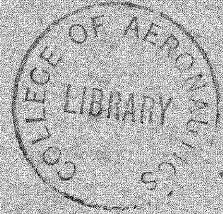


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DETERMINATION OF DYNAMICAL MODELS FOR  
ADAPTIVE CONTROL SYSTEMS

by

R. J. A. Paul

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Determination of Dynamical Models for  
Adaptive Control Systems

- by -

R. J. A. Paul, B.Sc.(Eng.), A.M.I.E.E., A.M.I.Mech.E.

SUMMARY

A method is described for the synthesis of a dynamical model of a linear system based on the use of orthonormal functions. It is shown that if the nominal values of all poles of a system are known, and if only one pole changes from its nominal value, then this change may be detected. It is also demonstrated that the numerator terms of the transmission transfer function of the system may be found provided the denominator is known. Active networks are described, for the simulation of the relevant orthonormal functions.

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## 1. Introduction

The approximate determination of the dynamics of processes has been the subject of several recent papers.

One form of adaptive control systems<sup>(1)</sup>, in which the controller is automatically adjusted to maintain a particular performance index in the presence of random variations of the system parameters, is based on the realization of a dynamical model of the process. This model, under normal operating conditions of the process, must be capable of automatic adjustment to ensure that it remains a reasonable approximation of the process when the parameters of the latter are subject to random variation.

The approximation of a linear system, based on the use of orthogonal functions, has been considered in many papers, including those by Gilbert<sup>(2)</sup> and Kitamori<sup>(3)</sup>. A spectrum analyser has been described by Braun et al<sup>(4)</sup> and uses a set of orthogonal functions.

In these papers, very little knowledge, if any, is assumed of the process dynamics in the determination of the approximate model. However, in many applications the nominal values of some of the system parameters are known, and this knowledge may be used to realize the most economical model of the system. This approach is justified on the grounds that there is no virtue in constructing an adaptive system which does not make use of all available information concerning the process. This paper is concerned with such applications and a procedure is outlined, based on the use of orthonormal functions, for the determination of a dynamical model of the process.

## 2. Basic Method of Measurement

A null method is used and is similar to that described by Kitamori<sup>(3)</sup>. The arrangement is shown in Fig. 2.1.

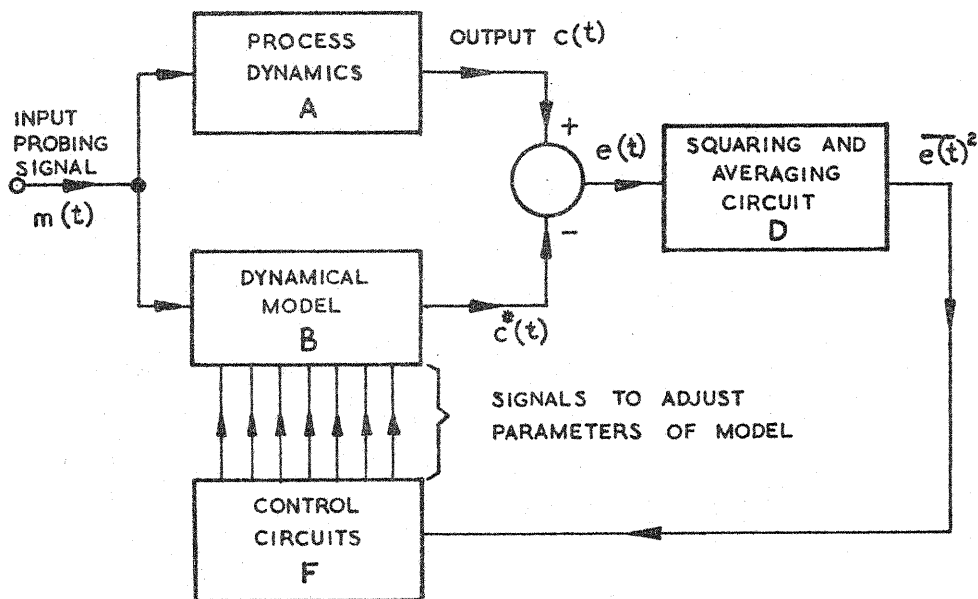


Fig. 2.1. Basic Arrangement

Block A represents the dynamics of the process to be controlled.

Block B represents the dynamical model consisting of active RC networks whose parameters are adjusted automatically to give a minimum value of mean square error  $\overline{e(t)^2}$ .

Block D represents a squaring and averaging circuit, the output of which represents the mean square error.

Block F represents the control circuits which control the adjustment of the parameters of the model. Only one parameter is adjusted at any time to give a minimum value of  $\overline{e(t)^2}$ .

$m(t)$  represents the common input probing signal to both the process and the model.

Although the process is subject to random variations, if it is assumed that these occur slowly compared with the response of the process, block A may be represented in mathematical form by its transmission transfer function  $G(s)$ , where  $s$  is the complex variable  $\sigma + j\omega$ . In other words the process is assumed to be time invariant during the period required to estimate its characteristics and to correct the model. If the process is assumed to be linear, then

$$C(s) = G(s) \cdot M(s) \quad (2.1)$$

$$\text{or } c(t) = \int_{0+}^{\infty} g(\lambda) \cdot m(t - \lambda) \cdot d\lambda, \quad (2.2)$$

where  $g(t)$  is the unit impulse response of the system.

If  $G^*(s)$  represents the transmission transfer function of the model, then,

$$C^*(s) = G^*(s) \cdot M(s) \quad (2.3)$$

$$\text{or } c^*(t) = \int_{0+}^{\infty} g^*(\tau) m(t - \tau) d\tau, \quad (2.4)$$

where  $g^*(\tau)$  is the unit impulse response of the model.

$$e(t) = c(t) - c^*(t) \quad (2.5)$$

$$\therefore \overline{e(t)} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [c(t) - c^*(t)]^2 dt. \quad (2.6)$$

$$\text{Thus } \overline{e(t)^2} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left[ \int_0^{\infty} g(\lambda) \cdot m(t - \lambda) d\lambda - \int_0^{\infty} g^*(\tau) \cdot m(t - \tau) d\tau \right]^2 dt. \quad (2.7)$$

$$\text{Let } f(t) = g(t) - g^*(t). \quad (2.8)$$

$$\text{Then } \overline{e(t)^2} = \int_0^{\infty} \int_0^{\infty} f(\lambda) \cdot f(\tau) \left[ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T m(t - \lambda) \cdot m(t - \tau) dt \right] d\lambda \cdot d\tau \quad (2.9)$$

$$\text{i. e. } \overline{e(t)^2} = \int_0^{\infty} \int_0^{\infty} f(\lambda) \cdot f(\tau) \phi(\lambda - \tau) d\lambda d\tau, \quad (2.10)$$

where  $\phi(\mu)$  is the auto correlation function of  $m(t)$ .

Now 
$$F(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t) dt \quad (2.11)$$

where  $F(j\omega)$  is the Fourier Transfer of  $f(t)$ .

Also 
$$\Phi_{mm}(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} \phi(\mu) d\mu \quad (2.12)$$

where  $\Phi_{mm}(\omega)$  is the spectral density of  $m(t)$ .

