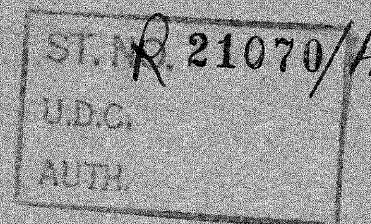
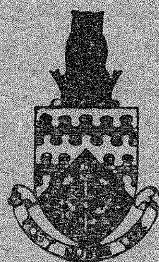


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A NONLINEAR TRANSPORTATION MODEL

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A Nonlinear Transportation Model :
A Step-by-step Solution

- by -

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SUMMARY

The model has linear restraints and a functional, to be maximised, consisting of linear first-order interactions of variables. General applications of the model are stated. It is shown that a direct solution is not generally possible, but iterative procedures, analogous to the 'Transportation' technique are developed both for integer and non-integer non-negative variables. A Tableau procedure is given, with suitable checks, to enable a solution to be systematically obtained, and finally a numerical example is treated.

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1. The Model

The model considered in this paper is :

Solve:

$$\left. \begin{aligned} \sum_{\alpha=1}^N x_{\alpha a} &= A_{oa} \quad (a = 1, 2, \dots, M) \\ \sum_{a=1}^M x_{\alpha a} &= A_{\alpha o} \quad (\alpha = 1, 2, \dots, N) \\ \sum_a A_{oa} &= \sum_{\alpha} A_{\alpha o} \end{aligned} \right\} \quad (1)$$

so as to maximise :

$$\phi = \sum_a \sum_{\beta=2}^N \sum_{\alpha=1}^{N-1} w_{\alpha\beta} x_{\alpha a} x_{\beta a} \quad (\alpha < \beta), (w_{\alpha\beta} \geq 0) \quad (2)$$

subject to:

$$x_{\alpha a} \geq 0 \quad (3)$$

where:

$x_{\alpha a}$ is the amount transported from source or origin α to destination a .

$A_{\alpha o}$ and A_{oa} are the amounts available and required at α and a respectively.

$w_{\alpha\beta}$ is the weight associated with transporting $x_{\alpha a}$ and $x_{\beta a}$ to destination a .

There are NM routes but, unlike a Linear Programme Transportation model, the contribution to the functional ϕ from transporting an amount along one is not independent of the other amounts having the same destination. The contribution from transporting all the amounts that have destination a is :

$$\sum_{\beta=2}^N \sum_{\alpha=1}^{N-1} w_{\alpha\beta} x_{\alpha a} x_{\beta a} \quad (\alpha < \beta) \quad (4)$$

This has $N C_2$ terms and as a result the functional has $M C_2$ terms. The weights however have been made independent of the destinations so that there are only $N C_2$ values of these. The relaxation of this assumption would lengthen the procedure of solution without changing the fundamental nature of the approach. In view of this the briefer model only is considered here.

Unlike a Linear Programme model the one considered here operates with increasing returns to scale, that is, the higher the level of operation the greater the marginal rate of return. The weights should not therefore be independent of the unit of time employed in an application of the model to a continuous system.

2. Applications

As a pure transportation model, provided it is sufficiently realistic, the functional could be in units of revenue or profit or whatever is required to be maximised.

Transportation models can be applied as has been generally recognised, in situations which are not purely transportation ones. The model of this paper could be used for the purpose of allocating service channels to different locations where it is required to minimise the amount of movement of 'customers' between locations. The assumptions of such an application are :

- (1) There are $A_{\alpha 0}$ service channels of type α . ($\alpha = 1, 2, \dots, N$)
- (2) There are A_{0a} places available for channels in location a .
($a = 1, 2, \dots, M$)
- (3) A channel serves one 'customer' at a time and 'customers' require service from two or more different types of channel.
- (4) A demand for the service of a channel of type α will be equally likely to be met by any one of $A_{\alpha 0}$.
- (5) The weight $w_{\alpha\beta}$ is proportional to the number of movements between a channel of type α and one of type β .

3. Requirements of an Optimum Solution

An optimum solution should first of all satisfy requirements (1) and (3) so as to be feasible. In order to satisfy the requirement (2) it is necessary that an interchange or transfer of any amount between routes should not decrease the value of the functional.

Let $[\beta \gamma a b]$ represent the operation:

Deduct an increment Δ from each of $x_{\beta a}$ and $x_{\gamma b}$ and add to each of $x_{\beta b}$ and $x_{\gamma a}$.

