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THERMAL STRESSES IN A BOX STRUCTURE

by

B. M. LEMPRIERE

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Thermal Stresses In A Box Structure

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B. M. Lempriere, B.Sc.(Eng.), M.Ae.E.

SUMMARY

This interim note presents an analysis, using an energy method, of the thermal stresses in a finite length box structure resulting from uniform skin heating. The solution depends upon an eighth order differential equation with constant coefficients. Numerical solutions are given for comparison with existing and projected experiments.

SYMBOLS

A, C,	Constants depending on temperature distribution and material properties.
a	Half width of I-section (= half web spacing)
d	Half depth of I-section (= half box depth)
E, G.	Moduli of Elasticity
f	$9F/2\Lambda$
F_1, F_2, $F_3(=F)$	Stress Functions in axial variable
l	Length of box
T, $\bar{T}$	Local and weighted average temperatures
r	Ratio of width to depth (= a/d)
t_s, t_w	Skin and web thickness
u	Non dimensional axial co-ordinate (= x/d)
v	Non dimensional transverse co-ordinate (= y/d)
w	Non dimensional lateral co-ordinate (= z/a)
x	Axial co-ordinate
y	Transverse co-ordinate see fig. 2
z	Lateral co-ordinate
α	Coefficient of linear expansion
ν	Poissons Ratio
σ_x, σ_y	Direct stress
τ	Shear stress
Ψ	Ratio of skin area to web area ($= 2at_s/dt_w$)

Introduction.

The analysis of thermal stresses due to kinetic heating in multi-web box wing structures was first discussed by Hoff,[1] who considered a structure of infinite length. This reduced the problem to a uniaxial system, and thus led to an extremely simple solution.

Further unpublished work has been done by Calkin[2] who has generalised the infinite solution for unsymmetric cases, etc., and who also attempted an experimental investigation. This however showed no agreement with the analysis, the differences being put down to the simplicity of the theory, mainly in neglecting end effects in a short box.

The present analysis accounts for finite length of box, and could be extended to practical cases of wings with root fixings, etc. Further experimental work is under way, being directed at establishing means of measuring thermal stresses.

Analysis.

An element of the box beam of Fig. 1 is idealised to one I section.

With uniform skin temperatures, the temperature distribution at any cross-section (Fig. 2) after a short time may be approximated by :-

$$T_{skin} = T_{max}$$

$$T_{web} = T_{min}\left[1 - (y/d)^2\right] + T_{max}(y/d)^2 \qquad \ldots\ldots 1$$

The temperature is constant axially so that there is only axial stress due to differential expansion. The total end load to completely suppress expansion is

$$P = -\int E\alpha T dA$$

so that, assuming the properties are constant, an overall 'average' temperature which would require the same restraint load, may be defined as

$$\overline{T} = \frac{1}{A_{skin} + A_{web}}\left[T_{skin}A_{skin} + \int T_{web}dA\right] \qquad \ldots\ldots 2$$

A thermal stress system due to the above temperature distribution, and having no resultant load is thus

$$\sigma_x = - E\alpha(T_{max} - T_{min})\left[(y/d)^2 - \frac{(1 + 3\Psi)}{3(1 + \Psi)}\right] \qquad .....3$$

$$\sigma_y = \tau = o$$

where Ψ is ratio of skin area to web area ($= 2\, t_s a/t_w d$).

There is then a constant stress in the skin and a parabolic stress distribution in the web. For a finite box with free ends a correction system must be superimposed on this having equal and opposite edge stresses, i.e.

$$x = o,\ l : \sigma_x = E\alpha(T_{max} - T_{min})\left[(y/d)^2 - \frac{(1 + 3\Psi)}{3(1 + \Psi)}\right]$$

$$\tau = o$$

Considering the web and skins separately as plane stress systems and treating only half the skin as there will be a shear discontinuity at the junction, the boundary conditions for all the edges of the two elements may be written as :

Web: $x = o,\ l : \sigma_x = A\left[v^2 - \frac{(1 + 3\Psi)}{3(1 + \Psi)}\right]$

$\tau = o$

$v = \pm 1 : \sigma_y = o$

$\tau = \tau_w(\pm d)$

Skin : $x = o,\ l : \sigma_x = 2A / 3(1 + \Psi)$

$\tau = o$

$w = o : \tau = \tau_s(o)$

$w = 1 : \sigma_z = o$

$\tau = o \qquad .....4$

where (i) the transverse co-ordinate y is replaced by $v = y/d$

(ii) the lateral " z " " " $w = z/a$

(iii) the web axial stress is parabolic, with $A = E\alpha\,(T_{max} - T_{min})$

(iv) the skin is assumed to have no bending rigidity so that it cannot resist normal loads from the web.

