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CRANFIELD

OPTIMUM DESIGN OF A MULTICELL BOX SUBJECTED TO A GIVEN BENDING MOMENT AND TEMPERATURE DISTRIBUTION

by

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Optimum Design of a Multicell Box Subjected to a Given Bending Moment and Temperature Distribution.

- by -

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SUMMARY

The optimum geometry of a multicell box of given depth, under a given bending moment and temperature distribution, is obtained. The method is general enough to permit the skin thickness to be either specified, e.g. by stiffness requirements, or not.

CONTENTS

LIST OF SYMBOLS

A	Area of I-section = $A_S + A_W$
A_S	Area of skin panels
A_W	Area of web
b_S	Width of skin panels
b_W	Depth of box
B	Structural chord of beam, defined in eq. 2.1.
c	Heat capacity per unit volume
C	Constant in eq. 5.2.
d	$\frac{b_W}{2}$
D	Constant in eq. A.2
E	Youngs Modulus
E_S	Secant Modulus = $\frac{\sigma}{\epsilon}$
E_T	Tangent Modulus = $\frac{d\sigma}{d\epsilon}$
f	Distance from web mid-plane to the near edge of the rivet shanks in a web angle
I	Moment of inertia for each cell
k	Thermal conductivity
K	A constant used for determining the buckling stress σ_c, defined in eq. 5.11
K_S	A constant used in Ref. 3 for determining the buckling stress σ_c, defined in eq. 5.12
K_W	A constant used in Ref. 4 for determining the web crushing stress, introduced in eq. A.2
m	Bending moment per unit chordwise length applied on section
M	Bending moment on each cell = mbs

List of Symbols continued.

p	Number of bays into which structural chord B is split
P	Web material maximum shear stress
q	Penetration depth. (See eq. 2.4)
r_b	Ratio of depth/width of each cell = b_W/b_S
r_t	Ratio of web thickness/skin thickness = t_W/t_S
R	$\frac{1}{2}(3 - \frac{q}{d})$ a function defined after eq. 5.7
s	Shear force per unit chordwise length
S	Shear force on each cell = sb_S
t	Time
t_1	Transit time, defined in eq. 2.5
t_S	Skin thickness
t_W	Web thickness
T_y	Temperature at distance y from neutral axis of I beam
T_m	Minimum temperature in the web
T_s	Maximum skin temperature
$\overline{T}$	$\frac{2}{3} E \alpha (T_s - T_m)$, a function introduced after eq. 5.10
w	$\frac{W}{b_S}$
W	Weight of each cell, per unit length spanwise $\rho(2b_S t_S + b_W t_W)$
y	Co-ordinate in plane of I-section measured along the web from the neutral axis (see Fig. 2).

List of Symbols continued.

α	Coefficient of expansion
ϵ	Strain
ϵ_2	Strain at which $E_T = \frac{1}{2} E_S$
μ	Poissons Ratio
ρ	density of the material
η	Plasticity correction factor (see eq. 5.12)
σ_C	Buckling stress of skin panels
σ_B	Bending stress in skin (compressive)
σ_T	Thermal stress in skin (compressive)
σ_W	Thermal stress in web (Tensile)
σ_S	$\sigma_B + \sigma_T$
σ_y	Stress at distance y from neutral axis
σ_2	Stress at which $E_T = \frac{1}{2} E_S$
θ	$\frac{1}{2} r_b r_t = \frac{A_W}{A_S}$ defined after eq. 5.19
ψ	$\dfrac{1 + \frac{1}{2} r_b r_t}{1 + {}^1/6\, r_b r_t}$

Optimum Design of a Multicell Box Subjected to Bending and Thermal Stresses.

1. Introduction.

Many authors have considered the optimisation of multicell wing structures in which no allowance was made for thermal effects. The results of various such investigations will be discussed first and the influence of thermal effects on the optimum designs will be discussed qualitatively.

A simplified analysis will then be presented from which the quantitative effects of kinetic heating on optimum design can be assessed.

The present analysis will be concerned only with multicell beams composed of thin skin members and full depth webs. This will probably be a practical method of construction for wings in which thermal effects are not too severe. No suggestions are made for ways to alleviate thermal stresses, nor will the advantage of post and/or stringer stabilisation of skins compared with full depth webs be shown. It should be noted, of course, that a vast field of structural optimisation awaits the ingenious designer of the future who will undoubtedly consider any optimisation from the standpoint of high temperature materials versus insulation versus heat suppression, diversion and removal techniques (Ref. 1). He will also have to consider alternative methods of construction.

Such an analysis will be formidable and, unless the basic information used is exact, and the assumptions made are realistic, false conclusions may be drawn. Many studies and investigations have been made to determine individual and combined effects of some of the above considerations and it is hoped that this note will also help to elucidate some of the problems of structural design.

2. Previous Analyses

2.1. Bending Stresses.

The maximum strength and structural efficiency of multicell structures, ignoring thermal effects, has been studied previously by other investigators (Refs. 2 - 6). Their findings will not be discussed in detail here but some comments may be opportune.

Gerard's analysis[2] for integral beams showed that optimum design exists when $r_t = 0.4\, r_b$. He also concluded that the optimum number of webs is given by the equation

$$p.(4p + 1) = 5\left(\frac{B}{bw}\right)^2 \qquad 2.1.$$

where p is the number of bays into which the structural chord B is split i.e. B = p.bs.

It can be shown that, with little error, for practical wing shapes, eq. 2.1 reduces to give

$$p \doteq \frac{B}{b_W} \doteq p. \frac{b_S}{b_W} \qquad 2.2$$

Therefore $r_b = 1$ and $r_t = 0.4$. A limiting assumption of this analysis is that the web buckling and skin buckling stresses are equal. Ref. 3 also dealt with integral beams and is noteworthy in that it delineates ranges of the parameters r_b, r_t for which the phenomenon of web buckling occurs. This phenomenon results in low values of the buckling stress coefficient K_S, values of which are shown in Fig. 1 (taken from Ref. 3).

Rosen[4] has made a most thorough analysis of the ultimate strength of multicell wings and has considered the deleterious effects of web buckling and web crushing on the strength of the structure. To avoid the former, he suggested as a criterion that $r_b \leqslant 3r_t$, but later showed this criterion to be conservative. For the latter phenomenon he showed the ranges of r_b and r_t for which web crushing is more critical than web buckling.

The analysis is quite general in that skin thickness is an independent parameter which may be specified by stiffness requirements. Although Rosen has considered integral beams, in general, he discussed the effects of adding heavy attachment members between web and skin, or using formed channel web members. He showed, that when such attachment members are added to an optimum design (based on his original analysis), it results in certain circumstances in a more efficient design.

Semonian and Anderson[5] have considered the use of formed channel webs in more detail and they have shown the large effects that flexibility of the web attachment flanges can have on the stability and ultimate strength of multicell beams. They have demonstrated, experimentally and theoretically, that very low values of the coefficient K_S result as the f-distance increases. This effective rivet offset distance may be defined as the distance from the web mid-plane to the near edge of the rivet shanks.

It was shown that, in order to achieve the buckling stresses predicted by the integral beam theory of Ref. 3, the quantity $\frac{f}{b_S \, r_t}$ must be less than 0.18, otherwise, failure will occur in the wrinkling mode rather than in local buckling.

More recently, Houghton and Chan[6] have reconsidered the analysis of Ref.3 and, using the values of the buckling stress coefficient K_S from Ref. 3, they have produced a more systematic, non-iterative process for determining the optimum geometry and weight of multicell beams under pure bending. Their analysis effectively implied an assumption of integral beams but is applicable to beams in which $\frac{f}{b_S r_t} < 0.18$.

For convenience, the present analysis will adopt the same notation and presentation as Ref. 6, and have similar limiting assumptions.

2.2. Thermal Stresses.

Previous analyses of thermal stress on multicell structures will not be discussed here since the majority of them do not present the thermal stress distributions in a form suitable for the needs of the present analysis. Since the basis of any thermal stress analysis is knowledge of the temperature distribution, it is proposed to adopt the analysis of Biot[7] since this offers a relatively simple expression for the temperature distribution in a section.

Biot actually considered the problems of a uniform slab and a typical integral I-section. The increased complexity of the latter solution does not encourage its use and it is believed that little error will accrue if his solution to the slab problem is followed.