We would have from any solution satisfying (1) a further solution that satisfies it.

Assuming that $w_{aa} = 0$ for all a and $w_{a\beta} = w_{\beta a}$ for all $\beta < a$, we have that the increase in the contribution of destination a to the functional as a result of $[\beta \gamma a b]$ is :

$$\sum_a (w_{\gamma a} - w_{\beta a}) x_{a a} \Delta - \Delta^2 w_{\beta \gamma} \quad (5)$$

and simultaneously the increase in the contribution of destination b is :

$$\sum_a (w_{\beta a} - w_{\gamma a}) x_{a b} \Delta - \Delta^2 w_{\beta \gamma} \quad (6)$$

Summing these we have that the total increase in the value of the functional as a result of $[\beta \gamma a b]$ is :

$$\left\{ \sum_a (w_{\gamma a} - w_{\beta a}) (x_{a a} - x_{a b}) \right\} \Delta - 2 \Delta^2 w_{\beta \gamma} \quad (7a)$$

This will be stated more briefly as :

$$\left\{ \beta \gamma a b \right\} \Delta - 2 w_{\beta \gamma} \Delta^2 \quad (7b)$$

The function (7) indicates that the value $2 w_{\beta \gamma}$, which will be termed 'the factor', is independent of the destinations involved in a transfer and also of the direction of the transfer itself.

The reverse transfer itself, i.e. $[\gamma \beta a b]$ yields an increase to the functional of :

$$-\left\{ \beta \gamma a b \right\} \Delta - 2 w_{\beta \gamma} \Delta^2 \quad (8)$$

and indicates that the calculation of $\left\{ \beta \gamma a b \right\}$ serves for evaluating the two transfers $[\beta \gamma a b]$ and $[\gamma \beta a b]$, two that differ only in the direction of the transfer.

We have, too, that :

$$\left\{ \beta \gamma a b \right\} = \left\{ \delta \gamma a b \right\} - \left\{ \delta \beta a b \right\} \quad (9)$$

and that :

$$\left\{ \beta \gamma a b \right\} = \left\{ \beta \gamma a c \right\} - \left\{ \beta \gamma b c \right\} \quad (10)$$

Thus, though there are $2^{N_C} 2^{M_C}$ transfer patterns for any solution, given Δ , the evaluation of them is dependent on only $(N-1)(M-1)$ independent values of $\left\{ \beta \gamma a b \right\}$ for variations in β , γ , a and b .

An optimum solution requires generally that :

$$\left| \left\{ \beta \gamma a b \right\} \right| \leq 2 w_{\beta\gamma} \Delta \quad (11)$$

If we let $\Delta \rightarrow 0$ then we could obtain a direct solution from any $(N-1)(M-1)$ independent series of the general equations :

$$\left\{ \beta \gamma a b \right\} = 0 \quad (12)$$

and the further $M + N - 1$ independent equation that would be provided by (1). However such a solution would not generally satisfy (3) and might require, in addition, integer values for the variables. A direct algebraic solution is not accordingly a general possibility and an iterative procedure has to be employed.

4. Step-by-step Procedure

The solution of the problem by this means requires one to start with a feasible solution and by stages to improve upon it. Each step requires the general calculation of $\left\{ \beta \gamma a b \right\}$ and this enables one to modify a solution and at the same time obtain an improved value of the functional.

Since the functional is convex the solution will not generally be at an extreme point of the solutions space as in a Linear Programme model. This implies that more than $M + N - 1$ of the variables could be non-zero. Further, in view of the interaction of the variables in the functional, it is necessary to evaluate for a transfer routes in use as well as those not in use to determine whether a particular feasible solution is optimum or not.

However, a route not in use cannot have an amount transferred from it if the result is to be feasible. Thus, if at any stage, $x_{\beta a} = 0$, then

positive values of $\{\beta \gamma a b\}$ or $\{\gamma \beta b a\}$ need not be considered, and in like manner negative values of $\{\gamma \beta a b\}$ or $\{\beta \gamma b a\}$, for variations in γ and b . Thus there is the possibility that a number of calculations need not be considered even if they were positive or negative. They will not therefore require to be undertaken. Others might indicate without completion that they would have a sign which would prevent them being considered. One needs however to consider the fact that only $(N-1)(M-1)$ full calculations need to be undertaken the remaining ones being derived simply from these.