(v) the shear stresses at the junction are to be considered later.

(vi) the box beam is considered to be symmetric so that there is no shear between adjacent I sections.

(vii) the lateral skin stress is taken to be zero at the edge of the I-section (a minimum is implied in vi).

Note: The two halves of the skin element are identical so the lateral stress is continuous at the web line and a final boundary condition on this stress is the zero value at the outer edge of the opposite half.

The simplest axial stress system which satisfies the boundary conditions 4 and varies axially is

$$\text{web} : \sigma_x = F_1(x) + v^2 F_2(x)$$

$$\text{skin} : \sigma_x = F_3(x) \qquad \ldots\ldots 5$$

The equilibrium equations for plane stress with no body forces (the thermal 'body forces' are not to be considered in this correction stress system) are satisfied by taking the other components of stress as

$$\text{web} : \tau = -vdF_1' - \frac{1}{3}v^3 dF_2' - dF_4'$$

$$\sigma_y = \frac{1}{2}v^2 d^2 F_1'' + \frac{1}{12} v^4 d^2 F_2'' + vd^2 F_4'' + d^2 F_5''$$

$$\text{skin} : \tau = -waF_3' - aF_6'$$

$$\sigma_z = \frac{1}{2}w^2 a^2 F_3'' + wa^2 F_6'' + a^2 F_7'' \qquad \ldots\ldots 6$$

where dashes denote differentiation with respect to x, and where the functions of integration F_4 to F_7 are in x, and are chosen in a convenient form.

Satisfying all the boundary conditions 4 (except the shears at the junction) requires

$$F_1(o) = F_1(l) = -\frac{A(1+3\Psi)}{3(1+\Psi)}\ ; \quad F_1'(o) = F_1'(l) = o$$

$$F_2(o) = F_2(l) = A\ ; \quad F_2'(o) = F_2'(l) = o$$

$$F_3(o) = F_3(l) = \frac{2A}{3(1+\Psi)}\ ; \quad F_3'(o) = F_3'(l) = 0$$

$$F_4'(o) = F_4'(l) = o\ ; \quad F_4'' = o \quad \therefore \quad F_4' = \text{const.} = o$$

$$F_5'' = \frac{1}{2}F'' - \frac{1}{12}F''$$

$$F_6'(o) = F_6'(l) = o\ ; \quad F_6' = -F_3'$$

$$F_7'' = -\frac{1}{2}F_3'' - F_6'' = \frac{1}{2}F_3'' \qquad \ldots.7$$

The stresses now become :

$$\text{web} \quad : \quad \sigma_x = F_1 + v^2F_2$$

$$\tau = -vdF_1' - \frac{1}{3}v^3dF_2'$$

$$\sigma_y = -\frac{d^2}{2}(1-v^2)F_1'' - \frac{d^2}{12}(1-v^4)F_2''$$

$$\text{skin} \quad : \quad \sigma_x = F_3$$

$$\tau = a(1-w)F_3'$$

$$\sigma_z = \frac{a^2}{2}(1-w)^2F_3'' \qquad \ldots.8$$

As there are three unknown functions, two conditions relating these may be imposed at the junction. These are equilibrium of shear and compatibility of axial strain. Noting that there are two halves of skin, the first of these conditions requires that

$$t_s aF_3' = -\frac{1}{2}t_w d(F_1' + \frac{1}{3}F_2')$$

This becomes on integration

$$F_1 + \frac{1}{3}F_2 = -\Psi F_3 + \text{Const. } C \qquad \ldots.9$$

For the second condition, the value of Young's Modulus for the web and skin may be taken to be the same, so that an equation of axial strain requires that

$$F_1 + F_2 = F_3 - \frac{\nu a^2}{2} F_3'' \qquad \ldots.10$$

Solving equations 9 and 10 gives

$$F_1 = -\frac{1}{2}(1+3\psi)F_3 + \frac{1}{4}\nu a^2 F_3'' + \frac{3}{2}C$$

$$F_2 = \frac{3}{2}(1+\psi)F_3 - \frac{3}{4}\nu a^2 F_3'' - \frac{3}{2}C \qquad \ldots..11$$

