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Shock wave structure in highly rarefied flows

- by -

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SUMMARY

The Boltzmann equation is written in terms of two functions associated with the gain and loss of a certain type of molecule due to collisions. Its integral form is then applied to the problem of normal shock structure, and an iteration technique is used to determine the solution. The first approximation to the velocity distribution function of the Chapman-Enskog sequence, which leads to the Navier-Stokes equations, is used to initiate the iteration scheme. Expressions for the distribution function and the flow parameters pertinent to the first iteration are derived and show that the B-G-K model results can be obtained as a special case. This model is found to be valid in the continuum regime only, and is consequently limited to the study of strong shocks. In the present treatment the iteration is carried out on the distribution function and the analysis indicates that the method is equally valid for variations in both Mach and Knudsen numbers. Finally, the results of the first approximation are simplified, and expressed in a form suitable for numerical computation, and the range of their validity is discussed. The method should be equally suitable for other flow problems of linear or nonlinear nature.

1. Introduction

The problem of plane shock wave structure has been studied in the past by various methods. The structure of weak shocks in a continuum medium is well understood and several attempts to describe the stronger shocks have been made. However, the position of the shock problem in rarefied flows is still not clear and its simple geometry and freedom from solid boundaries still do not necessarily imply that the strong shock results are applicable in this case. The predictions of the so-called higher-continuum approximations have not been supported by the experimental work available on this problem, and these approaches have been meeting growing criticism. The usual definition, in the shock problem, of the Knudsen number based on the geometry of the shock thickness, is quite arbitrary and does not offer a measure of the degree of the flow rarefaction throughout a wide range. Indeed, the more diffuse the shock becomes, the more this arbitrariness tends to be significant.

The aim of the present work is to study the shock problem under conditions of increasing rarefaction and variable shock strength. The problem is approached through the kinetic theory and the full Boltzmann equation is used, without actually linearising it. A solution by iteration is then adopted due to the success it has met in other problems and to its suitability for use in conjunction with the Boltzmann equation. To start the iteration scheme, which is performed on the velocity distribution function, the first approximation in the Chapman-Enskog sequence leading to the Navier-Stokes equations, is considered in view of its relative simplicity and validity. In addition, the parameters appearing in this distribution function can be associated with the local degree of rarefaction.

Since this work was undertaken, Liepmann et al (1962) and Chahine (1963) have studied the shock problem and computed profiles using the B-G-K model in an iteration scheme starting from the Navier-Stokes solution. This method is suitable for strong shocks, and can be obtained as a special case from the results of the present treatment. The study by Liepmann et al, has, however, touched upon several aspects of interest to the present study and to the shock problem in general.

In section II the integral form of the Boltzmann equation is transformed and suitably expressed for an iterative solution, and in section III the equation is applied to the shock wave problem with its appropriate boundary conditions.

In section IV the iteration technique is initiated using the Navier-Stokes approximation, and the first iteration to the distribution function is determined. In section V the equations for the flow parameters are derived and brought into a form which can be solved numerically. A simplified form of these equations is also determined. The study is then concluded by a discussion.

II General formulation

Consider the steady state one-dimensional Boltzmann integro-differential equation, (subsequently B.E.) for the single particle velocity distribution function $f(\vec{c},x)$, in the absence of external forces.

$$u\frac{df}{dx} = \iint \{ f(\vec{c}',x) f(\vec{c}_1',x) - f(\vec{c},x) f(\vec{c}_1,x) \} k_1(g,b)\, d\vec{k}\, d\vec{c}_1 \tag{2.1}$$

in which $k_1(g,b)$ is a positive scalar function of the relative speed and the encounter variable and $d\vec{k}$ contains the geometrical variables specifying a collision and is independent of the velocities.

The function f, in the case of plane shock wave in a uniform stream, varies in a continuous manner as the flow progresses downstream and its form could be described to undergo continuous 'stretching and distortion', consequently the function f is represented by

$$f(\vec{c},x) = f^{(0)}(\vec{c},x)\left[1 + \phi(\vec{c},x)\right] \tag{2.2}$$

$f^{(0)}$ is the local Maxwellian distribution function given by,

$$f^{(0)}(\vec{c},x) = n(x)\left(\frac{\beta(x)}{\pi}\right)^{3/2} \exp\{-\beta(x)(\vec{c}-\bar{u}(x))^2\};\ \beta(x) = \frac{m}{2kT(x)} \tag{2.3}$$

where n, $\bar{u}$, T are given by the actual local conditions, and the deviation function from Maxwellian, ϕ, to be determined, is such that f is a solution of the B.E. The function ϕ is not necessarily small, but when the random speed $C \to \infty$ the product $f^{(0)}\phi \to 0$. Though $f^{(0)}$ is not a solution of the B.E., it clearly obeys the condition.

$$f^{(0)'} f_1^{(0)'} = f^{(0)} f_1^{(0)}$$

Using the representation (2), and assuming that the collision term can be separated, the B.E. becomes:

$$u\frac{df}{dx} + f\iint f_1^{(0)}(1+\phi_1)k_1\, d\vec{k}\, d\vec{c}_1 = f^{(0)}\iint f_1^{(0)} k_1\, d\vec{k}\, d\vec{c}_1 + f^{(0)}\iint f_1^{(0)}(\phi' + \phi_1' + \phi'\phi_1')k_1\, d\vec{k}\, d\vec{c}_1 \tag{2.4}$$

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Equation (4) can now be transformed (Chapman and Cowling, 1960, Waldmann 1958), and an outline of the derivation is given.

With ψ and ϵ being polar angles specifying the direction of the unit vector $\vec{k}$ (apse line) about $\vec{g}$ as axis, and k_1 being a function of the relative speed g and the angle ψ between $\vec{g}$ and $\vec{k}$, part 1 and 2 of eq. (4) may be expressed as

$$\frac{1}{f^{(0)}}\int f^{(0)} f_1^{(0)} \left\{ \iint k_1(g,\psi) \sin\psi \, d\psi \, d\epsilon \right\} d\vec{c}_1 = \frac{1}{f^{(0)}} \delta_0(\vec{c}) \tag{2.5}$$

$$\frac{1}{f^{(0)}}\int \phi_1 \left\{ f^{(0)} f_1^{(0)} \iint k_1(g,\psi) \sin\psi \, d\psi \, d\epsilon \right\} d\vec{c}_1 = \frac{1}{f^{(0)}} \int \phi_1 \delta_1(\vec{c},\vec{c}_1) \, d\vec{c}_1 \tag{2.6}$$

where $\delta_1(\vec{c},\vec{c}_1)$ is a symmetrical function of $\vec{c}$, $\vec{c}_1$.

Regarding part 3 of eq. (4), we first note the relations of $\vec{c}'$, $\vec{c}_1'$ in terms of $\vec{c}$, $\vec{c}_1$,

$$\vec{c}_1' = \vec{c}_1 - (\vec{g}\cdot\vec{k})\vec{k} \; ; \; \vec{c}' = \vec{c} + (\vec{g}\cdot\vec{k})\vec{k} \; ; \; \vec{g} = \vec{c}_1 - \vec{c} \tag{2.7}$$

As $d\vec{c}_1 = d\vec{g}$, let $\vec{g} = g\vec{n}$, where $\vec{n}$ is a unit vector, then $d\vec{g} = g^2 dg d\vec{n}$. Again, let $\vec{R} = g\vec{k}$, then similarly $d\vec{R} = g^2 dg d\vec{k}$. Next we change the variable $\vec{R}$ to $\vec{K}$ where

$$\vec{K} = (\vec{g}\cdot\vec{k})\vec{k} = g\cos\psi\,\vec{k} = \cos\psi\,\vec{R}$$

then $d\vec{K} = \cos^3\psi d\vec{R}$ and $K = g\cos\psi$. A final change from $\vec{K}$ to $\vec{c}'$ can now be made.