Thus 
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} F(j\omega) d\omega \quad (2.13)$$

and 
$$\phi(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega\mu} \Phi_{mm}(\omega) d\omega \quad (2.14)$$

$$\therefore \phi(\lambda - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(\lambda - \tau)} \Phi_{mm}(\omega) d\omega \quad (2.15)$$

Thus 
$$\overline{e(t)^2} = \int_0^{\infty} \int_0^{\infty} f(\lambda) \cdot f(\tau) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(\lambda - \tau)} \Phi_{mm}(\omega) d\omega \right] d\lambda d\tau \quad (2.16)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{mm}(\omega) \left\{ \left[ \int_0^{\infty} f(\lambda) e^{j\omega\lambda} d\lambda \right] \left[ \int_0^{\infty} f(\tau) e^{-j\omega\tau} d\tau \right] \right\} d\omega \quad (2.17)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{mm}(\omega) F(j\omega) \cdot F(-j\omega) d\omega \quad (2.18)$$

i. e. 
$$\overline{e(t)^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \Phi_{mm}(\omega) d\omega \quad (2.19)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega) - G^*(j\omega)|^2 \Phi_{mm}(\omega) d\omega \quad (2.20)$$

Thus for a given spectral density of the input probing signal  $m(t)$ , the mean square error  $\overline{e(t)^2}$  is minimised as  $G^*(j\omega)$  becomes a closer approximation to  $G(j\omega)$ .

The frequency response function  $G^*(j\omega)$  is synthesized by a combination of active RC networks which represent a set of orthogonal functions.

### 3. Orthogonal Functions

An orthogonal set of functions  $O_1(s), O_2(s) \dots O_r(s)$ , may be derived from any set of linearly independent functions  $L_1(s); L_2(s) \dots L_r(s)$ , by making  $O_r(s)$  a suitable linear combination of  $L_r(s)$ , with

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} O_q(j\omega) \cdot O_r(-j\omega) \Phi_{mm}(\omega) d\omega = \begin{cases} 1 & \text{for } q = r \\ 0 & \text{" } q \neq r \end{cases} \quad (3.1)$$

$O_r(s)$  may be represented by the equations (3.5).

$$O_r(s) = \frac{\begin{vmatrix} L_r(s) & L_{r-1}(s) & \dots & L_1(s) \\ P_{r-1,r} & P_{r-1,r-1} & \dots & P_{r-1,1} \\ P_{r-2,r} & P_{r-2,r-1} & \dots & P_{r-2,1} \\ \dots & \dots & \dots & \dots \\ P_{1,r} & P_{1,r-1} & \dots & P_{1,1} \end{vmatrix}}{\sqrt{\begin{vmatrix} P_{r-1,r-1} & P_{r-1,r-2} & \dots & P_{r-1,1} \\ P_{r-2,r-1} & P_{r-2,r-2} & \dots & P_{r-2,1} \\ \dots & \dots & \dots & \dots \\ P_{1,r-1} & P_{1,r-2} & \dots & P_{1,1} \end{vmatrix} \begin{vmatrix} P_{rr} & P_{r,r-1} & \dots & P_{r,1} \\ P_{r-1,r} & P_{r-1,r-1} & \dots & P_{r-1,1} \\ \dots & \dots & \dots & \dots \\ P_{1,r} & P_{1,r-1} & \dots & P_{1,1} \end{vmatrix}}} \dots (3.2)$$

where  $P_{q,r} = \frac{1}{2\pi} \int_{-\infty}^{\infty} L_q(j\omega) L_r(-j\omega) \Phi_{mm}(\omega) d\omega$  . (3.3)

Thus any set of orthogonal functions is dependent on the spectral density of the input signal  $m(t)$ .

Consider the particular case when  $\Phi_{mm}(\omega)$  is a constant which may be normalized to unity. Such a spectral density represents that of a white noise signal. Although the latter is not physically realizable, it may be shown, that, provided  $\Phi(\omega)$  is constant over a bandwidth which is much greater than that of the process under investigation, it is a reasonable approximation to assume that  $\Phi(\omega)$  is constant over an infinite range of frequency.

If, therefore,  $\Phi_{mm}(\omega)$  is assumed to be unity then equation 3.3 may be expressed as

$$P_{qr} = \frac{1}{2\pi} \int_{-\infty}^{\infty} L_q(j\omega) \cdot L_r(-j\omega) d\omega$$
 . (3.4)

If  $L_q(j\omega) \cdot L_r(-j\omega)$  approaches zero more rapidly than  $\frac{1}{j\omega}$  as  $\omega \rightarrow \infty$ , then equation 3.4 may be expressed as a contour integral where the contour encircles the entire right half s-plane.

Thus,

$$\frac{1}{2\pi j} \int_c L_q(s) \cdot L_r(-s) ds = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} L_q(s) \cdot L_r(-s) ds$$

$$= \sum \text{residues of } L_q(s) \cdot L_r(-s) \text{ in right half s-plane} .$$
 (3.5)

If a linear, stable, discrete parameter system is now considered, its transmission transfer function is characterized by its poles and zeros, which may be real or conjugate complex.

For such a system the set of linearly independent functions may be expressed in the form:-

$$L_k(s) = \frac{1}{s + p_k} \quad \text{for } k = 1 \dots m \quad (3.6)$$

$$\begin{aligned} L_{m+2r-1}(s) &= \frac{1}{s^2 + 2\alpha_r s + \alpha_r^2 + \beta_r^2} & ) \\ & & ) \\ & & ) \quad \text{for } r=m+1 \dots n \\ \text{and } L_{m+2r}(s) &= s L_{m+2r-1}(s) & ) \\ & & ) \\ & & ) \quad \dots \quad (3.7) \end{aligned}$$

where  $p_k$ ,  $\alpha_r$  and  $\beta_r$  are positive constants.

If this set of functions is substituted in equation 3.1 the following orthonormal functions may be derived.

$$O_1(s) = \frac{\sqrt{2p_1}}{s + p_1} ; \quad O_2(s) = \frac{\sqrt{2p_2}}{s + p_2} \cdot \frac{s - p_1}{s + p_1} \quad (3.8)$$

and in general for  $k \leq m$

$$O_k(s) = \frac{\sqrt{2p_k}}{s + p_k} \cdot \frac{s - p_{k-1}}{s + p_{k-1}} \dots \frac{s - p_2}{s + p_2} \cdot \frac{s - p_1}{s + p_1} \quad (3.9)$$

If  $r > m$

$$\begin{aligned} O_{m+2r-1}(s) &= \frac{\sqrt{4\alpha_r} \sqrt{\alpha_r^2 + \beta_r^2}}{s^2 + 2\alpha_r s + \alpha_r^2 + \beta_r^2} \cdot \frac{s^2 - 2\alpha_{r-1} s + \alpha_{r-1}^2 + \beta_{r-1}^2}{s + 2\alpha_{r-1} s + \alpha_{r-1}^2 + \beta_{r-1}^2} \dots \\ & \dots \frac{s^2 - 2\alpha_1 s + \alpha_1^2 + \beta_1^2}{s^2 + 2\alpha_1 s + \alpha_1^2 + \beta_1^2} \cdot \frac{s - p_m}{s + p_m} \dots \frac{s - p_1}{s + p_1} \quad (3.10) \end{aligned}$$

$$= \frac{\sqrt{4\alpha_r} \sqrt{\alpha_r^2 + \beta_r^2}}{s^2 + 2\alpha_r s + \alpha_r^2 + \beta_r^2} \cdot N(s) \quad (3.11)$$

$$\text{and } O_{m+2r}(s) = \frac{s \sqrt{4\alpha_r}}{s^2 + 2\alpha_r s + \alpha_r^2 + \beta_r^2} \cdot N(s) \quad (3.12)$$