From the first three conditions of equations 7 these lead to the following conditions:

$$F_3(o) = F_3(l) = 2A / 3(1+\psi)$$

$$F_3'(o) = F_3'(\ell) = F_3''(o) = F_3''(\ell) = F_3'''(o) = F_3'''(\ell) = 0$$

$$C = 0 \qquad \ldots..12$$

The stress system may now be written in terms of the one unknown F_3, which will be abbreviated to F:

web: $$\sigma_x = -\frac{1}{2}\left\{(1+3\psi) - 3(1+\psi)v^2\right\}F + \frac{1}{4}\nu a^2(1-3v^2)F''$$

$$\tau = \frac{1}{2}d\left\{(1+3\psi)v - (1+\psi)v^3\right\}F' - \frac{1}{4}\nu a^2 d(v-v^3)F'''$$

$$\sigma_y = \frac{1}{4}d^2\left\{(1+3\psi)(1-v^2) - \frac{1}{2}(1+\psi)(1-v^4)\right\}F''$$
$$-\frac{1}{8}\nu a^2 d^2\left\{(1-v^2) - \frac{1}{2}(1-v^4)\right\}F''''$$

skin: $$\sigma_x = F$$

$$\tau = a(1-w)F'$$

$$\sigma_z = \frac{a^2}{2}(1-w)^2 F'' \qquad \ldots..13$$

The problem is now reduced to finding the function F for this system, satisfying the conditions 12. The internal energy may be found and minimised to yield a differential equation in F

Since energy is quadratic in the stresses, the highest order term will be a fourth derivative squared, which on minimisation will lead to an eighth order term. Thus the equation will require 8 conditions for a solution, and these are given in equations 12. There will be no need to establish boundary conditions in the minimisation process, so that the boundary terms will not be evaluated.

The strain energy is given by

$$U = \frac{1}{2E} \int \left[\sigma_x^2 + \sigma_y^2 + 2(1+\nu)\tau^2 - 2\nu\,\sigma_x\sigma_y \right] t\,dA \qquad \ldots.14$$

the integration being taken over the whole area of web and skins. These may be considered over the ranges $o<x<l$, and $o<v<1$, for half the web; and the skin over the ranges $o<x<l$ and $o<w<1$ for half of one skin. Then the former must be doubled and the latter quadrupled.

The system is finally non-dimensionalised by substituting for the axial co-ordinate from

$$x = ud \qquad \ldots.15$$

so that $d\frac{\partial}{\partial x} = \frac{\partial}{\partial u}$ etc.

The non-dimensional parameter $r = a/d$ is introduced also.

The expression for the energy becomes

$$\begin{aligned}
\frac{2EU}{t_w d^2} = \int_o^{l/d} du \Big\{ & \tfrac{1}{5}(1+7\psi+6\psi^2)F^2 \\
& - \left[\frac{1}{15}\nu r^2(3+8\psi) - \frac{2\nu}{105}(2+18\psi+51\psi^2) \right] F''F \\
& - \frac{1}{105}\nu^2 r^2(2+9\psi)F''''F \\
& + \tfrac{2}{3}(1+\nu) \left[\frac{1}{35}(2+18\psi+51\psi^2) + r^2\psi \right] F'F' \\
& - \frac{2(1+\nu)}{105}\nu r^2(2+9\psi)F'''F' \\
& + \left[\frac{2}{315}(1+11\psi+31\psi^2) + \frac{1}{20}r^4(\nu^2+\psi) - \frac{1}{105}\nu^2 r^2(2+9\psi) \right] F''F'' \\
& - \left[\frac{1}{315}\nu r^2(2+11\psi) - \frac{1}{105}\nu^3 r^4 \right] F''''F'' \\
& + \frac{(1+\nu)}{105}\nu^2 r^4 F'''F''' + \frac{1}{630}\nu^2 r^4 F''''F'''' \Big\}
\end{aligned} \qquad \ldots.16$$

where dashes now denote differentiation with respect to u.

