Technical Note No. 3

The College of Aeronautics

DEPARTMENT OF ELECTRICAL AND CONTROL ENGINEERING

Explicit formulae for ladder 2-ports

S. R. Deards

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Summary

Explicit formulae are obtained for the network functions of ladder 2-ports.

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Explicit formulae for ladder 2-ports

By S. R. Deards

Consider the tandem chain of n identical bilateral L-sections represented in Fig. 1. Each section has transfer matrix

$$A = \begin{bmatrix} 1 + ZY & Z \\ Y & 1 \end{bmatrix} \quad (1)$$

hence the tandem chain has transfer matrix

$$A_c = A^n = \phi(A) \quad (2)$$

By Sylvester's interpolation formula, if a square matrix M has order m and distinct latent roots $\lambda_1, \lambda_2, \ldots \lambda_m$ then

$$\phi(M) = \sum_{r=1}^{m} \phi(\lambda_r) \prod_{\substack{1 \leq k \leq m \\ k \neq r}} \frac{I\lambda_k - M}{\lambda_k - \lambda_r} \quad (3)$$

where I is the unit matrix of order m. Thus, the transfer matrix A_c of the tandem chain in Fig. 1 has the representation

$$A_c = A^n = \frac{A(\lambda_2^n - \lambda_1^n) - I\lambda_1\lambda_2(\lambda_2^{n-1} - \lambda_1^{n-1})}{\lambda_2 - \lambda_1} \qquad (4)$$

The trace of A is $a_{11} + a_{22}$ and the determinant of A is unity. Since the trace and determinant of A are preserved under a similarity transformation which converts A into the diagonal matrix with components λ_1 and λ_2 we have

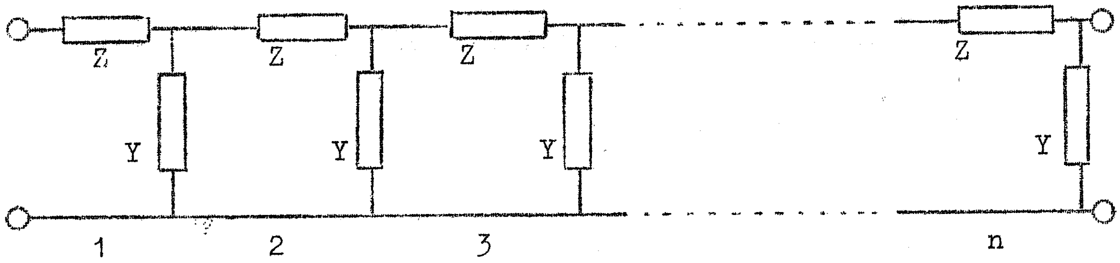


Fig. 1

$$\lambda_1 + \lambda_2 = a_{11} + a_{22} \qquad (5)$$

and

$$\lambda_1\lambda_2 = 1 \qquad (6)$$

Let

$$\left.\begin{aligned} \lambda_1 &= \varepsilon^{j\theta} \\ \lambda_2 &= \varepsilon^{-j\theta} \end{aligned}\right\} \qquad (7)$$

so that (4) can be written

$$A_c = A\frac{\sin n\theta}{\sin\theta} - I\frac{\sin(n-1)\theta}{\sin\theta} \qquad (8)$$

In view of (5) and (7)

$$a_{11} + a_{22} = 2\cos\theta \qquad (9)$$

If we put

$$\mu = \cos\theta \tag{10}$$

then

$$\frac{\sin n\theta}{\sin\theta} = \frac{\sin n\theta}{\sqrt{1-\cos^2\theta}} = \frac{\sin(n \operatorname{arc}\cos\mu)}{\sqrt{1-\mu^2}} = U_{n-1}(\mu) \tag{11}$$

which is the Chebyshev function of the second kind. Thus, (8) has the representation

$$A_c = A.U_{n-1}(\mu) - I.U_{n-2}(\mu)$$

$$= \begin{bmatrix} (1+ZY).U_{n-1}(\mu) - U_{n-2}(\mu) & Z.U_{n-1}(\mu) \\ Y.U_{n-1}(\mu) & U_{n-1}(\mu) - U_{n-2}(\mu) \end{bmatrix} \tag{12}$$

where

$$\mu = 1 + \tfrac{1}{2}ZY \tag{13}$$

Let us represent the Chebyshev function in the form

$$U_{n-1}(\mu) = \sum_{r=0}^{n-1} \binom{n+r}{2r+1} 2^r(\mu-1)^r \tag{14}$$

or, in view of (13),

$$U_{n-1}(\mu) = \sum_{r=0}^{n-1} \binom{n+r}{2r+1} Z^rY^r \tag{15}$$

whence follows

$$U_{n-1}(\mu) - U_{n-2}(\mu) = \sum_{r=0}^{n-1} \binom{n+r}{2r+1} Z^rY^r - \sum_{r=0}^{n-2} \binom{n+r-1}{2r+1} Z^rY^r$$