The speed C_1 appearing in $f_1^{(0)}$ is expressed next in terms of C'. By making use of eq. (7) and defining angles θ between $\vec{C}'$ and $\vec{K}$ and ϵ_1 between the planes of $\vec{C}'$ and $\vec{K}$ and $\vec{K}$ and $\vec{n}$ respectively, and then using a formula from spherical trigonometry, we obtain the relation

$$C_1^2 = C'^2 + 2KC'\sin\theta\tan\psi\cos\epsilon_1 + K^2\tan^2\psi \, .$$

Finally, the orientation of $\vec{n}$ about $\vec{K}$ is given by the polar angles ψ and ϵ_1, thus $d\vec{n} = \sin\psi d\psi d\epsilon_1$, and part 3 of eq. (4) becomes

$$f^{(0)}\iint \phi' f_1^{(0)} k_1 \, d\vec{k} \, d\vec{c}_1 = \int \phi' \left\{ 2\pi f^{(0)} f'^{(0)} \int_0^{\pi/2} \exp(-\beta K^2 \tan^2\psi) \, J_0(2i\beta |\vec{C} \wedge \vec{C}'| \cdot \tan\psi) \, k_1(K\sec\psi,\psi) \sec^3\psi \sin\psi \, d\psi \right\} d\vec{c}' = \int \phi' \delta_2(\vec{c},\vec{c}') \, d\vec{c}' \tag{2.8}$$

where J_0 is Bessel's zero function, and as one can easily verify, $\delta_2(\vec{c}, \vec{c}')$ is a symmetric function of $\vec{c}, \vec{c}'$. Changing now $\vec{c}'$ to $\vec{c}_1$, we accordingly have

$$f^{(0)\prime} \to f_1^{(0)} \; ; \; \phi' \to \phi_1 \; ; \; K \to g \; ; \; \delta_2(\vec{c}, \vec{c}') \to \delta_2(\vec{c}, \vec{c}_1)$$

Part 4 of eq. (4) is considered next, and in transforming this integral we use a new unit vector $\vec{K}_1$, whose direction is that of $\vec{g}-(\vec{g}.\vec{k})\,\vec{k}$ and the polar angle giving its orientation about $\vec{g}$ as axis are $\pi/2 - \psi$ and $\epsilon + \pi$. The transformation now proceeds as for part 3 of eq. (4), where this time we put

$$K_1 = \vec{c}_1' - \vec{c} = \vec{g} - (\vec{g}\cdot\vec{k})\,\vec{k} = \sin\psi\,\vec{R}_1$$

and finally the variable is changed from $\vec{K}_1$ to $\vec{c}_1'$. The speed C_1 in $f_1^{(0)}$ is now expressed in terms of C_1' and angles θ_1 and ϵ_2 are defined in a similar way to part 3 of eq. (4), and again by using eq. (7), we have

$$C_1^2 = C_1'^2 + K_1 \cot^2\psi + 2C_1' K_1 \sin\theta \cot\psi \cos\epsilon_2$$

and also $d\vec{n} = \cos\psi \, d\psi \, d\epsilon_2$

After integrating with respect to ϵ_2, putting $\psi' = \frac{\pi}{2} - \psi$ and then identifying ψ' with ψ, part 4 of eq. (4) becomes,

$$f^{(0)} \iint \phi_1' f_1^{(0)} k_1 \, d\vec{k} \, d\vec{c}_1 = \int \phi_1' \Big\{ 2\pi f^{(0)} f_1^{(0)} \int_0^{\pi/2} \exp[-\beta K_1^2 \tan^2\psi] .$$

$$. J_0(2i\beta |\vec{C} \wedge \vec{C}_1'| \tan\psi)\, k_1(K_1 \sec\psi, \tfrac{\pi}{2} - \psi) \sec^2\psi \, d\psi \Big\} \, d\vec{c}_1' = \int \phi_1' \delta_3(\vec{c}, \vec{c}_1') \, d\vec{c}_1' \qquad (2.9)$$

where $\delta_3(\vec{c}, \vec{c}_1')$ is a symmetrical function of $\vec{c}, \vec{c}_1'$.

Finally, changing $\vec{c}_1'$ to $\vec{c}_1$, we accordingly have

$$f_1^{(0)\prime} \to f_1^{(0)} \; ; \; \phi_1' \to \phi_1 \; ; \; K_1 \to g \; : \; \delta_3(\vec{c}, \vec{c}_1') \to \delta_3(\vec{c}, \vec{c}_1).$$

Part 5 of eq. (4) is represented for the time being by

$$\Omega = f^{(0)} \iint f_1^{(0)} \phi' \phi_1' k_1 \, d\vec{k} \, d\vec{c}_1 \qquad (2.10)$$

Combining now these results, eq. (4) will have the form

$$a \frac{df}{dx} + \frac{f}{f^{(0)}} \Big[\delta_0(\vec{c}) + \int \phi_1 \delta_1(\vec{c}, \vec{c}_1) \, d\vec{c}_1 \Big] = \delta_0(\vec{c}) + \int \phi_1 \{ \delta_2(\vec{c}, \vec{c}_1) + \delta_3(\vec{c}, \vec{c}_1) \} \, d\vec{c}_1 + \Omega \qquad (2.11)$$

The positive scalar $k_1(g,\psi)$ is given by

$$k_1(g,\psi) = \frac{gb}{\sin\psi}\left|\frac{\partial b}{\partial\psi}\right| = 4g\cos\psi\,\bar{k}_1(g,\psi) \qquad (2.12)$$

where b is the encounter variable and $\bar{k}_1$ is the collision cross section defined by

$$\bar{k}_1(g,\psi) = \frac{b}{2\sin 2\psi}\left|\frac{\partial b}{\partial\psi}\right|$$

For hard sphere molecules, with a molecular diameter σ

$$b = \sigma\sin\psi \; ; \; \bar{k}_1 = \frac{\sigma^2}{4}$$

Introducing $\bar{k}_1(g,\psi)$ in the $\int$'s, and putting $\chi = \pi - 2\psi$ and then upon substitution of the resulting coefficients in eq. (11), the B.E. is made dimensionless on multiplying throughout by the local $L\beta^{1/2}(x)$ One obtains the result:

$$\lambda \mathscr{C}_x \frac{df}{d\hat{x}} + f\left(K_0 + \int_{-\infty}^{\infty} \phi_1 K_1 \exp(-\mathscr{C}_1^2)\, d\vec{\mathscr{C}}_1\right) = f^{(0)}\left(K_0 + \int_{-\infty}^{\infty} \phi_1 K_2 \exp(-\mathscr{C}_1^2)\, d\vec{\mathscr{C}}_1 + \frac{L\beta^{1/2}\Omega}{f^{(0)}}\right) \qquad (2.13)$$

where

$$K_0(\vec{\mathscr{C}}) = 2\pi^{-\frac{1}{2}}\Delta \int_{-\infty}^{\infty} \mathscr{G} \exp(-\mathscr{C}_1^2)\left\{\int_0^{\pi} \bar{k}_1(g,\chi)\sin\chi\, d\chi\right\} d\vec{\mathscr{C}}_1 \qquad (2.14a)$$

$$K_1(\vec{\mathscr{C}},\vec{\mathscr{C}}_1) = 2\pi^{-\frac{1}{2}}\Delta\, \mathscr{G} \int_0^{\pi} \bar{k}_1(g,\chi)\sin\chi\, d\chi \qquad (2.14b)$$

$$K_2(\vec{\mathscr{C}},\vec{\mathscr{C}}_1) = 2\pi^{-1/2}\Delta\, \mathscr{G} \int_0^{\pi} \left\{\bar{k}_1\left(\frac{g}{\sin\chi/2},\chi\right) + \bar{k}_1\left(\frac{g}{\sin\frac{\chi}{2}},\pi-\chi\right)\right\} \frac{\exp(-\mathscr{G}^2\cot^2\chi/2)}{\sin^4\chi/2} \cdot \qquad (2.14c)$$

$$\cdot J_0\left(2i|\vec{\mathscr{C}}\wedge\vec{\mathscr{C}}_1|\cot\frac{\chi}{2}\right)\sin\chi\, d\chi .$$

and

Molecular velocity $\vec{\mathscr{C}}_r(\mathscr{C}_x,\mathscr{C}_y,\mathscr{C}_z) = \sqrt{\beta}(x)\,\vec{c}(u,v,w)$

