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RESEARCH ARTICLE

Gain-scheduled H_1 Control via Parameter-Dependent Lyapunov Functions

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Synthesizing a gain-scheduled output feedback H_1 controller via parameter-dependent Lyapunov functions for linear parameter-varying (LPV) plant models involves solving an infinite number of linear matrix inequal-

et al. 2009, Corno et al. 2008, Lu et al. 2006), function substitution (Shin et al. 2002), or state transformation (Balas 2002, Shamma and Cloutier 1993), are normally used to derive an LPV model from the original nonlinear model. Moreover, in the literature, there are several different varieties of LPV models, e.g. grid LPV model (Corno et al. 2008, Lu et al. 2006, Wu et al. 2002, 1996), affine LPV model (Apkarian and Adams 1998, Apkarian 1997, Apkarian et al. 1995) (or polytopic LPV model), tensor-product (TP) convex polytopic model (Baranyi et al. 2006, Baranyi 2005, 2004), etc.; these have been introduced for gain-scheduled H_1 control synthesis and analysis which is usually based on single quadratic Lyapunov function (Apkarian et al. 1995, Becker and Packard 1994) or parameter-dependent Lyapunov function, e.g. parameter-dependent (Corno et al. 2008, Wu et al. 2002, Apkarian and Adams 1998, Wu et al. 1996), affine parameter-dependent (Gahinet et al. 1996), piecewise-affine parameter-dependent (Lim and How 2003, 2002), blending parameter-dependent (Shin et al. 2002), multiple parameter-dependent Lyapunov functions (Lu et al. 2006, Lu and Wu 2004).

Using a single quadratic Lyapunov function, for both the affine LPV model (Apkarian et al. 1995) and the TP convex polytopic model (Chumalee and Whidborne 2009) cases, a finite number of LMIs need only to be evaluated at all vertices while, for the grid LPV models (Wu et al. 1995) case, an infinite number of LMIs have to be evaluated at all points over the entire parameter space in order to determine a pair of positive definite symmetric matrices $(X;Y)$. However, in practice, the symmetric matrices $(X;Y)$ can be determined from a finite number of LMIs by gridding the entire parameter space with a non-dense set of grid points. Having determined the symmetric matrices $(X;Y)$, a more dense grid points set can be tested with these determined symmetric matrices $(X;Y)$ to check whether the LMIs are still satisfied (Wu et al. 2002, 1996). If not, this process is repeated with a denser grid until the symmetric matrices $(X;Y)$, that satisfy the LMIs for all points over the entire parameters space, are obtained (Wu et al. 2002, 1996, Lim and How 2003). However the result of such a heuristic gridding technique is not necessarily reliable and the analysis result is dependent on the choice of gridding points (Wang and Balakrishnan 2002). In addition, for a grid LPV model case, the resulting gain-scheduled controller has high computational on-line complexity at the gain-scheduling level (Wu et al. 1995) while, for the other two cases, the gain-scheduled controller is constructed as an affine matrix-valued function in the polytopic coordinates of the scheduled parameters (Apkarian et al. 1995, Chumalee and Whidborne 2009).

In general, the single quadratic Lyapunov function is more conservative than the parameter-dependent Lyapunov function when the parameters are time-invariant or slowly varying (Gahinet et al. 1996). In addition, when the parameters have a large variation, the piecewise-affine parameter-dependent (Lim and How 2003, 2002), blending parameter-dependent (Shin et al. 2002), and multiple parameter-dependent Lyapunov functions (Lu et al. 2006, Lu and Wu 2004) are less conservative than the parameter-dependent Lyapunov functions. This result from the fact that an LPV model with a large parameter variation can be modelled as a switching linear parameter-varying (SLPV) system which can be made discontinuous along the switching surface by dividing the entire parameters spaces into parameters subsets that are small variation regions. Solving LMIs with the parameters subsets improve the performance measure (). Moreover, Yan and Ozbay (2007) provide sufficient conditions to guarantee the stability of the SLPV systems in terms of the dwell time and the average dwell time, where a switching signal of the SLPV systems with dwell time means a time interval between any two consecutive switching from one model to another model is equal to or greater than . However, using parameter-dependent (Corno et al. 2008, Wu et al. 2002, Apkarian and Adams 1998, Wu et al. 1996), blending parameter-dependent (Shin et al. 2002), and multiple parameter-dependent Lyapunov functions (Lu et al. 2006, Lu and Wu 2004), an infinite number of LMIs have to be evaluated at all points over the entire space of parameters. Furthermore, the resulting gain-scheduled controller requires more complex on-line computations at the gain-scheduling level (Corno et al. 2008, Lu et al. 2006, Lu and Wu 2004, Shin et al. 2002, Wu et al. 2002, Apkarian and Adams 1998).