It must be realised that there are two distinct phases to any heating problem in a section such as an I-beam. In the first phase the temperature has not yet begun to rise at the centre of the web (a symmetrical beam with symmetry of heating is assumed) and everything occurs as if the web depth were infinite. During this phase the temperature distribution is approximated by

$$T_y = T_s \left[1 - \left(\frac{d-y}{q}\right)\right]^2 \quad \text{for } (d-y) < q$$
$$T_y = 0 \quad \text{for } (d-y) > q \qquad 2.3$$

where the temperatures are measured above the initial strainless level and q is called the "penetration depth" at which the temperature is just beginning to rise. Biot showed that the expression for the penetration depth is

$$q = 3.36\sqrt{\frac{kt}{c}} \qquad 2.4$$

and he defined the time at which the temperature at the centre of the web begins to rise as the "transit time" t_1 i.e. when q = d

$$\text{or} \quad t_1 = 0.0885 \frac{cd^2}{k} \qquad 2.5$$

In the second phase of heating the temperature at the centre of the web is denoted by T_m and the temperature distribution is approximated by

$$T_y = T_s \left(\frac{y}{d}\right)^2 + T_m \left[1 - \left(\frac{y}{d}\right)^2\right] . \qquad 2.6$$

This latter expression has been assumed by Lempriere[8] and is, of course, only applicable for times greater than the transit time.

Using Biot's analysis[7] it can be shown that a convenient expression for T_m for $t > t_1$ when T_s is an arbitrary function of time is

$$T_m + 4.57\, t_1 \dot{T}_m = T_s - 1.075\, t_1 \dot{T}_s \qquad 2.7$$

T_s is determined neglecting the presence of the web and T_m is therefore obtained using eq. 2.7. These values will prove sufficiently accurate for most project studies.

Finally, it should be emphasised, that the temperature of the skin is a function of the skin thickness t_s. No exact theory will be presented here for the determination of skin temperature under arbitrary flight conditions since this has adequately been covered in the literature.

3. Qualitative Assessment of the Thermal Problem

In the past, wing structures have been designed on the strength and stiffness criteria appropriate to their various flight histories. In general this has meant that a certain maximum bending moment and shear force loading has had to be satisfied and the structural designer has not been particularly concerned at what time of the flight the worst conditions arose. In designs however where thermal effects must be considered, the time element may be all important.

In such designs it will be necessary to consider together the variation with time of both the normal manoeuvre load stresses and thermal stresses. Hence, the worst design conditions for the structure can be assessed at various times. In other words it will not do to add the most severe thermal and bending stresses unless they occur simultaneously. From this one can visualise a structure, designed on manoeuvre loads alone, negotiating satisfactorily a flight programme which produces high thermal stresses, providing these stresses occur at a time when the manoeuvre loads are low. This might possibly occur in a long range, high speed interceptor which would experience high thermal stresses early in its flight at a time of low manoeuvre; and at interception - a time of high manoeuvre, the whole structure would have reached its equilibrium temperature, the thermal stresses would be negligible and the only thermal effect would be that of deterioration of material properties.

Because of such considerations, no attempt will be made to consider the thermal stress patterns arising from specific flight programmes. Instead, the thermal stress terms will be introduced quite generally as functions only of the maximum skin temperature, T_s, and the minimum web temperature, T_m.

4. Assumptions.

The assumptions made in the following analysis are :-

4A. The design criterion for the wing is one of buckling stability of the compression skin under bending and thermal stresses The skin thickness is specified and left independent. This enables other, stiffness criteria to be satisfied. Shear strength and stiffness is assumed to be covered.

4B. The section is idealised as rectangular with its depth prefixed by aerodynamic considerations.

4C. Both the skins and the webs are fully effective in taking bending and the stress is distributed according to the Engineers Theory. The effects of any angles, which might make the skin to web joint with a formed channel web, on the weight and stiffness of the section have been neglected.

4D. The width of the box is sufficiently large in comparison with the depth for the panel buckling characteristics to be assumed to be the same as that of a box of infinite width. This assumption enables Fig. 1 taken from Ref. 3 to be used. Evidence given by Ref. 9 suggests this assumption is valid provided the box has at least 3 cells.

4E. The top and bottom skins are of the same thickness and therefore the neutral axis is central. This latter assumption also applies when thermal stresses and increases in temperature are applied to the section.

4F. The same material is used for the skins and the webs.

The limitations placed on the analysis by certain of the assumptions are discussed in the appendices.

5. Analysis.

5.1. Bending Stresses due to Manoeuvre Loads.

Because of assumption 4D it is convenient to take as our typical structural element an I section from the beam of Fig. 2.

Let the bending moment applied on each element be M, and the moment per unit length (chordwise) be m.

$$\therefore \quad M = mbs.$$

Then following the assumptions of 4C and 4E, the compression stress in the skin due to bending is given by

$$\sigma_B = \frac{M}{I} \frac{b_W}{2}.$$

The moment of inertia of each element is

$$I = \frac{1}{12} b_W^3 t_W + 2 b_S t_S \left(\frac{b_W}{2} \right)^2$$

$$\text{or} \quad I = \frac{1}{2} b_W^2 b_S t_S \left(1 + \frac{1}{6} r_b r_t \right).$$

Hence $$\sigma_B = \frac{m b_S b_W}{2 I} = \frac{m}{b_W t_S (1 + \frac{1}{6} r_b r_t)}. \qquad 5.1$$

5.2. Thermal Stress.

5.2.1. First Heating Phase ($t < t_1$)

With uniform skin temperature, the temperature distribution through the I beam is given by

$$\left.\begin{aligned} T_y &= T_s \left[1 - \left(\frac{d-y}{q} \right) \right]^2 && \text{for } (d-y) < q \\ &= 0 && \text{for } (d-y) \geq q \end{aligned}\right\}. \qquad 2.3$$

The temperature is assumed constant axially so that there is only axial stress due to differential expansion. It is assumed also that the beam considered is long, and the section shown in Fig. 2 is free from any end effects.

To determine the thermal stress distribution, Hookes Law is assumed, then, if plane sections remain plane the strain across the section is given by

$$\epsilon_y = \frac{\sigma_y}{E} + \alpha T_y = \text{Constant} = C \, . \qquad 5.2$$

For zero net thrust on the section,

$$\int_A \sigma_y \, dA = 0 \, , \qquad 5.3$$

where dA is elemental area of I beam of total area A.

From eq. 5.2, $\sigma_y = EC - E\alpha T_y$, which, when substituted into eq. 5.3, gives

$$C = \frac{\int_A E \alpha T_y \, dA}{\int_A E \, dA} \, .$$

Hence

$$\sigma_y = E \left[\frac{\int_A E \alpha T_y \, dA}{\int_A E \, dA} - \alpha T_y \right] \, . \qquad 5.4$$

Eq. 5.4 is perfectly general and allows for variation of the properties E, α with temperature or with y.

If the material properties are assumed constant and independent of y and T_y eq. 5.4 becomes

$$\sigma_y = \frac{E\alpha}{A} \left[\int_A T_y \, dA - A \, T_y \right] \, . \qquad 5.5$$

Therefore, using eq. 2.3

$$\sigma_y = \frac{E\alpha}{A} \left[T_s \left\{ 2\, b_S\, t_S + \frac{2}{3} t_W\, q \right\} - A\, T_s \left\{ 1 - \left(\frac{d-y}{q}\right) \right\}^2 \right]$$

or
$$\sigma_y = \frac{E\alpha T_s}{A} \left[2\, b_S t_S + \frac{2}{3} t_W q - A \left\{ 1 - \left(\frac{d-y}{q}\right) \right\}^2 \right] . \qquad 5.6$$

Hence, for the skin, the compressive thermal stress is

$$\sigma_T = \frac{2}{3}\, E\, \alpha\, T_s \cdot \frac{A_W}{A} \cdot R , \qquad 5.7$$

where
$$R = \frac{1}{2}\left(3 - \frac{q}{d}\right) .$$

5.2.2. Second Heating Phase $t > t_1$

The temperature distribution may be approximated by

$$T_y = T_s\left(\frac{y}{d}\right)^2 + T_m \left[1 - \left(\frac{y}{d}\right)^2 \right] . \qquad 2.6$$

