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NOTES ON THE THEORY OF PLANAR ELECTRIC NETWORKS

by

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S. R. Deards

SUMMARY

The primary purpose of these notes is to sketch an approach to the elementary algebraic theory of planar electric networks which conduces to a formal dual correspondence between the so-called mesh and nodal representations of planar networks that is complete in every respect.

It is also shown that if general variables are employed in the analysis of a network the number of voltage or current equations that serve to describe the network is infinite.

Finally, it is demonstrated that the impedance or admittance matrix of a passive network is not necessarily symmetrical.

CONTENTS

1. Planar linear graphs

Our definition of a planar linear graph will be related to the connectivity of the two-dimensional orientable closed surface on which we shall suppose it to be delineated. A surface is said to have connectivity ε if the maximum number of Jordan curves * that can be traced upon it without separating it into two or more regions is $\varepsilon-1$. A surface of unit connectivity is called simply-connected while surfaces of connectivity greater than unity are called multiply-connected. If a linear graph is mappable on a simply-connected surface without the crossing of branches (Ref.1) it is similarly mappable on a plane and will therefore be called planar.

2. Sagittal graphs

The use of a linear graph as a device for charting the flow of electric currents in an electric network requires that we associate with each branch the notion of direction. If this is done by placing an arbitrarily directed arrow-head on every branch, the graph will be called sagittal (Ref.2). The direction associated with a branch will also be taken to connote the idea of polarity. If branch k is directed away from node v (Ref.1) and toward node w then node v will be regarded as the positive node and node w the negative node of branch k.

3. Mesh tie-sets and node cut-sets

The mapping of a planar linear graph on a simply-connected surface (that of a sphere, for example) divides the surface into a number of regions. These regions will be called meshes and the set of branches bounding a mesh will be described as the tie-set ⊖ of the mesh. If a planar graph contains b branches and n nodes, it divides the surface into m meshes (and therefore contains m mesh tie-sets) according to Euler's relation between the faces, edges, and vertices of a polyhedron of characteristic 2, thus

$$m = b - n + 2 \tag{3.1}$$

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* A closed curve that does not intersect itself; the homeomorph of a circle.

⊖ The term tie-set is due to Professor E.A. Guillemin.

The set of branches meeting at a node will be described as the cut-set (Ref.1) of the node. If a planar graph contains n nodes it contains n node cut-sets.

An example of a planar sagittal graph containing six mesh tie-sets and five node cut-sets is shown in Fig.3.1. It should be noted that, because the graph is here mapped on a plane surface, it is necessary to suppose the area surrounding the figure to represent one of the six meshes. The meshes are numbered 1,2, ..., 6.

With each mesh tie-set of a planar graph we shall associate the idea of direction. For this purpose we suppose all the branches not included in the tie-set belonging to mesh q to dilate to the point of rupture so that all meshes other than mesh q unite to form one. Taking the direction as clockwise around the tie-set, the result will be called the directed tie-set of mesh q. The six directed mesh tie-sets of the graph of Fig.3.1, for example, are represented in Fig.3.2. It should be explained that directed tie-set 6 is that belonging to the mesh corresponding to the area surrounding the figure in Fig.3.1. and is therefore reproduced as if viewed from the opposite side of the closed surface on which the graph is supposed to be drawn. It is evident that each branch of a planar graph is a member of two mesh tie-sets; a similarly directed set and an oppositely directed set.

With each node cut-set of a planar graph we shall associate the idea of polarity. For this purpose we suppose all the branches not included in the cut-set belonging to node q to contract to the point when the meshes bounded by these branches vanish so that all nodes other than node q unite to form one. Taking the polarity as that with node q positive and the coalesced nodes negative, the result will be called the polarized cut-set of node q. The five polarized cut-sets of the graph of Fig.3.1 are represented in Fig.3.3. It is evident that each branch of a planar graph is a member of two node cut-sets; a similarly polarized set and an oppositely polarized set.

4. Algebraic representation

A planar sagittal graph may be characterized by either of two matrices. The first of these is a rectangular matrix $\underline{C}$ of order m by b with components:

A bar below a letter will be employed to indicate a matrix.

$$c_{rs} = \begin{cases} 1 & \text{if the tie-set of mesh r has the same direction as branch s} \\ -1 & \text{if the tie-set of mesh r has the opposite direction to branch s} \\ 0 & \text{if the tie-set of mesh r does not contain branch s} \end{cases}$$

The matrix $\underline{C}$ will be called the mesh-branch incidence matrix of the graph; it has rank $c = m - 1$ and column nullity $p = n - 1$. For the graph of Fig.3.1,

$$\underline{C} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad (4.1)$$

Row r of $\underline{C}$ enumerates and orientates the branches belonging to the directed tie-set of mesh r and column s of $\underline{C}$ identifies the two mesh tie-sets, one similarly directed and the other oppositely directed, that contain branch s. Thus, the columns of $\underline{C}$ contain two non-zero entries; 1 and -1.

The second matrix $\underline{A}$ is also rectangular with order n by b and components:

$$a_{rs} = \begin{cases} 1 & \text{if the cut-set of node r has the same polarity as branch s} \\ -1 & \text{if the cut-set of node r has the opposite polarity to branch s} \\ 0 & \text{if the cut-set of node r does not contain branch s} \end{cases}$$

The matrix $\underline{A}$ will be called the node-branch incidence matrix of the graph; it has rank $p = n - 1$ and column nullity $c = m - 1$. For the graph of Fig.3.1,

$$\underline{A} = \begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix} \quad (4.2)$$

Row r of $\underline{A}$ enumerates and orientates the branches belonging to the polarized cut-set of node r and column s of $\underline{A}$ identifies the two node cut-sets, one similarly polarized and the other oppositely polarized, that contain branch s. Thus, as in C, the columns of $\underline{A}$ contain two non-zero entries; 1 and -1.

5. The dual of a planar sagittal graph

If $\underline{C}$ is the m by b mesh-branch incidence matrix and $\underline{A}$ the n by b node-branch incidence matrix of a planar sagittal graph G_1, the sagittal graph G_2 that has $\underline{C}$ for node-branch incidence matrix and $\underline{A}$ for mesh-branch incidence matrix is defined as the dual of G_1. This definition arises from the recognition of the duality between mesh and node. A branch is a self-dual. Hence the graph G_2 contains b branches, n meshes, and m nodes.

The following rules facilitate the construction of the graph G_2 from the graph G_1 :

(i) On the reverse side of the surface on which the graph G_1 is drawn, indicate one node on every mesh. Number the nodes according to the corresponding meshes.

(ii) Interconnect the nodes with branches in such a way that each branch crosses a branch of the original graph. Number the branches according to those in the original graph to which they correspond. Number the meshes thus formed according to the corresponding nodes in the original graph.

(iii) If, in the original graph, branch k belongs to the tie-set of mesh r and has the same (respectively opposite) direction then branch k in the dual belongs to the cut-set of node r and has the same (respectively opposite) polarity.

The procedure is illustrated in Fig.5.1 (a) in which the dual of the sagittal graph of Fig.3.1 is constructed; the dual is reproduced in Fig.5. (1(b).

Rule (i) above differs from that usually given but if the two sides of the surface on which the graph is drawn are regarded as duals, the application of the rules to the dual sagittal graph G_2 will result in the original sagittal graph G_1. Otherwise it is necessary to distinguish between a graph and its dual by employing oppositely directed mesh tie-sets when passing from the dual to the original. Such a distinction is incompatible with the concept of duality.

6. Topological independence

We now identify (i) the directions and (ii) the polarities associated with the branches of a sagittal graph with (i) the directions of the instantaneous electric currents in and (ii) the polarities of the instantaneous potential differences between the terminals of the branches in the physical network.