In this paper, we consider the design problem of gain-scheduled controllers with guaranteed L_2-gain performance for a class of affine LPV systems using parameter-dependent Lyapunov functions. By simple manipulations on the parameter-dependent Lyapunov functions or the bounded real lemma (Apkarian et al. 1995) inequality, a symmetric matrix polytope form of these inequalities is obtained. Hence, the LMIs need only to be evaluated at all vertices of the system state-space model matrices and the variation rate of the scheduled parameters. In addition,

Fig. 2. Spectrogram of the generated OFDM signal for $E_b=E_0 = 15$ dB

As is displayed in Fig. 2, the generated OFDM signal demonstrates its characteristics, such as 14 tones, central frequency of 15000Hz, bandwidth of 10000Hz, etc., that match the desired configuration. The periodic intervals during the transmission signals are observable, which proves the success of embedding spectrum holes using the simulation framework presented in Fig. 1.

III. SPECTRAL CORRELATION FUNCTION

With the assumption of periodicity in spectrum holes, cyclostationary feature sensing is expected to outperform conventional methods, e.g. energy detection, especially in low SNR regions due to better noise immunity [21]. Spectral correlation function (SCF) is one of the most common methods to extract cyclic features in the frequency domain and is introduced in this section.

According to [22], the cyclic autocorrelation function (CAF) $R_X[\ ;\ _i]$ of a discrete signal $X[n]$ is defined in the time domain as follows:

$$R_X[\ ;\ _i] = \lim_{L\to\infty} \frac{1}{L} \sum_{n=-L/2}^{L/2} R_X[n;\ ]\, e^{-j2\ _i n} \qquad (1)$$

where is cyclic frequency belonging to the cyclic frequency set A; is the time lag; corresponds to the time interval; L is the length of the signal.

SCF of $x[n]$ is the Fourier transform (FT) of CAF [23] with the expression as: $_X(f;\ ) = F[R_X[\ ;\ ]]$. Practical method of estimating SCF is based on using an averaged periodogram, hence the estimated SCF is given as follows [24]:

$$\hat{}_X(f;\ ) = \frac{1}{K}\sum_{k=0}^{K-1} X^k_{n_w}[f + (1\ \ )\ ]\, X^k_{n_w}[f\ \ ] \qquad (2)$$

where $X^k_{n_w}$ is obtained from short time Fourier transform (STFT) of $X[n]$ with an averaging window defined by coefficients k_{th}; $K = \lceil[(L\ \ n_w)/R] + 1$ with L is signal length, n_w is window width, and R is a sample index allowing overlaps. $\lceil\ \rceil$ is a ceil function that returns the smallest integer value that is bigger or equal to the input value.

The three dimensional representation of the SCF of the simulated communication signal (generated using the system in Fig. 1) is shown in Fig. 3.

Fig. 3. Spectral correlation function of the generated OFDM signal

As one can see from the Fig. 3, the CAF of the desired communication signal has an observable pattern in a form of multiple peaks. We can easily extract some other characteristics, e.g. the central frequency is around 15000 Hz, bandwidth is 10000 Hz, number of tone (presented as peaks) is 14, and cyclic frequency is 88 Hz. The above results are in accordance to our desired configuration and this validates the effectiveness of SCF method in extraction of cyclic patterns of the simulated signal.

IV. LEARNING BASED DETECTION SOLUTION

In this paper, we propose to use a learning based scheme to detect and extract spectrum holes, where R-CNN is considered as a intelligent detector for spectrum sensing because of its efficiency in 'recognition using regions' [18]. The framework of the proposed LB solution is presented in Fig. 4.

Fig. 4. The framework of R-CNN based detection

As illustrated in Fig. 4, the framework of the proposed LB solution contains the following parts: data stream generator (presented in Sec. II), the cyclostationary detector (SCF based sensing explained in Sec. III), a learning based detector (R-CNN is explained in Sec. IV-B) and the extractor of spectrum hole parameters (explained below).

The first step of our LB solution is to use the SCF, defined by equation (2), to extract the feature map of the processed signal. At the second step, by performing the deep learning based detection, the presence of spectrum hole is detected and its characteristics are extracted.