The transmission transfer function of the model may be expressed as

$$G^*(s) = \sum_{i=1}^N k_i O_i(s) \quad (3.13)$$

where

$$k_i O_i(s) = \frac{k_i \sqrt{2p_i}}{s+p_i} \cdot \frac{s-p_{i-1}}{s+p_{i-1}} \cdots \frac{s-p_1}{s+p_1} \quad \text{for } i \leq m \quad (3.14)$$

and

$$k_i O_i(s) = \frac{2\sqrt{\alpha_i} \left[ k_i s + k_i \sqrt{\alpha_i^2 + \beta_i^2} \right]}{s^2 + 2\alpha_i s + \alpha_i^2 + \beta_i^2} \cdot \frac{s^2 - 2\alpha_1 s + \alpha_1^2 + \beta_1^2}{s^2 + 2\alpha_1 s + \alpha_1^2 + \beta_1^2} \cdots \frac{s-p_m}{s+p_m} \cdots \frac{s-p_1}{s+p_1} \quad \text{for } i > m \quad (3.15)$$

From equation 2.20

$$\overline{e(t)^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega) - G^*(j\omega)|^2 \Phi_{mm}(\omega) d\omega \quad (3.16)$$

$$\text{With } \Phi_{mm}(\omega) = 1 \quad (3.17)$$

$$\overline{e(t)^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega) - G^*(j\omega)|^2 d\omega \quad (3.18)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} [G(j\omega) - G^*(j\omega)] [G(-j\omega) - G^*(-j\omega)] d\omega \quad (3.19)$$

Substituting equation 3.13 into equation 3.19

$$\overline{e(t)^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ |G(j\omega)|^2 - \left[ G(j\omega) \sum_{i=1}^N k_i O_i(-j\omega) + G(-j\omega) \sum_{i=1}^N k_i O_i(j\omega) \right] \right\} d\omega + \sum_{i=1}^N k_i^2 \quad (3.20)$$

For the minimum value of  $\overline{e(t)}$  by adjustment of  $k_i$

$$\frac{\partial \overline{e(t)^2}}{\partial k_i} = \frac{1}{2\pi} \int_{-\infty}^{\infty} -2 \left[ G(j\omega) O_i(-j\omega) \right] d\omega + 2k_i = 0 \quad (3.21)$$

$$\therefore k_i = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) O_i(-j\omega) d\omega \quad (3.22)$$

with  $i = 1 \text{ ----- } N$

Since  $G(j\omega) O_i(-j\omega)$  approaches zero more rapidly than  $\frac{1}{j\omega}$  as  $\omega \rightarrow \infty$ , (3.21) may be expressed as

$$k_i = \sum \left. \begin{array}{l} \text{Residues in right half } s\text{-plane of } G(s) O_i(-s) \\ \text{Residues in right half } s\text{-plane of } G(-s) O_i(s) \end{array} \right\} \quad (3.23)$$

for  $i = 1, 2, \dots, N$ .

#### 4. Synthesis of the dynamical model

The transmission transfer function of a linear, discrete parameter system may be expressed by the partial fraction expression :-

$$G(s) = \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + \dots + \frac{a_k}{s+p_k} + \dots + \frac{a_m}{s+p_m} + \frac{b_1 s + c_1}{s^2 + 2\alpha_1 s + \alpha_1^2 + \beta_1^2} + \dots + \frac{b_N s + c_N}{s^2 + 2\alpha_N s + \alpha_N^2 + \beta_N^2} \quad (4.1)$$

where the parameters  $p_1, \dots, p_m, \alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N$  are positive constants.  $a_1, \dots, a_m, b_1, \dots, b_N, c_1, \dots, c_N$  are unknown constants which may have a positive or negative sign.

##### 4.1. Determination of the denominator terms

If, at any time, all but one, say  $p_k$ , of the denominator coefficients are known, a model may be constructed in a way such that the minimization of the mean square error, by variation of one pole position, gives the value of the unknown parameter. Consider the model whose transmission transfer function is given by

$$G^*(s) = \frac{\sqrt{2\gamma}}{s+\gamma} \cdot \frac{s-p_1}{s+p_1} \dots \frac{s-p_{k-1}}{s+p_{k-1}} \cdot \frac{s-p_{k+1}}{s+p_{k+1}} \dots \frac{s-p_m}{s+p_m} \cdot \frac{s^2 - 2\alpha_1 s + \alpha_1^2 + \beta_1^2}{s^2 + 2\alpha_1 s + \alpha_1^2 + \beta_1^2} \dots \frac{s^2 - 2\alpha_N s + \alpha_N^2 + \beta_N^2}{s^2 + 2\alpha_N s + \alpha_N^2 + \beta_N^2} \quad (4.2)$$

Under these conditions, for  $\overline{e(t)^2}$  to be a minimum by adjustment of  $\gamma$ , then from (3.19)

$$\frac{\partial \overline{e(t)^2}}{\partial \gamma} = \frac{1}{2\pi} \int_{-\infty}^{\infty} -\frac{\partial}{\partial \gamma} \left[ G(j\omega) G^*(-j\omega) + G(-j\omega) G^*(+j\omega) \right] d\omega = 0 \quad (4.3)$$

$$= -2 \frac{\partial}{\partial \gamma} \sum \text{Residues in the right half } s\text{-plane of } G(-s) G^*(s) \quad (4.4)$$

Hence

$$\frac{\partial}{\partial \gamma} \left( \frac{\sqrt{2\gamma}}{p_k + \gamma} \right) = 0 \quad (4.5)$$

i. e.  $(p_k + \gamma) \left( \frac{1}{\sqrt{2\gamma}} \right) - \sqrt{2\gamma} = 0 \quad (4.6)$

i. e.  $p_k = \gamma \quad (4.7)$

Thus if  $\overline{e(t)^2}$  is minimized, the value of  $\gamma$  required is equal to the unknown parameter  $\gamma$ .

If now the nominal values of all the poles of the system are known but in a period, which is long compared with the measurement time, one pole can change from its nominal value, this change may be detected by a slight rearrangement of the model, i. e.

$$G^*(s) = \frac{\sqrt{2\gamma}}{s + \gamma} \cdot \frac{s - p_1}{s + p_1} \cdots \frac{s - p_{k-1}}{s + p_{k-1}} \cdot \frac{s - p_k}{s + p_k} \cdot \frac{s - p_{k+1}}{s + p_{k+1}} \cdots \frac{s - p_m}{s + p_m} \cdot \frac{s^2 - 2\alpha_1 s + \alpha_1^2 + \beta_1^2}{s^2 + 2\alpha_1 s + \alpha_1^2 + \beta_1^2} \cdots \frac{s^2 - 2\alpha_N s + \alpha_N^2 + \beta_N^2}{s^2 + 2\alpha_N s + \alpha_N^2 + \beta_N^2} \quad (4.8)$$

The model has now an excess real pole compared with the system. If any one of the real poles (say  $p_k$ ) is subject to variation, then the minimization of

$\frac{\partial}{\partial \gamma} \overline{e(t)^2}$  will again give the condition that  $p_k = \gamma$ . In this case, however, the particular pole which has changed from its nominal value is not known and a procedure is necessary to give this information. This may be achieved in the following manner. After the minimization of  $\overline{e(t)^2}$  the factor  $\left( \frac{s - p_1}{s + p_1} \right)$

function of the model. If no change in the minimum condition for  $\overline{e(t)^2}$  occurs as a result of this operation then it is obvious that  $p_1$  is the variable pole. If however,

a change occurs, the factor  $\left( \frac{s - p_1}{s + p_1} \right)$  is re-inserted and the process repeated with  $\left( \frac{s - p_2}{s + p_2} \right)$ . This sequence of operations is repeated until the removal of a particular factor  $\left( \frac{s - p_r}{s + p_r} \right)$  causes no change. The pole  $p_r$  then represents the nominal value of the variable pole of the system and the redundant factor  $\frac{s - p_r}{s + p_r}$  may be

removed from the model. The validity of this procedure is based on the assumption, as already stated, that the change in the pole position during the time of measurement is negligible.