$$= \sum_{r=0}^{n-2} \left[\binom{n+r}{2r+1} - \binom{n+r-1}{2r+1} \right] Z^rY^r + Z^{n-1}Y^{n-1}$$

$$= \sum_{r=0}^{n-2} \binom{n+r-1}{2r} Z^rY^r + Z^{n-1}Y^{n-1}$$

$$= \sum_{r=0}^{n-1} \binom{n+r-1}{2r} Z^rY^r \tag{16}$$

Furthermore

$$(1 + ZY).U_{n-1}(\mu) - U_{n-2}(\mu) = U_{n-1}(\mu) - U_{n-2}(\mu) + ZY.U_{n-1}(\mu)$$

$$= \sum_{r=0}^{n-1} \binom{n+r-1}{2r} Z^r Y^r + \sum_{r=0}^{n-1} \binom{n + r}{2r + 1} Z^{r+1} Y^{r+1}$$

$$= \binom{n-1}{0} + \sum_{r=1}^{n-1} \binom{n+r-1}{2r} Z^r Y^r + \sum_{r=0}^{n-2} \binom{n + r}{2r + 1} Z^{r+1} Y^{r+1} + \binom{2n-1}{2n-1} Z^n Y^n \qquad (17)$$

Consider the second and third terms in (17)

$$\sum_{r=1}^{n-1} \binom{n+r-1}{2r} Z^r Y^r + \sum_{r=0}^{n-2} \binom{n+r}{2r+1} Z^{r+1} Y^{r+1} = \sum_{r=0}^{n-2} \left[\binom{n+r}{2r+2} + \binom{n+r}{2r+1} \right] Z^{r+1} Y^{r+1}$$

$$= \sum_{r=0}^{n-2} \binom{n+r+1}{2r + 2} Z^{r+1} Y^{r+1} \qquad (18)$$

Thus, (17) may be written

$$(1 + ZY).U_{n-1}(\mu) - U_{n-2}(\mu) = 1 + \sum_{r=0}^{n-2} \binom{n+r+1}{2r+2} Z^{r+1} Y^{r+1} + Z^n Y^n$$

$$= \sum_{r=0}^{n} \binom{n+r}{2r} Z^r Y^r \qquad (19)$$

By substituting (15), (16), and (19) in (12) we obtain the explicit form of the transfer matrix of the n-section ladder 2-port, thus

$$A_c = \begin{bmatrix} \sum_{r=0}^{n} \binom{n+r}{2r} Z^r Y^r & \sum_{r=0}^{n-1} \binom{n+r}{2r+1} Z^{r+1} Y^r \\ \sum_{r=0}^{n-1} \binom{n+r}{2r+1} Z^r Y^{r+1} & \sum_{r=0}^{n-1} \binom{n+r-1}{2r} Z^r Y^r \end{bmatrix} \qquad (20)$$

The components of A_c for n=1 to 12 are given in the tables below.

Coefficients of a_{11}

n		ZY	Z^2Y^2	Z^3Y^3	Z^4Y^4	Z^5Y^5	Z^6Y^6	Z^7Y^7	Z^8Y^8	Z^9Y^9	$Z^{10}Y^{10}$	$Z^{11}Y^{11}$	$Z^{12}Y^{12}$
1	1	1											
2	1	3	1										
3	1	6	5	1									
4	1	10	15	7	1								
5	1	15	35	28	9	1							
6	1	21	70	84	45	11	1						
7	1	28	126	210	165	66	13	1					
8	1	36	210	462	495	286	91	15	1				
9	1	45	330	924	1287	1001	455	120	17	1			
10	1	55	495	1716	3003	3003	1820	680	153	19	1		
11	1	66	715	3003	6435	8008	6188	3060	969	190	21	1	
12	1	78	1001	5005	12870	19448	18564	11628	4845	1330	231	23	1