The representation of R-CNN output (called bounding box) is commonly defined as rectangle described by parameters $(t; l; w; h)$, where t and l are the top and left corner indices, and w and h are the width and height of the rectangle, respectively.

equations

$$\begin{matrix} \underline{x} \\ \underline{x}_k \end{matrix} = A_{cl}(\ ;\ \bar{\ }) \begin{matrix} x \\ x_k \end{matrix} + B_{cl}(\ )w$$

$$z = C_{cl}(\ ) \begin{matrix} x \\ x_k \end{matrix} + D_{cl}(\ )w \qquad (25)$$

where

$$A_{cl}(\ ;\ \bar{\ }) = \begin{matrix} A(\ ) + B_2 D_k(\ ) C_2 & B_2 C_k(\ ) \\ B_k(\ ) C_2 & A_k(\ ;\ \bar{\ }) \end{matrix}$$

(28), we get

$$\begin{pmatrix} \underline{\mathbf{X}}(\theta) + \left(\mathbf{X}(\theta)\mathbf{A}(\theta) + \hat{\mathbf{B}}_{\mathrm{k}}(\theta)\mathbf{C}_2 + (\star)\right) & \star & \star & \star \\ \hat{\mathbf{A}}_{\mathrm{k}}^{\mathrm{T}}(\theta) + \mathbf{A}(\theta) + \mathbf{B}_2\mathbf{D}_{\mathrm{k}}(\theta)\mathbf{C}_2 & -\underline{\mathbf{Y}}(\theta) + \left(\mathbf{A}(\theta)\mathbf{Y}(\theta) + \mathbf{B}_2\hat{\mathbf{C}}_{\mathrm{k}}(\theta) + (\star)\right) & \star & \star \\ \mathbf{B}_1^{\mathrm{T}}(\theta)\mathbf{X}(\theta) + \mathbf{D}_{21}^{\mathrm{T}}\hat{\mathbf{B}}_{\mathrm{k}}^{\mathrm{T}}(\theta) & \mathbf{B}_1^{\mathrm{T}}(\theta) + \mathbf{D}_{21}^{\mathrm{T}}\mathbf{D}_{\mathrm{k}}^{\mathrm{T}}(\theta)\mathbf{B}_2^{\mathrm{T}} & -\gamma\mathbf{I} & \star \\ \mathbf{C}_1(\theta) + \mathbf{D}_{12}\mathbf{D}_{\mathrm{k}}(\theta)\mathbf{C}_2 & \mathbf{C}_1(\theta)\mathbf{Y}(\theta) + \mathbf{D}_{12}\hat{\mathbf{C}}_{\mathrm{k}}(\theta) & \mathbf{D}_{11}(\theta) + \mathbf{D}_{12}\mathbf{D}_{\mathrm{k}}(\theta)\mathbf{D}_{21} & -\gamma\mathbf{I} \end{pmatrix} < 0 \qquad (44)$$

where the notation $\star$ represents a symmetric matrix block. Moreover, substituting (22) and (35){(42) in (44), we have

$$\sum_{i=1}^{m}\sum_{k=1}^{m}\alpha_i^2\beta_k \begin{pmatrix} \tilde{\mathbf{X}}_k + \left(\hat{\mathbf{X}}_i\hat{\mathbf{A}}_i + \tilde{\mathbf{B}}_{\mathrm{k}_i}\mathbf{C}_2 + (\star)\right) & \star & \star & \star \\ \tilde{\mathbf{A}}_{\mathrm{k}_i}^{\mathrm{T}} + \hat{\mathbf{A}}_i + \mathbf{B}_2\tilde{\mathbf{D}}_{\mathrm{k}_i}\mathbf{C}_2 & -\tilde{\mathbf{Y}}_k + \left(\hat{\mathbf{A}}_i\hat{\mathbf{Y}}_i + \mathbf{B}_2\tilde{\mathbf{C}}_{\mathrm{k}_i} + (\star)\right) & \star & \star \\ \hat{\mathbf{B}}_{1_i}^{\mathrm{T}}\hat{\mathbf{X}}_i + \mathbf{D}_{21}^{\mathrm{T}}\tilde{\mathbf{B}}_{\mathrm{k}_i}^{\mathrm{T}} & \hat{\mathbf{B}}_{1_i}^{\mathrm{T}} + \mathbf{D}_{21}^{\mathrm{T}}\tilde{\mathbf{D}}_{\mathrm{k}_i}^{\mathrm{T}}\mathbf{B}_2^{\mathrm{T}} & -\gamma\mathbf{I} & \star \\ \hat{\mathbf{C}}_{1_i} + \mathbf{D}_{12}\tilde{\mathbf{D}}_{\mathrm{k}_i}\mathbf{C}_2 & \hat{\mathbf{C}}_{1_i}\hat{\mathbf{Y}}_i + \mathbf{D}_{12}\tilde{\mathbf{C}}_{\mathrm{k}_i} & \hat{\mathbf{D}}_{11_i} + \mathbf{D}_{12}\tilde{\mathbf{D}}_{\mathrm{k}_i}\mathbf{D}_{21} & -\gamma\mathbf{I} \end{pmatrix}$$