An alternative situation may arise, however, in that the variation may occur in either of the two nominal parameter values of a particular pair of conjugate complex

poles. If it is known that a particular pair of conjugate complex poles has a damping factor which is subject to variation, this variation may be detected by a slight rearrangement of the model with a transfer function of the form

$$G^*(s) = \frac{2\sqrt{\mu_1}\omega_{o_1}(s + \omega_{o_1})}{s^2 + 2\mu_1\omega_{o_1}s + \omega_{o_1}^2} \cdot \frac{s^2 - 2\zeta_2\omega_{o_2}s + \omega_{o_2}^2}{s^2 + 2\zeta_2\omega_{o_2}s + \omega_{o_2}^2} \cdots \frac{s^2 - 2\zeta_N\omega_{o_N}s + \omega_{o_N}^2}{s^2 + 2\zeta_N\omega_{o_N}s + \omega_{o_N}^2} \cdots \frac{s - p_1}{s + p_1} \cdots \frac{s - p_m}{s + p_m} \quad (4.9)$$

where  $\mu_1$  represents the variable parameter and all other parameters are known.

$$\text{Note } \zeta_r \omega_{o_r} = \alpha_r \quad \text{and} \quad \omega_{o_r} = \sqrt{\alpha_r^2 + \beta_r^2} \quad (4.10)$$

Under these conditions for  $\overline{e(t)^2}$  to be a minimum by adjustment of  $\mu_1$  then

$$\frac{\partial \overline{e(t)^2}}{\partial \mu_1} = -2 \frac{\partial}{\partial \mu_1} \sum \text{Residues in right half s-plane of } G(-s) G^*(s) = 0 \quad (4.11)$$

$$\therefore \sum \text{Residues in right half s-plane of } G(-s) \frac{\partial}{\partial \mu_1} G^*(s) = 0 \quad (4.12)$$

$$\text{i.e. } \sum \text{Residues in right half s-plane of } \frac{G(-s) \cdot \frac{\partial}{\partial \mu_1} \frac{(s + \omega_{o_1})(s^2 - 2\mu_1\omega_{o_1}s + \omega_{o_1}^2)}{\mu_1 (s^2 + 2\mu_1\omega_{o_1}s + \omega_{o_1}^2)^2}}{\mu_1} = 0$$

$$\text{i.e. } \sum \text{Residues of } \frac{s^2 - 2\mu_1\omega_{o_1}s + \omega_{o_1}^2}{(s^2 + 2\mu_1\omega_{o_1}s + \omega_{o_1}^2)^2} \cdots \quad (4.13)$$

$$\text{i.e. } \sum \text{Residues of } \frac{1}{s^2 - 2\zeta_1\omega_{o_1}s + \omega_{o_1}^2} = 0 \quad (4.14)$$

This results in the condition that,

$$\zeta_1 = \mu_1 \quad (4.15)$$

Hence the model has followed the variation in  $\zeta_1$  by minimizing  $\overline{e(t)^2}$  by variation of  $\mu_1$ .



4.2 Determination of the numerator terms

The transmission transfer function of the system as expressed by 4.1 may be re-written as :-

$$\begin{aligned}
 G(s) = & \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} \cdot \frac{s-p_1}{s+p_1} + \dots + \frac{A_k}{s+p_k} \cdot \frac{s-p_{k-1}}{s+p_{k-1}} \dots \frac{s-p_1}{s+p_1} \dots \frac{A_m}{s+p_m} \cdot \frac{s-p_{m-1}}{s+p_{m-1}} \dots \frac{s-p_1}{s+p_1} \\
 & + \frac{B_1 s + C_1}{s^2 + s2\zeta_1 \omega_{O_1} + \omega_{O_1}^2} \cdot \frac{s-p_m}{s+p_m} \cdot \frac{s-p_{m-1}}{s+p_{m-1}} \dots \frac{s-p_1}{s+p_1} \\
 & + \frac{B_r s + C_r}{s^2 + s2\zeta_r \omega_{O_r} + \omega_{O_r}^2} \cdot \frac{(s^2 - s2\zeta_{r-1} \omega_{O_{r-1}} + \omega_{O_{r-1}}^2)}{(s^2 + s2\zeta_{r-1} \omega_{O_{r-1}} + \omega_{O_{r-1}}^2)} \dots \frac{(s^2 - s2\zeta_1 \omega_{O_1} + \omega_{O_1}^2)}{(s^2 + s2\zeta_1 \omega_{O_1} + \omega_{O_1}^2)} \cdot \frac{s-p_m}{s+p_m} \cdot \frac{s-p_{m-1}}{s+p_{m-1}} \\
 & \dots \frac{s-p_1}{s+p_1} \\
 & + \frac{B_N s + C_N}{s^2 + s2\zeta_N \omega_{O_N} + \omega_{O_N}^2} \cdot \frac{(s^2 - s2\zeta_{N-1} \omega_{O_{N-1}} + \omega_{O_{N-1}}^2)}{(s^2 + s2\zeta_{N-1} \omega_{O_{N-1}} + \omega_{O_{N-1}}^2)} \dots \frac{(s^2 - s2\zeta_1 \omega_{O_1} + \omega_{O_1}^2)}{(s^2 + s2\zeta_1 \omega_{O_1} + \omega_{O_1}^2)} \cdot \frac{s-p_m}{s+p_m} \cdot \frac{s-p_{m-1}}{s+p_{m-1}} \\
 & \dots \frac{s-p_1}{s+p_1} , \\
 & \dots \dots \dots \quad (4.16)
 \end{aligned}$$

where  $\zeta_r \omega_{O_r} = \alpha_r$  and  $\omega_{O_r} = \sqrt{\alpha_r^2 + \beta_r^2}$  , (4.17)

i.e.  $G(s) = \sum_1^N \ell_i O_i(s)$  where  $\ell_i$  is a constant for  $i = 1 \dots N$  . (4.18)

With the transmission transfer function of the model in this case given by :-

$$G^*(s) = \sum_1^N k_i O_i(s) \quad (4.19)$$

the object is to determine the coefficients  $A_1, \dots, A_m, B_1, \dots, B_N$  and  $C_1, \dots, C_N$ . All these coefficients may be subject to variation but it is assumed that only one coefficient at any time changes significantly from its nominal value, in the period required for measuring the change.

Note  $O_N(s)$  contains all the poles of the system which are assumed to be known. From equation 3.22, it is obvious that  $k_i = \ell_i$  for the condition  $\frac{\partial e(t)}{\partial k_i} = 0$ .