This energy expression can be minimised by the variational calculus, resulting in integrals of variations of derivatives. A typical term is

$$\delta \int_0^{\ell/d} F'F''' \, du = \int_0^{\ell/d} (F'\delta F''' + F''' \delta F') \, du$$

Integrating by parts this becomes

$$\Big[F' \delta F'' - F'' \delta F' + 2F''' \delta F \Big]_0^{\ell/d} - 2 \int_0^{\ell/d} F''' \delta F du$$

where δF is an arbitrary variation in the function F.
Now all the boundary terms will be zero, as every term contains one of the first three derivatives or its variation, and these are zero from the boundary conditions 12. Then since δF is arbitrary, all the integral terms may be collected together and equated to zero.

For the particular case of a box with square cells, having the skin thickness the same as, and twice, that of the web, respectively (i.e. r = 1, ψ = 2 and 4 resp.) this results in the following differential equations for F in terms of u:

$$\psi = 2: F^{v'''} - 166F^{v'} + 8065F'''' - 47,000F'' + 546,000F = 0$$

$$\psi = 4: F^{v'''} - 313F^{v'} + 27,000F'''' - 209,000F'' + 175,000F = 0 \quad \ldots .17$$

The solutions of these equations satisfying the boundary conditions of equations 12 will yield the correction stresses, from equation 13, to be imposed on the infinite system of equation 3.

The general solution of these equations is

$$F = C_1 e^{m_1 u} + C_2 e^{-m_1 u} + C_3 e^{m_2 u} + C_4 e^{-m_2 u} + (C_5 \cos m_3 u + C_6 \sin m_3 u) e^{m_4 u}$$

$$+ (C_7 \cos m_3 u + C_8 \sin m_3 u) e^{-m_4 u} \quad \ldots\ldots\ldots\ldots\ldots\ldots .18$$

with the indices having the values tabulated below:

ψ	m_1	m_2	m_3	m_4
2	1.252	2.227	1.395	9.045
4	.976	2.744	1.439	12.416

For a semi-infinite box (i.e. with one free end at x = 0 and one at x = ∞) all positive order exponentials must vanish, so that only four coefficients remain, to be determined from the boundary conditions for u = 0.

These coefficients for the two cases are :

ψ	C_2	C_4	C_6	C_8	
2	3.054	-2.217	.168	.286	$\times$ .222A
4	1.824	-.902	.078	-.191	$\times$ .133A

........19

where A is defined in equations 4 note iii, and represents the temperature difference and properties, whilst the odd coefficients are zero.

For a short free-ended box, the system is symmetric, and the origin of axes may be taken at the centre, with the boundary conditions expressed at $\pm \ell/2$. The general solution 18 may be written in terms of even functions only, with four coefficients determined at either edge.

The solutions are:

$$F = C_1' \cosh m_1 u + C_2' \cosh m_2 u + C_3' \cosh m_3 u . \cos m_4 u + C_4' \sinh m_3 u . \sin m_4 u$$

....20

where the values of the m are as before, and where the C' for different box sizes are as below:

ψ	ℓ/d	C_1'	C_2'	C_3'	C_4'	
2	2	+1.222	-.297	$+3.59\times10^{-5}$	$+1.40\times10^{-5}$	$\times$ .222A
	4	+ .4875	-.0498	$+1.57\times10^{-9}$	-8.65×10^{-9}	
	7	+ .0791	-.00186	$+9.33\times10^{-15}$	$+7.93\times10^{-15}$	
4	2	+ .9827	-.0675	$+7.737\times10^{-7}$	$+2.097\times10^{-7}$	$\times$ .133A
	4	+ .4897	-.00685	$-.7595\times10^{-12}$	$+6.041\times10^{-12}$	

.....21

The functions of equations 18 and 20, with the coefficients 19 and 21 are plotted in Fig. 3.

