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THE COLLEGE OF AERONAUTICS

DEPARTMENT OF ELECTRICAL AND CONTROL ENGINEERING

The 2-port as an analogue of the Lorentz
transformation of special relativity theory

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Summary

If the variables associated with a linear resistance 2-port are identified with the variables of special relativity theory, it is shown that a resistance 2-port transforms its port variables according to the Lorentz equations.

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transformation of special relativity

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If the linear 2-port represented in Fig. 1 is purely resistive, bilateral, and symmetrical then the port variables are related in

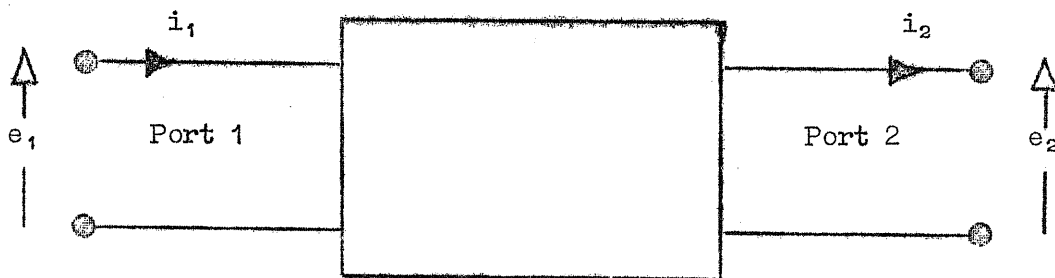


Fig. 1

any one of the following three equations.

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} r_{11} & -r_{21} \\ r_{21} & -r_{11} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} g_{11} & -g_{21} \\ g_{21} & -g_{11} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad (2)$$

$$\begin{bmatrix} e_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} e_2 \\ i_2 \end{bmatrix} \quad (3)$$

If R represents the transformation (resistance) matrix in (1) then R has determinant

$$\det R = r_{21}^2 - r_{11}^2 \quad (4)$$

Equation (2) is the inverse of equation (1) and conversely. Thus, the leading component of the transformation (conductance) matrix in (2) can be written

$$g_{11} = -\frac{r_{11}}{\det R} \quad (5)$$

whence

$$\det R = -\frac{r_{11}}{g_{11}} \quad (6)$$

From (4) and (6)

$$r_{21} = \sqrt{r_{11} \left(r_{11} - \frac{1}{g_{11}} \right)} \quad (7)$$

in which the non-physically realizable negative root is ignored.

In the notation of attenuator theory

$$\left. \begin{aligned} r_{11} &= r_o = \text{resistance at either port with the other port open} \\ \frac{1}{g_{11}} &= r_s = \text{resistance at either port with the other port closed} \end{aligned} \right\} \quad (8)$$

(6) and (7) can therefore be written

$$\det R = -r_o r_s \quad (9)$$

$$r_{21} = \sqrt{r_o(r_o - r_s)} \quad (10)$$

The transformation (transfer) matrix in (3) is easily expressed in terms of the components of R. Thus, in view of (8), (9), and (10),

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{11} \end{bmatrix} = \begin{bmatrix} \frac{r_{11}}{r_{21}} & -\frac{\det R}{r_{21}} \\ \frac{1}{r_{21}} & \frac{r_{11}}{r_{21}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1 - \frac{r_s}{r_o}}} & \frac{r_s}{\sqrt{1 - \frac{r_s}{r_o}}} \\ \frac{1}{r_o \sqrt{1 - \frac{r_s}{r_o}}} & \frac{1}{\sqrt{1 - \frac{r_s}{r_o}}} \end{bmatrix} \quad (11)$$

If a load resistance r_c is now connected across port 2, (3) becomes

$$\begin{bmatrix} e_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} r_c \\ 1 \end{bmatrix} i_2 \quad (12)$$

whence the resistance r_1 at port 1,

$$r_1 = \frac{e_1}{i_1} = \frac{a_{11}r_c + a_{12}}{a_{21}r_c + a_{11}} \quad (13)$$

If r_1 is put equal to r_0 then (13) yields

$$r_0 = \sqrt{\frac{a_{12}}{a_{21}}} \quad (14)$$

or, substituting from (11)

$$r_0 = \sqrt{-\det R} = \sqrt{r_s r_0} \quad (15)$$

Thus, if the load resistance at port 2 has the value given by (15) then the resistance at port 1 will have the same value. The resistance r_0 as given by (15) is the well known 2-port characteristic resistance of symmetrical attenuator theory. From (15),

$$r_0 = \frac{r^2}{r_s} \quad (16)$$

If (16) is substituted in (11) then equation (3) can be written

$$\begin{bmatrix} e_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1 - \frac{r_s^2}{r_c^2}}} & \frac{r_s}{\sqrt{1 - \frac{r_s^2}{r_c^2}}} \\ \frac{r_s}{r_c^2 \sqrt{1 - \frac{r_s^2}{r_c^2}}} & \frac{1}{\sqrt{1 - \frac{r_s^2}{r_c^2}}} \end{bmatrix} \begin{bmatrix} e_2 \\ i_2 \end{bmatrix} \quad (17)$$

Evidently, for a physically realizable 2-port, r_0 is the upper limit of r_s .

