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NOTES ON THE PROBLEM OF THE OPTIMUM DESIGN OF STRUCTURES

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SUMMARY

The urgent need for a systematic approach to the problems of the optimum design of structures is stressed and ideal formulations of these problems are considered. Differential equations and a variational principle are derived for the case of plates loaded in their own planes; these can form the basis for approximate solutions, in the form of optimum distributions of plate thickness and the corresponding stress distributions which are required to equilibrate given systems of external loads.

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Introduction

The real problem of aircraft structural design is the disposal of material in such a way, that it can safely equilibrate given systems of applied forces and at the same time weigh as little as possible. Practical considerations relating to manufacture, maintenance or function will force a departure from such an ideal solution, but a knowledge of ideal optima is clearly of great value as a control. Almost all* of the content of the Theory of Structures is concerned with the strength or stiffness of a given structure subjected to given loads and the designer is left to do the best that he can, using his native wit and the processes of trial and error. This is unsatisfactory, since there is no means of telling how far from the ideal solution any given practical construction lies. This has always been the case, but in view of the very severe loading conditions on modern aeroplanes and missiles, and the vital need to minimise their structure weight, in order to achieve competitive performance, there would seem to be at present a special need for developing in a systematic way, the study of optimum structures. This note is written in the hope that it may contribute towards the encouragement of such developments.

Ideal Formulation of the Problem

Suppose that we have the problem of designing an optimum structure to carry a system of loads, which can be specified as forces distributed through given volumes or over given surfaces in space. Suppose further that we have at our disposal a material, available in a continuous range of densities from zero upwards, and such that the moduli of elasticity and the yield stress increase monotonically with the density (e.g. they might be proportional to the density). A typical optimum design problem would then be to determine that distribution of material density throughout space, such that the given loads can be equilibrated by that material without the yield conditions being exceeded at any point and such that the total weight of material is as small as possible.

* Exceptions include Refs. 1, 2.

This is perhaps the simplest formulation. In some cases it may be necessary to add a requirement for stability of equilibrium. In others, consideration must be given to several different loading systems and stiffness requirements, and to thermal effects as well.

The problem can be made as complex as one likes, but even in its simplest form, it brings with it a number of formidable mathematical difficulties. Its equations are non-linear, since the relations between stress and strain have coefficients depending upon the unknown density and the yield condition in its usual Mises-Hencky form is a quadratic relation between the stresses. The yield condition is also an inequality, not an equation and if it is replaced by an equation, making the material yield at all points, then this completely determines the solution, without any explicit reference to minimum weight. A further difficulty arises in cases where the applied loads are not continuously distributed (e.g. when they act on separate pieces of surface). In this case the "shape" of part of the structural surface is one of the unknowns of the problem.

In view of these difficulties we shall confine ourselves in what follows to the two-dimensional case of plates loaded in their own planes. We shall take the thickness of the plate as our unknown rather than the density and shall consider a material of constant moduli and yield stress. This is clearly equivalent, in this special case, to our general formulation and has been adopted since its equations present a more familiar appearance. Strictly speaking, of course, the variation of thickness invalidates the assumptions of "plane stress", but since the equivalent variable density formulation avoids this objection, we shall disregard it and proceed on the basis of "conventional plate theory". We shall further assume the yield condition to be satisfied at every point of the plate and shall examine, where possible, the relation of this restriction to the condition of minimum weight. Finally, in view of the great difficulty of even the two dimensional equations, we shall seek alternative variational formulation.

Equations for the Two-Dimensional Problem

Consider a plate referred to axes $O(x,y)$ in its plane. Denote the stresses due to loads applied to the edges, in the plane of the plate by f_{xx}, f_{yy} and f_{xy}. Then if "t" is the variable thickness, equilibrium demands the existence of a stress function ϕ such that

$$tf_{xx} = \frac{\partial^2\phi}{\partial y^2}\ , \quad tf_{yy} = \frac{\partial^2\phi}{\partial x^2}\ , \quad tf_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} \qquad \ldots (1)$$

The condition of compatibility for the strains can be written using the strain-stress relations as,

$$\frac{\partial^2}{\partial x^2}(f_{yy} - \nu f_{xx}) + \frac{\partial^2}{\partial y^2}(f_{xx} - \nu f_{yy}) = 2(1+\nu)\frac{\partial^2 f_{xy}}{\partial x\partial y} \qquad \ldots (2)$$

where ν is Poisson's ratio. Finally the Mises-Hencky yield criterion, which is assumed satisfied everywhere, is

$$f_{xx}^2 + f_{yy}^2 - f_{xx}f_{yy} + 3f_{xy}^2 = 3q^2 \qquad \ldots (3)$$

where q is the yield stress for pure shear.