Alternatively from 3.23,

$$\begin{aligned}
 k_i &= \sum \text{Residues in right half } s\text{-plane of } G(s) O_i(-s) & (4.20) \\
 \therefore k_k &= \sum \text{Residues in right half } s\text{-plane of } G(s) \frac{\sqrt{2p_k}}{s-p_k} \cdot \frac{s+p_{k-1}}{s-p_{k-1}} \dots \frac{s+p_1}{s-p_1} & \dots \dots (4.21)
 \end{aligned}$$

$$\begin{aligned}
 \text{i.e. } k_k &= \frac{\sqrt{2p_k} A_k}{2p_k} & ) \\
 & & ) \\
 \text{or } k_k \sqrt{2p_k} &= A_k & ) & (4.22)
 \end{aligned}$$

$$\text{Also } k'_r = \text{Residues of } \left[ \frac{B_r s + C_r}{s^2 + s2\zeta_r \omega_{or} + \omega_{or}^2} \cdot \frac{2\sqrt{\zeta_r \omega_{or}} (-s)}{s^2 - s2\zeta_r \omega_{or} + \omega_{or}^2} \right] & (4.23)$$

$$\begin{aligned}
 \text{i.e. } k'_r &= \frac{B_r}{2\sqrt{\zeta_r \omega_{or}}} & ) \\
 & & ) \\
 \text{or } k'_r 2\sqrt{\zeta_r \omega_{or}} &= B_r & ) & (4.24)
 \end{aligned}$$

$$\text{and } k''_r = \text{Residues of } \left[ \frac{B_r s + C_r}{s^2 + s2\zeta_r \omega_{or} + \omega_{or}^2} \cdot \frac{2\sqrt{\zeta_r \omega_{or}} \cdot \omega_{or}}{s^2 - s2\zeta_r \omega_{or} + \omega_{or}^2} \right] & (4.25)$$

$$\begin{aligned}
 \text{i.e. } k''_r &= \frac{C_r}{2\omega_{or} \sqrt{\zeta_r \omega_{or}}} & ) \\
 & & ) \\
 \text{or } 2\sqrt{\zeta_r \omega_{or}} \cdot \omega_{or} k''_r &= C_r & ) & (4.26)
 \end{aligned}$$

The above analysis indicates that all numerator coefficients may be found provided that the poles of  $G(s)$  are identical to  $G^*(s)$ . Thus the search procedure for the model adaptation should be that the pole variation is first determined before an attempt is made to follow the changes in the numerator coefficients.

5. Simulation of the Dynamical Model

5.1. The Basic Arrangement

It is required to generate the orthonormal functions so far described.

Use is made of a computing or operational amplifier associated with passive linear RC two port networks. The use of one-port networks may be considered as a special case.

The general arrangement is shown in Fig. 5.1.

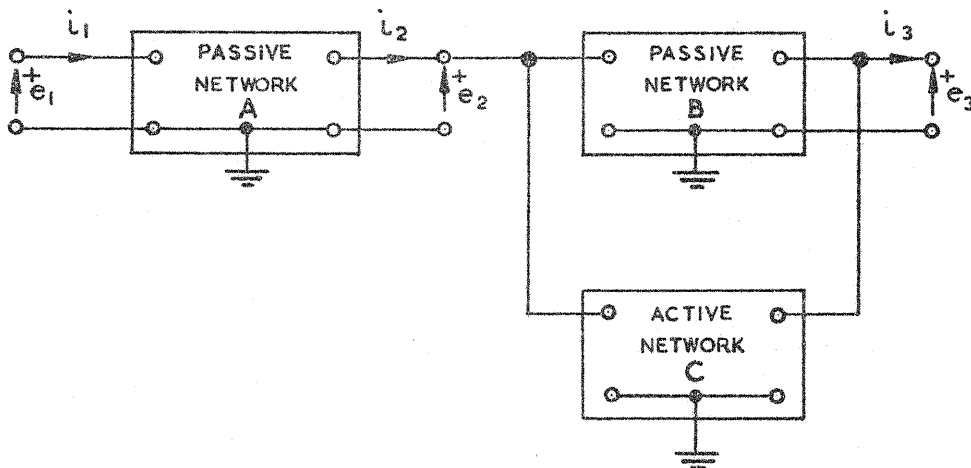


Fig. 5.1.

Network B is a two port network placed in parallel with the computing amplifier i. e. active network C.

It may be shown<sup>(6,7)</sup> that,

$$\frac{E_3(s)}{E_1(s)} = \frac{y_{21}^A(s)}{y_{12}^B(s)} \quad (5.1)$$

where

$$y_{21}^A(s) = \frac{I_2(s)}{E_1(s)} \quad \left. \begin{array}{l} \text{forward short circuit} \\ \text{transfer admittance of} \\ \text{network A} \end{array} \right|_{E_2(s)=0} \quad (5.2)$$

and

$$y_{12}^B(s) = \frac{I_2(s)}{E_3(s)} \quad \left. \begin{array}{l} \text{- reverse short circuit} \\ \text{transfer admittance of} \\ \text{network B} \end{array} \right|_{E_2(s)=0}$$

provided that the following assumptions are valid.

- (a) Networks A and B are initially relaxed (i. e. zero charge on all capacitors) .
- (b) The forward open circuit voltage transfer function of the amplifier  $-K G_1(s) \rightarrow -\infty$  over the frequency range of interest. (K is a positive constant and  $G_1(s)$  is a rational function of the complex frequency  $s$ ) .
- (c)  $y_{11}^C(s) = 0$ , i. e. amplifier input impedance is infinite .
- (d)  $y_{22}^C(s) = -\infty$ , i. e. amplifier output impedance is zero .
- (e)  $y_{12}^B(s) > \frac{y_{22}^A(s) - y_{11}^B(s)}{K}$  for the frequency range of interest .
- (f) If  $y_{22}^A(s)$  or  $y_{11}^B(s) \rightarrow \infty$  as  $s \rightarrow \infty$  that  $y_{12}^B(s)$  should approach infinity as  $s \rightarrow \infty$  .

$$(y_{22}^A(s) \equiv \text{short circuit admittance of output port of network A}) .$$

$$(y_{11}^B(s) \equiv \text{short circuit admittance of input port of network B}) .$$

## 5.2. Use of one-port networks

With networks A and B representing combinations of resistors and capacitors between input and output ports, the conventional analogue computing circuits are given.

i. e. if A represents a resistance R and B represents a capacitance C

$$\text{then } \frac{E_2(s)}{E_1(s)} = - \frac{1}{sCR} , \quad (5.3)$$

which is the transfer function of an integrator.

$$\text{Also } \frac{E_2(s)}{E_1(s)} = - \frac{R_2}{R_1} , \quad (5.4)$$

if network A represents a resistance  $R_1$  and B a resistance  $R_2$ .

Network A may also represent several one port networks to each of which is connected an input voltage.

For this case,

$$E_2(s) = - \sum_{i=1}^N E_i \frac{R_2}{R_1} . \quad (5.5)$$



5.3. Use of two-port networks

The parallel combination of standard T-networks to represent networks A and B results in transfer functions where coefficients are capable of independent adjustment<sup>(7)</sup>.

The forms of T networks together with their respective short circuit transfer admittance functions are given in Table 5.1.

TABLE 5.1.