If $\underline{v}$ and $\underline{i}$ represent respectively column matrices of the branch potential differences $v_1, v_2, \ldots, v_b$ and the branch currents $i_1, i_2, \ldots, i_b$ in the physical network then the application of the Kirchhoff voltage law to every mesh tie-set of the network has the representation

$$\underline{C}\underline{v} = \underline{o} \qquad (6.1)$$

and the application of the Kirchhoff current law to every node cut-set of the network has the expression

$$\underline{A}\underline{i} = \underline{o} \qquad (6.2)$$

Let the clockwise directions associated with the mesh tie-sets of the sagittal graph be identified with those of hypothetical tie-set currents $i'_1, i'_2, \ldots, i'_m$. If $\underline{i}'$ represents the column matrix of these currents then $\underline{i}$ and $\underline{i}'$ are related in the equation

$$\underline{i} = \underline{C}_{(t)}\underline{i}' \qquad (6.3)$$

where $\underline{C}_{(t)}$ is the transpose of $\underline{C}$. Similarly, let the polarities associated with the node cut-sets of the sagittal graph be identified with those of hypothetical cut-set potential differences $v_1', v_2', \ldots, v_n'$. If $\underline{v}'$ represents the column matrix of these potential differences then $\underline{v}$ and $\underline{v}'$ are related in the equation

$$\underline{v} = \underline{A}_{(t)}\underline{v}' \qquad (6.4)$$

where $\underline{A}_{(t)}$ is the transpose of $\underline{A}$.

However, the rank of $\underline{C}$ is $c = m - 1$ and the rank of $\underline{A}$ is $p = m - 1$. That is, the ranks of $\underline{C}$ and $\underline{A}$ are each one less than the number of rows indicating that the rows are not linearly independent; indeed, any row of $\underline{C}$ or $\underline{A}$ is the negative of the sum of all the other rows. The set of mesh tie-sets represented by any c of the m rows of $\underline{C}$ and the set of node cut-sets represented by any p of the n rows of $\underline{A}$ are each said to be topologically linearly independent; either set completely characterizes the configuration of the system. Thus, of the m directed mesh tie-sets that may be defined on a planar sagittal graph, one is redundant for the complete specification of the system on a mesh basis. The mesh associated with the redundant tie-set will be called the datum mesh and designated mesh 0. Similarly, of the n polarized node cut-sets that may be defined, one is redundant for the complete specification of the system on a nodal basis. The node associated with the redundant cut-set will be called the datum node and designated node 0.

It follows that any c of the m mesh tie-set currents in the column matrix $\underline{i}'$ are sufficient to represent the b branch currents. Likewise, any p of the n cut-set potential differences in the column matrix $\underline{v}'$ are sufficient to represent the b branch potential differences.

The rank of the incidence matrix $\underline{C}$ or, which is equivalent, the column nullity of the incidence matrix $\underline{A}$ will be called the cyclomatic index and the rank of the incidence matrix $\underline{A}$ or, which is equivalent, the column nullity of the incidence matrix $\underline{C}$ we shall take the liberty of calling the nodalic index of the corresponding linear graph.† If (3.1) is written as

$$(m - 1) = b - (n - 1) \tag{6.5}$$

we have the well-known equation relating the cyclomatic index c and the nodalic index p, thus

$$c = b - p \tag{6.6}$$

It is evident that the cyclomatic index and the nodalic index are duals.

7. Mesh tie-set and node cut-set transformations

Considering first the mesh basis representation, let us suppose that mesh 3 in the graph of Fig.3.1 is selected as the datum mesh and the remaining meshes renumbered as in Fig.7.1. The corresponding incidence matrix $\underline{C}_1$ is obtainable from that in (4.1) by suppressing row3, thus

$$\underline{C}_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{7.1}$$

Let it now be required to derive from $\underline{C}_1$ the matrix $\underline{C}_2$ corresponding to a change of datum mesh from mesh 0 to mesh 5 with mesh 0 renumbered 5 and mesh 5 renumbered 0. The matrix $\underline{C}_2$ may be obtained from $\underline{C}_1$ by exchanging row 5 for row 3 of $\underline{C}$. Row 3 of $\underline{C}$ is, however, the negative of the sum of all the rows of $\underline{C}_1$. The matrix $\underline{C}_2$ may therefore be obtained from $\underline{C}_1$ (according to the theory of elementary matrices) by premultiplying it by the

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† The terms cyclomatic index (J.B.Listing) and nodalic index are adopted in preference to Whitney's terms nullity and rank because the column nullity of $\underline{A}$ is equal to the rank of $\underline{C}$ and vice versa.

matrix $\underline{P}$ obtained from the 5 by 5 unit matrix by substituting in the place of row 5 the negative of the sum of all the rows, thus

$$\underline{C}_2 = \underline{P}\underline{C}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix} \quad \underline{C}_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \qquad (7.2)$$

We have, therefore,

> Theorem I: If $\underline{C}_j$ is the c by b mesh-branch incidence matrix of a planar sagittal graph and if $\underline{P}$ is the c by c unit matrix with the qth row exchanged for a row of (-1)'s, the c by b mesh-branch incidence matrix $\underline{C}_k = \underline{P}\,\underline{C}_j$ represents the graph after the datum mesh has been changed from mesh 0 to mesh q with mesh 0 renumbered q and mesh q renumbered 0.

If $\underline{i}'_j$ is the column matrix of mesh tie-set currents $i'_1, i'_2, \ldots, i'_c$ corresponding to a choice of c topologically independent tie-sets with mesh-branch incidence matrix $\underline{C}_j$, the equation

$$\underline{i} = \underline{C}_{j(t)}\,\underline{i}'_j \qquad (7.3)$$

is an adequate representation of the b branch currents. If $\underline{i}'_k$ is the column matrix of tie-set currents after a change of datum mesh according to incidence matrix $\underline{C}_k = \underline{P}\,\underline{C}_j$, then $\underline{i}'_j$ and $\underline{i}'_k$ are related in the equation

$$\underline{i}'_j = \underline{P}_{(t)}\,\underline{i}'_k \qquad (7.4)$$

The determinant, det $\underline{P}$ of the transformation matrix $\underline{P}$ is -1. Furthermore, $\underline{P}$ is such that $\underline{P}^2$ is the unit matrix; that is, $\underline{P}$ is involuntory. It follows, therefore, that

$$\underline{i}'_k = \underline{P}_{(t)}\ \underline{i}'_j \tag{7.5}$$

Considering now the nodal basis representation, let us suppose that node 3 in the graph of Fig.3.1 is selected as the datum node and the remaining nodes renumbered as in Fig.7.2. The incidence matrix $\underline{A}_1$ is obtainable from that in (4.2) by suppressing row 3, thus

$$\underline{A}_1 = \begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix} \tag{7.6}$$

Let it now be required to derive from $\underline{A}_1$ the matrix $\underline{A}_2$ corresponding to a change of datum node from node 0 to node 4 with node 0 renumbered 4 and node 4 renumbered 0. The matrix $\underline{A}_2$ may be obtained from $\underline{A}_1$ by exchanging row 4 for row 3 of $\underline{A}$. Row 3 of $\underline{A}$ is, however, the negative of the sum of all the rows of $\underline{A}_1$. The matrix $\underline{A}_2$ may therefore be obtained from $\underline{A}_1$ by premultiplying it by the matrix $\underline{P}$ obtained from the 4 by 4 unit matrix by substituting in the place of row 4 the negative of the sum of all the rows, thus

$$\underline{A}_2 = \underline{P}\,\underline{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix} \quad \underline{A}_1 = \begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \tag{7.7}$$

We have, therefore,

Theorem II: If $\underline{A}_j$ is the p by b node-branch incidence matrix of a planar sagittal graph and if $\underline{P}$ is the p by p unit matrix with the qth row exchanged for a row of (-1)'s, the p by b node-branch incidence matrix $\underline{A}_k = \underline{P}\ \underline{C}_j$ represents the graph after the datum node has been changed from node 0 to node q with node 0 renumbered q and node q renumbered 0.