Following an analysis identical to that in the first heating phase, the thermal stress distribution is given by

$$\sigma_y = -E\, \alpha\, (T_s - T_m) \left[\left(\frac{y}{d}\right)^2 - \frac{A_W + 3\, A_S}{3A} \right] . \qquad 5.8$$

Hence, for the skin, the compressive thermal stress is

$$\sigma_T = \frac{2}{3}\, E\, \alpha\, (T_s - T_m) \frac{A_W}{A} . \qquad 5.9$$

5.2.3. Summary. The following general expression defines the compressive thermal stress in the skin at all times

$$\sigma_T = \bar{T}\, R\, \frac{A_W}{A} ,$$

or
$$\sigma_T = \bar{T}\, R\, \frac{\frac{1}{2} r_b r_t}{1 + \frac{1}{2} r_b r_t} , \qquad 5.10$$

where
$$\bar{T} = \frac{2}{3}\, E\, \alpha\, (T_s - T_m) ,$$

and for $t < t_1$ $\quad T_m = 0, \quad R = \frac{1}{2}\left(3 - \frac{q}{d}\right)$

and for $t > t_1$ $\quad T_m \neq 0, \quad R = 1$.

A similar but more detailed analysis has been made by Hoff[10]. He considered a beam in which skin and web material were dissimilar, and he allowed for the decrease in skin temperature at the junction with the web. Unfortunately his results cannot be presented in a concise form, but for the case of a uniform beam his results do not differ greatly from those above. At time $t = 0$ the analyses correspond exactly and at the transit time (beyond which Hoffs analysis is not valid) the discrepancy is, for all likely configurations, less than 10%. Because of the simplicity of the present analysis this discrepancy will be ignored.

5.3. Buckling Stress of Skin Panels.

The buckling stress σ_c for the skin panel is usually given as

$$\sigma_c = K E_S \left(\frac{t_S}{b_S}\right)^2, \qquad 5.11$$

where E_S is the secant modulus ($= \frac{\sigma}{\epsilon}$) usually used in connection with this type of buckling (Ref.11) and K is a constant depending only on the panel configuration.

The critical stress is given in Ref 3 as

$$\sigma_c = \frac{K_S \pi^2 \eta E}{12(1-\mu^2)} \left(\frac{t_S}{b_S}\right)^2. \qquad 5.12$$

The value of ηE in eq. 5.12 corresponds approximately to the secant modulus E_S. Therefore by comparing eqs. 5.11 and 5.12

$$K = \frac{\pi^2 K_S}{12(1-\mu^2)} \qquad 5.13$$

Fig. 1 reproduces the curves that give the values of K_S from Ref. 3 and Table 1 tabulates the calculated values of K (from eq. 5.13) for a range of parameters r_b, r_t. (assuming Poissons Ratio, μ, = 0.3).

5.4. Weight of Box.

The weight (per unit length spanwise) of each cell is

$$W = \rho\,(2\, b_S t_S + b_W t_W).$$

Dividing this by the width of the cell b_S gives

$$w = \frac{W}{b_S} = 2\rho\, t_S(1 + \tfrac{1}{2} r_b r_t), \qquad 5.14$$

and $$\frac{w}{2\rho b_W} = \frac{t_S}{b_W}\,(1 + \tfrac{1}{2} r_b r_t). \qquad 5.15$$

6. Optimisation.

The object of this investigation is to obtain the skin thickness t_S, web thickness t_W, and web spacing b_S, for a box of given depth b_W, to resist applied bending stresses and thermal stresses so that the weight W is a minimum.

Alternatively, if skin thickness is specified by stiffness requirements, the following equations present the correct approach to satisfy strength and stiffness requirements and give the required beam proportions.

It is recognised that failure will occur soon after the total compression stress, σ_S, in the skin reaches the buckling stress σ_C given in eq. 5.11. Therefore σ_C is used as the maximum permissible value σ_S.

Using eqs. 5.1 and 5.10 the total compressive stress in the skin is given by

$$\sigma_S = \sigma_B + \sigma_T,$$

$$\text{or} \quad \sigma_S = \frac{m}{b_W t_S(1+\frac{1}{6}r_b r_t)} + \frac{\bar{T} R \frac{1}{2} r_b r_t}{(1 + \frac{1}{2} r_b r_t)}, \qquad 5.16$$

and the value of σ_C is

$$\sigma_C = K E_S \frac{t_S}{b_S} . \qquad 5.11$$

Rearranging eqs. 5.11 and 5.16 gives

$$\frac{\sigma_C}{E_S} = \epsilon = K r_b^2 \left(\frac{t_S}{b_W}\right)^2 \qquad 5.17$$

$$\sigma_S = \left(\frac{m}{b_W^2}\right)\left(\frac{b_W}{t_S}\right)\frac{1}{(1 + \frac{1}{6} r_b r_t)} + \frac{\bar{T} R \frac{1}{2} r_b r_t}{(1 + \frac{1}{2} r_b r_t)} \qquad 5.18$$

$$= \frac{1}{1+\theta}\left\{\left(\frac{m}{b_W^2}\right)\left(\frac{b_W}{t_S}\right)\psi + \bar{T} R.\theta\right\} \qquad 5.19$$

where

$$\theta = \tfrac{1}{2} r_b r_t$$

$$\psi = \frac{1 + \frac{1}{2} r_b r_t}{1 + \frac{1}{6} r_b r_t}$$

The procedure therefore will be (Given $\frac{m}{b_W^2}$, R)

(a) Assume values of $\frac{t_S}{b_W}$, r_b, r_t

(b) Hence determine K, from Table 1, and ϵ from eq. 5.17

(c) From the compressive stress strain curve for the material, determine σ corresponding to ϵ

(d) The value of t_S assumed is used to determine $\overline{T}$.

(e) Substituting values of σ, $\frac{t_S}{b_W}$, r_b, r_t, $\overline{T}$, R into eq. 5.19 solve for $\left(\frac{m}{b_W^2}\right)$

Then keeping $\frac{t_S}{b_W}$ and the product $r_b\, r_t$ constant, new values of r_b, r_t enable a new value of K to be found and the above procedure repeated. Therefore, since

$$\frac{W}{2\rho\, b_W} = \frac{t_S}{b_W}\left(1 + \tfrac{1}{2} r_b r_t\right), \qquad 5.15$$

the above procedure with $\frac{t_S}{b_W}$, and the product $r_b r_t$, constant means that $\frac{W}{2\rho\, b_W}$ is also constant. Therefore the above procedure enables the variation of $\left(\frac{m}{b_W^2}\right)$ with r_b (or r_t) to be obtained for constant weight. Hence the "most efficient" structure for given parameters $\left(\frac{t_S}{b_W}\right)$, $(r_b r_t)$ is obtained.

Similar analyses using different values of the product $r_b r_t$ enable the "optimum" structure, for a given $\frac{t_S}{b_W}$, to be determined.

The "optimum" structure is that "most efficient" structure which has a maximum value of $\left(\frac{m}{b_W^2}\right)$ equal to the applied value.

If the value of t_S is not specified, the above processes are repeated for various values of $\frac{t_S}{b_W}$. The combinations of $\frac{t_S}{b_W}$ and the product $r_b r_t$, for various "optimum" structures enables the optimum $\frac{t_S}{b_W}$ and other beam parameters to be found. To apply the above method an example will now be performed.

7. Example

A light alloy (DTD.687) structure is considered. The idealised stress strain curve for the material is given in Fig. 3 and it is assumed that the temperatures and heating times involved in this analysis are sufficiently low to neglect effects due to deterioration of the material properties.

Let it be given that the beam has a value of $\frac{t_S}{b_W} = \frac{1}{40}$ specified by stiffness requirements; a flight programme is assumed which produces a value of $\overline{TR} = 20,000$ lb/in² at the time considered. The problem is to find the optimum structure which will simultaneously sustain a value of $\left(\frac{m}{b_W{}^2}\right) = 1400$ lb/in².

Tables 2, 3 and Fig. 4 present the results of the calculations using eqs. 5.17 and 5.19 as described in Sect. 6. Also shown in Table 3 are the values of $\left(\frac{m}{b_W{}^2}\right)$ attainable in the absence of thermal stresses. The results are plotted in Fig. 5.