$$+2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\sum_{k=1}^{m}\alpha_i\alpha_j\beta_k \begin{pmatrix} \tilde{\mathbf{X}}_k + \frac{1}{2}\left(\hat{\mathbf{X}}_j\hat{\mathbf{A}}_i + \tilde{\mathbf{B}}_{\mathrm{k}_j}\mathbf{C}_2 + \hat{\mathbf{X}}_i\hat{\mathbf{A}}_j + \tilde{\mathbf{B}}_{\mathrm{k}_i}\mathbf{C}_2 + (\star)\right) & \star & \star & \star \\ \frac{1}{2}\left(\tilde{\mathbf{A}}_{\mathrm{k}_j}^{\mathrm{T}} + \hat{\mathbf{A}}_i + \mathbf{B}_2\tilde{\mathbf{D}}_{\mathrm{k}_j}\mathbf{C}_2 + \tilde{\mathbf{A}}_{\mathrm{k}_i}^{\mathrm{T}} + \hat{\mathbf{A}}_j + \mathbf{B}_2\tilde{\mathbf{D}}_{\mathrm{k}_i}\mathbf{C}_2\right) & -\tilde{\mathbf{Y}}_k + \frac{1}{2}\left(\hat{\mathbf{A}}_i\hat{\mathbf{Y}}_j + \mathbf{B}_2\tilde{\mathbf{C}}_{\mathrm{k}_j} + \hat{\mathbf{A}}_j\hat{\mathbf{Y}}_i + \mathbf{B}_2\tilde{\mathbf{C}}_{\mathrm{k}_i} + (\star)\right) & \star & \star \\ \frac{1}{2}\left(\hat{\mathbf{B}}_{1_i}^{\mathrm{T}}\hat{\mathbf{X}}_j + \mathbf{D}_{21}^{\mathrm{T}}\tilde{\mathbf{B}}_{\mathrm{k}_j}^{\mathrm{T}} + \hat{\mathbf{B}}_{1_j}^{\mathrm{T}}\hat{\mathbf{X}}_i + \mathbf{D}_{21}^{\mathrm{T}}\tilde{\mathbf{B}}_{\mathrm{k}_i}^{\mathrm{T}}\right) & \frac{1}{2}\left(\hat{\mathbf{B}}_{1_i}^{\mathrm{T}} + \mathbf{D}_{21}^{\mathrm{T}}\tilde{\mathbf{D}}_{\mathrm{k}_j}^{\mathrm{T}}\mathbf{B}_2^{\mathrm{T}} + \hat{\mathbf{B}}_{1_j}^{\mathrm{T}} + \mathbf{D}_{21}^{\mathrm{T}}\tilde{\mathbf{D}}_{\mathrm{k}_i}^{\mathrm{T}}\mathbf{B}_2^{\mathrm{T}}\right) & -\gamma\mathbf{I} & \star \\ \frac{1}{2}\left(\hat{\mathbf{C}}_{1_i} + \mathbf{D}_{12}\tilde{\mathbf{D}}_{\mathrm{k}_j}\mathbf{C}_2 + \hat{\mathbf{C}}_{1_j} + \mathbf{D}_{12}\tilde{\mathbf{D}}_{\mathrm{k}_i}\mathbf{C}_2\right) & \frac{1}{2}\left(\hat{\mathbf{C}}_{1_i}\hat{\mathbf{Y}}_j + \mathbf{D}_{12}\tilde{\mathbf{C}}_{\mathrm{k}_j} + \hat{\mathbf{C}}_{1_j}\hat{\mathbf{Y}}_i + \mathbf{D}_{12}\tilde{\mathbf{C}}_{\mathrm{k}_i}\right) & \frac{1}{2}\left(\hat{\mathbf{D}}_{11_i} + \mathbf{D}_{12}\tilde{\mathbf{D}}_{\mathrm{k}_j}\mathbf{D}_{21} + \hat{\mathbf{D}}_{11_j} + \mathbf{D}_{12}\tilde{\mathbf{D}}_{\mathrm{k}_i}\mathbf{D}_{21}\right) & -\gamma\mathbf{I} \end{pmatrix} < 0 \qquad (45)$$