If $\underline{v}'_j$ is the column matrix of node cut-set potential differences $v'_1, v'_2, \ldots, v'_p$ corresponding to a choice of p topologically independent cut-sets with node-branch incidence matrix $\underline{A}_j$, the equation

$$\underline{v} = \underline{A}_{j(t)}\ \underline{v}'_j \tag{7.8}$$

is an adequate representation of the b branch potential differences. If $\underline{v}'_k$ is the column matrix of cut-set potential differences after a change of datum node according to incidence matrix $\underline{A}_k = \underline{P}\ \underline{A}_j$, then $\underline{v}'_j$ and $\underline{v}'_k$ are related in the equation

$$\underline{v}'_j = \underline{P}_{(t)}\ \underline{v}'_k \tag{7.9}$$

whence

$$\underline{v}'_k = \underline{P}_{(t)}\ \underline{v}'_j \tag{7.10}$$

8. Network equations

The k th branch of the general b-branch electric network has the alternative representations shown in Fig.8.1. It will be supposed that either (a) or (b) is used exclusively in the representation of the complete network according as the nodalic index p of its linear graph is greater or less than the cyclomatic index c.

Evidently, from the Helmholtz branch representation in Fig.8. 1(a), the column matrix $\underline{v}$ can be expressed as

$$\underline{v} = \underline{u} - \underline{e} \tag{8.1}$$

where $\underline{u}$ is the column matrix of branch impedance potential differences $u_1, u_2, \ldots, u_b$ with the same polarities as those of the branch potential differences and $\underline{e}$ is the column matrix of branch impressed electromotive forces $e_1, e_2, \ldots, e_b$ with polarities opposite to those of the branch potential differences.

If $\underline{Z}$ is the square non-singular b by b matrix representing the impedances of the network branches with components:

$$z_{rs} = \begin{cases} \text{the impedance of branch } r \text{ for } s = r \\ \text{the impedance coupling branches } r \text{ and } s \text{ for } s \neq r \end{cases}$$

then

$$\underline{Z}\,\underline{i} = \underline{u} \tag{8.2}$$

In general, z_{sr} is not the same as z_{rs}. By substituting (7.3) in (8.2) and premultiplying by $\underline{C}_j$ we obtain, in view of (8.1),

$$\underline{C}_j\,\underline{Z}\,\underline{C}_{j(t)}\,\underline{i}'_j = \underline{C}_j\underline{u} = \underline{C}_j\underline{v} + \underline{C}_j\underline{e} \tag{8.3}$$

but, according to (6.1), $\underline{C}_j\underline{v} = \underline{o}$ so that (8.3) may be written

$$\underline{Z}'_{jj}\,\underline{i}'_j = \underline{e}'_j \tag{8.4}$$

where

$$\underline{Z}'_{jj} = \underline{C}_j\,\underline{Z}\,\underline{C}_{j(t)} \tag{8.5}$$

and

$$\underline{e}'_j = \underline{C}_j\,\underline{e} \tag{8.6}$$

$\underline{Z}'_{jj}$ is the non-singular c by c mesh tie-set impedance matrix corresponding to the c independent mesh tie-sets represented by $\underline{C}_j$. If the network is passive $\underline{Z}'_{jj}$ is symmetrical but if the network contains unilateral impedances, such as thermionic amplifying valves, then $\underline{Z}'_{jj}$ is asymmetrical. The column matrix $\underline{e}'_j$ is that of the mesh tie-set impressed electromotive forces e'_1, e'_2, ... e'_c corresponding to the same c independent mesh tie-sets.

By premultiplying (8.4) by the inverse of $\underline{Z}'_{jj}$ we obtain

$$\underline{i}'_j = \underline{Z}'^{-1}_{jj} \; \underline{e}'_j$$

hence, by (7.3)

$$\underline{i} = \underline{C}_{j(t)} \, \underline{i}'_j = \underline{C}_{j(t)} \, \underline{Z}'^{-1}_{jj} \; \underline{e}'_j$$

that is

$$\underline{i} = \underline{C}_{j(t)} \, (\underline{C}_j \, \underline{Z} \, \underline{C}_{j(t)})^{-1} \, \underline{C}_j \, \underline{e} \tag{8.7}$$

which is Kron's equation (Ref.3) giving the branch current column matrix $\underline{i}$ in terms of the column matrix of branch impressed electromotive forces $\underline{e}$, the branch impedance matrix $\underline{Z}$, and the mesh-branch incidence matrix $\underline{C}_j$.

Now let the datum mesh be changed from that according to the incidence matrix $\underline{C}_j$ to that according to the incidence matrix $\underline{C}_k = \underline{P} \, \underline{C}_j$. By substituting (7.4) in (8.4) and premultiplying the result by $\underline{P}$ there is obtained

$$\underline{P} \, \underline{Z}'_{jj} \, \underline{P}_{(t)} \underline{i}'_k = \underline{P} \, \underline{e}'_j \tag{8.8}$$

which may be written as

$$\underline{Z}'_{kk} \, \underline{i}'_k = \underline{e}'_k \tag{8.9}$$

where

$$\underline{Z}'_{kk} = \underline{P} \, \underline{Z}'_{jj} \, \underline{P}_{(t)} \tag{8.10}$$

and

$$\underline{e}'_k = \underline{P} \, \underline{e}'_j \tag{8.11}$$

$\underline{Z}'_{kk}$ is the non-singular c by c mesh tie-set impedance matrix corresponding to the c independent mesh tie-sets represented by $\underline{C}_k$. From (8.10) it is evident that the relation between $\underline{Z}'_{kk}$ and $\underline{Z}'_{jj}$ is one of congruence. The column matrix $\underline{e}'_k$ is that of the mesh tie-set electromotive forces corresponding to the same c independent mesh tie-sets. By substituting (8.5) in (8.10) we see that

$$\underline{Z}'_{kk} = \underline{P}\,\underline{C}_j\,\underline{Z}\,\underline{C}_{j(t)}\,\underline{P}_{(t)} = \underline{P}\,\underline{C}_j\,\underline{Z}\,(\underline{P}\,\underline{C}_j)_{(t)} = \underline{C}_k\,\underline{Z}\,\underline{C}_{k(t)} \qquad (8.12)$$

If (8.9) is premultiplied by the inverse of $\underline{Z}'_{kk}$ we obtain

$$\underline{i}'_k = \underline{Z}'^{-1}_{kk}\ \underline{e}'_k$$

which leads to the branch current equation

$$\underline{i} = \underline{C}_{k(t)}\ (\underline{C}_k\,\underline{Z}\,\underline{C}_{k(t)})^{-1}\ \underline{C}_k\,\underline{e} \qquad (8.13)$$

From the foregoing we have

Theorem III: If the equation $\underline{Z}'_{jj}\,\underline{i}'_j = \underline{e}'_j$ is the mesh tie-set voltage equation of a given network, the matrix $\underline{Z}'_{kk}$ in the equation $\underline{Z}'_{kk}\,\underline{i}'_k = \underline{e}'_k$ representing the network after the datum mesh has been changed from mesh 0 to mesh q (with mesh 0 renumbered q and mesh q renumbered 0) may be obtained from the matrix $\underline{Z}'_{jj}$ in the original equation by substituting for the q th column of $\underline{Z}'_{jj}$ the negative of the sum of all its columns and then substituting the negative of the sum of all the rows of the resulting matrix for the q th row. The column matrix $\underline{e}'_k$ in the new equation may be obtained from the matrix $\underline{e}'_j$ in the original equation by substituting for the q th component of $\underline{e}'_j$ the negative of the sum of all its components.