Random velocity $\vec{\mathscr{C}}(\mathcal{U},\mathcal{V},\mathcal{W}) = \sqrt{\beta}(x)\,\vec{C}(U,V,W)$

Mean speed $\bar{\mathcal{U}}(x) = \sqrt{\beta}(x)\,\bar{u}(x)$

Knudsen number $\lambda(x) = \frac{\bar{l}(x)}{L} = \frac{\Delta}{nL} \; ; \; \Delta = \frac{1}{\sqrt{2}\,\pi\sigma^2}$

The expression for the mean free path $\bar{\ell}$ is chosen arbitrarily, L is a typical length and $\hat{x} = \frac{x}{L}$

Equation (13) can now be integrated with respect to $\hat{x}$, yielding

$$f(\vec{\mathscr{C}}_r,\hat{x}) = f(\vec{\mathscr{C}}_r,\hat{x}_o)\exp\left(-\int_{\hat{x}_o}^{\hat{x}}\frac{D(\vec{\mathscr{C}}_r,x)}{\lambda(x)\,\mathscr{C}_x}dx\right)+\int_{\hat{x}_o}^{\hat{x}}\exp\left(-\int_{\hat{x}'}^{\hat{x}}\frac{D(\vec{\mathscr{C}}_r,x)dx}{\lambda(x)\,\mathscr{C}_x}\right)f^{(o)}\frac{P(\vec{\mathscr{C}}_r,\hat{x}')}{\lambda(\hat{x}')\,\mathscr{C}_x}d\hat{x}' \tag{2.15}$$

where

$$D(\vec{\mathscr{C}}_r,x) = K_o + \int_{-\infty}^{\infty}\phi_1 K_1 \exp(-\mathscr{C}_1^2)\,d\vec{\mathscr{C}}_1 = K_o + \overline{\phi K_1} \tag{2.16}$$

$$P(\vec{\mathscr{C}}_r,x) = K_o + \int_{-\infty}^{\infty}\phi_1 K_2 \exp(-\mathscr{C}_1^2)\,d\vec{\mathscr{C}}_1 + L\sqrt{\beta}\,\frac{\Omega}{f^{(o)}} = K_o + \overline{\phi K_2} + \overline{\phi K_4} \tag{2.17}$$

and $\hat{x}_o$ is a fixed value of $\hat{x}$

Equations (15-17) have been rendered dimensionless using the local flow parameters and they replace the normal integral form of the one-dimensional B.E. The functions D and P are, as we know, associated with the disappearance and production of molecules by encounters in $d\vec{c}$ about $\vec{c}$, and here they are given in terms of various average weighting functions, themselves dependent on the actual form of the distribution function f, and on coefficients related to the particular collision cross section.

The determination of the coefficients K_o, K_1 and K_2 require the specification of the type of encounter between the molecules, and the formulation of the problem is restricted now to hard sphere molecules. For this case the collision cross section $\bar{k}_1(g,\psi)$ is simply a constant equal to $\frac{\sigma^2}{4}$ and the integrations in eq. (14) can be performed analytically, yielding

$$K_o(\vec{\mathscr{C}}) = (2\pi)^{-\frac{1}{2}}\left[\exp(-\mathscr{C}^2) + \left(2\mathscr{C}+\frac{1}{\mathscr{C}}\right)\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(\mathscr{C})\right] \tag{2.18a}$$

$$K_1(\vec{\mathscr{C}},\vec{\mathscr{C}}_1) = 2^{-1/2}\pi^{-3/2}\,\mathscr{G} \tag{2.18b}$$

$$K_2(\vec{\mathscr{C}},\vec{\mathscr{C}}_1) = 2^{-1/2}\pi^{-3/2}\left[\frac{2}{\mathscr{G}}\exp\left(\left\{\frac{|\vec{\mathscr{C}}_1\wedge\vec{\mathscr{C}}|}{\mathscr{G}}\right\}^2\right)\right] \tag{2.18c}$$

When the dimensionless random speed $\mathscr{C}$ equals one, K_0 is very nearly equal to one also, a fact which will be useful later on when comparing the Krook model with the B.E.

Equations (15-17) express the B.E. in a form suitable for solution by an iterative process, but first they are applied to the normal shock wave problem.

III Shock wave structure

Consider a normal shock wave in a uniform free stream and fix the co-ordinate system at its center. The B.E. (2.15) is now applied to this case noting that the boundary conditions at $\pm\infty$ are such that the distribution function is Maxwellian. The flow parameters n_1, $\bar{u}_1$, T_1 upstream at $-\infty$ and n_2, $\bar{u}_2$, T_2 downstream at $+\infty$ are given a priori, but satisfy the Rankine-Hugoniot conditions. The application of these boundary conditions to the B.E. necessitate the definition of four representations in the half spaces $x \gtrless 0$, $\mathscr{C}_x \gtrless 0$ but as the boundaries are at infinity, two half range representations in the velocity space are sufficient. From eq. (2.15) these are

$$f^{(+)}(\vec{\mathscr{C}}_r, \hat{x}) = \int_{-\infty}^{\hat{x}} \exp\left(-\int_{\hat{x}'}^{\hat{x}} \frac{D(\vec{\mathscr{C}}_r, x)}{\lambda(x)\,\mathscr{C}_x} dx\right) \frac{\overset{(o)}{f}(\vec{\mathscr{C}}_r, \hat{x}')}{\lambda(\hat{x}')\,\mathscr{C}_x} P(\vec{\mathscr{C}}_r, \hat{x}')\, d\hat{x}' \qquad \mathscr{C}_x > 0 \tag{3.1}$$

$$f^{(-)}(\vec{\mathscr{C}}_r, \hat{x}) = \int_{\infty}^{\hat{x}} \exp\left(-\int_{\hat{x}'}^{\hat{x}} \frac{D(\vec{\mathscr{C}}_r, x)}{\lambda(x)\,\mathscr{C}_x} dx\right) \frac{\overset{(o)}{f}(\vec{\mathscr{C}}_r, \hat{x}')}{\lambda(\hat{x}')\,\mathscr{C}_x} P(\vec{\mathscr{C}}_r, \hat{x}')\, d\hat{x}' \qquad \mathscr{C}_x < 0 \tag{3.2}$$

At large distances from the shock the local flow parameters $n(x)$, $\bar{u}(x)$ and $T(x)$ tend to the respective constant values at the boundary, the deviation from Maxwellian, ϕ , becomes increasingly small, and at the limit the distribution function is identical to the Maxwellian and the boundary conditions are satisfied.

The unknown local flow parameters are obtained from the relations

$$n(x) = \iiint_{-\infty}^{\infty\;0} f^{(-)} d\vec{c} + \iiint_{-\infty\;0}^{\infty} f^{(+)} d\vec{c} \tag{3.3a}$$

$$n(x)\,\bar{u}(x) = \iiint_{-\infty}^{\infty\;0} u f^{(-)} d\vec{c} + \iiint_{-\infty\;0}^{\infty} u f^{(+)} d\vec{c} \tag{3.3b}$$

$$3\,\frac{k}{m}\,n(x)T(x) = \iiint\limits_{-\infty}^{\infty\;0}\left[\vec{c}-\bar{u}(x)\right]^2 f^{(-)}d\vec{c} + \iiint\limits_{-\infty\;0}^{\infty}\left[\vec{c}-\bar{u}(x)\right]^2 f^{(+)}d\vec{c} \tag{3.3c}$$

where, it should be noted, we have assumed that the solutions n(x), $\bar{u}(x)$, T(x) of the system (3.3) do actually exist and are differentiable and satisfy the boundary conditions.

In the integral eqs.(1) and (2), the distribution function f is basically the only unknown and a solution by iteration is proposed. First a guess of the function ϕ is made and the corresponding parameters, n, $\bar{u}$ and T are found from the solution of the equations of molecular transport of mass, momentum and energy (Patterson 1956). With the knowledge of these zeroth approximations of ϕ and the flow parameters, the first iterate of f can, in principal, be found from eqs.(1) and (2) and the corresponding n, $\bar{u}$, T are determined from eq. (3). The first iterate of ϕ is then found from eq. (2.2) and the process is carried on. The problem as such reduces to one of convergence of this process and the determination of a zeroth ϕ for the initiation of the iterative process. The problem of convergence is not dealt with in this paper, but somewhat similar set of conditions to those obtained by Willis (1961), in his treatment of the linearized Couette Flow could probably be derived. Alternatively, the study of the convergence could be made on the results of a computing machine.