5. Integer Solution

Here we require a feasible solution that generally satisfies :

$$\left| \left\{ \beta \gamma a b \right\} \right| \leq 2 w_{\beta\gamma} \quad (13)$$

Having a feasible solution the test (13) can be applied to discover if an improvement can be made. If any evaluation does not satisfy (13) then an unit transfer can be made provided it is feasible, the direction depending on the sign of $\{\beta \gamma a b\}$. More than one interchange can be made simultaneously if they do not involve common destinations.

Further, if it is feasible, it may be beneficial to carry out more than a unit transfer in a particular situation. If λ units are transferred then the functional will be increased by :

$$\left| \left\{ \beta \gamma a b \right\} \right| \lambda - 2 \lambda^2 w_{\beta\gamma} \quad (14)$$

for the particular values of β, γ, a and b . This has a maximum with respect to λ at :

$$\lambda = \left| \left\{ \beta \gamma a b \right\} \right| / 2 (2 w_{\beta\gamma}) \quad (15)$$

and yields an increase in the value of the functional of :

$$\left\{ \beta \gamma a b \right\}^2 / 4 (2 w_{\beta\gamma}) \quad (16)$$

Since however λ has to be an integer value it may be necessary to choose $\lambda = \lambda_1$, or $\lambda_1 + 1$, the nearest integer values respectively less than or greater than the value obtained by (15) according as :

$$\left| \left\{ \beta \gamma a b \right\} \right| - 2 w_{\beta\gamma} (1 + 2 \lambda_1) \leq 0 \quad (17)$$

If this is so, then (14) and not (16) will give the increase in the value of the functional.

6. Non-Integer Solution

Here we require a feasible solution that generally satisfies :

$$\left| \left\{ \beta \gamma a b \right\} \right| \leq 2 w_{\beta\gamma} \Delta, \quad (\Delta < 1) \quad (18)$$

Two procedures can be adopted here :

(i) Replace A_{oa} , A_{ao} , x_{aa} and ϕ by A'_{oa} , A'_{ao} , x'_{aa} and ϕ' respectively

where :

$$A'_{oa} = A_{oa} / \Delta, \quad A'_{ao} = A_{ao} / \Delta$$

$$x'_{aa} = x_{aa} / \Delta, \quad \phi' = \phi / \Delta^2$$

and solve as an integer problem. It is assumed that A'_{oa} and A'_{ao} are integer values. The variables and the functional of the solution can then be transformed to satisfy the original problem. Such a procedure assumes generally that the unit load is divisible into a number of parts given by the denominator of the fractional value Δ .

(ii) The amounts transferred are reduced at each step until a solution is reached where a further step would involve an incremental transfer less than the value of Δ .

A non-zero value of any $\left\{ \beta \gamma a b \right\}$ of a feasible solution implies that a transfer, provided it is itself feasible, can be beneficially made. There may be a number of these and one would choose the largest of the values :

$$\Delta_1 = \left| \left\{ \beta \gamma a b \right\} \right| / 2 (2 w_{\beta\gamma}) \quad (19)$$

At the next stage, when this transfer has been completed the value of that particular $\left\{ \beta \gamma a b \right\}$ would be zero. The increase in the value of the functional as a result would be :

$$\left\{ \beta \gamma a b \right\}^2 / 4 (2 w_{\beta\gamma}) \quad (20)$$

Starting with an integer solution, the optimum solution would require generally that the unit load is divisible into a number of parts given by

the L.C.M. of the denominators of the actual values of Δ_1 employed.

At each stage of the second procedure it may be possible to carry out more than one transfer simultaneously. Provided they are feasible and do not involve common destinations this will be in order. This could give rise to more than one value of Δ_1 at a particular step. However it is still the largest value that determines the stage of adjustment that has been reached but the assumed divisibility of the unit load will generally be increased.

In fact, the second method could begin with any non-integer solution that was feasible and this could itself prove to be optimum. It is the difference between the variables that the second method is concerned with rather than the divisibility of the variable unit so that a number of different optimum solutions are possible by it.

It will be observed that in using the second method the value of Δ_1 need not be less than unity at the commencement. The final value of Δ_1 employed could be determined by such as the rate of improvement of the functional. It is not therefore necessary to stipulate the value of Δ at the start.

7. Technique of Solution

The procedure of solution is indicated in the Tables A to E, pages 11, 12. The values of $w_{\alpha\beta}$ are contained in Table A and the remaining original information represented by $A_{\alpha\alpha}$ and $A_{\alpha\alpha}$ is contained in Table C. Table B is derived from Table A and Table D is derived from the feasible solution contained in Table C. Table E is derived from Tables B and D.