The stress components can be eliminated from (1),(2),(3) to yield a pair of equations for ϕ and t, namely,

$$\left(\frac{1}{t}\right)\nabla^4\phi + 2\frac{\partial}{\partial x}\left(\frac{1}{t}\right)\frac{\partial}{\partial x}\nabla^2\phi + 2\frac{\partial}{\partial y}\left(\frac{1}{t}\right)\frac{\partial}{\partial y}\nabla^2\phi$$
$$+ \frac{\partial^2}{\partial x^2}\left(\frac{1}{t}\right)\left(\frac{\partial^2\phi}{\partial x^2} - \nu\frac{\partial^2\phi}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{t}\right)\left(\frac{\partial^2\phi}{\partial y^2} - \nu\frac{\partial^2\phi}{\partial x^2}\right)$$
$$+ 2(1+\nu)\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{t}\right).\frac{\partial^2\phi}{\partial x\partial y} = 0 \qquad \ldots (4)$$

and,

$$(\nabla^2\phi)^2 - 3\left\{\frac{\partial^2\phi}{\partial x^2}\cdot\frac{\partial^2\phi}{\partial y^2} - \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right\} = 3q^2t^2 \qquad \ldots (5)$$

This would seem to be the simplest formulation of our problem. The variable t can be eliminated, since it is given explicitly by (5), but the result is hardly worth writing. The resulting equation is of fourth order in ϕ and must be solved subject to the usual plate boundary conditions, which are definite if the external loading is given on a closed curve and lightening holes are not considered! The problem is clearly very difficult, but might yield to a numerical approach, like "relaxation", using successive approximations for t, beginning with t = constant, defined by the preceeding approximation for ϕ by equation (5).

A Simple Example

Since the general equations are clearly very difficult, one turns naturally to the simplest possible case of the problem, which is provided by a circle radius "a" loaded by uniform radial tension T per unit length. Let us fill the circle with a plate of thickness t which is a function of the polar coordinate r measured from the centre. Then if V is the radial displacement the stresses f_{rr}, $f_{\theta\theta}$ are given by,

$$f_{rr} = \frac{E}{(1-\nu^2)}\left(\frac{dV}{dr} + \nu\frac{V}{r}\right), \quad f_{\theta\theta} = \frac{E}{(1-\nu^2)}\left(\frac{V}{r} + \nu\frac{dV}{dr}\right) \qquad \ldots (6)$$

where E is Young's Modulus. The condition of equilibrium is,

$$\frac{d}{dr}(t\, r\, f_{rr}) = t\, f_{\theta\theta} \qquad \ldots (7)$$

and the boundary conditions are,

$$(t\, f_{rr})_{r=a} = T \qquad \ldots (8)$$

and $\quad dV/dr \;,\; V/r \;$ finite at $r = 0 \qquad \ldots (9)$

Finally to avoid non-linearity we use the maximum shear stress condition of yielding, which can be written,

$$\left| f_{rr} - f_{\theta\theta} \right| = 2q \qquad \ldots (10a)$$

$$\text{or} \quad \left| f_{rr} \right| = 2q \qquad \ldots (10b)$$

$$\text{or} \quad \left| f_{\theta\theta} \right| = 2q \qquad \ldots (10c)$$

Substitution from (6) in (10) and integration for V yields the solutions,

$$V = \pm \frac{2q(1+\nu)}{E} r \log r + C_1 r \qquad \ldots (11a)$$

$$V = \pm \frac{2q(1-\nu)}{E} r + C_2 r^{-\nu} \qquad \ldots (11b)$$

$$V = \pm \frac{2q(1-\nu)}{E} r + C_3 r^{-\frac{1}{\nu}} \qquad \ldots (11c)$$

where C_1, C_2, C_3 are constants of integration. Solution (11a) is incompatible with (9) and so (10a) cannot be used near $r = 0$. Solutions (11b),(11c) satisfy (a) if $C_2 = C_3 = 0$. They are thus identical and yield constant isotropic strains and stresses which by (6) are given by,

$$f_{rr} = f_{\theta\theta} = 2q \qquad \ldots (12)$$

where the positive sign must clearly be taken.
Substitution in (7) yields $t = \text{const.}$, which is perhaps not unexpected!
The solution of (12) must apply near $r = 0$. Since V and f_{rr} must be continuous it follows from (6) that $f_{\theta\theta}$ is continuous as well and so, at any boundary where (12) ceases to apply, we have $|f_{rr} - f_{\theta\theta}| = 0$ as a boundary condition for the remaining portion of the circle.
This means that (10a) cannot be valid for this region and so (12), which follows from (10b) and (10c), must be valid everywhere.
Equation (8) then gives,

$$t = T/2q \qquad \ldots (13)$$

which is our "optimum design"! This solution although "trivial" illustrates the sort of considerations involved in these design problems.