IV The first approximation

To start the iteration procedure, a guess, and a correct one, of the distribution function must be made, and the Navier-Stokes approximation (subsequently N.S.) would appear to be the obvious choice to consider here, in view of the support it has had from recent experimental work. Let us therefore choose for the deviation function from Maxwellian, ϕ, the first approximation of the Chapman-Enskog solution of the B.E. (Chapman and Cowling, 1960). Its one-dimensional form is given by:

$$\phi_0(\vec{\mathscr{C}},\hat{x}) = -\frac{\theta}{t_x}\,\mathcal{U}\left(\mathscr{C}^2-\frac{5}{2}\right) - \frac{\theta}{t_{xx}}\left(\mathcal{U}^2-\frac{\mathscr{C}^2}{3}\right) \tag{4.1}$$

where

$$\theta = \frac{\mu}{p\sqrt{\beta}\,L}\;;\quad t_x = \frac{1}{\frac{3}{2}\,\frac{d\log T}{d\hat{x}}}\;;\quad t_{xx} = \frac{1}{2\sqrt{\beta}\,\frac{d\bar{u}}{d\hat{x}}} \tag{4.2}$$

The coefficient θ is, in this case, equal to the local Knudsen number multiplied by a factor of order unity. It is also possible (Sherman 1958) to relate θ with the mean free time, and t_x and t_{xx} with characteristic times for $\bar{u}$ and T to undergo a change.

The corresponding distribution function, f_o, to eq. (1) leads, as we know, to the one-dimensional N.S. equations, with solutions $n_o(x)$, $\bar{u}_o(x)$ and $T_o(x)$ which we are considering here as known. These solutions together with eq. (1) enable us to determine the zeroth approximation D_o and P_o from eqs. (2 .16, 17), and subsequently the first iteration of the distribution function from eqs. (3 . 1, 2). In the following the index (o) will be dropped.

Upon substitution in D of ϕ_o, and the coefficients K_o and K_1 from eqs. (2.. 18a, b), and performing the integration involved, we obtain

$$D(\vec{\mathcal{C}},\hat{x}) = K_o + \overline{\phi K_1} = (2\pi)^{-\frac{1}{2}}\Big[\exp(-\mathcal{C}^2)\Big\{1 - \frac{\theta}{4t_{xx}}\frac{1}{\mathcal{C}^2}\Big\} + \Big\{2\mathcal{C} + \frac{1}{\mathcal{C}}\Big(1 - \frac{\theta}{6t_{xx}}\Big) + \frac{\theta}{4t_{xx}}\frac{1}{\mathcal{C}^3}\Big\}\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(\mathcal{C})\Big] \quad (4.3)$$

Similarly, and by using the coefficient K_2, eq. (2.18c), the integration of the expression $\overline{\Phi K_2}$ of P, eq. (2.17) can be performed analytically after few substitutions and gives

$$\overline{\phi K_2}(\vec{\mathcal{C}},\hat{x}) = (2\pi)^{-\frac{1}{2}}\frac{\theta}{6t_{xx}}\Big[\exp(-\mathcal{C}^2)\Big\{\mathcal{C}^2 - 1 - \frac{3}{4\mathcal{C}^2}\Big\} + \Big\{2\mathcal{C}^3 - \mathcal{C} + \frac{1}{2\mathcal{C}} + \frac{3}{4\mathcal{C}^3}\Big\}\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(\mathcal{C})\Big] \quad (4.4)$$

The nonlinear part $\overline{\Phi K_4}$ of P is considered next, where Ω is given in eq. (2.10), and in which we substitute $k_1 = g\sigma^2\cos\psi$ and $d\vec{k} = \sin\psi\, d\psi\, d\epsilon$. In addition, ϕ' and ϕ_1' are obtained from eq. (1), where for ϕ', for example, we have

$$\phi' = -\theta\Big[\frac{u'}{t_x}\Big(\mathcal{C}'^2 - \frac{5}{2}\Big) + \frac{1}{t_{xx}}\Big(u'^2 - \frac{\mathcal{C}'^2}{3}\Big)\Big]$$

Finally, the velocity components after the collision can be found in terms of the given initial components,(7, p. 70) with $\delta = 2\psi$, and the integrations involved in $\overline{\Phi K_4}$ can now be performed. The determination of this part is long and tedious but an analytical result can be obtained and this is given in Appendix 1.

Combining the results of K_o, $\overline{\phi K_2}$ and $\overline{\phi K_4}$, from eqs. (2.18a), (4) and (Al.1) respectively, we obtain

$$
\begin{aligned}
P(\vec{\mathcal{C}}, \hat{x}) = {} & K_0 + \overline{\Phi K_2} + \overline{\Phi K_4} = 2\pi^{-\frac{1}{2}} \Big[\exp(-\mathcal{C}^2) \Big\{ 1 + \\
& + \frac{\theta}{6t_{xx}} \left(\mathcal{C}^2 - 1 - \frac{3}{4\mathcal{C}^2} \right) + \frac{\theta^2}{4} \Big(N_{-4} \frac{1}{\mathcal{C}^4} + N_{-2} \frac{1}{\mathcal{C}^2} + N_0 + N_2 \mathcal{C}^2 + \\
& + N_4 \mathcal{C}^4 + N_6 \mathcal{C}^6 \Big) \Big\} + \Big\{ 2\mathcal{C} + \frac{1}{\mathcal{C}} + \frac{\theta}{6t_{xx}} \Big(2\mathcal{C}^3 - \mathcal{C} + \frac{1}{2\mathcal{C}} + \\
& + \frac{3}{4} \frac{1}{\mathcal{C}^3} \Big) + \frac{\theta^2}{4} \Big(N_{-5} \frac{1}{\mathcal{C}^5} + N_{-3} \frac{1}{\mathcal{C}^3} + N_{-1} \frac{1}{\mathcal{C}} + N_1 \mathcal{C} + \\
& + N_3 \mathcal{C}^3 + N_5 \mathcal{C}^5 + N_7 \mathcal{C}^7 \Big) \Big\} \frac{\sqrt{\pi}}{2} \operatorname{erf}(\mathcal{C}) \Big]
\end{aligned}
\tag{4.5}
$$

where the coefficients N_{-5} .... N_7 are functions of $\hat{x}$ and are given in Appendix 1.

The dependance of the distribution function, eq. (3.1, 2), on the molecular velocity is now known, but to determine its dependance on $\hat{x}$ we require the N.S. analytical solutions. However, as we are only interested in certain moments, the entire determination of the distribution function is not necessary.

It is noted that for the continuum case $\theta = \lambda \approx 0$ and from eqs. (3, 5) $D = P = K_0$, where K_0, as we mentioned earlier, is of order unity when $\mathcal{C}$ is of order unity, and the distribution function (3.1,2) reduces to that of the B-G-K model. The additional terms we have obtained in D and P are dependent on the Knudsen number and the velocity and temperature gradients and could all be regarded as corrections, becoming of importance as the flow medium becomes more rarefied.

As one would expect, if we consider the flow parameters appearing in eq. (3.1,2) as constant, the resulting distribution function will be Maxwellian, with uniform parameters. Regarding D and P, a close examination when $\mathcal{C} \rightarrow$ o, reveals no singularities.