Now the direct web stresses of equation 13 added to the infinite system of equation 3, for the two cases of $\psi = 2$ and 4, may be written in the following forms, to give the actual stresses in the short box:

$$\sigma_x = A\left[\left\{.777(1-f) + .0166f''\right\} - \left\{(1-f) + .050f''\ v^2\right\}\right] \quad (\psi = 2)$$

$$= A\left[\left\{.866(1-f) + .010\ f''\right\} - \left\{(1-f) + .030f''\ v^2\right\}\right] \quad (\psi = 4)$$

$$\sigma_y = A\left\{(.306f'' - .0042f^{iv}) - (.381f'' - .0083f^{iv})v^2 + (.083f'' - .0042f^{iv})v^4\right\} \quad (\psi = 2)$$

$$= A\left\{(.233f'' - .0025f^{iv}) - (.433f'' - .0050f^{iv})v^2 + (.083f'' - .0025f^{iv})v^4\right\} \quad (\psi = 4)$$

......22

where $f = \frac{9F}{2A}$ and $= \frac{15F}{2A}$ for $\psi = 2$ and 4 respectively. The remaining stresses are not considered here, but may be obtained from equation 13.

For purposes of comparison with experiment, the difference between axial and lateral web strain for the case of $\psi = 4$, will be calculated as this corresponds to the results obtained by Calkin[2]. He assumed that the lateral stresses were negligible, so that two strain gauges placed perpendicularly on the web would give the axial stress only, the gauges being temperature compensated. The apparent axial stress would then be taken as:

$$\sigma_{app} = (\epsilon_x - \epsilon_y)E/(1 + \nu)$$

However this would really become

$$\sigma_{app} = \sigma_x - \sigma_y$$

For the case corresponding to Calkin's tests (i.e. $\psi = 4$, $\ell/d = 4$, $r = 1$, and taking $\nu = 0.3$) the present theory gives

$$\sigma_{app} = (.350 - .350v^2 - .034v^4)\ E\ \alpha(T_{max} - T_{min}) \quad ......23$$

Values of this stress are calculated using Calkin's measured temperature distributions, (assuming these to be parabolic) and with the same material properties, namely $E\alpha = 303$ p.s.i./°C. These are plotted in Fig. 4 together with the earlier experimental results and infinite theory calculations, and with the corrected axial stress of equation 23.

Conclusions

1. According to the present theory for short boxes, the axial stresses are about half those predicted by the infinite theory, for the box examined.

2. A short initial period occurs before the heat flow from the skin penetrates to the centre of the web, so that in this period the web temperature distribution is not as assumed, namely parabolic. Hence the present theory cannot be reliable then.

3. The experimental results and the calculated values do not agree well having different forms.

4. The weakness of the present theory probably lies in the fact that it is based on a simplified temperature distribution. While the web parabolic approximation may be justified in some cases, the skin temperature would always have a significant drop at the web, and the maximum web temperature would be different from the skin temperature, due to thermal resistance at the joint. The theory is also only a second approximation to the true stress distribution, even if the idealisations are justified. However a more refined solution would be impractical and probably unnecessary.

5. In a long box, the infinite theory would be accurate for positions more than about $2\frac{1}{2}$ depths ($\ell/d = 5$) from a free end.

6. For boxes of typical proportions, the stress distribution is only slightly affected by the ratio of skin area to web area.

References:

1. Hoff, N.J. Structural problems of future aircraft; Third Anglo-American Conference, 1951.

2. Calkin, P. An analytic and experimental study of some problems of thermal stresses; College of Aeronautics Diploma Thesis, 1956. (Unpublished).