From Figs. 4,5 graphs are constructed (Fig. 6), showing the variation of maximum $\left(\frac{m}{b_W{}^2}\right)$ with $r_b r_t$, and with r_b. From Fig. 6 we may deduce the optimum structure with $\frac{t_S}{b_W} = \frac{1}{40}$, to sustain the bending and thermal stresses, and the bending stresses alone. The results are given in Table 4.

Similar calculations have also been performed with $\frac{t_S}{b_W} = \frac{1}{41}$ and the results are presented in Figs. 7 - 9 and Tables 3 and 4.

It is interesting to compare the above results with those obtained by the analysis of Ref. 6 for bending stresses only. In Ref. 6 skin thickness was not specified and the optimum structure was obtained with the optimum t_S/b_W. The results of Ref. 6 for $\left(\frac{m}{b_W{}^2}\right) = 1400$ lb/in² are also shown in Table 4.

8. Discussion.

An iterative method has been proposed by which the optimum beam structure to sustain bending and thermal stresses can be found. A review of previous work in the field of structural optimisation ignoring thermal effects suggests that Refs. 4 and 6 offer the best approach for preliminary design. Ref. 4 is more general than Ref. 6 but the latter is a simpler, more systematic analysis.

The present analysis is a logical extension of the work of Ref. 6 and the introduction of the thermal stress terms makes the analysis less simple, but it is still systematic. The chief merit in this analysis is that skin thickness may or may not be specified initially by stiffness requirements. In the former case the solution is more easily obtained.

An examination of the results obtained for the typical example shows several interesting results.

From Table 4 it can be seen that,

(a) as thermal stresses are added to the structure of Row 2, the new optimum structure in Row 1, has thicker webs, more closely spaced. The same result follows from Rows 4 and 3. This result is a little surprising as it might have been expected that <u>thinner</u> webs, more closely spaced would be required.

(b) The structures with the thicker skin are marginally lighter than those with the thinner skin. Simultaneously, the webs are thinner and more widely spaced. The actual differences in the numbers in Table 4, for $\frac{W}{2\rho b_W}$, in Rows 1 - 4, are so small, and the graphical method of solution suspect to error, that it is safer to deduce that the variation of structure weight with skin thickness is a fairly flat function.

(c) The optimum structure calculated in Ref. 6 shows considerable differences from the geometries for the optimum structures of the present analysis. Again, it is deduced that the weight is fairly insensitive to quite large changes in the optimum geometry.

To summarise, the method of analysis presented in this note has been shown to be relatively simple and systematic, and to offer a convenient means of deciding structural shapes in the project stage, when thermal stresses are present.

9. List of References

1. Meyer, J.H. — Thermoelastic Distortion and Wing Structural Design. Aero. Eng. Review Vol.16 No.9 pp.46-53. Sept. 1957.

2. Gerard, G. — Optimum Number of Webs required for a Multicell Box Under Bending. Jour.Aero.Sci. Vol.15. No.1. pp.53-56 Jan.1948.

3. Schuette, E. McCulloch, J. — Charts for the Minimum Weight Design of Multiweb Wings in Bending. NACA T.N.1323, 1947.

4. Rosen, B.W. — Analysis of the Ultimate Strength and Optimum Proportions of Multiweb Wing Structures. NACA T.N.3633. 1955.

5. Semonian, J.W. Anderson, R.A. — An Analysis of the Stability and Ultimate Bending Strength of Multiweb Beams with Formed Channel Webs. NACA T.N.3232. 1954.

6. Houghton, D.S. Chan, A.S.L. — The Design of a Multicell Box in Pure Bending for Minimum Weight. College of Aeronautics Note No.74.

7. Biot, M.A. — New Methods in Heat Flow Analysis with Application to Flight Structures. Jour. Aero.Sci. Vol.24 No. 12 pp. 857-873 Dec. 1957.

8. Lempriere, B.M. — Thermal Stresses in a Box Structure. College of Aeronautics Note (To be published).

List of References continued.

9. Eggwertz, S.F. Buckling Stress of Box Beams under Pure Bending. F.F.A. (The Aeronautical Research Institute of Sweden) Report 33.

10. Hoff, N.J. Thermal Buckling of Supersonic Wing Panels. Jour. Aero. Sci. Vol. 23 pp. 1019 - 1028 Nov. 1956.

11. Stowell, E.Z. A Unified Theory of Plastic Buckling of Columns and Plates. NACA T.N. 1556.

12. Monaghan, R.J. Formulae and Approximations for Aerodynamic Heating Rates in High Speed Flight. R.A.E. Tech. Note AERO 2407 Oct. 1955.

APPENDIX I

Limitations on the Analysis by Various Assumptions.

The assumptions which probably have the greatest influence on the validity of this analysis will be listed and their effects on the analysis discussed.

A.1. Idealised Structure to be Thin-skinned with Full Depth webs only as stabilisers :

On low speed conventional wing structures, wing depth is so large that all the shear loads are adequately taken with only a few full depth webs and wing skin stabilisation is provided by stringers and/or posts. Such a structure is unlikely to experience severe thermal stresses because in order to reduce drag and achieve speeds where thermal effects are important the wing depth must be reduced to a point where it is more efficient to achieve stabilisation by full depth webs only. A structure of this type has been assumed in this analysis.

For very high speed structures, wing depth is often so low that more shear carrying material must be put into the wing than is necessary for stabilisation alone. Obviously this analysis will not assist the design of such structures unless the following condition is imposed on the results.

A.2. The Effect of Shear.

Neglecting combined effects of direct and shear stresses in the web, the shear strength criterion to be satisfied is that

$$\frac{S}{b_W\ t_W} \leqslant P ,$$ where P is the maximum shear stress and S is the shear force per element.

If the shear force per unit length chordwise is $s = \dfrac{S}{b_S}$

the above criterion reduces to

$$r_b\ r_t \geqslant \left(\frac{b_W}{t_S}\right)\left(\frac{s}{P\ b_W}\right) \qquad \text{A.1}$$

Therefore in the procedure described in Section 6, where in any particular calculation b_W and t_S are constant, the minimum permissible value of $\dfrac{s}{b_W}$

$r_b r_t$ is easily determined from eq. A.1.

The introduction of such a limitation will only affect the optimisation if the optimum value of $r_b r_t$ for any given value of $\frac{t_S}{b_W}$ is <u>less</u> than that given by eq. A.1.

If this should happen, the values of r_b and r_t, which give the required product of $r_b r_t$, are chosen to give the largest possible value of $\left(\frac{m}{b_W^2}\right)$. Hence for a given $\frac{t_S}{b_W}$ the lightest structure satisfying eq. A.1 may give a value of $\left(\frac{m}{b_W^2}\right)$ larger than required. If $\frac{t_S}{b_W}$ is not specified the optimum structure giving the required value of $\frac{m}{b_W}$ may still be found.

A.3. The Brazier Effect

In the analysis no allowance has been made for web crushing, or the Brazier effect. Rosen in his analysis[4] has considered web crushing and he has developed a criterion for this problem, viz.

$$r_t \geq D^{\frac{1}{3}} \left(\frac{b_W}{t_S}\right)^{\frac{1}{3}},$$

$$\text{where} \quad D = \frac{3.5}{K_W} \frac{\sigma_2 (\epsilon_2)^{\frac{1}{2}}}{E}.$$

In the above expression for D, K_W the non-dimensional web crushing stress coefficient was taken by Rosen to equal 3 and σ_2 and ϵ_2 are the co-ordinates of the point on a compressive stress-strain curve at which the tangent modulus is equal to one-half the secant modulus for a material. Hence, for any given material and value of $\frac{b_W}{t_S}$, $r_t \geq$ constant.

The introduction of such a limitation into the optimisation follows readily.

A.4. The Justification for Using The Buckling Criterion of Ref. 3.

The values of K assumed in this analysis have been determined from Ref. 3 and, for structures in which buckling must be precluded, Ref. 3 offers the best available information. It should be pointed out however that for structures in which buckling, particularly web buckling, is permissible this present analysis is possibly conservative. Rosen[4] has investigated the ultimate strength of multicell beams and developed an apparently satisfactory criterion. He showed that buckling itself need not constitute failure and that beams can sustain as much as twice the applied moment needed to initiate web buckling. The use of Ref. 3 is probably justified therefore in studies for non-buckling structures up to, say, the proof load; but for the determination of ultimate strengths Rosen's analysis is more applicable.