V. The flow parameters

A. Derivation

The function D, eq. (4.3) is very nearly unity when the dimensionless random speed $\mathcal{C}$ is equal to one, and this is true even for large values of $\left|\frac{\Theta}{t_{xx}}\right|$ of order 10. We can therefore expand the exponential in eq. (3.1,2) as a power series in the neighbourhood of this value of D, and consider only the first two terms of the expansion. The resulting distribution function is then used in eq. (3.3) and the first approximation to the flow parameters is obtained, namely:

$$n^{(1)}(\hat{x}) = \int_{-\infty}^{\hat{x}} \frac{n(\hat{x}')}{\sqrt{\pi}\,\lambda(\hat{x}')} J_0^{(+)}(\hat{x},\hat{x}')\,d\hat{x}' + \int_{\hat{x}}^{\infty} \frac{n(\hat{x}')}{\sqrt{\pi}\,\lambda(\hat{x}')} J_0^{(-)}(\hat{x},\hat{x}')\,d\hat{x}' = \mathcal{F}_0^{(1)}(\hat{x}) \tag{5.1a}$$

$$n^{(1)}(\hat{x})\,\bar{u}^{(1)}(\hat{x}) = \int_{-\infty}^{\hat{x}} \frac{n(\hat{x}')}{\sqrt{\pi}\,\lambda(\hat{x}')\beta^{1/2}(\hat{x}')} J_1^{(+)}(\hat{x},\hat{x}')\,d\hat{x}' + \int_{\hat{x}}^{\infty} \frac{n(\hat{x}')}{\sqrt{\pi}\,\lambda(\hat{x}')\beta^{1/2}(\hat{x}')} J_1^{(-)}(\hat{x},\hat{x}')\,d\hat{x}' = \mathcal{F}_1^{(1)}(\hat{x}) \tag{5.1b}$$

$$\frac{3k}{m} n^{(1)}(\hat{x})T^{(1)}(\hat{x}) + n^{(1)}(\hat{x})\bar{u}^{(1)2}(\hat{x}) = \int_{-\infty}^{\hat{x}} \frac{n(\hat{x}')}{\sqrt{\pi}\,\lambda(\hat{x}')\beta(\hat{x}')} \left[J_2^{(+)}(\hat{x},\hat{x}') + I_0^{(+)}(\hat{x},\hat{x}')\right] d\hat{x}' +$$

$$+ \int_{\hat{x}}^{\infty} \frac{n(\hat{x}')}{\sqrt{\pi}\,\lambda(\hat{x}')\beta(\hat{x}')} \left[J_2^{(-)}(\hat{x},\hat{x}') + I_0^{(-)}(\hat{x},\hat{x}')\right] d\hat{x}' = \mathcal{F}_2^{(1)}(\hat{x}) \tag{5.1c}$$

where

$$J_n^{(\pm)}(\hat{x},\hat{x}') = \int_0^{\pm\infty} \exp\left(-\frac{\delta(\hat{x},\hat{x}')}{\mathcal{C}_x} - \left(\mathcal{C}_x - \bar{u}(\hat{x}')\right)^2\right) M_1(\mathcal{C}_x,\hat{x},\hat{x}')\,\mathcal{C}_x^{n-1}\,d\mathcal{C}_x \tag{5.2}$$

and

$$I_n^{(\pm)}(\hat{x},\hat{x}') = \int_0^{\pm\infty} \exp\left(-\frac{\delta(\hat{x},\hat{x}')}{\mathcal{C}_x} - \left(\mathcal{C}_x - \bar{u}(\hat{x}')\right)^2\right) M_2(\mathcal{C}_x,\hat{x},\hat{x}')\,\mathcal{C}_x^{n-1}\,d\mathcal{C}_x \tag{5.3}$$

$$M_1(\mathscr{C}_x, \hat{x}, \hat{x}') = \frac{1}{\pi}\iint_{-\infty}^{+\infty} \exp(-\mathscr{C}_y^2 - \mathscr{C}_z^2) P(\vec{\mathscr{C}}, \hat{x}') \left\{1 - \frac{\epsilon(\vec{\mathscr{C}}, \hat{x}, \hat{x}')}{\mathscr{C}_x}\right\} d\mathscr{C}_y \, d\mathscr{C}_z \quad (5.4)$$

$$M_2(\mathscr{C}_x, \hat{x}, \hat{x}') = \frac{1}{\pi}\iint_{-\infty}^{+\infty} (\mathscr{C}_y^2 + \mathscr{C}_z^2) \exp(-\mathscr{C}_y^2 - \mathscr{C}_z^2) P(\vec{\mathscr{C}}, \hat{x}') \left\{1 - \frac{\epsilon(\vec{\mathscr{C}}, \hat{x}, \hat{x}')}{\mathscr{C}_x}\right\} d\mathscr{C}_y \, d\mathscr{C}_z \quad (5.5)$$

$$\delta(\hat{x}, \hat{x}') = \int_{\hat{x}'}^{\hat{x}} \sqrt{\frac{\beta(\hat{x}')}{\beta(x)}} \frac{D(1,x)}{\lambda(x)} \, dx \quad (5.6)$$

$$\epsilon(\vec{\mathscr{C}}, \hat{x}, \hat{x}') = \int_{\hat{x}'}^{\hat{x}} \sqrt{\frac{\beta(\hat{x}')}{\beta(x)}} \frac{1}{\lambda(x)} \left\{D(\vec{\mathscr{C}}, x) - D(1,x)\right\} dx \quad (5.7)$$

At this level, the present equations can be compared again with the B-G-K model results (Darrozès, 1963) for which, we find, that $M_1 = M_2 = D = 1$ and consequently $I_o = J_o$.

With the functions D and P known from eqs. (4.3, 5), M_1 and M_2 can be determined. On integrating the part which involves ϵ, it is assumed that U, the random speed in the flow direction, is equal to zero, which is an acceptable assumption, particularly as the contribution of the corresponding integral is basically small. Carrying out the integrations in M_1 and M_2 yields:

$$\begin{aligned} M_i(\mathscr{C}_x, \hat{x}, \hat{x}') = (2\pi)^{-\frac{1}{2}} \Big[& \exp(-\mathcal{U}^2) \sum_{\nu/2=0}^{3} a_{i\nu} \mathcal{U}^\nu + \frac{\sqrt{\pi}}{2} \operatorname{erf}(\mathcal{U}) \sum_{\frac{\nu+1}{2}=0}^{4} b_{i\nu} \mathcal{U}^\nu + \\ & + \exp(+\mathcal{U}^2) \frac{\pi}{4} \left\{1 - (\operatorname{erf} \mathcal{U})^2\right\} \sum_{\frac{\nu}{2}=0}^{1} c_{i\nu} \mathcal{U}^\nu + \frac{d_i}{\mathcal{U}^2}\left(\exp(-\mathcal{U}^2) - \frac{1}{\mathcal{U}} \operatorname{erf}(\mathcal{U})\right) + \\ & + \frac{\epsilon_i(\hat{x}, \hat{x}')}{\mathscr{C}_x} \Big] \quad ; \quad i = 1,2 \end{aligned} \quad (5.8)$$

where the coefficients $a_{i\nu}$, $b_{i\nu}$, $c_{i\nu}$ and d_i are functions of $\hat{x}'$ only and are given together with ϵ_i in Appendix 2. It is worth noting that if we put in these coefficients $U = \theta = \lambda = \epsilon_1 = \epsilon_2 = 0$ we find that for this continuum flow case $M_1 = 1.026$ and $M_2 = 1.225$; in addition we find $D = 1.041$, where as we have seen, according to the B-G-K model these coefficients are equal to one.

The functions M_1 and M_2 can be introduced now in eqs. (2.3 ; for the determination of J_n and I_n. However, the integrations involved in these functions cannot be performed analytically, though a considerable reduction in their number is possible. It is now convenient to express $J_2 + I_o$ by the symbol J_2 only, the resulting functions J_n have the form

$$J_n^{(\pm)}(\hat{x},\hat{x}') = (2\pi)^{-\frac{1}{2}} \int_0^{\pm\infty} \exp\left(-\frac{\delta}{\mathcal{C}_x}\right)\Big[\exp\left(-2\{\mathcal{C}_x-\bar{U}\}^2\right)\left(\frac{D_n}{\mathcal{C}_x}+E_n+F_n\mathcal{C}_x\right)+$$
$$+\exp\left(-\{\mathcal{C}_x-\bar{U}\}^2\right)\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(\mathcal{C}_x-\bar{U})\left\{\frac{A_n}{\mathcal{C}_x-\bar{U}}+\frac{W_n}{\mathcal{C}_x}+V_n+T_n\mathcal{C}_x\right\}+\frac{\pi}{4}\left(1-\left[\mathrm{erf}(\mathcal{C}_x-\bar{U})\right]^2\right)B_n\Big]d\mathcal{C}_x \qquad (5.9)$$

The determination of the twenty four coefficients appearing in eq. (9) is tedious and their final form is lengthy. However, ultimately they are all functions of $\bar{U}$, δ, $\frac{\theta}{t_x}$, $\frac{\theta}{t_{xx}}$ and their products. In order to keep the size of this paper within limits, it was decided to omit these coefficients from it.