in which the inequality (45) can be also rewritten as

$$\sum_{i=1}^{m}\sum_{k=1}^{m}\alpha_i^2\beta_k\left(\mathbf{\Psi}_{\mathsf{cl}_{ii}}+\mathcal{Q}^{\mathsf{T}}\hat{\mathbf{K}}_i^{\mathsf{T}}\mathcal{P}+\mathcal{P}^{\mathsf{T}}\hat{\mathbf{K}}_i\mathcal{Q}\right)+2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\sum_{k=1}^{m}\alpha_i\alpha_j\beta_k\left(\frac{1}{2}\left(\mathbf{\Psi}_{\mathsf{cl}_{ij}}\right.\right.$$

$$\left.\left.+\mathcal{Q}^{\mathsf{T}}\hat{\mathbf{K}}_i^{\mathsf{T}}\mathcal{P}+\mathcal{P}^{\mathsf{T}}\hat{\mathbf{K}}_i\mathcal{Q}+\mathbf{\Psi}_{\mathsf{cl}_{ji}}+\mathcal{Q}^{\mathsf{T}}\hat{\mathbf{K}}_j^{\mathsf{T}}\mathcal{P}+\mathcal{P}^{\mathsf{T}}\hat{\mathbf{K}}_j\mathcal{Q}\right)\right)<0 \qquad (46)$$

where

$$\mathbf{\Psi}_{\mathsf{cl}_{ii}}=\begin{pmatrix}\tilde{\mathbf{X}}_k+\hat{\mathbf{X}}_i\hat{\mathbf{A}}_i+(\star) & \star & \star & \star\\ \hat{\mathbf{A}}_i & -\tilde{\mathbf{Y}}_k+\hat{\mathbf{A}}_i\hat{\mathbf{Y}}_i+(\star) & \star & \star\\ \hat{\mathbf{B}}_{1_i}^{\mathsf{T}}\hat{\mathbf{X}}_i & \hat{\mathbf{B}}_{1_i}^{\mathsf{T}} & -\gamma\mathbf{I} & \star\\ \hat{\mathbf{C}}_{1_i} & \hat{\mathbf{C}}_{1_i}\hat{\mathbf{Y}}_i & \hat{\mathbf{D}}_{11_i} & -\gamma\mathbf{I}\end{pmatrix}$$

$$\mathbf{\Psi}_{\mathsf{cl}_{ij}}=\begin{pmatrix}\tilde{\mathbf{X}}_k+\hat{\mathbf{X}}_j\hat{\mathbf{A}}_i+(\star) & \star & \star & \star\\ \hat{\mathbf{A}}_i & -\tilde{\mathbf{Y}}_k+\hat{\mathbf{A}}_i\hat{\mathbf{Y}}_j+(\star) & \star & \star\\ \hat{\mathbf{B}}_{1_i}^{\mathsf{T}}\hat{\mathbf{X}}_j & \hat{\mathbf{B}}_{1_i}^{\mathsf{T}} & -\gamma\mathbf{I} & \star\\ \hat{\mathbf{C}}_{1_i} & \hat{\mathbf{C}}_{1_i}\hat{\mathbf{Y}}_j & \hat{\mathbf{D}}_{11_i} & -\gamma\mathbf{I}\end{pmatrix}$$

$$\mathbf{\Psi}_{\mathsf{cl}_{ji}}=\begin{pmatrix}\tilde{\mathbf{X}}_k+\hat{\mathbf{X}}_i\hat{\mathbf{A}}_j+(\star) & \star & \star & \star\\ \hat{\mathbf{A}}_j & -\tilde{\mathbf{Y}}_k+\hat{\mathbf{A}}_j\hat{\mathbf{Y}}_i+(\star) & \star & \star\\ \hat{\mathbf{B}}_{1_j}^{\mathsf{T}}\hat{\mathbf{X}}_i & \hat{\mathbf{B}}_{1_j}^{\mathsf{T}} & -\gamma\mathbf{I} & \star\\ \hat{\mathbf{C}}_{1_j} & \hat{\mathbf{C}}_{1_j}\hat{\mathbf{Y}}_i & \hat{\mathbf{D}}_{11_j} & -\gamma\mathbf{I}\end{pmatrix}$$

$$\mathcal{Q}=\left[\mathcal{C},\quad \mathcal{D}_{21},\quad \mathbf{0}_{(p+q_2)\times q_1}\right],\ \mathcal{P}=\left[\tilde{\mathbf{B}}^{\mathsf{T}},\quad \mathbf{0}_{(p+m_2)\times m_1},\quad \mathcal{D}_{12}^{\mathsf{T}}\right]$$