NETWORK	$Y_{21}(s) \quad (= -Y_{12}(s))$
	$\frac{1}{2R} \cdot \frac{s^2 C^2 R^2}{1 + sCR}$
	$\frac{1}{2R} \cdot \frac{2sCR}{1 + sCR}$
	$\frac{1}{2R} \cdot \frac{1}{1 + sCR}$

Network A or network B may be arranged to comprise each of these networks multiplied by a scaling constant  $k$ .

Thus for example if

$$\begin{aligned}
 y_{21}^A(s) &= \frac{1}{2R} \cdot \frac{k_1 s^2 C^2 R^2 + k_2 2sCR + k_3}{(1 + sCR)} \quad , \\
 &= \frac{1}{2R} \cdot \frac{k_1 s^2 T^2 + k_2 2sT + k_3}{(1 + sT)} \quad ,
 \end{aligned}
 \tag{5.6}$$

and

$$y_{12}^B(s) = - \frac{1}{2R} \cdot \frac{k_4 s^2 T^2 + k_5 2sT + k_6}{(1 + sT)} \quad .
 \tag{5.7}$$

then 
$$\frac{E_3(s)}{E_1(s)} = - \frac{k_1 s^2 T^2 + k_2 2sT + k_3}{k_4 s^2 T^2 + k_5 2sT + k_6} \quad (5.8)$$

where  $T = RC$  and the  $k$ 's are  $\leq 1$  (5.9)

5.4. Simulation of the relevant orthonormal functions

(i) 
$$G^*(s) = \frac{\sqrt{2a}}{s+a} \quad (5.10)$$

where  $a$  is to be capable of adjustment and may be greater or less than unity .

If 
$$a = \frac{1}{kT} \quad (5.11)$$

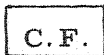
where  $T$  is a positive pre-determined integer and  $k$  is a variable coefficient  $\leq 1$ ,

then 
$$G^*(s) = \frac{\sqrt{2kT}}{1 + skT} \quad (5.12)$$

The active network arrangement <sup>(7)</sup> which satisfies equation 5.12 is shown in Fig. 5.2 in which



represents a computing amplifier



represents a cathode follower

and  $T = RC$  .

Note: The magnitude only of  $T$  is used in specifying resistor values  $\frac{R}{\sqrt{2T}}$  and  $0.5\sqrt{2T}$ , etc.

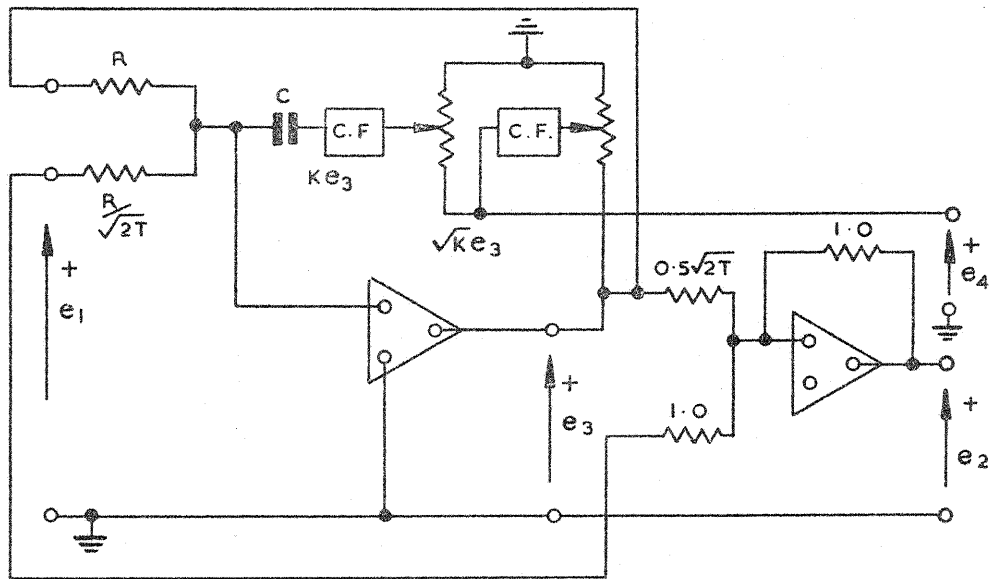


Fig. 5.2.



From Fig. 5.2.

$$\frac{E_4(s)}{E_1(s)} = - \frac{\sqrt{2kT}}{1 + skT} = - \frac{\sqrt{2a}}{s + a} \quad (5.13)$$

Also

$$\frac{E_2(s)}{E_1(s)} = - \left( 1 - \frac{2\sqrt{2T}}{2T(1 + skT)} \right) = - \frac{(s - a)}{(s + a)} \quad (5.14)$$

$$(ii) \quad G^*(s) = - 2\sqrt{\zeta}\omega_0 \left[ \frac{s}{s^2 + s2\zeta\omega_0 + \omega_0^2} + \frac{\omega_0}{s^2 + s2\zeta\omega_0 + \omega_0^2} \right] \quad (5.15)$$

where  $\omega_0$  may be greater or less than unity and  $\zeta$  is a positive coefficient which cannot exceed unity. It is required that  $\zeta$  may be capable of independent adjustment.

$$\text{If } \omega_0 = \frac{1}{T} \quad (5.16)$$

$$\text{then } G^*(s) = 2\sqrt{T} \left[ - \frac{\sqrt{\zeta} \cdot sT}{1 + s2\zeta T + s^2 T^2} - \frac{\sqrt{\zeta}}{1 + s2\zeta T + s^2 T^2} \right] \quad (5.17)$$

$$= 2\sqrt{T} [G_1(s) + G_2(s)] \quad (5.18)$$

where

$$G_1(s) = - \frac{\sqrt{\zeta} sT}{1 + s2\zeta T + s^2 T^2} \quad (5.19)$$

and

$$G_2(s) = - \frac{\sqrt{\zeta}}{1 + s2\zeta T + s^2 T^2}$$

The instrumentation of  $G_1(s)$  and  $G_2(s)$  is shown in Fig. 5.3.

$$\text{If } F(s) = - \frac{s^2 T^2 - s2\zeta T + 1}{s^2 T^2 + s2\zeta T + 1} \quad (5.20)$$

This may be expressed as

$$F(s) = - \left( 1 - \frac{s4\zeta T}{1 + s2\zeta T + s^2 T^2} \right) \quad (5.21)$$

$$= - (1 + 4\sqrt{\zeta} \cdot G_1(s)) \quad (5.22)$$

The instrumentation of equation 5.21 is also shown in Fig. 5.3. With reference to the latter,

$$T = RC \quad (5.23)$$

at output A

$$\sqrt{\zeta} E_2(s) = - \frac{\sqrt{\zeta} \cdot sT}{1 + s2\zeta T + s^2 T^2} \cdot E_1(s) \quad (5.24)$$

$$\text{i. e. } \frac{E_A(s)}{E(s)} = G_1(s) . \quad (5.25)$$

At output B,

$$E_B(s) = \sqrt{\zeta} E_4(s) = - \frac{\sqrt{\zeta}}{1 + s2\zeta T + s^2 T^2} E_1(s) , \quad (5.26)$$

$$\text{i. e. } \frac{E_B(s)}{E_1(s)} = G_2(s) . \quad (5.27)$$

At output C,

$$E_3(s) = - (4\zeta E_2(s) + E_1(s)) , \quad (5.28)$$

$$\begin{aligned} \text{i. e. } \frac{E_3(s)}{E_1(s)} &= - \left( 1 - \frac{4\zeta s T}{1 + s2\zeta T + s^2 T^2} \right) , \\ &= - \left( \frac{1 - s2\zeta T + s^2 T^2}{1 + s2\zeta T + s^2 T^2} \right) . \end{aligned} \quad (5.29)$$

If the voltages  $E_A(s)$  and  $E_B(s)$ , respectively, are multiplied by the coefficient  $2\sqrt{T}$  the required orthonormal functions are obtained. It should be noted that a four gang potentiometer is required to give the coefficients dependent on  $\zeta$ .