The part of this theorem that refers to the impedance matrix is the dual of Shekel's first theorem for the case of planar networks (Ref.4)

From (8.10),

$$\det \underline{Z}'_{kk} = \det \underline{P} \cdot \det \underline{Z}'_{jj} \cdot \det \underline{P}_{(t)} \tag{8.14}$$

but $\det \underline{P} = \det \underline{P}_{(t)} = -1$ so that

$$\det \underline{Z}'_{kk} = \det \underline{Z}'_{jj} \tag{8.15}$$

hence,

> <u>Theorem IV</u>: The determinant of the mesh tie-set impedance matrix of a planar network is invariant for a change of datum mesh.

This is the dual of Shekel's second theorem for the case of planar networks (Ref.4)

Turning now to the Norton branch representation in Fig.8.1(b), it is evident that the column matrix $\underline{i}$ can be expressed as

$$\underline{i} = \underline{j} - \underline{a} \tag{8.16}$$

where $\underline{j}$ is the column matrix of branch admittance currents $j_1, j_2, \ldots, j_b$ with directions the same as those of the branch currents and $\underline{a}$ is the column matrix of impressed branch currents $a_1, a_2, \ldots, a_b$ with directions opposite to those of the branch currents.

If $\underline{Y}$ is the square non-singular b by b matrix representing the admittances of the network branches with components:

$$y_{rs} = \begin{cases} \text{the admittance of branch } r \text{ for } s = r \\ \text{the admittance coupling branches } r \text{ and } s \text{ for } s \neq r \end{cases}$$

then

$$\underline{Y}\,\underline{v} = \underline{j} \tag{8.17}$$

In general, y_{sr} is not the same as y_{rs}. By substituting (7.8) in (8.17) and premultiplying by $\underline{A}_j$ we obtain, in view of (8.16),

$$\underline{A}_j\,\underline{Y}\,\underline{A}_{j(t)}\,\underline{v}'_j = \underline{A}_j\,\underline{a} + \underline{A}_j\,\underline{i} \tag{8.18}$$

but according to (6.2), $\underline{A}_j\,\underline{i} = \underline{o}$ as that (8.18) may be written

$$\underline{Y}'_{jj}\,\underline{v}'_j = \underline{a}'_j \tag{8.19}$$

where

$$\underline{Y}'_{jj} = \underline{A}_j\,\underline{Y}\,\underline{A}_{j(t)} \tag{8.20}$$

and

$$\underline{a}'_j = \underline{A}_j\,\underline{a} \tag{8.21}$$

$\underline{Y}'_{jj}$ is the non-singular p by p node cut-set admittance matrix corresponding to the p independent node cut-sets represented by $\underline{A}_j$. If the network is passive $\underline{Y}'_{jj}$ is symmetrical but if the network contains unilateral admittances then $\underline{Y}'_{jj}$ is asymmetrical. The column matrix $\underline{a}'_j$ is that of the node cut-set impressed currents a'_1, a'_2, ..., a'_p corresponding to the same p independent node cut-sets.

By multiplying (8.19) by the inverse of $\underline{Y}'_{jj}$ we obtain

$$\underline{v}'_j = \underline{Y}'^{-1}_{jj}\,\underline{a}'_j$$

hence, by (7.8)

$$\underline{v} = \underline{A}_{j(t)}\,\underline{v}'_j = \underline{A}_{j(t)}\,\underline{Y}'^{-1}_{jj}\,\underline{a}'_j$$

that is

$$\underline{v} = \underline{A}_{j(t)}\,(\underline{A}_j\,\underline{Y}\,\underline{A}_{j(t)})^{-1}\,\underline{A}_j\,\underline{a} \tag{8.22}$$

which is Kron's equation (Ref.3) giving the branch potential difference matrix $\underline{v}$ in terms of the column matrix of impressed currents $\underline{a}$, the branch admittance matrix $\underline{Y}$, and the node-branch incidence matrix $\underline{A}_j$.

Let us now change the datum node from that according to the incidence matrix $\underline{A}_j$ to that according to the incidence matrix $\underline{A}_k = \underline{P}\,\underline{A}_j$. By substituting (7.9) in (8.19) and premultiplying the result by $\underline{P}$ we obtain

$$\underline{P}\,\underline{Y}'_{jj}\,\underline{P}_{(t)}\,\underline{v}'_k = \underline{P}\,\underline{a}'_j \qquad (8.23)$$

which may be written

$$\underline{Y}'_{kk}\,\underline{v}'_k = \underline{a}'_k \qquad (8.24)$$

where

$$\underline{Y}'_{kk} = \underline{P}\,\underline{Y}'_{jj}\,\underline{P}_{(t)} \qquad (8.25)$$

and

$$\underline{a}'_k = \underline{P}\,\underline{a}'_j \qquad (8.26)$$

$\underline{Y}'_{kk}$ is the non-singular p by p node cut-set admittance matrix corresponding to the p independent node cut-sets represented by $\underline{A}_k$. From (8.25) it is evident that the relation between $\underline{Y}'_{kk}$ and $\underline{Y}_{jj}$ is one of congruence. The column matrix $\underline{a}'_k$ is that of the node cut-set impressed currents corresponding to the same p independent node cut-sets. By substituting (8.20) in (8.25) we see that

$$\underline{Y}'_{kk} = \underline{P}\,\underline{A}_j\,\underline{Y}\,\underline{A}_{j(t)}\,\underline{P}_{(t)} = \underline{P}\,\underline{A}_j\,\underline{Y}\,(\underline{P}\,\underline{A}_j)_{(t)} = \underline{A}_k\,\underline{Y}\,\underline{A}_{k(t)} \qquad (8.27)$$

By premultiplying (8.24) by the inverse of $\underline{Y}_{kk}$ we obtain

$$\underline{v}'_k = \underline{Y}'^{-1}_{kk}\,\underline{a}'_k$$

which leads to the branch potential difference equation

$$\underline{v} = \underline{A}_{k(t)}\,(\underline{A}_k\,\underline{Y}\,\underline{A}_{k(t)})^{-1}\,\underline{A}_k\,\underline{a} \qquad (8.28)$$

From the foregoing we have

<u>Theorem V</u> : If the equation $\underline{Y}'_{jj}\ \underline{v}'_j = \underline{a}'_j$ is the node cut-set current equation of a given network, the matrix $\underline{Y}'_{kk}$ in the equation $\underline{Y}'_{kk}\ \underline{v}'_k = \underline{a}'_k$ representing the network after the datum node has been changed from node 0 to node q (with node 0 renumbered q and node q renumbered 0) may be obtained from the matrix $\underline{Y}'_{jj}$ in the original equation by substituting for the q th column of $\underline{Y}'_{jj}$ the negative of the sum of all its columns and then substituting the negative of the sum of all the rows of the resulting matrix for the q th row. The column matrix $\underline{a}'_k$ in the new equation may be obtained from the matrix $\underline{a}'_j$ in the original equation by substituting for the q th component of $\underline{a}'_j$ the negative of the sum of all its components.

The part of the theorem that refers to the admittance matrix is Shekel's first theorem for the case of planar networks (Ref.4).

From (8.25)

$$\det \underline{Y}'_{kk} = \det \underline{P}.\ \det \underline{Y}'_{jj}.\ \det \underline{P}_{(t)} \tag{8.29}$$

but $\det \underline{P} = \det \underline{P}_{(t)} = -1$ so that

$$\det \underline{Y}'_{kk} = \det \underline{Y}'_{jj} \tag{8.30}$$

hence,

<u>Theorem VI</u> : The determinant of the node cut-set admittance matrix of a planar network is invariant for a change of datum node.