$$\hat{\mathbf{K}}_i=\begin{pmatrix}\tilde{\mathbf{A}}_{\mathsf{k}_i} & \tilde{\mathbf{B}}_{\mathsf{k}_i}\\ \tilde{\mathbf{C}}_{\mathsf{k}_i} & \tilde{\mathbf{D}}_{\mathsf{k}_i}\end{pmatrix},\ \tilde{\mathbf{B}}=\begin{bmatrix}\mathbf{I}_p & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_2\end{bmatrix}$$

$$\mathcal{C}=\begin{bmatrix}\mathbf{0} & \mathbf{I}_p\\ \mathbf{C}_2 & \mathbf{0}\end{bmatrix},\ \mathcal{D}_{12}=\begin{bmatrix}\mathbf{0} & \mathbf{D}_{12}\end{bmatrix},\ \mathcal{D}_{21}=\begin{bmatrix}\mathbf{0}\\ \mathbf{D}_{21}\end{bmatrix} \qquad (47)$$

By Lemma 2.1 and knowing the matrix vertices $(\hat{\mathbf{X}}_i, \hat{\mathbf{Y}}_i)$, $i = 1, 2, \ldots, m$, the system matrix vertices $\hat{\mathbf{K}}_i$ can be determined from (46), that is an LMI in $\hat{\mathbf{K}}_i$, at all vertices for which $(\hat{\mathbf{K}}_1, \hat{\mathbf{K}}_2, \ldots, \hat{\mathbf{K}}_m)$ have to satisfy all of $m^2(m+1)/2$ LMIs. Furthermore, knowing $\tilde{\mathbf{A}}_{\mathsf{k}_i}, \ldots, \tilde{\mathbf{D}}_{\mathsf{k}_i}$, the controller system matrices $\mathbf{A}_{\mathsf{k}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}), \ldots, \mathbf{D}_{\mathsf{k}}(\boldsymbol{\theta})$ can be computed on-line in real-time using (29)–(31) and (38) with instantaneous measurement values of $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$, where the proposed intermediate controller variables $\hat{\mathbf{A}}_{\mathsf{k}}(\boldsymbol{\theta})$, $\hat{\mathbf{B}}_{\mathsf{k}}(\boldsymbol{\theta})$, $\hat{\mathbf{C}}_{\mathsf{k}}(\boldsymbol{\theta})$ and $\mathbf{D}_{\mathsf{k}}(\boldsymbol{\theta})$, and $\left(\mathbf{X}(\boldsymbol{\theta}), \mathbf{Y}(\boldsymbol{\theta})\right)$ depend affinely on the parameters $\boldsymbol{\theta}$ and they can be computed on-line in real-time using (35)–(40). Hence, the proposed method reduces computational burden and eases controller implementation compared to the explicit controller formulas (Apkarian and Adams 1998, Gahinet 1996).

However, usually, the parameter derivatives either are not available or are difficult to estimate during system operation (Apkarian and Adams 1998). To avoid using the measured value of $\dot{\boldsymbol{\theta}}$, we can constrain either $\mathbf{X}(\boldsymbol{\theta})$ or $\mathbf{Y}(\boldsymbol{\theta})$ to depend affinely on $\boldsymbol{\theta}$. This yields $\dot{\mathbf{X}}(\boldsymbol{\theta})\mathbf{Y}(\boldsymbol{\theta}) + \dot{\mathbf{N}}(\boldsymbol{\theta})\mathbf{M}^{\mathsf{T}}(\boldsymbol{\theta}) = -(\mathbf{X}(\boldsymbol{\theta})\dot{\mathbf{Y}}(\boldsymbol{\theta}) + \mathbf{N}(\boldsymbol{\theta})\dot{\mathbf{M}}^{\mathsf{T}}(\boldsymbol{\theta})) = 0$ (Apkarian and Adams 1998), hence equation (29) becomes

$$\mathbf{A}_{\mathsf{k}}(\boldsymbol{\theta})=\mathbf{N}^{-1}(\boldsymbol{\theta})\left(\hat{\mathbf{A}}_{\mathsf{k}}(\boldsymbol{\theta})-\mathbf{X}(\boldsymbol{\theta})\left(\mathbf{A}(\boldsymbol{\theta})-\mathbf{B}_2\mathbf{D}_{\mathsf{k}}(\boldsymbol{\theta})\mathbf{C}_2\right)\mathbf{Y}(\boldsymbol{\theta})\right.$$

$$\left.-\hat{\mathbf{B}}_{\mathsf{k}}(\boldsymbol{\theta})\mathbf{C}_2\mathbf{Y}(\boldsymbol{\theta})-\mathbf{X}(\boldsymbol{\theta})\mathbf{B}_2\hat{\mathbf{C}}_{\mathsf{k}}(\boldsymbol{\theta})\right)\mathbf{M}^{-\mathsf{T}}(\boldsymbol{\theta}) \qquad (48)$$