Tables A and B remain unaltered for each feasible solution but Tables C, D and E will be subject to alteration though this will not necessarily be so for each particular cell value.

Since only $(N-1)(M-1)$ independent values of $\{\beta \gamma a b\}$ are required in the first place we can conveniently keep β and a at one particular value and we shall use $\beta = a = 1$. The independent values of $\{1 \gamma 1 b\}$ for variations in γ and b are contained in the top left hand corner of E within the broken lines. The remaining values of Table E are derived by employing relationships (9) and (10). The factor values $2 w_{\beta\gamma}$ are placed at the top of each column of Table E for the purposes of comparison according to formula (13) or (18).

The Tables are labelled at the head of the rows and columns with the appropriate subscript values. This is to assist correct procedure. Checks however can be made on the calculations themselves. The check on the values of Table C is obvious if a solution is to be feasible. In fact, we have checks for the columns and rows in this case. There is an overall check on the values of Table B making use of Table A. We have :

$$\sum_{\alpha=1}^N \sum_{\gamma=2}^N (w_{\gamma\alpha} - w_{1\alpha}) = 2 \left(\sum_{\alpha=2}^{N-1} \sum_{\beta=3}^N w_{\alpha\beta} - \sum_{\beta=2}^N w_{1\beta} \right) \dots (21)$$

As a check on Table D we make use of Table C :

$$\sum_{\alpha=1}^N (x_{\alpha 1} - x_{1\alpha}) = A_{01} - A_{0b} \quad (22)$$

We have in fact a check on each column of D involving the sums of the values of each column. They are included at the base of Table D. Though Table D changes the check values remain constant.

For Table E we have that the bottom right hand corner can be obtained either from the rows of the bottom left hand corner or from the columns of the top right hand corner. This will not ensure that the independent values have been correctly obtained and if this is required then it could be accomplished by extending Table D to include variations in a.

8. A Numerical Example

Consider the following quantitative values for the model :

$A_{\alpha 0} = 40, 36, 36, 12$: $\alpha = 1, 2, 3, 4$ respectively

$A_{0\alpha} = 8, 28, 88$: $\alpha = 1, 2, 3$ respectively

Table A :

α	β		
	2	3	4
1	10	7	1
2		3	4
3			6

From Table A we obtain Table B :

α	β		
	2	3	4
1	10	7	1
2	-10	-7	-6
3	-4	-7	-1
4	3	5	-1

Using (21) as a means of checking Table B, we have :

$$\text{Total of Table B} = 2 (3 + 4 + 6) - (10 + 7 + 1) = -10$$

The ensuing procedure is shown in the Tableaux on page 13. Table C of Tableau 1 contains a feasible integer solution which has a functional with a value of 20208. From this Table we obtain Table D and then using Table B we obtain Table E. Because certain routes of the solution are empty the only calculation of the general function $\{\beta \gamma a b\}$ that need have been attempted are :

$$\begin{array}{ccccc} \{1313\} & \{1423\} & \{2313\} & \{2423\} & \{3412\} \\ \{1323\} & & \{2323\} & & \{3423\} \end{array}$$

and only $\{3412\}$ can be considered if it is positive. It occurs that only this one is in fact positive and employing (15) we have that $\lambda = (24/2(12)) = 1$. Thus an unit transfer would be beneficial in this case and by (16) it would increase the value of the functional by 12.

The improved integer solution is shown in Tableau II and on testing it proves to be the optimum integer solution.

Suppose we wish to proceed to a non-integer solution. Tableau II indicates that only $\{3123\}$ is both feasible and beneficial. The value of Δ_1 is given by (19) and we have $\Delta_1 = (8/2(14)) = 2/7$. Using (20) we have that this transfer would increase the value of the functional by $1 \frac{1}{7}$. The new solution is given in Tableau III. An improvement to this would be given by only $\{3412\}$, one that has featured previously, and the value of $\Delta_1 = (24/(7)(2)(12)) = 1/7$. The increased in the value of the functional as a result of this transfer is $12/49$. The new solution has not however been recorded in a new Tableau.

It will be observed that A_{ao} and A_{oa} have an H.C.F. of 4. If these values were accordingly reduced to one quarter of their original values we would have values of $\Delta_1 = \frac{1}{4}$, $\frac{1}{14}$ and $\frac{1}{28}$ for the subsequent steps of the iterative procedure. Tableau II would then become a non-integer solution (the values reduced to 25% of those in Table C) according to procedure (i) and with $\Delta = \frac{1}{4}$.