The number of integrals in the expression J_n can be reduced further by allowing the use of an operator which involves differentiations with respect to δ, eq. (6), or effectively with respect to $\hat{x}$. Carrying out this operation, we obtain

$$J_n^{(\pm)}(\hat{x},\hat{x}') = (2\pi)^{-\frac{1}{2}}\Big[A_n\int_0^{\pm\infty}\exp\left(-\frac{\delta}{\mathcal{C}_x}-(\mathcal{C}_x-\bar{U})^2\right)\frac{\sqrt{\pi}}{2}\,\frac{\mathrm{erf}(\mathcal{C}_x-\bar{U})}{\mathcal{C}_x-\bar{U}}\,d\mathcal{C}_x+$$
$$+\mathcal{D}_n\int_0^{\pm\infty}\exp\left(-\frac{\delta}{\mathcal{C}_x}\right)\frac{\pi}{4}\left(1-\{\mathrm{erf}(\mathcal{C}_x-\bar{U})\}^2\right)\Big]d\mathcal{C}_x \qquad (5.10)$$

where the operator $\mathcal{D}_n(\hat{x},\hat{x}')$ is given by

$$\mathcal{D}_n(\hat{x},\hat{x}') = -2F_n\int_\infty^\delta d\delta' + \left(E_n+(\delta-\bar{U})F_n+\frac{T_n}{2}+B_n\right)+\left(-E_n\delta+\bar{U}\delta F_n-\right.$$
$$\left.-T_n\frac{\delta}{2}\right)\frac{d}{d\delta}+\left(D_n(\bar{U}+\delta)-\bar{U}\delta E_n-\frac{W_n}{2}+V_n\frac{\delta}{2}\right)\frac{d^2}{d\delta^2}+\left(D_n(1+\bar{U}\delta)-\right.$$
$$\left.-E_n\delta-W_n\frac{\delta}{2}+F_n\frac{\delta^2}{2}\right)\frac{d^3}{d\delta^3}+\left(2\delta D_n-E_n\frac{\delta^2}{2}\right)\frac{d^4}{d\delta^4}+D_n\frac{\delta^2}{2}\frac{d^5}{d\delta^5} \qquad (5.11)$$

and it should be noted, differentiation with respect to δ is made at constant $\hat{x}'$. Finally, eq.(1), defining the flow parameters, can be expressed in the following form

$$\mathcal{F}_n^{(1)}(\hat{x}) = \int_{-\infty}^{\hat{x}} \frac{n(\hat{x}')}{\sqrt{\pi}\,\lambda(\hat{x}')} \frac{J_n^{(+)}(\hat{x},\hat{x}')}{\beta^{\frac{n}{2}}(\hat{x}')} d\hat{x}' + \int_{\hat{x}}^{\infty} \frac{n(\hat{x}')}{\sqrt{\pi}\,\lambda(\hat{x}')} \frac{J_n^{(-)}(\hat{x},\hat{x}')}{\beta^{\frac{n}{2}}(\hat{x}')} d\hat{x}' \qquad (5.12)$$

in which n = 1,2,3 and subscript (1) refers to the first approximation, and the functions under the integral signs are the N.S. solutions. The integrals in eqs. (9, 10) do not appear to be solvable analytically. However, expansions for some limiting values of $\mathcal{U}$ should be possible. (Chahine and Narashima, 1963). It is noted that some of these integrals are singular for certain values of δ, but the form of eq. (12) is such that these singularities cancel out and $\mathcal{F}_n^{(1)}(\hat{x})$ is always finite.

With the knowledge of the N.S. solutions, the present equations are in a form basically suitable for programming on a computing machine.

B. The simplified equations

An analytic solution of $J_n(\hat{x},\hat{x}')$, eq. (10), does not appear to be possible and even a numerical solution would be lengthy and involved. However, before attempting such a task, an insight into the basic factors affecting the problem can be gained by introducing a simplification in the form of the functions M_i, (eq. (8)). It is noted, that most of the contribution of M_i comes from the neighbourhood of the region where $\mathcal{U}$ is equal to zero. To obtain the dominant terms, we put in M_i that $\mathcal{U} = \epsilon_i = 0$, and get

$$M_1(x) = (2\pi)^{-\frac{1}{2}}\left(a_{10} + b_{-11} + \frac{\pi}{4} c_{10}\right) = (2\pi)^{-\frac{1}{2}}\left[1 + \frac{\pi}{2} + \frac{\Theta}{3t_{xx}} - \frac{\Theta^2}{4}\left(\frac{7}{15}\frac{1}{t_x^2} + \frac{359}{2160}\frac{1}{t_{xx}^2}\right)\right] \qquad (5.13a)$$

$$M_2(x) = (2\pi)^{-\frac{1}{2}}\left(a_{20} + b_{-21} + \frac{\pi}{4} c_{20}\right) = (2\pi)^{-\frac{1}{2}}\left[\frac{3}{2} + \frac{2\pi}{4} + \left(\frac{1}{3} + \frac{\pi}{6}\right)\frac{\Theta}{t_{xx}} + \frac{\Theta^2}{4}\left(\frac{1}{15}\frac{1}{t_x^2} + \frac{41}{135}\frac{1}{t_{xx}^2}\right)\right] \qquad (5.13b)$$

and

$$D(1,x) = (2\pi)^{-\frac{1}{2}}\left[\exp(-1)\left\{1 - \frac{\Theta}{4t_{xx}}\right\} + \left\{3 + \frac{1}{12}\frac{\Theta}{t_{xx}}\right\}\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(1)\right] \qquad (5.13c)$$

where M_i is now a function of the displacement only, and the coefficients appearing in eq. (13) are given in Appendix 2.

By using θ, t_x and t_{xx} from eq. (4.2), and the expression relating the mean free path with the viscosity coefficient for the hard sphere molecules, at the N.S. approximation level, the dimensional form of the flow parameters, (1), becomes:

$$n^{(1)}(x) = \int_{-\infty}^{x} \frac{\beta^{\frac{1}{2}}(x')\, n(x')}{\sqrt{\pi}} \frac{5\sqrt{\pi}}{8} \frac{p(x')}{\mu(x')} M_1(x')\, j_0^{(+)}(x,x')\, dx' + \tag{5.14a}$$

$$+ \int_{x}^{\infty} \frac{\beta^{\frac{1}{2}}(x')\, n(x')}{\sqrt{\pi}} \frac{5\sqrt{\pi}}{8} \frac{p(x')}{\mu(x')} M_1(x')\, j_0^{(-)}(x,x')\, dx'.$$

$$n^{(1)}(x)\,\bar{u}^{(1)}(x) = \int_{-\infty}^{x} \frac{n(x')}{\sqrt{\pi}} \frac{5\sqrt{\pi}}{8} \frac{p(x')}{\mu(x')} M_1(x')\, j_1^{(+)}(x,x')\, dx' + \tag{5.14b}$$

$$+ \int_{x}^{\infty} \frac{n(x')}{\sqrt{\pi}} \frac{5\sqrt{\pi}}{8} \frac{p(x')}{\mu(x')} M_1(x')\, j_1^{(-)}(x,x')\, dx'.$$

$$\frac{3k}{m} n^{(1)}(x)\, T^{(1)}(x) + n^{(1)}(x)\, \bar{u}^{(1)2}(x) = \int_{-\infty}^{x} \frac{n(x')}{\sqrt{\beta(x')}\sqrt{\pi}} \frac{5\sqrt{\pi}}{8} \frac{p(x')}{\mu(x')} \Big[M_1(x')\, j_2^{(+)}(x,x') +$$

$$+ M_2(x')\, j_0^{(+)}(x,x') \Big] dx' + \int_{x}^{\infty} \frac{n(x')}{\sqrt{\beta(x')}\sqrt{\pi}} \frac{5\sqrt{\pi}}{8} \frac{p(x')}{\mu(x')} \Big[M_1(x')\, j_2^{(-)}(x,x') + M_2(x')\, j_0^{(-)}(x,x') \Big] dx'. \tag{5.14c}$$

in which the functions under the integral sign refer to the N.S. solutions, and

$$j_n^{(\pm)}(x,x') = \beta^{\frac{n}{2}}(x') \int_{0}^{\pm\infty} \exp\left(-\frac{\delta(x,x')}{u} - \beta(u-\bar{u})^2\right) u^{n-1}\, du \tag{5.15}$$

where $\delta(x, x')$ is given now by

$$\delta(x,x') = \int_{x'}^{x} \frac{5\sqrt{\pi}}{8} \frac{p(y)}{\mu(y)} D(1,y)\, dy \tag{5.16}$$