Lemma 3.1: (Gahinet and Apkarian 1994, Projection lemma) Given an inequality problem of the form

$$\boldsymbol{\Psi} + \mathbf{Q}^{\mathrm{T}}\mathbf{K}^{\mathrm{T}}\mathbf{P} + \mathbf{P}^{\mathrm{T}}\mathbf{K}\mathbf{Q} < 0 \tag{49}$$

where $\boldsymbol{\Psi} \in \mathbb{R}^{m\times m}$ is a symmetric matrix, $\mathbf{Q}$ and $\mathbf{P}$ are matrices with column dimension m. Let $\mathbf{Q}_\perp$ and $\mathbf{P}_\perp$ be any matrices whose columns form bases of the null spaces of $\mathbf{Q}$ and $\mathbf{P}$ respectively; the above problem is solvable for a matrix $\mathbf{K}$ of compatible dimensions if and only if

$$\mathbf{Q}_\perp^{\mathrm{T}}\boldsymbol{\Psi}\mathbf{Q}_\perp < 0, \quad \mathbf{P}_\perp^{\mathrm{T}}\boldsymbol{\Psi}\mathbf{P}_\perp < 0 \tag{50}$$

By Lemmas 2.1 and 3.1, the LMIs of (46) are solvable at all vertices for $\hat{\mathbf{K}}_i$ if and only if there exist a pair of positive definite symmetric matrices $\big(\mathbf{X}(\boldsymbol{\theta}), \mathbf{Y}(\boldsymbol{\theta})\big)$ that satisfy the following theorem.

Theorem 3.2: There exists an LPV controller $\mathbf{K}(\boldsymbol{\theta})$ guaranteeing the closed-loop system (21) and (23), quadratic H_∞ performance γ along all possible parameter trajectories, $\forall(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \boldsymbol{\Theta}\times\boldsymbol{\Phi}$, if and only if the following LMI conditions hold for some positive definite symmetric matrices $(\mathbf{X}(\boldsymbol{\theta}), \mathbf{Y}(\boldsymbol{\theta}))$, which further satisfy Rank$(\mathbf{X}(\boldsymbol{\theta}) - \mathbf{Y}^{-1}(\boldsymbol{\theta})) \le p$.

$$\begin{pmatrix} \mathbf{N}_{\mathrm{X}} & 0 \\ 0 & \mathbf{I} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \hat{\mathbf{A}}_i^{\mathrm{T}}\hat{\mathbf{X}}_i + \hat{\mathbf{X}}_i\hat{\mathbf{A}}_i + \dot{\mathbf{X}}_k & \hat{\mathbf{X}}_i\hat{\mathbf{B}}_{1_i} & \hat{\mathbf{C}}_{1_i}^{\mathrm{T}} \\ \hat{\mathbf{B}}_{1_i}^{\mathrm{T}}\hat{\mathbf{X}}_i & -\gamma\mathbf{I} & \hat{\mathbf{D}}_{11_i}^{\mathrm{T}} \\ \hat{\mathbf{C}}_{1_i} & \hat{\mathbf{D}}_{11_i} & -\gamma\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{N}_{\mathrm{X}} & 0 \\ 0 & \mathbf{I} \end{pmatrix} < 0 \tag{51}$$

$$\begin{pmatrix} \mathbf{N}_{\mathrm{Y}} & 0 \\ 0 & \mathbf{I} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \hat{\mathbf{A}}_i\hat{\mathbf{Y}}_i + \hat{\mathbf{Y}}_i\hat{\mathbf{A}}_i^{\mathrm{T}} - \dot{\mathbf{Y}}_k & \hat{\mathbf{Y}}_i\hat{\mathbf{C}}_{1_i}^{\mathrm{T}} & \hat{\mathbf{B}}_{1_i} \\ \hat{\mathbf{C}}_{1_i}\hat{\mathbf{Y}}_i & -\gamma\mathbf{I} & \hat{\mathbf{D}}_{11_i} \\ \hat{\mathbf{B}}_{1_i}^{\mathrm{T}} & \hat{\mathbf{D}}_{11_i}^{\mathrm{T}} & -\gamma\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{N}_{\mathrm{Y}} & 0 \\ 0 & \mathbf{I} \end{pmatrix} < 0 \tag{52}$$