For problems involving mainly thermal stresses it may be unreasonable to apply the results of Ref. 3 for the main reason that the basic stress distributions differ in the bending and thermal problems. Hoff (Ref.10) assumed the skin buckling stress coefficient for a simply-supported plate ($K \rightarrow 3.62$). Since, in general, the results of Ref. 3 are lower than the plate results, the use of Ref. 3 in combined bending thermal problems is probably conservative.

A.5. The Design Criterion for the Wing being that of Compression Stability of the Skin under Bending and Thermal Stresses.

It was assumed in the analysis that the critical design condition for the beam occurred in the heating phase and that the combined bending and thermal stresses in the skin on the compression side constituted the major problem.

However, simultaneously with the compressive thermal stresses in the skin there are tensile thermal stresses in the web. A general expression for the maximum tensile thermal stresses in the web is given from eqs. 5.6 and 5.8 as;

$$\sigma_W = \frac{E\,\alpha\,(T_s - T_m)}{A}\,(A_S + \tfrac{1}{3} A_W \tfrac{g}{d}) ,$$

or $$\sigma_W = \overline{T}\,G\,\frac{A_S}{A} , \qquad \text{A.2}$$

where $G = \frac{1}{2}(3 + \frac{g}{d}\theta)$, and corresponds to R in section 5.2.

Therefore at any given time, we must compare the maximum tensile stress in the web with the ultimate tensile strength of the material, and if the web tensile stress is greater, skin buckling is precluded and the analysis invalid. If it should happen that web tensile stress is the critical factory, a more simple optimisation is possible.

(a) At time $t = 0$, $T_m = 0$ $q = 0$ and the thermal tensile stress in the web is constant throughout, and given by

$$\sigma_W = \bar{T} \quad \frac{A_S}{A} \times \frac{3}{2} \quad , \text{ from eq. A.2.}$$

Therefore the maximum tensile stress in the web is given by

$$\sigma = \bar{T} \quad \frac{A_S}{A} \times \frac{3}{2} + \left(\frac{m}{b_W^2}\right)\left(\frac{b_W}{t_S}\right)\left(\frac{1}{1 + \frac{1}{6} r_b r_t}\right) , \quad \text{— eq. 5,18}$$

or $$\sigma = \frac{3}{2} \bar{T} \left(\frac{1}{1+\theta}\right) + \left(\frac{m}{b_W^2}\right)\left(\frac{b_W}{t_S}\right)\left(\frac{3}{3+\theta}\right) . \qquad \text{A.3}$$

Therefore, assuming that $\bar{T}$, $\left(\frac{m}{b_W^2}\right)$ and $\left(\frac{b_W}{t_S}\right)$ are given, the required beam geometry is decided by the value of θ which satisfies the equation

$$\frac{3}{2} \bar{T} \left(\frac{1}{1+\theta}\right) + \left(\frac{m}{b_W^2}\right)\left(\frac{b_W}{t_S}\right)\left(\frac{3}{3+\theta}\right) = \sigma_{ult} , \qquad \text{A.4}$$

where σ_{ult} is the ultimate tensile strength of the material.

If $\frac{t_S}{b_W}$ is not specified, the optimum structure is given when the weight $\frac{W}{2\rho\, b_W}$ is a minimum,

where $$\frac{W}{2\rho\, b_W} = \frac{t_S}{b_W}\left(1 + \frac{1}{2} r_b r_t\right) = \frac{t_S}{b_W}(1 + \theta), \; \ldots\ldots\; 5.15$$

$$\therefore \quad \frac{W}{2\rho\, b_W} = (1 + \theta)\left[\frac{\left(\frac{m}{b_W^2}\right)\left(\frac{3}{3+\theta}\right)}{\sigma_{ult} - \frac{3\,\bar{T}}{2}\left(\frac{1}{1+\theta}\right)}\right] \quad \text{using eq. A.4}$$

or $$\frac{W}{2\rho\, b_W} = \frac{(1+\theta)^2}{(3+\theta)}\left[\frac{6\left(\frac{m}{b_W^{\;2}}\right)}{2\,\sigma_{ult}(1+\theta) - 3\,\bar{T}}\right] . \qquad \text{A.5}$$

Eqn. A.5 must then be minimised with respect to θ.

Comparing eq. A.3 with eq. 5.18 it is seen that the difference between the maximum stresses in web and skin is given by

$$\sigma_W - \sigma_T = \frac{3}{2}\,\overline{T}\,\frac{A_S - A_W}{A} = \frac{3}{2}\overline{T}\left(\frac{1-\theta}{1+\theta}\right) . \qquad \text{A.6}$$

Therefore the condition that web tensile failure is critical is given by

$$\frac{3}{2}\,\overline{T}\left(\frac{1-\theta}{1+\theta}\right) + \sigma_S < \sigma_{ult} , \qquad \text{A.7}$$

where σ_S is the value of the maximum compressive stress in the skin, quoted in Table 3.

(b) At a time $t \geqslant t_1 \quad T_m \geqslant 0 \quad q = d$

In this case the thermal stress distribution across the web depth is parabolic and the maximum tensile stress occurs at a distance y from the neutral axis where the combined bending and thermal stresses are

$$\sigma_y = \frac{My}{I} - E\,\alpha\,(T_s - T_m)\left[\left(\frac{y}{d}\right)^2 - \frac{1}{3}\left(\frac{3+\theta}{1+\theta}\right)\right] . \qquad \text{from eq. 5.8}$$

Differentiation of this equation with respect to y yields the position and value of the maximum tensile stress in the web as

$$\frac{y}{d} = \frac{Md}{2\,I\,E\,\alpha\,(T_s - T_m)} , \qquad \text{A.8}$$

and

$$\sigma = \left[\frac{1}{4}\,\frac{M^2d^2}{I^2} + E^2\,\alpha^2\,(T_s - T_m)^2\,\frac{1}{3}\left(\frac{3+\theta}{1+\theta}\right)\right]\frac{1}{E\,\alpha(T_s - T_m)} . \qquad \text{A.9}$$

This equation can be rewritten as,

$$\sigma = \frac{1}{2}\,\overline{T}\left(\frac{3+\theta}{1+\theta}\right) + \frac{1}{6\,\overline{T}}\left(\frac{m}{b_W^2}\right)^2\left(\frac{b_W}{t_S}\right)^2\frac{9}{(3+\theta)^2} . \qquad \text{A.10}$$

As in section (a) above the beam geometry can now be decided by solution of the following equation for θ. If $\frac{t_S}{b_W}$ is not specified the optimisation follows similarly

$$\frac{1}{2\,\overline{T}}\left[\overline{T}^2\left(\frac{3+\theta}{1+\theta}\right) + \left(\frac{m}{b_W^2}\right)^2\left(\frac{b_W}{t_S}\right)^2\frac{3}{(3+\theta)^2}\right] = \sigma_{ult} \qquad \text{A.11}$$

Obviously the condition that web tensile failure is critical is given by eq. A.11. Insertion into eq. A.11 of the parameters determined by the analysis of section 6 and tabulated in Table 3 will show whether, or not, the condition is satisfied.

TABLE 1.