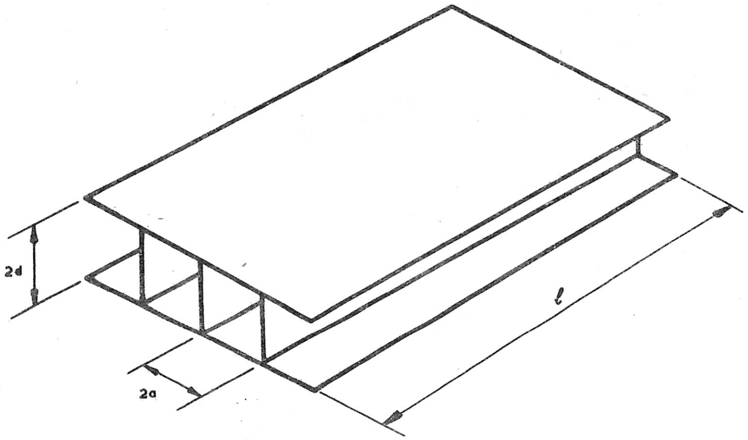

FIG. 1 MULTICELL BOX STRUCTURE.

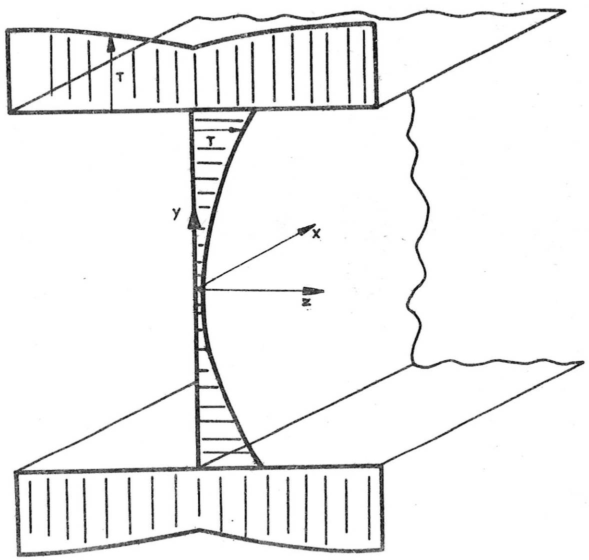

FIG 2 IDEALISED I-SECTION WITH TYPICAL TEMPERATURE DISTRIBUTION.