Further, with

$$\frac{\Theta}{t_{xx}} = \frac{2\mu}{p}\frac{d\bar{u}}{dx} \quad , \quad \frac{\Theta}{t_x} = \frac{3\mu}{p}\frac{d\sqrt{2RT}}{dx}$$

eq. (13) becomes

$$M_1(x) = 1.0255 + 0.2659\frac{\mu}{p}\frac{d\bar{u}}{dx} - \frac{\mu^2}{p^2}\left\{0.0663\left(\frac{d\bar{u}}{dx}\right)^2 + 0.4188\left(\frac{d\sqrt{2RT}}{dx}\right)^2\right\} \qquad (5.17a)$$

$$M_2(x) = 1.2249 + 0.6837\frac{\mu}{p}\frac{d\bar{u}}{dx} + \frac{\mu^2}{p^2}\left\{0.1211\left(\frac{d\bar{u}}{dx}\right)^2 + 0.0598\left(\frac{d\sqrt{2RT}}{dx}\right)^2\right\} \qquad (5.17b)$$

and from eq. (13c)

$$D(1,x) = 1.0404 - 0.0242\frac{\mu}{p}\frac{d\bar{u}}{dx} \qquad (5.17c)$$

Equations (14), for the flow parameters, together with eqs. (15 - 17) make the system complete for the determination of the first approximations $n^{(1)}(x)$, $\bar{u}^{(1)}(x)$ and $T^{(1)}(x)$, provided, of course, that the N.S. solutions are known. Once these data are available, the equations can be programmed on a computing machine, and the normal shock profiles obtained. Regarding the N.S. profiles themselves, these can be determined for example by the saddle-point singularity method proposed by Gilbarg and Paolucci (1953) and used by Chahine (1963). With the knowledge of the functions D and P, eqs. (4.3, 5) the first approximation to the distribution function $f^{(1)}$, eqs. (3.1, 2), is known as well, and so the flow parameters to the same degree of approximation. The second approximation can then be attempted, first by finding $\phi^{(1)}$ from eq. (2.2), and then $D^{(1)}$ and $P^{(1)}$ from eq. (2.16, 17), where the coefficients K_o, K_1 and K_2 remain unaltered as they are independent of the distribution function. From this information the second approximation $f^{(2)}$ is then found from eq. (3.1, 2), and consequently the corresponding flow parameters are known.

IV Discussion and conclusions

The problem of normal shock wave structure has been studied by Liepmann et al (1962) and Chahine (1963) using the B-G-K model. From section II, it is seen that this model can actually be derived from the B.E. itself, by regarding f', f_1, f_1', to be local Maxwellian, and equating the collision operator K_o, eq. (2.18a), to unity. Liepmann et al and Chahine approached the shock problem by solving by iteration the resulting equations of the flow parameters, considering the N.S. approximation as the initial solution. In the present work the problem is tackled by using the B.E. itself, and a solution by iteration is proposed for f. The process is initiated by choosing as the first solution, the distribution function leading to the Navier-Stokes equations and substituting this for f', f_1 and f_1'. The resulting B.E. is then formally applied to the shock problem.

Equations (2.15-17) express the integral form of B.E. in terms of D and P, which are functions associated with the gain and loss of a certain type of molecule due to collisions. When D and P are considered unity, the B-G-K model is obtained as a special case of the B.E. Furthermore, with θ or $\frac{\mu}{p}$ being simply related to the Knudsen number, the integrated form of D and P, eqs. (4.3,5) based on the distribution function of the N.S. equations, shows that when the continuum case is dealt with these coefficients are very nearly equal to unity (1.02, 1.04 respectively), see also eq. (5.17). A result implying that the B-G-K model is valid in the continuum case only. *

In the present study the iteration is basically performed on the distribution function, and the knowledge of which determines the flow parameters, whereas by Liepmann et al and Chahine's treatment the iteration is carried out on the flow parameters themselves. In fact, this latter way of iteration is a special one of the former, special in the sense that the distribution function remains unaltered in form throughout the whole iteration process. This type of iteration can be used, for example, on the simplified equations of the flow parameters (5.14), without further recourse being made to the distribution function, and indeed for a constant form of f, and in the continuum regime, the higher iterations will be identical to the B-G-K model results. However, it should be noted that in the present study no restriction was imposed on the Mach number or the Knudsen number, and the accuracy and range of validity of the results depend on the number of iterations performed only, assuming of course that the process is convergent. On the other hand, as the B-G-K- model is found to be restricted to continuum media,* the higher iterations on the flow parameters themselves are suitable for stronger shocks only.

An important aspect of the present study is the choice of the distribution function of N.S. equations as the starting step for the iteration scheme. This choice is, in fact, a natural one, due to the relative simplicity of the distribution function, and the confidence in its validity. The use of the so-called higher continuum approximations, based on the Navier-Stokes, is not necessary, and moreover, could prove to be unsatisfactory. As mentioned above, the present method basically seeks higher iterations of the distribution function, and in this sense it is similar to Chapman-Enskog method. Liepmann et al and Chahine noted that in their special case, their procedure was essentially different from the Chapman-Enskog sequence and their results converged more rapidly. From the present point of view this essential difference lies in the two different approaches to solve the same

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* The model is, of course, also valid in the free molecule flow where the distribution function is uniform Maxwellian.

equation, namely the B.E. This question is indeed fundamental and requires further study, however, in the present equations there appears the term $\exp(-\frac{\delta}{u})$, where δ and u vary independently between zero and infinity, and therefore its approximation by an expansion of finite terms may not be suitable, or permissible, in this case.

It is interesting to note in the equations the appearance of the coefficients

$$\frac{\mu}{p}\,\frac{d\bar{u}}{dx}\,;\quad \frac{\mu}{p}\,\frac{d\sqrt{2RT}}{dx}$$

which can be interpreted as the local ratio of the mean free time to the time of variation of a parameter of the flow, or as a local Knudsen number based on an actual characteristic length. It is seen that two such ratios can be defined and associated with the local state of the flow. Now, as the flow becomes more rarefied $\frac{\mu}{p}$ increases but the gradients of $\bar{u}$ and the most probable speed $\sqrt{2RT}$ decrease , and at the limit the above expressions will tend to a limit $\neq \infty$. To obtain an idea of the order of magnitude of this limit, let us assume that

$$\left|\frac{d\bar{u}}{dx}\right| = \frac{d\sqrt{2RT}}{dx}$$

and without recourse to higher iterations, consider eq. (5.17a) when $M_1 = 0$ i.e. when the number density is actually zero. The solution of the resulting quadratic equation gives

$$\text{Limit}\ \frac{\mu}{p}\left|\frac{d\bar{u}}{dx}\right| = 1.2$$

in which case there is no distinct shock which can be studied. More generally, with the assumption of constant total enthalpy across the shock, this limit will depend upon the Mach number and will vary between roughly 2 for $M = 1$ and zero for $M = \infty$. The result of Liepmann et al in this respect is noteworthy here, who found, in a rather interesting manner, that the maximum value of this parameter only is equal to 1.125 and occurs upstream at the point of maximum stress. This may mean, therefore, that the first iteration we have found could prove to be valid for a fairly pronounced degree of rarefaction, and for stronger shocks also, the iteration could be continued on the flow parameters themselves, as in the case of Liepmann et al and Chahine.