Values of K. (Taken from Ref. 6).

r_b \ r_t	.25	.40	.50	.60	.80	1.00
0.5	3.62	3.71	3.80	3.89	4.18	4.45
1.0	2.35	3.62	3.70	3.78	4.07	4.37
1.5	0.80	2.67	3.32	3.62	3.93	4.29
2.0	-	1.45	2.25	2.96	3.71	4.18
2.5	-	-	1.23	2.00	2.95	3.75
3.0	-	-	-	-	2.09	3.04

TABLE 2. EXAMPLE $\overline{T}$ R = 20,000 lb/in^2

BASIC PARAMETERS

$r_b r_t$	θ	ψ	r_b	r_t	K
1.00	0.50	9/7	1.00	1.00	4.37
			1.50	0.66°	3.72
			1.60	0.625	3.54
			1.66°	0.60	3.40
			1.75	0.571	3.14
			1.80	0.556	2.98
			2.00	0.50	2.25
0.90	0.45	29/23	1.00	0.90	4.22
			1.50	0.60	3.62
			1.60	0.562	3.35
			1.66°	0.540	3.14
			1.75	0.515	2.86
			1.80	0.50	2.62
			2.00	0.45	1.85
0.80	0.40	21/17	1.00	0.80	4.07
			1.33°	0.60	3.67
			1.50	0.53	3.42
			1.54	0.52	3.31
			1.60	0.50	3.11
			1.66°	0.48	2.82
			1.75	0.456	2.46
			2.00	0.40	1.45

TABLE 3. EXAMPLE: $\overline{T}$ R = 20,000 lb/in²

Calculation of $\frac{m}{b_W^2}$ for different values of $r_b r_t$ $\frac{t_S}{b_W}$

$r_b r_t$	r_b	$\frac{t_S}{b_W} = \frac{1}{40}$				$\frac{t_S}{b_W} = \frac{1}{41}$			
		ϵ	σ	$\frac{m}{b_W^2}$	$\frac{m}{b_W^2}$ *	ϵ	σ	$\frac{m}{b_W^2}$	$\frac{m}{b_W^2}$ *
1.00	1.00	27.3	27.3	600	796	27.0	27.0	580	769
	1.50	52.2	52.0	1320	1520	49.7	49.7	1225	1415
	1.60	56.6	55.7	1430	1630	53.9	53.2	1325	1515
	1.66°	59.0	57.5	1480	1680	56.2	55.4	1382	1575
	1.75	60.1	58.3	1505	1700	57.3	56.3	1412	1600
	1.80	60.1	58.3	1505	1700	57.3	56.3	1412	1600
	2.00	56.3	55.5	1425	1620	53.6	53.0	1320	1508
0.90	1.00	26.4	26.4	580	758	25.2	25.2	530	704
	1.50	51.0	50.9	1280	1463	48.6	48.6	1188	1360
	1.60	53.6	53.0	1340	1523	51.1	50.9	1250	1423
	1.66°	54.5	53.9	1370	1545	51.8	51.5	1268	1441
	1.75	54.7	54.0	1372	1552	52.1	51.9	1275	1451
	1.80	53.0	52.6	1330	1512	50.5	50.4	1235	1410
	2.00	46.3	46.3	1150	1330	41.1	41.1	974	1150
0.80	1.00	25.5	25.5	558	722	24.3	24.3	512	670
	1.33°	40.6	40.6	988	1150	38.7	38.7	910	1068
	1.50	48.1	48.1	1200	1362	45.8	45.8	1105	1265
	1.54	49.1	49.1	1230	1390	46.8	46.8	1135	1290
	1.60	49.8	49.8	1250	1410	47.5	47.5	1151	1310
	1.66°	48.9	48.9	1222	1385	46.6	46.6	1130	1286
	1.75	47.1	47.1	1170	1335	44.8	44.8	1075	1235
	2.00	36.3	36.3	866	1028	34.6	34.6	796	955

Units in lb, ins

$\epsilon \times 10^4$, $\sigma \times 10^{-3}$ lb/in² $\frac{m}{b_W^2}$ lb/in²

* When $\overline{T}$ R = 0

TABLE 4. EXAMPLE

GEOMETRY OF OPTIMUM STRUCTURE TO SUSTAIN $\frac{m}{b_W} = 1400$

Row	Source	$\bar{T}$ R	$\frac{t_S}{b_W}$	$r_b r_t$	r_b	r_t	$\frac{W}{2\ b_W}$	Remarks
1	Fig. 6	20,000	.025	.913	1.732	.527	.0364	$\frac{t_S}{b_W}$ specified
2	Fig. 6	0	.025	.793	1.587	.5	.0349	
3	Fig. 9	20,000	.0244	.99	1.772	.575	.0365	
4	Fig. 9	0	.0244	.862	1.689	.511	.0350	
5	Table 7 Ref. 6	0	.0237	.962	1.76	.55	.0350	OPTIMUM t_S/b_W

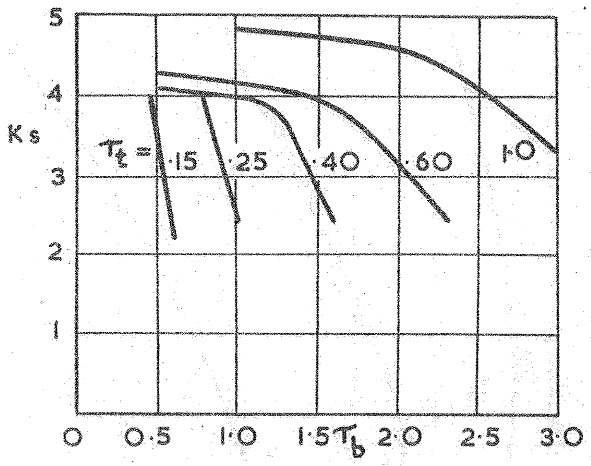

FIG.1 VALUES OF Ks.
(TAKEN FROM REF.3)