Alternative network arrangements may be derived in which independent adjustment of  $\omega_0$  is also available as described in Ref. 7.

### 5.5. The Dynamical Model

The dynamical model of the system may be constructed by connecting, in tandem, the networks described in 5.3.

One possible arrangement is shown in Fig. 5.4. With reference to Fig. 5.4., networks  $N_1 \dots N_m$  are of the form shown in Fig. 5.2 and networks  $N_{m+1} \dots N_N$  are of the form shown in Fig. 5.3.

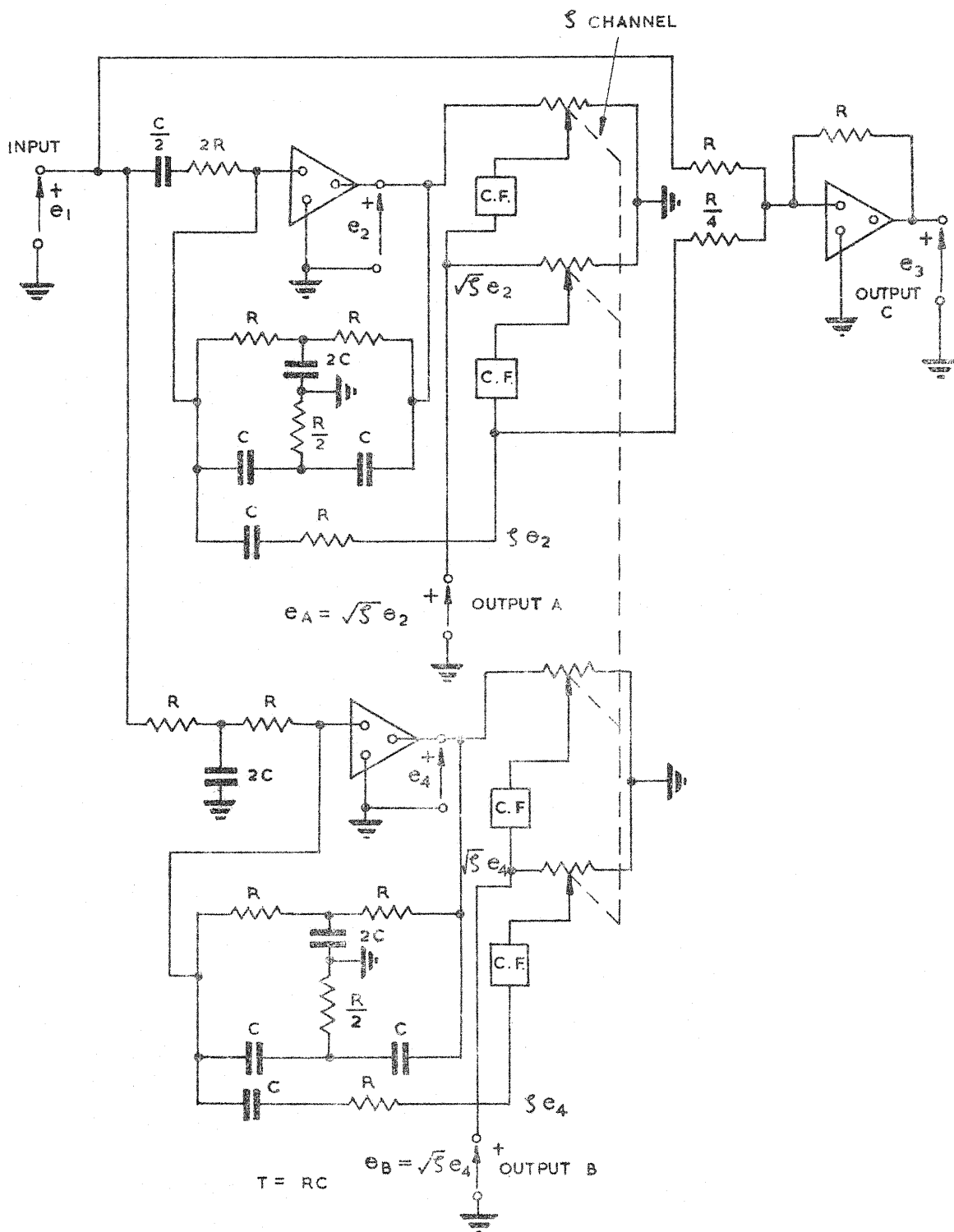


Fig. 5.3

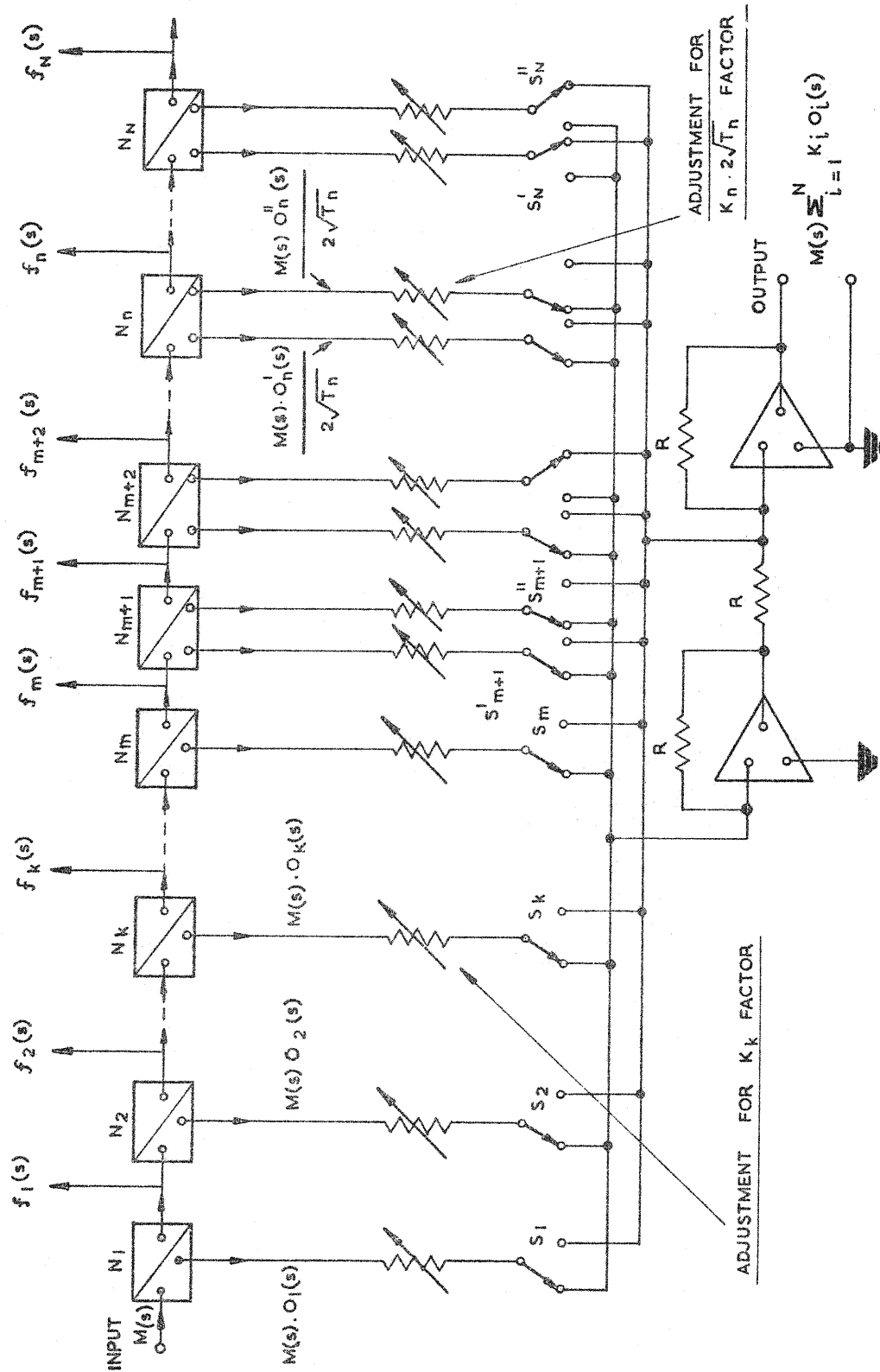


Fig. 5.4. DYNAMICAL MODEL

$$\begin{aligned}
 f_1(s) &= - \frac{s - p_1}{s + p_1} \cdot M(s) \\
 f_m(s) &= M(s) \prod_{k=1}^m - \frac{s - p_k}{s + p_k} \\
 f_{m+1}(s) &= - \frac{s^2 - 2\zeta_{m+1} \omega_{o_{m+1}} s + \omega_{o_{m+1}}^2}{s^2 + 2\zeta_{m+1} \omega_{o_{m+1}} s + \omega_{o_{m+1}}^2} \cdot f_m(s) \cdot M(s) \\
 f_N(s) &= M(s) \cdot f_m(s) \cdot \prod_{r=m+1}^N - \frac{s^2 - 2\zeta_r \omega_{o_r} s + \omega_{o_r}^2}{s^2 + 2\zeta_r \omega_{o_r} s + \omega_{o_r}^2} \\
 O_1(s) &= - \frac{\sqrt{2p_1}}{s + p_1} \cdot M(s) \\
 O_m(s) &= - \frac{\sqrt{2p_m}}{s + p_m} \cdot f_{m-1}(s) \cdot M(s) \\
 O'_N(s) &= M(s) \cdot \left( - \frac{2\sqrt{\zeta_N T_N} \cdot s T_N}{1 + s 2\zeta_N T_N + s^2 T_N^2} \right) \cdot f_{N-1}(s) \\
 &= M(s) \cdot \left( - \frac{2\sqrt{\zeta_N \omega_{o_N}} s}{s^2 + s 2\zeta_N \omega_{o_N} + \omega_{o_N}^2} \right) \cdot f_{N-1}(s) \\
 O''_N(s) &= M(s) \cdot \left( - \frac{2\sqrt{\zeta_N \omega_{o_N}} \cdot \omega_{o_N}}{s^2 + s 2\zeta_N \omega_{o_N} + \omega_{o_N}^2} \right) \cdot f_{N-1}(s)
 \end{aligned} \tag{5.30}$$

The transmission transfer function of the model is given by

$$G^*(s) = \sum_{i=1}^N K_i O_i(s) \tag{5.31}$$

where  $K_i$  is a scaling coefficient which may be positive or negative. In Fig. 5.4. this scaling is achieved by means of the variable resistors and in the case of networks  $N_{m+1} \dots N_N$  this scaling includes the factor  $2\sqrt{T_i}$ . Switches  $S_1 \dots S_N$  connect the outputs of the networks to the input of amplifier 1 or amplifier 2 to take account of the sign of  $K_i$  required.

With such a model each switch and the associated variable resistor would be operated sequentially to give a minimum value of  $\overline{e(t)^2}$ . Provision would be made for one servo-drive to operate all sections.

## 6. Practical Considerations

### 6.1. On-Line Operation

For on-line operation the adaptation of the model, as already described, is achieved by injecting the low level test signal into the input of the system in addition to the actuating signal. The system output is then correlated with the test signal to give the desired signal for comparison with the output of the model.

The test signal in this application has a flat power density spectrum over a finite bandwidth, which with the latter, say, ten times that of the system, is suggested as a reasonable approximation to white noise.