This is Shekel's second theorem for the case of planar networks (Ref.4).

9. General independent circuital current systems.

The analysis of a planar network is effectively accomplished on a voltage basis by proceeding from the selection of an appropriate independent mesh tie-set system in the corresponding linear graph but there appears to be no reason why independent circuital current systems other than independent mesh tie-set current systems should not be employed. Indeed, provided $\underline{P}$ is non-singular, its components may be selected at random from the field of real numbers and the transformation will correspond to an independent system of circuital currents that adequately represent the branch currents of the network.

Consider the planar network shown in Fig.9. 1(a) * For simplicity we shall suppose the branch impedances to be pure resistors of unit magnitude. According to the mesh tie-set system indicated in Fig.9.1(b), the equation corresponding to (8.4) comes out to be

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}_{11} \begin{bmatrix} i'_1 \\ i'_2 \\ i'_3 \end{bmatrix}_1 = \begin{bmatrix} e_1 \\ -e_4 \\ e_4 + e_5 \end{bmatrix}_1 \qquad (9.1)$$

If now the components of $\underline{P}$ are selected arbitrarily, say

$$\underline{P} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0.25 & 0 \\ 0 & 1 & 0.5 \end{bmatrix} \qquad \det \underline{P} \neq 0 \qquad (9.2)$$

---

* It will be observed that this network has cyclomatic index c = 3 and nodalic index p = 2. A nodal analysis is therefore preferable to the mesh analysis which is here employed for the purpose of illustration.

then the equation corresponding to (8.9) is

$$\frac{1}{16}\begin{bmatrix} 160 & -4 & -32 \\ -4 & 3 & 10 \\ -32 & 10 & 40 \end{bmatrix}_{22} \begin{bmatrix} i_1' \\ i_2' \\ i_3' \end{bmatrix}_2 = \begin{bmatrix} 2e_1 - e_4 - e_5 \\ -0.25\, e_4 \\ 0.5\,(e_5 - e_4) \end{bmatrix}_2 \qquad (9.3)$$

which represents the network equally well as (9.1). In view of (7.4) it is easily seen that the matrix $\underline{P}$ transforms the mesh tie-set current system of Fig.9.1(b) to the circuital current system represented in Fig.9.2. Thus, as has been shown by Le Corbeiller (Ref.5), if we admit every possible system of independent circuital currents, it follows that the totality of transformation matrices $\underline{P}$ constitute an infinite group with respect to multiplication. The identity element is the unit matrix.

We have shown that the determinant of the mesh tie-set impedance matrix is the same for every independent mesh tie-set system. An independent mesh tie-set current system is a special case of the more general independent circuital current system. Immediately we allow the use of all possible independent circuital current systems the determinant of the impedance matrix ceases to possess the property of invariance as we pass from one system to another except for those transformations for which det $\underline{P}$ has the value plus or minus one. It has been stated in the literature that the determinant of a mesh tie-set impedance matrix of a given planar network has the minimum value for that network. Evidently this is not so if the use of general circuital current systems is allowed. Consider, for example, the case cited above. The determinant of the mesh tie-set impedance matrix in (9.1) is

$$\det \underline{Z}_{11}' = 8 \qquad (9.4)$$

and the determinant of the impedance matrix in (9.3) is given by

$$\det \underline{Z}_{22}' = \det \underline{P}.\ \det \underline{Z}_{11}'.\ \det \underline{P}_{(t)}$$

Now, from (9.2), det $\underline{P}$ = 0.25 hence

$$\det \underline{Z}'_{22} = (0.25)\,(8)\,(0.25) = 0.5 \tag{9.5}$$

which is less than det $\underline{Z}'_{11}$ in (9.4).

10. General independent potential difference systems.

The analysis of a planar network is effectively accomplished on a current basis by proceeding from the selection of an appropriate independent node cut-set system but there appears to be no reason why independent potential difference systems other than independent node cut-set potential difference systems should not be employed. Indeed, a similar freedom of choice exists as does in the case of analysis on a voltage basis. The transformation matrix $\underline{P}$ belongs to an infinite group with respect to multiplication.

Consider the planar configuration shown in Fig.9.1(a). For simplicity we shall suppose the branch admittances to be pure conductors of unit magnitude. According to the node cut-set system indicated in Fig.9.1(b) the equation corresponding to (8.19) comes out to be

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}_{11} \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix}_1 = \begin{bmatrix} -a_1 \\ a_1 \\ -a_5 \end{bmatrix}_1 \tag{10.1}$$

If now the components of $\underline{P}$ are selected arbitrarily, say

$$\underline{P} = \begin{bmatrix} 0.5 & 0 & 1 \\ 3 & -1 & 0 \\ 0 & 0.25 & 6 \end{bmatrix} \qquad \det \underline{P} \neq 0 \tag{10.2}$$

the equation corresponding to (8.24) is

$$\frac{1}{16}\begin{bmatrix} 44 & 48 & 234 \\ 48 & 576 & -216 \\ 234 & -216 & 1683 \end{bmatrix}_{22} \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix}_2 = \begin{bmatrix} -(0.5a_1 + a_5) \\ -4a_1 \\ 0.25a_1 - 6a_5 \end{bmatrix}_2 \qquad (10.3)$$

In view of (7.9) it is easily seen that the matrix $\underline{P}$ transforms the node cut-set potential difference system of Fig.10.1(b) to the potential difference system represented in Fig.10.2.

We have shown that the determinant of the node cut-set admittance matrix is the same for every independent node cut-set system. An independent node cut-set potential difference system is a special case of the more general independent potential difference system. Immediately we allow the use of all possible independent potential difference systems the determinant of the admittance matrix ceases to possess the property of invariance as we pass from one system to another except for those transformations for which det $\underline{P}$ has the value plus or minus one. As in the case of voltage analysis, the determinant of the node cut-set admittance matrix of a given planar network is not the minimum determinant for that network if the use of general potential difference systems is allowed. In the case of the network of Fig.10.1, the determinant of $\underline{Y}'_{22}$ is, fortuitously, greater than that of $\underline{Y}'_{11}$. We have

$$\det \underline{Y}'_{11} = 16 \qquad (10.4)$$

Det $\underline{P}$ is -2.25, hence

$$\det \underline{Y}'_{22} = 81 \qquad (10.5)$$

If, instead of $\underline{P}$ in (10.2), we select

$$\underline{P} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

then, corresponding to (10.3), we obtain

$$\begin{bmatrix} 4 & & \\ & 4 & \\ & & 4 \end{bmatrix}_{33} \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix}_3 = \begin{bmatrix} -(a_1 + a_5) \\ 0 \\ a_2 - a_5 \end{bmatrix}_3$$

with det $\underline{Y}_{33}' = 64$. This is the dual of the example given by Seshu (Ref.6).