We now venture to propose the following analogies:

- $e \sim x$ (linear distance)
- $i \sim t$ (time)
- $r_0 \sim c$ (speed of light)
- $r_s \sim v$ (linear velocity)

and hence write, from (17),

$$\begin{bmatrix} x_1 \\ t_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} & \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} \\ \frac{v}{c^2 \sqrt{1 - \frac{v^2}{c^2}}} & \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \end{bmatrix} \begin{bmatrix} x_2 \\ t_2 \end{bmatrix} \quad (18)$$

Equation (18) will be recognized as the space-time Lorentz transformation of special relativity which transforms the description of events on the x_2 -axis of a Cartesian co-ordinate system S_2 to the x_1 -axis of a similar co-ordinate system S_1 , when S_2 is in motion relative to S_1 with constant velocity v . The transformation is such that the equation

$$x_1 = ct_1 \quad (19)$$

describing the behaviour of a light ray along the x_1 -axis of the system S_1 , becomes

$$x_2 = ct_2 \quad (20)$$

in the system S_2 .

In the case of the 2-port, port 2 and port 1 are analogous respectively to the co-ordinate systems S_2 and S_1 . Equation (17) is a voltage-current transformation which can be said to transform the description of events at port 2 to port 1 when the 2-port has "short-circuit" resistance r_s . The transformation is such that the equation

$$e_1 = r_c i_1 \quad (21)$$

describing the behaviour at port 1 when port 2 is terminated with a load equal to the 2-port characteristic resistance becomes

$$e_2 = r_o i_2 \quad (22)$$

at port 2. Thus, the study of light propagation in special relativity is analogous to the study of a resistance 2-port in iterative connexion.

It is evident that the Lorentz transformation (18) reverts to the Galilean transformation of classical physics if c is allowed to have an infinitely large value. Thus

$$\begin{bmatrix} x_1 \\ t_1 \end{bmatrix} = \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ t_2 \end{bmatrix} \quad (23)$$

Analogously if r_c is allowed to have an infinitely large value, the transformation (17) reverts to

$$\begin{bmatrix} e_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} 1 & r_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_2 \\ i_2 \end{bmatrix} \quad (24)$$

which corresponds to the 2-port represented in Fig. 2.

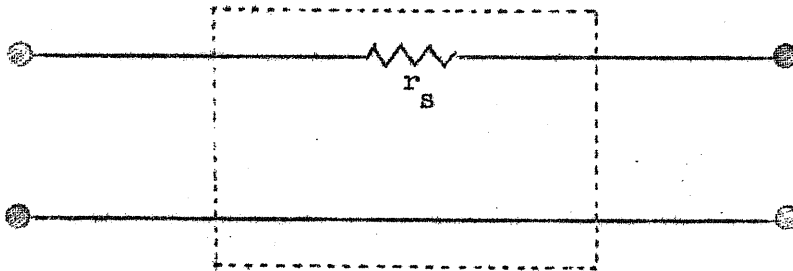


Fig. 2

The theory of special relativity consists in the results obtained from the application of the Lorentz transformation to the study of physical phenomena. These results differ from those of classical physics which is based on the Galilean transformation. As an example, let us consider the so-called Einstein theorem of addition for velocities in one direction.

Suppose a particle in S_2 is moving along the x_2 -axis with speed $w_2 = dx_2/dt_2$. According to (18), the speed of the particle as measured in S_1 will be

$$w_1 = \frac{dx_1}{dt_1} = \frac{dx_2 + v dt_2}{dt_2 + \frac{v}{c^2} dx_2} = \frac{w_2 + v}{1 + \frac{v w_2}{c^2}} \quad (25)$$

If the particle is a photon then $w_2 = c$ and (25) reduces

$$w_1 = \frac{c + v}{1 + \frac{v}{c}} = c \quad (26)$$

which is consistent with the postulate of special relativity; that is, the speed of light is the same in all inertial co-ordinate systems. If c is made infinitely large, (25) reverts to the Galilean theorem of addition for velocities, thus

$$w_1 = w_2 + v \quad (27)$$

Consider now the case of the 2-port. The resistance at port 2 is $r_2 = de_2/di_2$. According to (17), the resistance at port 1 is

$$r_1 = \frac{de_1}{di_1} = \frac{de_2 + r_s di_2}{di_1 + \frac{r_s}{r_c} de_2} = \frac{r_2 + r_s}{1 + \frac{r_s r_2}{r_c}} \quad (28)$$

If port 2 is terminated with a load equal to the characteristic resistance of the 2-port then $r_2 = r_c$ and (28) reduces to

$$r_1 = \frac{r_2 + r_s}{1 + \frac{r_s}{r_c}} = r_c \quad (29)$$

which is consistent with the theory of iteratively connected resistance 2-ports. The resistance is the same at all ports. If r_c is made infinitely large, (28) reverts to

$$r_1 = r_2 + r_s \quad (30)$$

corresponding to the 2-port of Fig. 2.

Similarly, other results obtainable from the Lorentz transformation can be shown to have analogous interpretations in the case of the 2-port. It is well known for instance, that the Lorentz transformation is equivalent to a rotation of the co-ordinate system in Minkowski space. The "angle of rotation" is $\text{arctanh} \frac{v}{c}$. The analogous quantity in the case of the 2-port is $\text{arctanh} \frac{r_s}{r_c}$ which is, of course, the 2-port attenuation constant.

We conclude, therefore, that the resistance 2-port transforms its port variables according to the theory of special relativity.