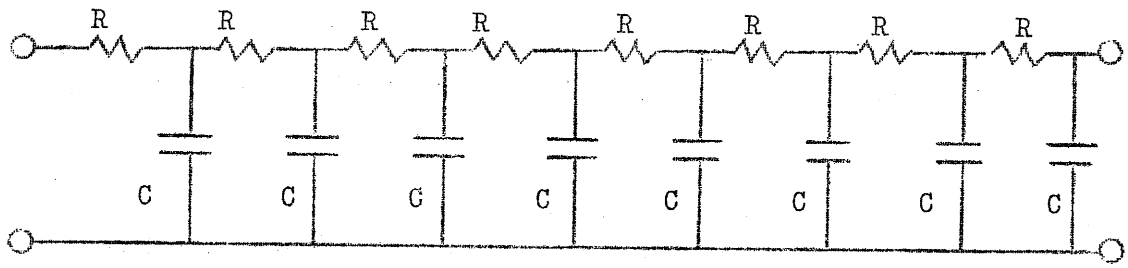
Coefficients of a_{12} and a_{21}

a_{12}	Z	Z^2Y	Z^3Y^2	Z^4Y^3	Z^5Y^4	Z^6Y^5	Z^7Y^6	Z^8Y^7	Z^9Y^8	$Z^{10}Y^9$	$Z^{11}Y^{10}$	$Z^{12}Y^{11}$
n												
1	1											
2	2	1										
3	3	4	1									
4	4	10	6	1								
5	5	20	21	8	1							
6	6	35	56	36	10	1						
7	7	56	126	120	55	12	1					
8	8	84	252	330	220	78	14	1				
9	9	120	462	792	715	364	105	16	1			
10	10	165	792	1716	2002	1265	560	136	18	1		
11	11	220	1287	3432	5005	4368	2380	816	171	20	1	
12	12	286	2002	6435	11440	12376	8568	3876	1140	210	22	1
a_{21}	Y	ZY^2	Z^2Y^3	Z^3Y^4	Z^4Y^5	Z^5Y^6	Z^6Y^7	Z^7Y^8	Z^8Y^9	Z^9Y^{10}	$Z^{10}Y^{11}$	$Z^{11}Y^{12}$

Coefficients of a_{22}

n		ZY	Z^2Y^2	Z^3Y^3	Z^4Y^4	Z^5Y^5	Z^6Y^6	Z^7Y^7	Z^8Y^8	Z^9Y^9	$Z^{10}Y^{10}$	$Z^{11}Y^{11}$
1	1											
2	1	1										
3	1	3	1									
4	1	6	5	1								
5	1	10	15	7	1							
6	1	15	35	28	9	1						
7	1	21	70	84	45	11	1					
8	1	28	126	210	165	66	13	1				
9	1	36	210	462	495	286	91	15	1			
10	1	45	330	924	1287	1001	455	120	17	1		
11	1	55	495	1716	3003	3003	1820	680	153	19	1	
12	1	66	715	3003	6435	8008	6188	3060	969	190	21	1

Example:



In the RC ladder 2-port shown, n = 8; ZY = sCR = sT if T = CR. We have at once, from the tables, the forward voltage transfer function

$$\frac{1}{a_{11}} = \frac{1}{1+36sT+210s^2T^2+462s^3T^3+495s^4T^4+286s^5T^5+91s^6T^6+15s^7T^7+s^8T^8}$$

and the short-circuit impedance function at port 1

$$\frac{a_{12}}{a_{22}} = R\,\frac{8+84sT+252s^2T^2+330s^3T^3+220s^4T^4+78s^5T^5+14s^6T^6+s^7T^7}{1+28sT+126s^2T^2+210s^3T^3+165s^4T^4+66s^5T^5+13s^6T^6+s^7T^7}$$

Similarly we may find

$\dfrac{a_{11}}{a_{21}}$ = open-circuit impedance at port 1

$\dfrac{a_{12}}{a_{11}}$ = short-circuit impedance at port 2

$\dfrac{a_{22}}{a_{21}}$ = open-circuit impedance at port 2

$\dfrac{1}{a_{21}}$ = open-circuit transfer impedance

a_{12} = short-circuit transfer impedance

With $s = j\omega$,

$$\frac{1}{2a_{21}}\left[(a_{11} - a_{22}) + \sqrt{(a_{11} + a_{22})^2 - 4}\right] = \text{forward iterative impedance}$$

$$\frac{1}{2a_{21}}\left[(a_{22} - a_{11}) + \sqrt{(a_{11} + a_{22})^2 - 4}\right] = \text{reverse iterative impedance}$$

$$\text{arc cosh}\,\frac{a_{11} + a_{22}}{2} = \text{iterative propagation function}$$

$$\sqrt{\frac{a_{11}a_{12}}{a_{22}a_{21}}} = \text{image impedance at port 1}$$

$$\sqrt{\frac{a_{22}a_{12}}{a_{11}a_{21}}} = \text{image impedance at port 2}$$

$$\ln(\sqrt{a_{11}a_{22}} + \sqrt{a_{12}a_{21}}) = \text{image propagation function}$$

$$\begin{matrix}\mathcal{R}\\ \mathcal{I}\end{matrix}\ \ln\frac{a_{11}Z_2 + a_{21}Z_2Z_1 + a_{12} + a_{22}Z_1}{Z_1 + Z_2} = \text{insertion}\ \begin{matrix}\text{loss}\\ \text{phase}\end{matrix}$$

where Z_k is the external impedance at port k.