$$\begin{pmatrix} \mathbf{N}_{\mathrm{X}} & 0 \\ 0 & \mathbf{I} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \hat{\mathbf{A}}_i^{\mathrm{T}}\hat{\mathbf{X}}_j + \hat{\mathbf{X}}_j\hat{\mathbf{A}}_i + \hat{\mathbf{A}}_j^{\mathrm{T}}\hat{\mathbf{X}}_i + \hat{\mathbf{X}}_i\hat{\mathbf{A}}_j + 2\dot{\mathbf{X}}_k & \hat{\mathbf{X}}_i\hat{\mathbf{B}}_{1_j} + \hat{\mathbf{X}}_j\hat{\mathbf{B}}_{1_i} & \hat{\mathbf{C}}_{1_i}^{\mathrm{T}} + \hat{\mathbf{C}}_{1_j}^{\mathrm{T}} \\ \hat{\mathbf{B}}_{1_i}^{\mathrm{T}}\hat{\mathbf{X}}_j + \hat{\mathbf{B}}_{1_j}^{\mathrm{T}}\hat{\mathbf{X}}_i & -2\gamma\mathbf{I} & \hat{\mathbf{D}}_{11_i}^{\mathrm{T}} + \hat{\mathbf{D}}_{11_j}^{\mathrm{T}} \\ \hat{\mathbf{C}}_{1_i} + \hat{\mathbf{C}}_{1_j} & \hat{\mathbf{D}}_{11_i} + \hat{\mathbf{D}}_{11_j} & -2\gamma\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{N}_{\mathrm{X}} & 0 \\ 0 & \mathbf{I} \end{pmatrix} < 0 \tag{53}$$

$$\begin{pmatrix} \mathbf{N}_{\mathrm{Y}} & 0 \\ 0 & \mathbf{I} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \hat{\mathbf{A}}_i\hat{\mathbf{Y}}_j + \hat{\mathbf{Y}}_j\hat{\mathbf{A}}_i^{\mathrm{T}} + \hat{\mathbf{A}}_j\hat{\mathbf{Y}}_i + \hat{\mathbf{Y}}_i\hat{\mathbf{A}}_j^{\mathrm{T}} - 2\dot{\mathbf{Y}}_k & \hat{\mathbf{Y}}_i\hat{\mathbf{C}}_{1_j}^{\mathrm{T}} + \hat{\mathbf{Y}}_j\hat{\mathbf{C}}_{1_i}^{\mathrm{T}} & \hat{\mathbf{B}}_{1_i} + \hat{\mathbf{B}}_{1_j} \\ \hat{\mathbf{C}}_{1_i}\hat{\mathbf{Y}}_j + \hat{\mathbf{C}}_{1_j}\hat{\mathbf{Y}}_i & -2\gamma\mathbf{I} & \hat{\mathbf{D}}_{11_i} + \hat{\mathbf{D}}_{11_j} \\ \hat{\mathbf{B}}_{1_i}^{\mathrm{T}} + \hat{\mathbf{B}}_{1_j}^{\mathrm{T}} & \hat{\mathbf{D}}_{11_i}^{\mathrm{T}} + \hat{\mathbf{D}}_{11_j}^{\mathrm{T}} & -2\gamma\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{N}_{\mathrm{Y}} & 0 \\ 0 & \mathbf{I} \end{pmatrix} < 0 \tag{54}$$

$$\begin{pmatrix} \mathbf{X}_i & \mathbf{I} \\ \mathbf{I} & \mathbf{Y}_i \end{pmatrix} > 0, \quad \text{for } i, k = 1, \ldots, m \text{ and } 1 \le i < j \le m \tag{55}$$

Note that Theorem 3.2 provides a new approach where an alternative to the multi-convexity approach (Apkarian and Tuan 2000) is given in Appendix B. In addition, $\mathbf{N}_{\mathrm{X}}$ and $\mathbf{N}_{\mathrm{Y}}$ denote bases of the null spaces of $[\mathbf{C}_2, \mathbf{D}_{21}]$ and $[\mathbf{B}_2^{\mathrm{T}}, \mathbf{D}_{12}^{\mathrm{T}}]$, respectively. The inequality (55) ensures $\mathbf{X}(\boldsymbol{\theta}), \mathbf{Y}(\boldsymbol{\theta}) > 0$ and $\mathbf{X}(\boldsymbol{\theta}) - \mathbf{Y}(\boldsymbol{\theta})^{-1} \ge 0$.

4. Numerical example

To demonstrate the effectiveness of the proposed approach, we consider the example of Leith and Leithead (1999). For this example it has been shown that for an LPV plant model derived from the Jacobian, a common approach, with an LPV controller synthesized using the method of Apkarian et al. (1995), the closed-loop system is only stable when the LPV controller is applied