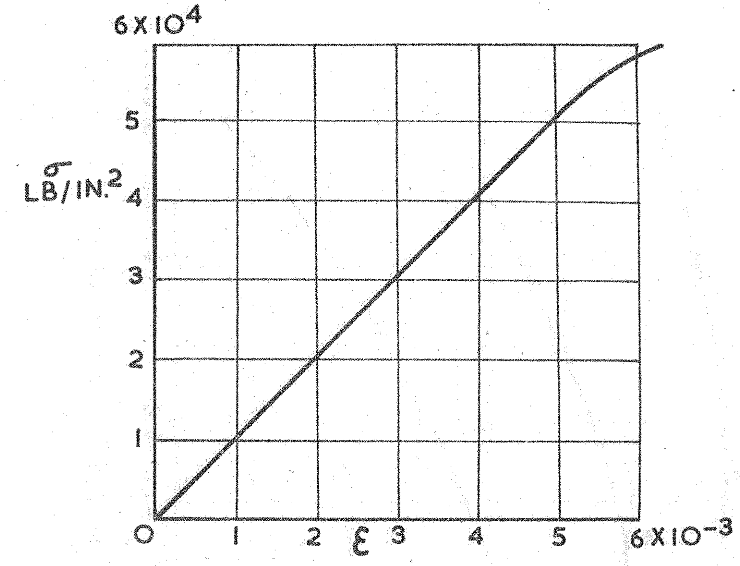

FIG. 3 STRESS-STRAIN CURVE
D.T.D. 687

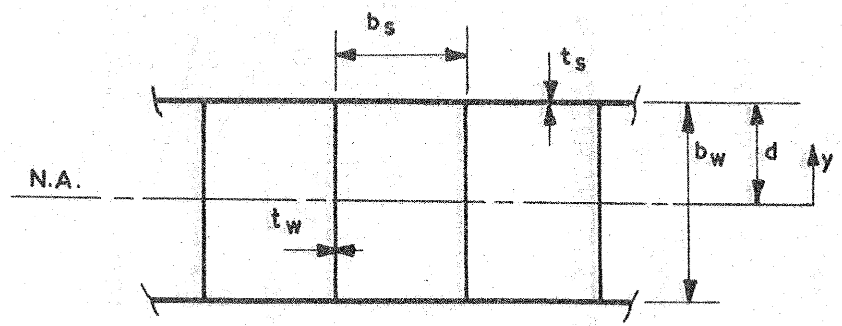

FIG. 2 IDEALISED MULTI-WEB BEAM

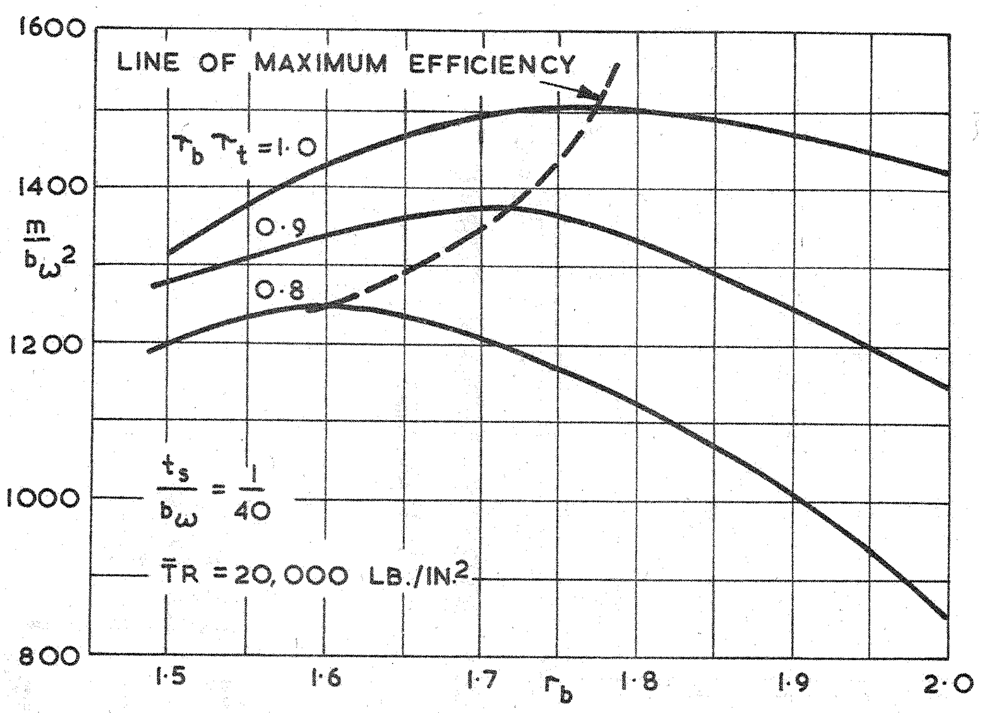

FIG.4 VARIATION OF $\frac{m}{b\omega^2}$ WITH τ_b AND $\tau_b \tau_t$

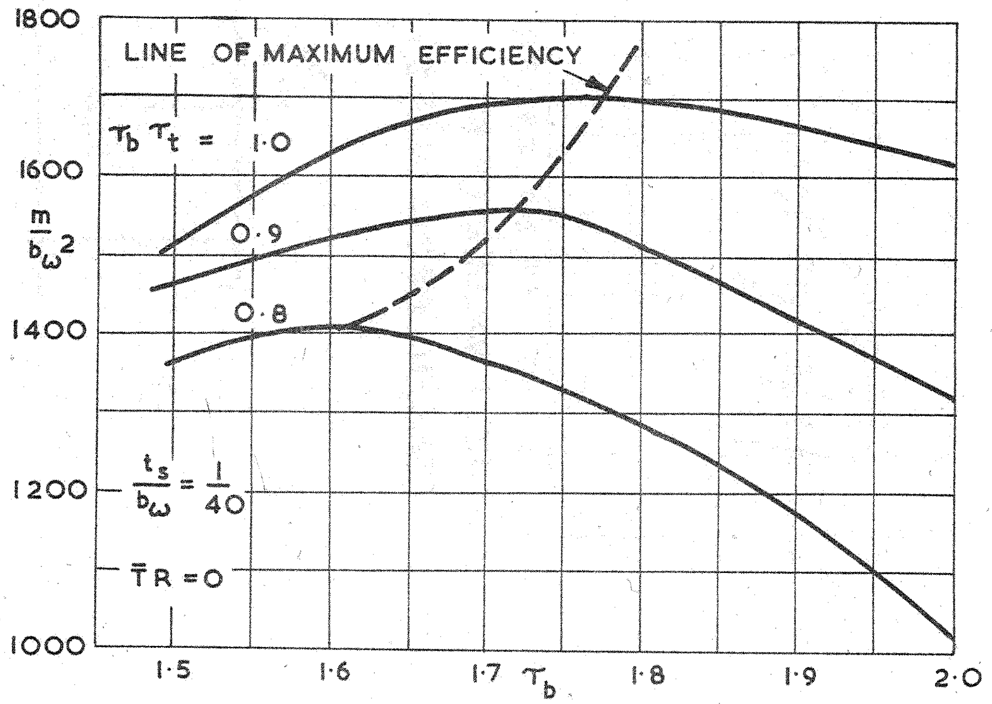

FIG.5 VARIATION OF $\frac{m}{b\omega^2}$ WITH τ_b AND $\tau_b \tau_t$

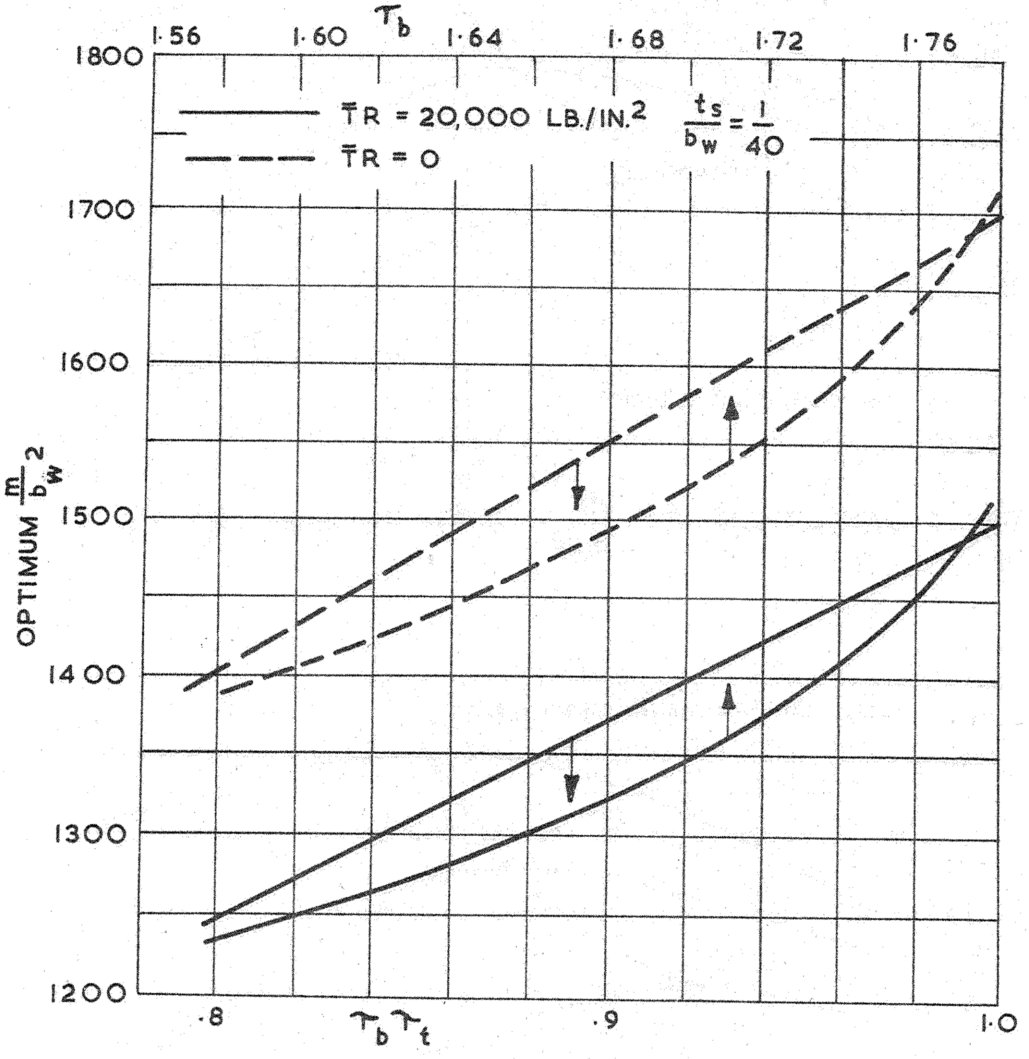

FIG. 6 VARIATION OF OPTIMUM $\frac{m}{b_w{}^2}$ WITH OPTIMUM VALUES OF τ_b AND $\tau_b \tau_t$