11. Hybrid systems.

In connexion with (8.8), it is interesting to note that if we had merely substituted (7.4) in (8.4) to obtain

$$\underline{Z}_{jj}' \, \underline{P}_{(t)} \, \underline{i}_k' = \underline{e}_j' \tag{11.1}$$

which may be written

$$\underline{Z}_{jk}' \, \underline{i}_k' = \underline{e}_j' \tag{11.2}$$

where

$$\underline{Z}_{jk}' = \underline{Z}_{jj}' \, \underline{P}_{(t)} \tag{11.3}$$

we would have produced a voltage equation according to the mesh tie-set system before the datum mesh is changed from mesh 0 to mesh q but in terms of the mesh tie-set currents after the datum mesh has been so changed. $\underline{Z}_{jk}'$ is the non-singular c by c mesh tie-set impedance matrix corresponding to a choice of c independent mesh tie-sets according to incidence matrix $\underline{C}_j$ and c independent mesh tie-set currents according to incidence matrix $\underline{C}_k$. By substituting (8.5) in (11.3) we see that

$$\underline{Z}_{jk}' = \underline{C}_j \, \underline{Z} \, \underline{C}_{j(t)} \, \underline{P}_{(t)} = \underline{C}_j \, \underline{Z} \, (\underline{P} \, \underline{C}_j)_{(t)} = \underline{C}_j \, \underline{Z} \, \underline{C}_{k(t)} \tag{11.4}$$

Unlike, $\underline{Z}'_{jj}$ and $\underline{Z}'_{kk}$, however, $\underline{Z}'_{jk}$ is, in general, asymmetrical for a passive network and possibly symmetrical for a network containing unilateral elements. Thus, as Guillemin has pointed out (Ref.7), the symmetry of a mesh tie-set impedance matrix does not necessarily indicate a passive network. From (11.3) it is evident that $\underline{Z}'_{jk}$ and $\underline{Z}'_{jj}$ are right associates. By premultiplying (11.2) by the inverse of $\underline{Z}'_{jk}$ we obtain

$$\underline{i}'_k = \underline{Z}'^{-1}_{jk} \; \underline{e}'_j$$

which leads to the branch current equation

$$\underline{i} = \underline{C}_{k(t)} \left(\underline{C}_j \, \underline{Z} \, \underline{C}_{k(t)}\right)^{-1} \underline{C}_j \, \underline{e} \qquad (11.5)$$

Alternatively, had we merely premultiplied (8.4) by $\underline{P}$ to obtain

$$\underline{P} \, \underline{Z}'_{jj} \, \underline{i}'_j = \underline{e}'_k \qquad (11.6)$$

which may be written

$$\underline{Z}'_{kj} \, \underline{i}'_j = \underline{e}'_k \qquad (11.7)$$

where

$$\underline{Z}'_{kj} = \underline{P} \, \underline{Z}'_{jj} \qquad (11.8)$$

we would have produced a voltage equation according to the mesh tie-set system after the datum mesh has been changed from mesh 0 to mesh q but in terms of the mesh tie-set currents before the datum mesh has been so changed. $\underline{Z}'_{kj}$ is the non-singular c by c mesh tie-set impedance matrix corresponding to a choice of c independent mesh tie-sets according to incidence matrix $\underline{C}_k$ and c independent mesh tie-set currents according to incidence matrix $\underline{C}_j$. By substituting (8.5) in (11.8) we see that

$$\underline{Z}'_{kj} = \underline{P} \, \underline{C}_j \, \underline{Z} \, \underline{C}_{j(t)} = \underline{C}_k \, \underline{Z} \, \underline{C}_{j(t)} \qquad (11.9)$$

As in the case of $\underline{Z}'_{jk}$, $\underline{Z}'_{kj}$ is, in general, asymmetrical for a passive network and possibly symmetrical for an active network. From (11.8) it is evident that $\underline{Z}'_{kj}$ and $\underline{Z}'_{jj}$ are left associates. By premultiplying (11.7) by the inverse of $\underline{Z}'_{kj}$ and making the appropriate substitutions we obtain the branch current equation

$$\underline{i} \; = \; \underline{C}_{j(t)} \, (\underline{C}_k \, \underline{Z} \, \underline{C}_{j(t)})^{-1} \;\; \underline{C}_k \, \underline{e} \qquad (11.10)$$

From (11.3)

$$\det \underline{Z}'_{jk} \; = \; \det \underline{Z}'_{jj} . \; \det \underline{P}_{(t)} \qquad (11.11)$$

and from (11.8)

$$\det \underline{Z}'_{kj} \; = \; \det \underline{P} . \;\; \det \underline{Z}'_{jj} \qquad (11.12)$$

hence, since $\det \underline{P} = -1$,

$$\det \underline{Z}'_{jk} = \det \underline{Z}'_{kj} \; = \; - \det \underline{Z}'_{jj} \qquad (11.13)$$

The foregoing discussion applies equally well to circuital current systems (except for the statement in (11.13)) other than mesh tie-set systems. For example, if we make the substitution

$$\begin{bmatrix} i'_1 \\ i'_2 \\ i'_3 \end{bmatrix}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.25 & 1 \\ -1 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} i'_1 \\ i'_2 \\ i'_3 \end{bmatrix}_2$$

in (9.1) we obtain the voltage equation

$$\begin{bmatrix} 4 & -0.25 & -1 \\ -1 & 0.75 & 2.5 \\ -2 & -0.25 & 0 \end{bmatrix}_{12} \begin{bmatrix} i'_1 \\ i'_2 \\ i'_3 \end{bmatrix}_2 = \begin{bmatrix} e_1 \\ -e_4 \\ e_4 + e_5 \end{bmatrix}_1$$

which, although the impedance matrix is asymmetrical, is a true representation of the passive network of Fig. 9.1(a).

The same reasoning applies when the network is analysed on a current basis. Referring to (8.23), if we had merely substituted (7.9) in (8.19) to obtain

$$\underline{Y}'_{jj}\ \underline{P}_{(t)}\ \underline{v}'_{k} = \underline{a}'_{j} \tag{11.14}$$

which may be written

$$\underline{Y}'_{jk}\ \underline{v}'_{k} = \underline{a}'_{j} \tag{11.15}$$

where

$$\underline{Y}'_{jk} = \underline{Y}'_{jj}\ \underline{P}_{(t)} \tag{11.16}$$

we would have produced a current equation according to the node cut-set system before the datum node is changed from node 0 to node q but in terms of the node cut-set potential differences after the datum node has been so changed. $\underline{Y}'_{jk}$ is the non-singular p by p node cut-set admittance matrix corresponding to a choice of p independent node cut-sets according to incidence matrix $\underline{A}_{j}$ and p independent node cut-set potential differences according to incidence matrix $\underline{A}_{k}$. By substituting (8.20) in (11.16) we see that

$$\underline{Y}'_{jk} = \underline{A}_{j}\ \underline{Y}\ \underline{A}_{j(k)}\ \underline{P}_{(t)} = \underline{A}_{j}\ \underline{Y}\ (\underline{P}\ \underline{A}_{j})_{(t)} = \underline{A}_{j}\ \underline{Y}\ \underline{A}_{k(t)} \tag{11.17}$$

Unlike $\underline{Y}'_{jj}$ and $\underline{Y}'_{kk}$ however, $\underline{Y}'_{jk}$ is, in general, asymmetrical for a passive network and possibly symmetrical for an active network. Thus, symmetry of the node cut-set admittance matrix does not necessarily indicate a passive network. From (11.16) it is evident that $\underline{Y}'_{jk}$ and $\underline{Y}'_{jj}$ are right associates. By premultiplying (11.15) by the inverse of $\underline{Y}'_{jk}$ we obtain

$$\underline{v}'_{k} = \underline{Y}'^{-1}_{jk}\ \underline{a}'_{j}$$

which leads to the branch potential difference equation

$$\underline{v} = \underline{A}_{k(t)} (\underline{A}_j \underline{Y} \underline{A}_{k(t)})^{-1} \underline{A}_j \underline{a}$$

