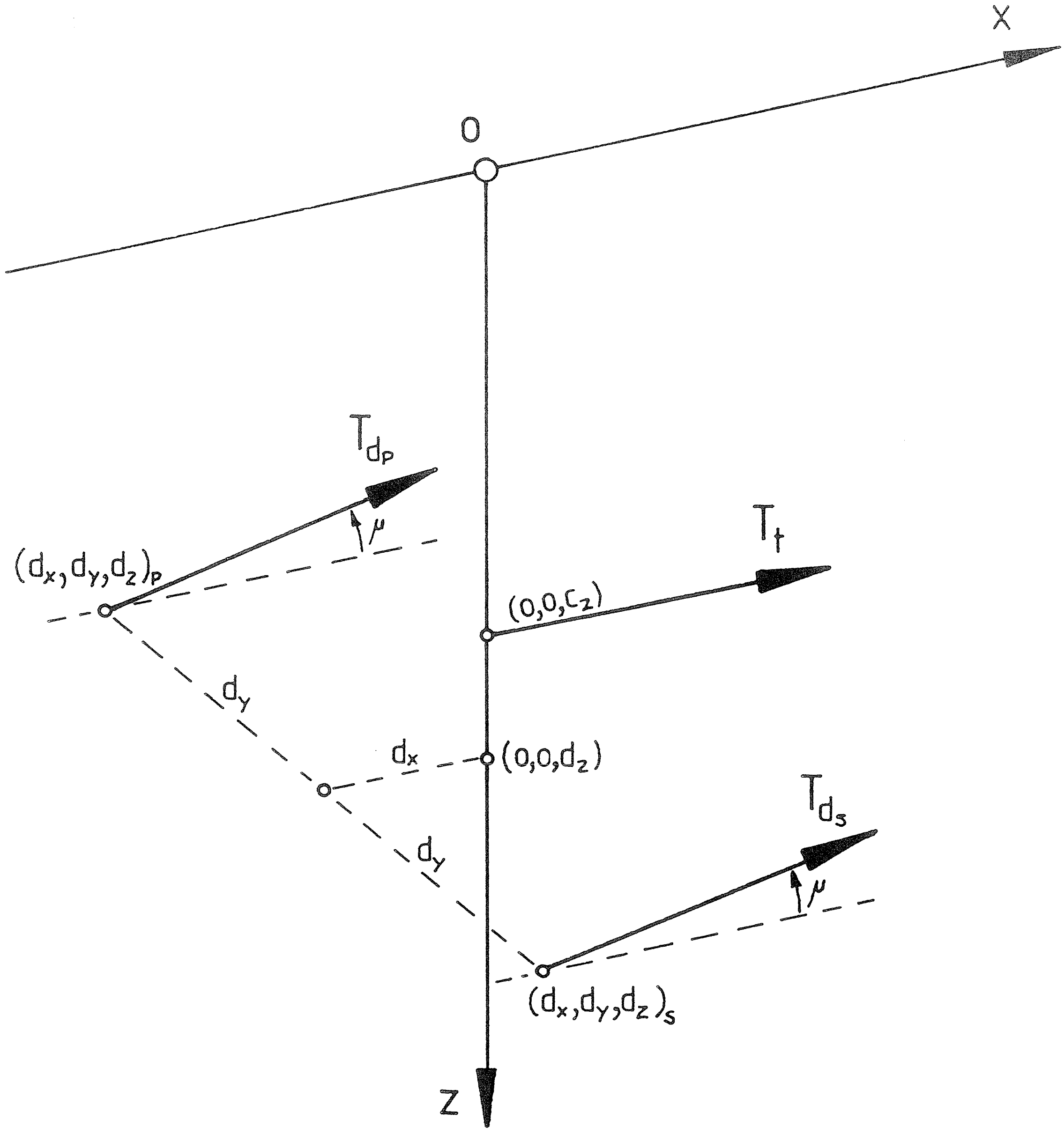

FIG. 3 PROPULSION SYSTEM GEOMETRY

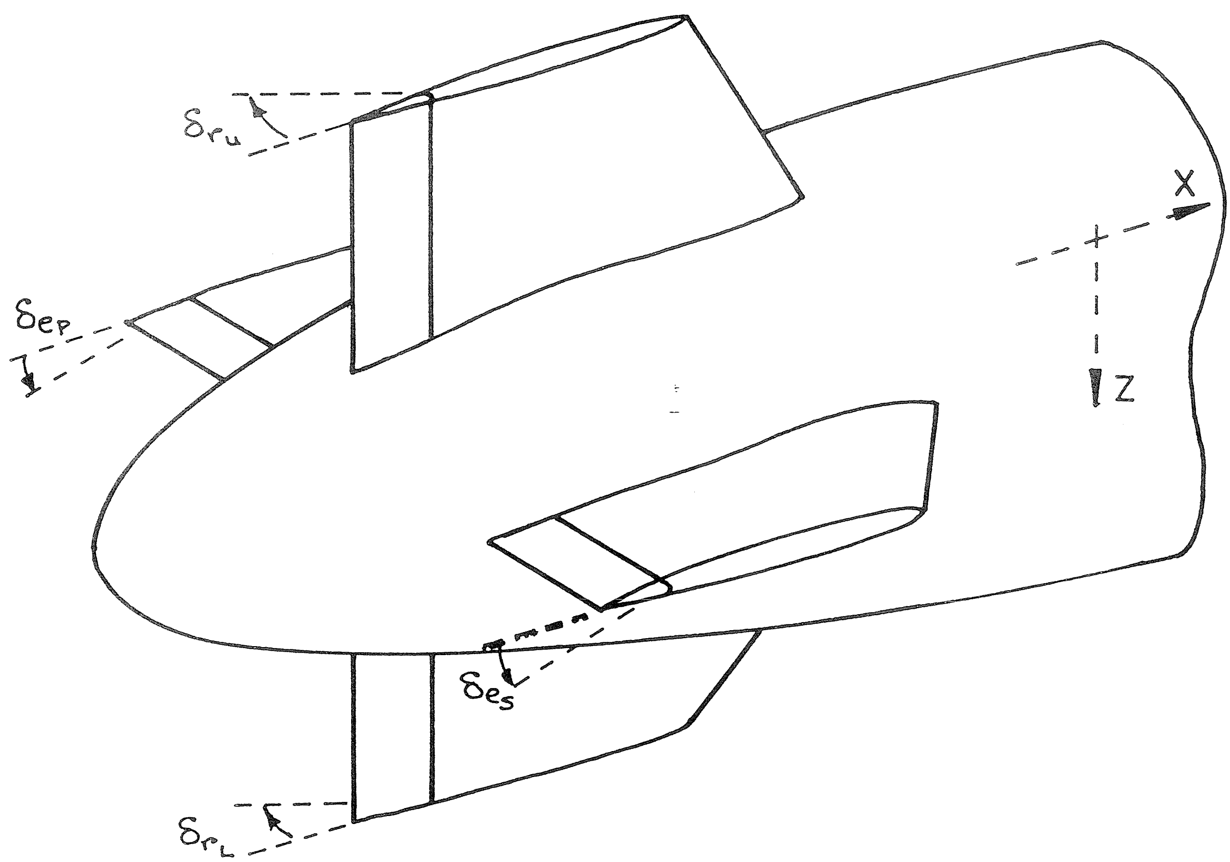

FIG. 4 CONTROL SURFACES NOTATION