Variational Formulation

Let us consider a variation in the stress distribution δf_{xx}, δf_{yy}, δf_{xy} and in the thickness δt, such that both the conditions of equilibrium as well as a yielding condition like (3) are satisfied in the varied state.

A well known calculation* then gives the relation

$$\iint\left\{e_{xx}\delta(tf_{xx}) + e_{yy}\delta(tf_{yy}) + 2e_{xy}\delta(tf_{xy})\right\}dxdy = 0 \qquad \ldots (14)$$

where e_{xx}, e_{yy} and e_{xy} are the components of the strain tensor.
Introducing the density of strain energy W per unit volume given by

$$\begin{aligned} W &= \frac{1}{2}(f_{xx}\, e_{xx} + f_{yy}\, e_{yy} + 2f_{xy}\, e_{xy}) \\ &= \frac{1}{2E}\left\{(f_{xx} + f_{yy})^2 + 2(1+\nu)(f_{xy}^2 - f_{xx}f_{yy})\right\} \\ &= \frac{(1-2\nu)}{6E}(f_{xx} + f_{yy})^2 + \frac{(1+\nu)}{3E}(f_{xx}^2 + f_{yy}^2 - f_{xx}f_{yy} + 3f_{xy}^2) \end{aligned} \qquad \ldots (15)$$

we can write (14) in the form,

$$\iint(t\,\delta W + 2W\,\delta t)dxdy = 0 \qquad \ldots (16)$$

This is the variational equation of our problem. We note that, if t is not varied, it reduces to the usual minimum energy principle.

Equation (16) can be used with any yielding hypothesis. Suppose for the moment that we are old fashioned and adopt the Haigh theory and write,

$$f_{xx}^2 + f_{yy}^2 - 2\nu f_{xx}f_{yy} + 2(1+\nu)f_{xy}^2 = 2(1+\nu)q^2 \qquad \ldots (17)$$

This is equivalent by (15) to W = constant. We see then that (16) becomes

$$\delta\iint t\,dxdy = 0 \qquad \ldots (18)$$

*If the formulae for the strains in terms of the displacements are substituted in (14), an application of Green's Theorem reduces this equation to the varied equilibrium equations.

i.e. a condition of minimum weight! Conversely, the condition of minimum weight (18) implies compatibility of strain as determined by (16), only if W = constant and so, if any hypothesis but that of Haigh is adopted for yielding, the weight of the optimum design is not a strict minimum.

If the Mises-Hencky yielding condition of (3) is used, then from (3) and (15) we find

$$W = \frac{(1-2\nu)}{6E}(f_{xx} + f_{yy})^2 + \frac{(1+\nu)}{E} q^2 \qquad \ldots (19)$$

and the variational equation (16) can be put into the special form

$$\iint \left[t(f_{xx}+f_{yy})(\delta f_{xx}+\delta f_{yy}) + \left\{ (f_{xx}+f_{yy})^2 + \frac{6(1+\nu)}{(1-2\nu)} q^2 \right\} \delta t \right] dxdy = 0 \qquad \ldots (20)$$

These variational equations may well be used to construct approximate solutions to optimum design problems. One might begin with a stress function ϕ which is chosen so that the boundary conditions are satisfied and which depends upon a number of unknown parameters or functions. A formula for t then follows from (5) and the stress components are obtainable from (1). Substititution in (20) will then yield by the usual processes of the calculus of variations a series of equations for the unknowns. These will hardly be simple, but since they may well have the form of algebraic or ordinary differential equations, they will probably yield more readily to treatment than equation (4).

Conclusions

(1) There is an urgent need for the development of systematic methods for the determination of optimum structural designs and their corresponding ideal weights.

(2) This means by and large the creation of a new branch of the Theory of Structures.

(3) The construction of ideal solutions using materials, with continuously varying density or thickness, transmitting given systems of loads, is of value but methods for the consideration of alternative loading and stiffness requirements must be developed.

(4) Techniques for positioning unloaded boundaries are also required.

(5) The extension of the theory to three dimensions is perhaps not terribly urgent for aircraft work, but extension to curved shells is clearly important.

(6) The equations (4) and (5) for the two-dimensional form of the simplest kind of problem are difficult, but will probably yield to a relaxation type of analysis.

(7) Exact solutions though of interest will doubtless be rather trivial.

(8) The equations of the problem can be given a variational form as in (16),(18) and (20). Equations of this kind may well be the main basis for approximate solution of problems.

References

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