Alternatively, had we merely premultiplied (8.19) by $\underline{P}$ to obtain

$$\underline{P} \underline{Y}'_{jj} \underline{v}'_j = \underline{a}'_j \tag{11.18}$$

which may be written

$$\underline{Y}'_{kj} \underline{v}'_j = \underline{a}'_j \tag{11.19}$$

where

$$\underline{Y}'_{kj} = \underline{P} \underline{Y}'_{jj} \tag{11.20}$$

we would have produced a current equation according to the node cut-set system after the datum node has been changed from node 0 to node q but in terms of the node cut-set potential differences before the datum node has been so changed. $\underline{Y}'_{kj}$ is the non-singular p by p node cut-set admittance matrix corresponding to a choice of p independent node cut-sets according to incidence matrix $\underline{A}_k$ and p independent node cut-set potential differences according to incidence matrix $\underline{A}_j$. By substituting (8.20) in (11.20) we see that

$$\underline{Y}'_{kj} = \underline{P} \underline{A}_j \underline{Y} \underline{A}_{j(t)} = \underline{A}_k \underline{Y} \underline{A}_{j(t)} \tag{11.21}$$

As in the case of $\underline{Y}'_{jk}$, $\underline{Y}'_{kj}$ is, in general, asymmetrical for a passive network and possibly symmetrical for an active network From (11.20) it is evident that $\underline{Y}'_{kj}$ and $\underline{Y}'_{jj}$ are left associates. By premultiplying (11.19) by the inverse of $\underline{Y}'_{kj}$ and making the appropriate substitutions we obtain the branch potential difference equation

$$\underline{v} = \underline{A}_{j(t)} (\underline{A}_k \underline{Y} \underline{A}_{j(t)})^{-1} \underline{A}_k \underline{a} \tag{11.22}$$

From (11.16)

$$\det \underline{Y}'_{jk} = \det \underline{Y}'_{jj} \cdot \det \underline{P}_{(t)} \qquad (11.23)$$

and from (11.20)

$$\det \underline{Y}'_{kj} = \det \underline{P} \cdot \det \underline{Y}'_{jj} \qquad (11.24)$$

hence, since det $\underline{P} = -1$,

$$\det \underline{Y}'_{jk} = \det \underline{Y}'_{kj} = -\det \underline{Y}'_{jj} \qquad (11.25)$$

The above discussion applies equally well to potential difference systems (except for the statement in (11.25)) other than node cut-set systems.

12. Conclusion

It has been shown that if the algebraic theory of planar electric networks proceeds from the identification of the meshes, branches, and nodes of a network respectively with the faces, edges, and vertices of a simple convex polyhedron the dual correspondence between the voltage and current representations of a planar network is complete in every respect. The duality of the concepts of mesh and node has long been recognised in circuit topology but, because of inconsistencies in the definitions of a mesh, its true significence has been obscured. It is for this reason, for instance, that the mesh counterparts in the planar case of the two nodal theorems recently proposed by Shekel (Ref.4) were not immediately recognised. The approach recommended here makes for a dualism that is both logical and elegant.

In the case of a complicated non-planar network the selection of a tree in the linear graph and the subsequent interpretation of the chord currents as circuital currents or, alternatively, the use of tree-branch potential differences ensures the independence of the variables. Whatever the complexity of a planar network however, the selection of an independent mesh tie-set current system or node cut-set potential difference system requires no effort and the network readily yields to analysis. Nevertheless, it has been shown above that if circuital current systems or potential difference systems other than these are allowed then there is no limit to the number of systems that may be used to represent a network. It follows therefore that a network has an infinitude of impedance or admittance matrices certain of which possess, respectively, the same determinant. The latter have recently been the subject of discussion in the literature.

It has also been shown that the symmetry or asymmetry of a network parameter matrix does not necessarily indicate the passivity or activity of the network.

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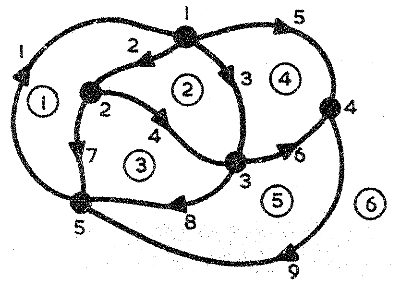


FIG. 1. PLANET SAGITTAL GRAPH.

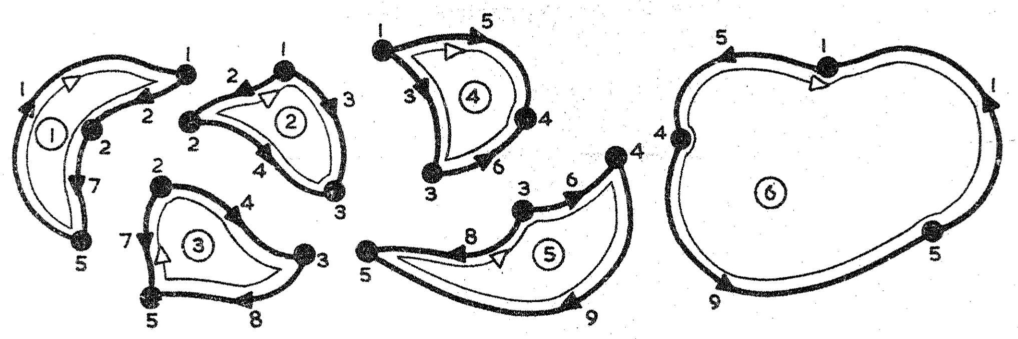


FIG. 2. DIRECTED MESH TIE-SETS.

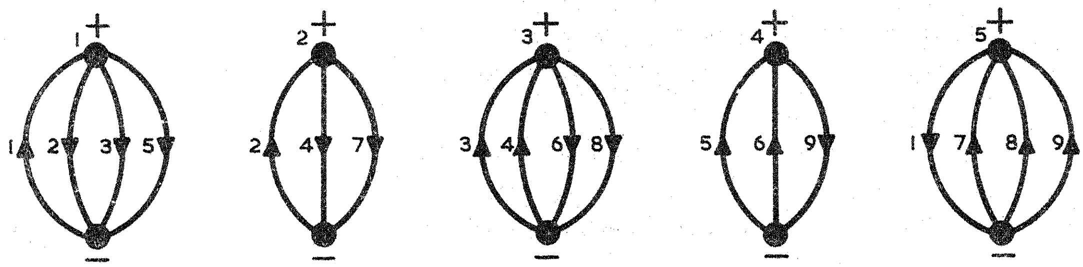
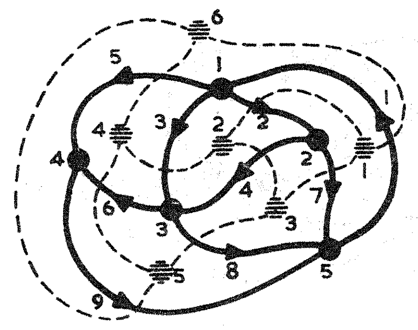
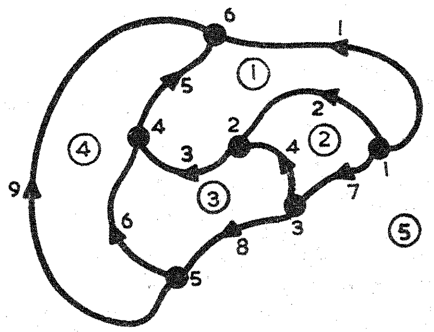


FIG. 3. POLARIZED NODE CUT-SETS.



(a.)



(b.)

FIG. 4. THE CONSTRUCTION OF THE DUAL OF A PLANET SAGITTAL GRAPH.

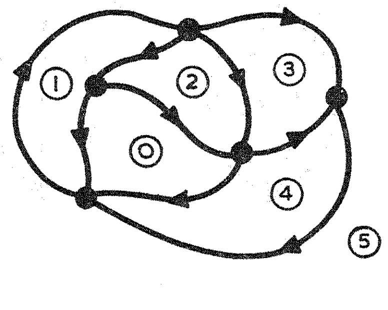


FIG. 5. TOPOLOGICALLY INDEPENDENT MESH TIE-SETS.