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TABLES

- A Values of $w_{\alpha\beta} \equiv \alpha\beta$ (given)
- B Values of $(w_{\gamma\alpha} - w_{\beta\alpha}) \equiv \alpha\beta\gamma$ for $\beta = 1$
- C Values of $A_{\alpha\alpha}$ and $A_{\alpha\alpha}$ (given) and values of $x_{\alpha\alpha} \equiv \alpha\alpha$
- D Values of $(x_{\alpha\alpha} - x_{\alpha\beta}) \equiv \alpha\alpha\beta$ for $\alpha = 1$
- E Values of $\left\{ \begin{matrix} \beta & \gamma & \alpha & \beta \end{matrix} \right\}$

		A	
α	β	2	3
1		12	13
2			23
N-1			
			N
			1N
			2N
			(N-1)N

		B	
α	γ	2	3
1		12	13
2		-12	213
3		312	-13
N		N12	N13
			-1N
			1N
			21N
			31N

		C	
α	α	1	2
1		11	12
N		N1	N2
			M
			1M
			NM
			$A_{\alpha\alpha}$
			$A_{\alpha\alpha}$
			$A_{\alpha\alpha}$
			$A_{\alpha\alpha}$

		D	
α	β	2	3
1		112	113
2		212	213
3		312	313
N		N12	N13
			M
			11M
			21M
			31M
			N1M
			$A_{\alpha\alpha} - A_{\alpha\alpha}$
			$A_{\alpha\alpha} - A_{\alpha\alpha}$
			$A_{\alpha\alpha} - A_{\alpha\alpha}$

Tables (Continued)

$\beta\gamma$		E							
		$2(w_{\beta\gamma})$		$1N$		$2N$		$(N-1)N$	
ab	12	13	$2(12)$	$2(13)$	$2(1N)$	$2(23)$	$2(2N)$	$2(N-1)N$	$(N-1)N12$
12	1212	1312	$1N12$	2312	$2N12$	$(N-1)N12$	$(N-1)N1M$	$(N-1)N23$	$(N-1)N23$
$1M$	$121M$	$131M$	$1N1M$	$231M$	$2N1M$	$(N-1)N1M$	$(N-1)N23$	$(N-1)N23$	$(N-1)N23$
23	1223	1323	$1N23$	2323	$2N23$	$(N-1)N23$	$(N-1)N23$	$(N-1)N23$	$(N-1)N23$
$(M-1)M$	$12(M-1)M$	$13(M-1)M$	$1N(M-1)M$	$23(M-1)M$	$2N(M-1)M$	$(N-1)N(M-1)M$	$(N-1)N(M-1)M$	$(N-1)N(M-1)M$	$(N-1)N(M-1)M$

TABLEAUX

C

D

E

I

$2(w)$
 ρ_Y

a	1	2	3	
1			40	40
2			36	36
3	8	16	12	36
4		12		12
	8	28	88	

b	2	3	
1	0	-40	
2	0	-36	
3	-8	-4	
4	-12	0	
	-20	-80	

	12	13	14	23	24	34
12	20	14	2	6	8	12
13	-4	-4	20	0	24	24
14	-24	0	180	24	204	180
23	-20	4	160	24	180	156

II

	1	2	3
1			40
2			36
3	7	17	12
4	1	11	

	2	3
1		-40
2		-36
3	-10	-5
4	-10	1

	12	13	14	23	24	34
12	10	20	20	10	10	0
13	-17	12	180	29	197	168
14	-27	-8	160	19	187	168

III

	1	2	3
1		$\frac{2}{7}$	$39\frac{5}{7}$
2			36
3	7	$16\frac{5}{7}$	$12\frac{2}{7}$
4	1	11	

	2	3
1	$-\frac{2}{7}$	$-39\frac{5}{7}$
2		-36
3	$-9\frac{5}{7}$	$-5\frac{2}{7}$
4	-10	1

	12	13	14	23	24	34
12	6	16	$19\frac{3}{7}$	10	$13\frac{3}{7}$	$3\frac{3}{7}$
13	-13	16	$180\frac{4}{7}$	29	$193\frac{4}{7}$	$164\frac{4}{7}$
14	-19	0	$161\frac{1}{7}$	19	$180\frac{1}{7}$	$161\frac{1}{7}$