The problem of convergence of the iteration scheme is not dealt with here, however, Chahine's computed results indicate that the process converges quite rapidly, particularly for weak shocks. One would expect, therefore, that this should also hold for a 'slight' degree of rarefaction.

The method used here, though applied to the shock structure, should also be useful for other problems of linear or nonlinear nature.

Appendix 1

Coefficients in connection with the first approximation $f^{(1)}$ to the distribution function

The integrations involved in the expression $\overline{\Phi K_4}$, eq. (2.17), yield

$$\overline{\Phi K_4} = (2\pi)^{-\frac{1}{2}} \frac{\theta^2}{4} \left[\exp(-\zeta^2) \sum_{-2}^{3} N_{2n} \zeta^{2n} + \frac{\sqrt{\pi}}{2} \operatorname{erf}(\zeta) \sum_{-3}^{3} N_{2n+1} \zeta^{2n+1} \right] \qquad \text{(A1.1)}$$

Note:- Identify ζ with $\mathscr{C}$

where

$$N_{-4} = -\frac{9}{64} \frac{1}{t_{xx}^2}$$

$$N_{-2} = \frac{1}{16} \frac{1}{t_x^2} - \frac{1}{6} \frac{1}{t_{xx}^2}$$

$$N_0 = -\frac{1}{3t_x^2} - \frac{67}{180} \frac{1}{t_{xx}^2}$$

$$N_2 = -\frac{19}{30} \frac{1}{t_x^2} + \frac{17}{270} \frac{1}{t_{xx}^2}$$

$$N_4 = \frac{8}{15t_x^2} + \frac{7}{540} \frac{1}{t_{xx}^2}$$

$$N_6 = -\frac{1}{15} \frac{1}{t_x^2}$$

$$N_{-5} = \frac{9}{64} \frac{1}{t_{xx}^2}$$

$$N_{-3} = -\frac{1}{16} \frac{1}{t_x^2} + \frac{7}{96} \frac{1}{t_{xx}^2}$$

$$N_{-1} = \frac{3}{8} \frac{1}{t_x^2} - \frac{5}{72} \frac{1}{t_{xx}^2}$$

$$N_1 = -\frac{2}{t_x^2} - \frac{1}{4t_{xx}^2}$$

$$N_3 = -\frac{2}{3} \frac{1}{t_x^2} + \frac{5}{36} \frac{1}{t_{xx}^2}$$

$$N_5 = \frac{1}{t_x^2} + \frac{7}{270} \frac{1}{t_{xx}^2}$$

$$N_7 = -\frac{2}{15} \frac{1}{t_x^2}$$

Appendix 2

Coefficients in connection with the first approximation to the flow parameters

The coefficients relating to eq. (5.8) are:

For M_1

$$a_{10} = 1 + \frac{\theta}{12t_{xx}} - \frac{\theta^2}{4}\left(\frac{41}{120}\frac{1}{t_x^2} + \frac{269}{2160}\frac{1}{t_{xx}^2}\right)$$

$$a_{12} = \frac{\theta}{6t_{xx}} + \frac{\theta^2}{4}\left(\frac{3}{20}\frac{1}{t_x^2} + \frac{103}{1080}\frac{1}{t_{xx}^2}\right)$$

$$a_{14} = \frac{\theta^2}{4}\left(\frac{3}{10}\frac{1}{t_x^2} + \frac{7}{540}\frac{1}{t_{xx}^2}\right)$$

$$a_{16} = -\frac{\theta^2}{4}\,\frac{1}{15}\,\frac{1}{t_x^2}$$

$$b_{-11} = \frac{\theta}{4t_{xx}} - \frac{\theta^2}{4}\left(\frac{1}{8t_x^2} + \frac{1}{24}\frac{1}{t_{xx}^2}\right)$$

$$b_{11} = 2 + \frac{\theta}{3t_{xx}} - \frac{\theta^2}{4}\left(\frac{1}{t_x^2} - \frac{1}{18}\frac{1}{t_{xx}^2}\right)$$

$$b_{13} = \frac{\theta}{3t_{xx}} + \frac{\theta^2}{4}\left(\frac{2}{3}\frac{1}{t_x^2} + \frac{11}{54}\frac{1}{t_{xx}^2}\right)$$

$$b_{15} = \frac{\theta^2}{4}\left(\frac{8}{15t_x^2} + \frac{7}{270t_{xx}^2}\right)$$

$$b_{17} = -\frac{\theta^2}{4}\,\frac{2}{15}\,\frac{1}{t_x^2}$$

$$c_{10} = 2$$

$$c_{12} = 0$$

$$d_1 = -\frac{\theta^2}{4} \quad \frac{3}{32}\,\frac{1}{t_{xx}^2}$$

$$\epsilon_1 = \Bigg(-\,0.1599 - 0.2499\,\frac{\theta}{t_{xx}} - \frac{\theta^2}{4}\Big\{0.2916\,\frac{1}{t_x^2} + 0.1799\,\frac{1}{t_{xx}^2}\Big\}$$

$$+\,\frac{\theta^3}{16t_{xx}}\Big\{0.0307\,\frac{1}{t_x^2} + 0.0037\,\frac{1}{t_{xx}^2}\Big\}\Bigg)\frac{1}{2\pi}\int_{\hat{x}'}^{\hat{x}}\frac{dx}{\lambda(x)}$$

For M_2

$$a_{20} = \frac{3}{2} + \frac{1}{3}\,\frac{\theta}{t_{xx}} + \frac{\theta^2}{4}\left(\frac{1}{15t_x^2} + \frac{251}{2160}\,\frac{1}{t_{xx}^2}\right)$$

$$a_{22} = \frac{\theta}{6t_{xx}} + \frac{\theta^2}{4}\left(\frac{7}{20}\,\frac{1}{t_x^2} + \frac{23}{180}\,\frac{1}{t_{xx}^2}\right)$$

$$a_{24} = \frac{\theta^2}{4}\left(\frac{1}{15}\,\frac{1}{t_x^2} + \frac{7}{540}\,\frac{1}{t_{xx}^2}\right)$$

$$a_{26} = -\frac{\theta^2}{4}\,\frac{1}{15}\,\frac{1}{t_x^2}$$

$$b_{-21} = \frac{\theta^2}{4}\,\frac{3}{16}\,\frac{1}{t_{xx}^2}$$

$$b_{21} = 4 + \frac{5}{6}\,\frac{\theta}{t_{xx}} + \frac{\theta^2}{4}\left(\frac{1}{4t_x^2} + \frac{11}{24}\,\frac{1}{t_{xx}^2}\right)$$

$$b_{23} = \frac{1}{3}\,\frac{\theta}{t_{xx}} + \frac{\theta^2}{4}\left(\frac{5}{6t_x^2} + \frac{29}{108}\,\frac{1}{t_{xx}^2}\right)$$

$$b_{25} = \frac{\theta^2}{4}\left(\frac{1}{15}\,\frac{1}{t_x^2} + \frac{7}{270}\,\frac{1}{t_{xx}^2}\right)$$

$$b_{27} = -\frac{\theta^2}{4}\,\frac{2}{15}\,\frac{1}{t_x^2}$$

$$c_{20} = 2 + \frac{2}{3}\,\frac{\theta}{t_{xx}}$$

$$c_{22} = -\,2$$

$$d_2 = 0$$

$$\epsilon_2 = \left\{ - 1.8284 - 0.9622 \frac{\theta}{t_{xx}} - \frac{\theta^2}{4}\left(\frac{0.2419}{t_x^2} + \frac{0.5993}{t_{xx}^2}\right) + \frac{\theta^3}{16t_{xx}}\left(\frac{0.0518}{t_x^2} + \frac{0.0273}{t_{xx}^2}\right)\right\} \frac{1}{2\pi}\int_{\hat{x}'}^{\hat{x}} \frac{dx}{\lambda(x)}$$

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