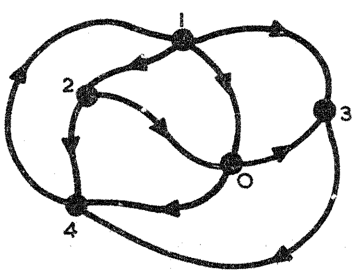
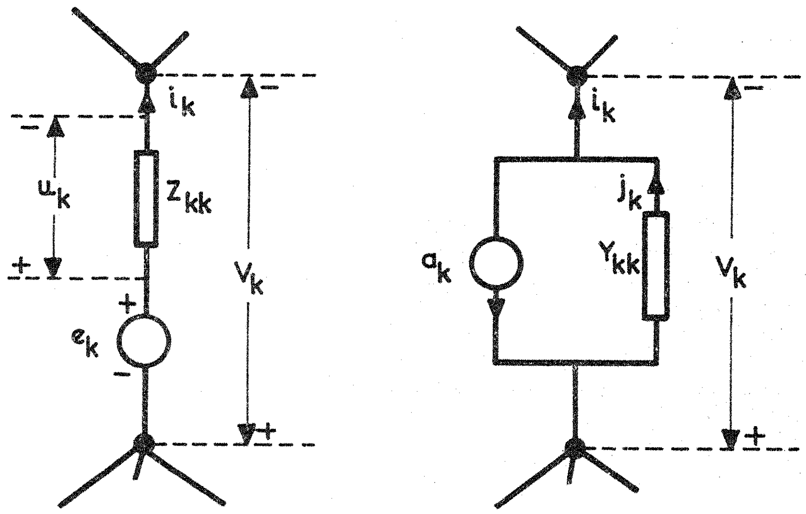


FIG. 6. TOPOLOGICALLY INDEPENDENT NODE CUT-SETS.



(a.) HELMHOLTZ REPRESENTATION

(b.) NORTON REPRESENTATION

FIG. 7. NETWORK BRANCH REPRESENTATIONS.

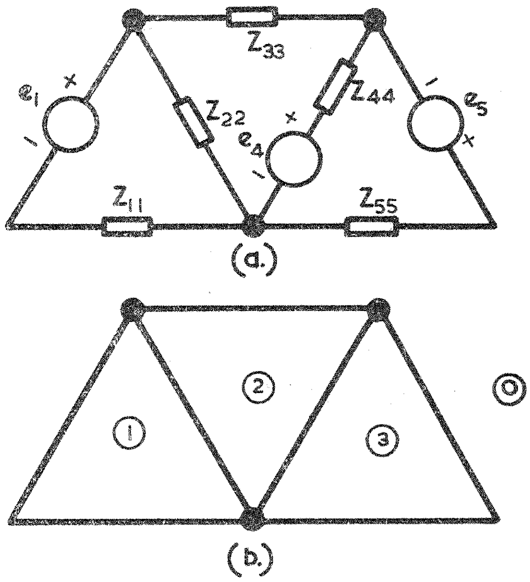


FIG. 8.

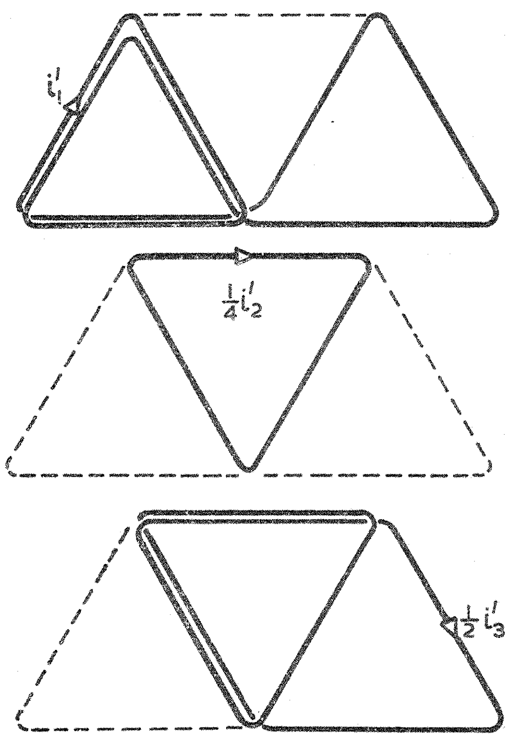


FIG. 9.

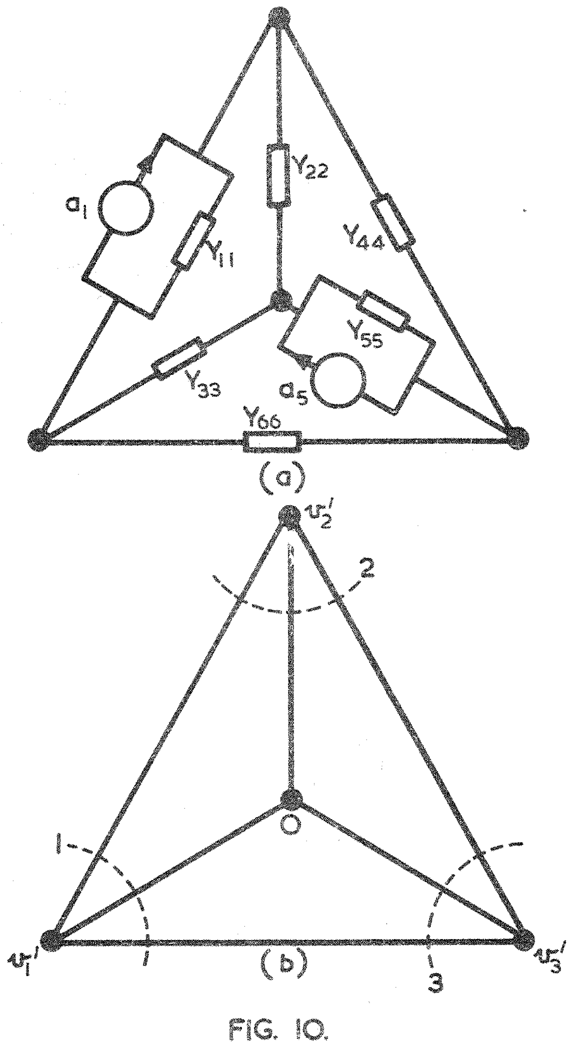


FIG. 10.

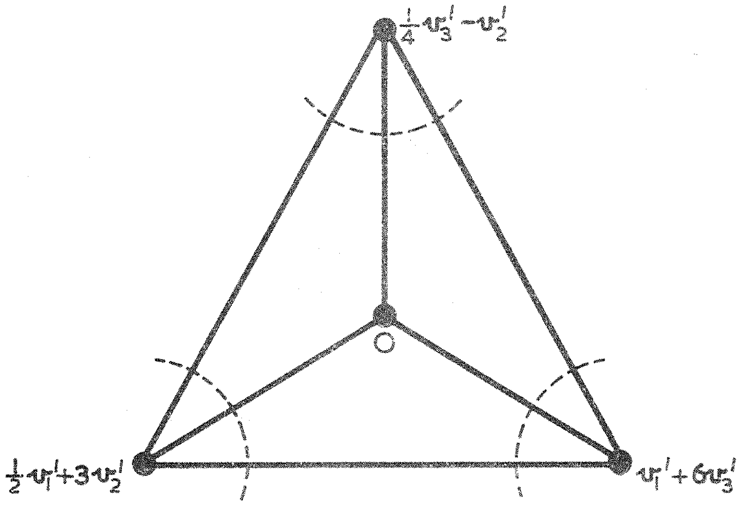


FIG. 11.