The requirement for the cross-correlation, however, is a distinct disadvantage since it involves the instrumentation of a time delay, multiplier and integrator. Moreover, the assumption is implied that the process is ergodic which is not strictly true with slowly-varying parameters. A compromise must also be made in choosing a finite time of integration instead of an infinite period, and this results in functions of the actuating signals, which in this case are noise components, to appear at the output of the correlator. To achieve a reasonable signal to noise ratio the integration period must be maintained at some value  $T$ , with the result that changes in the system parameters can only be detected after a finite number of periods  $T$ . This is due to the fact that several coefficients may have to be adjusted and each adjustment occupies a number of periods  $T$ . Again, in order to obtain sufficient information regarding the system response, several correlations having different delays are required.

These considerations suggest the need for sampling and quantizing the input and output signals. In this way the correlator can be considerably simplified<sup>(8, 9, 10)</sup>. Alternatively, in some applications, the actuating signal itself may be used as the test signal. However, as discussed in section 3, this means that the functions forming the model are no longer orthogonal and the adaptation of the model occupies a much longer time. Moreover, unless the signal approximates to white noise, there is no guarantee that the model, which gives a minimum error function, is a good approximation to the system.

Another factor is, that in a system having a large number of poles, there will be difficulty in detecting a minimum if say a low-pass filter is used as the averaging device. This is due to the fact that the output of the filter will be subject to random variation due to inadequate filtering.

The above considerations are severe restrictions on the use of the methods described in this paper. However, it appears that many of these restrictions are imposed on any method of model adaptation based on the use of test signals.

### 6.2. The effect of system non-linearities

Another disadvantage of any method based on test input signals is the fact that their presence may result in the system being operated in a non-linear mode over certain periods. For the case of simple non-linearities, such as well defined saturation effects, a possible cure might be the incorporation of identical non-linearities in the model. The success of this technique would depend on the



proportion of time the system was operating linearly to the time of non-linear operation. This technique has not been investigated by the author.

### 6.3. The injection of test signals at several points in the system

If access is available to several points in the system, it would seem to be desirable to construct models for parts of the system by using several test signals and correlating the available output with the respective input signal in each case.

In this way, each model is simplified and the time of adaptation is reduced as all the models may be adapted simultaneously. The disadvantage of this approach is the duplication of test equipment such as correlators. However, it may even be advantageous to adapt each component model sequentially with the same test equipment, if the time variation of the system parameter is sufficiently slow compared with the measurement time.

## 7. Conclusions

The synthesis of a model based on the use of orthonormal functions seems to have some merit.

The effect of non-linearities and inaccurate measurement techniques, however, warrants closer investigation to ensure that a reasonable approximation to system performance is achieved.

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