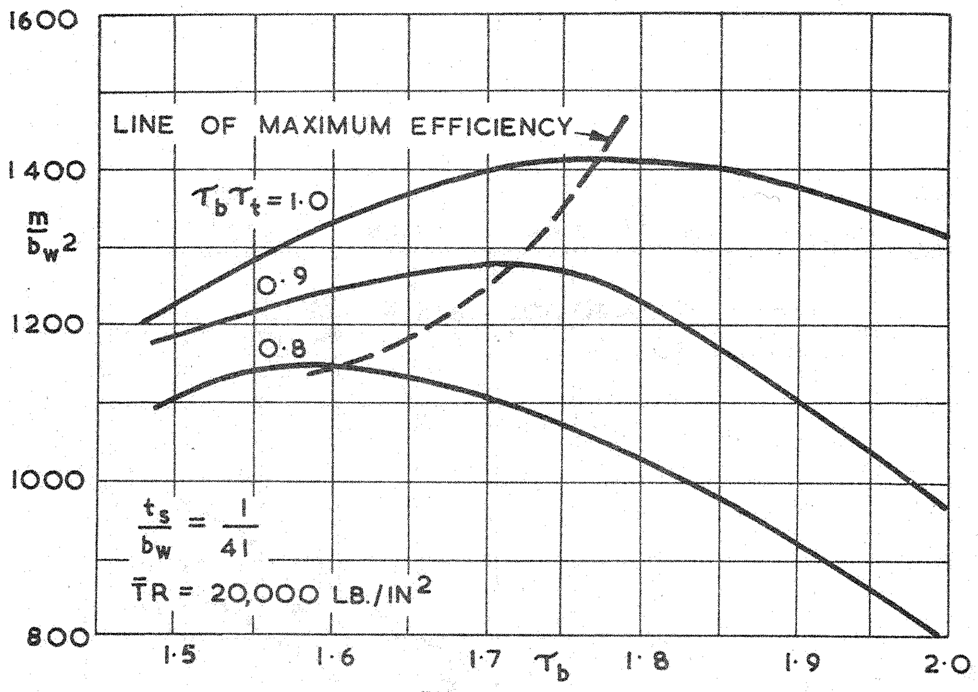

FIG.7 VARIATION OF $\frac{m}{b_w 2}$ WITH τ_b AND $\tau_b\ \tau_t$

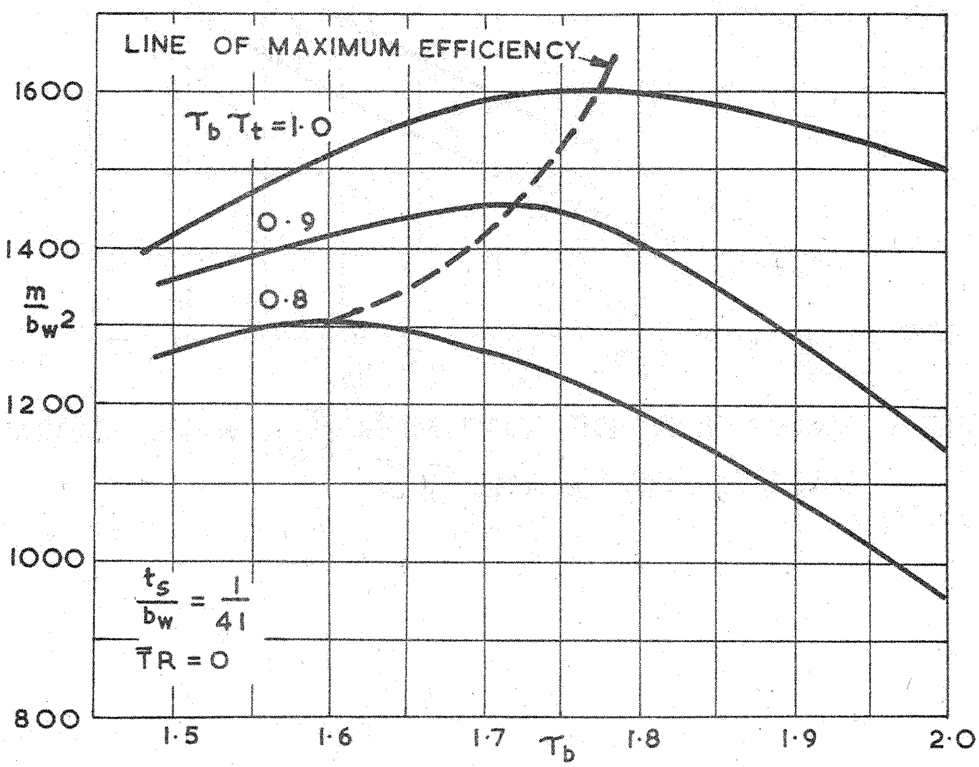

FIG. 8 VARIATION OF $\frac{m}{b_w 2}$ WITH τ_b AND $\tau_b\ \tau_t$

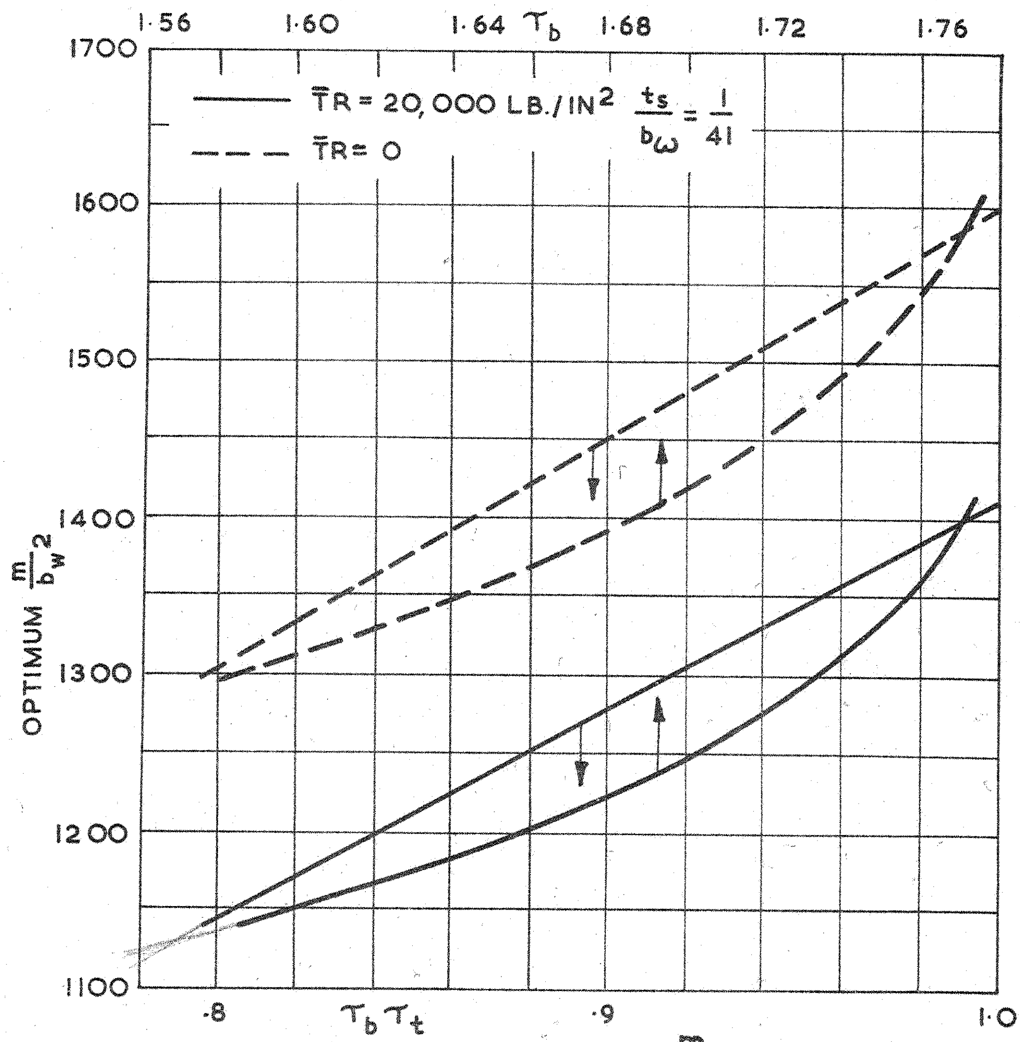

FIG. 9 VARIATION OF OPTIMUM $\frac{m}{b_w 2}$ WITH OPTIMUM VALUES OF τ_b AND $\tau_b \tau_t$