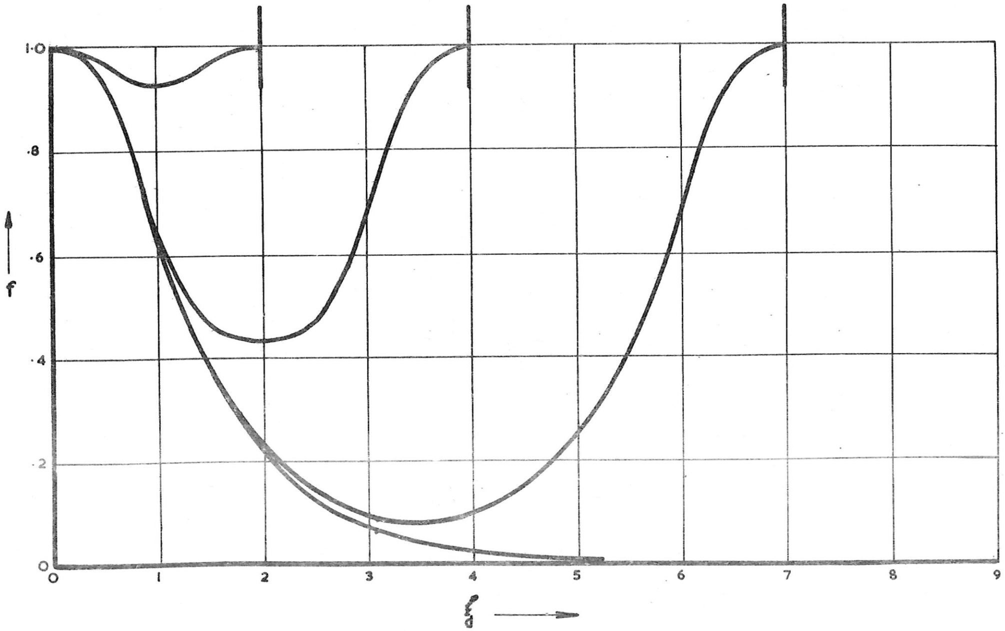

FIG. 3 VARIATION OF AXIAL STRESS FUNCTION FOR DIFFERENT BOX LENGTHS

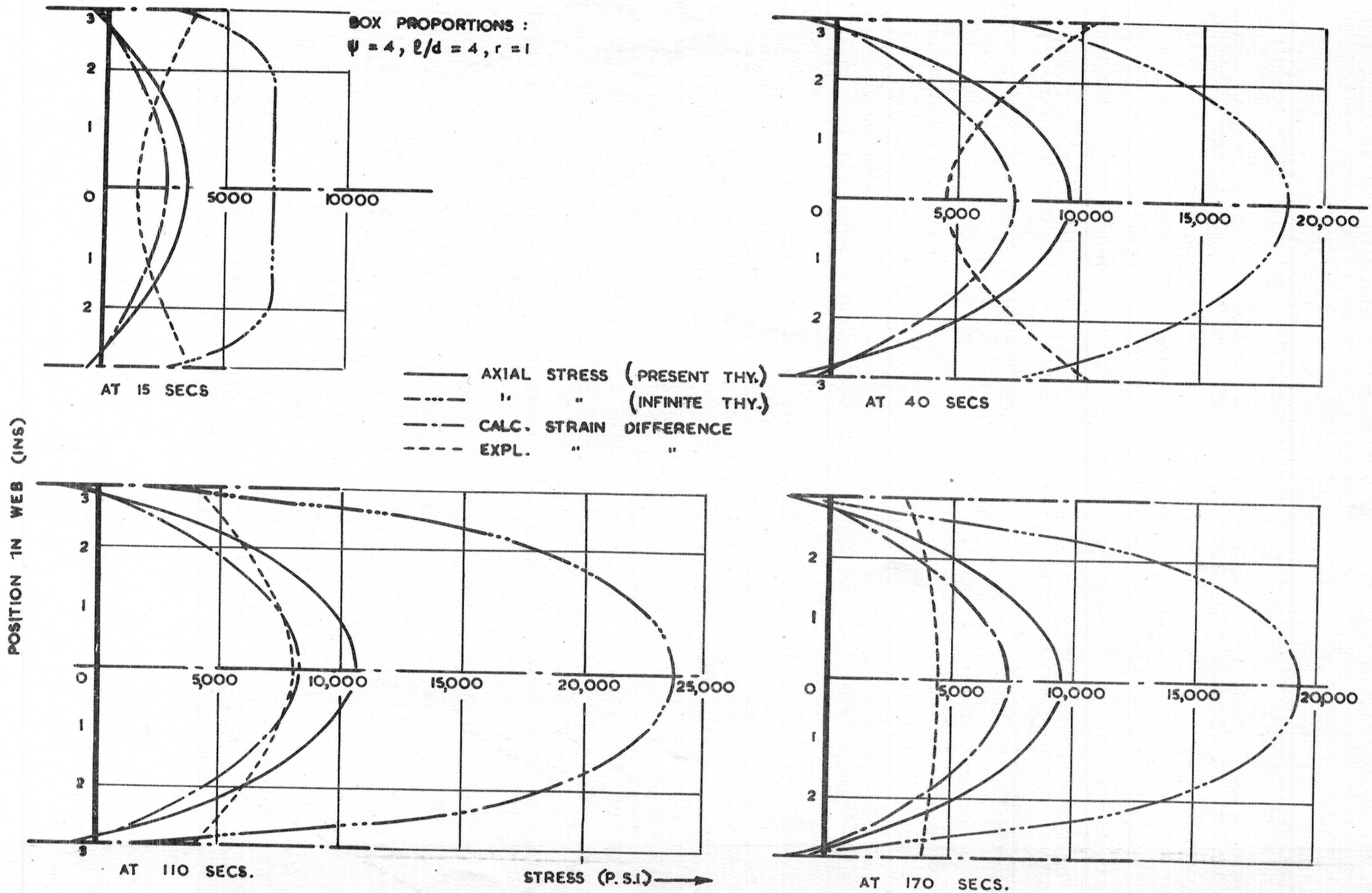

FIG. 4. WEB AXIAL THERMAL STRESSES ACROSS CENTRE LINE OF SHORT BOX