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A MATHEMATICAL MODEL FOR HANDLING IN A WAREHOUSE

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1. INTRODUCTION AND SUMMARY OF CONCLUSIONS

1.1. One of the projects, which were assigned to the M.H.R.U. by the Research Fellowship Committee, is the construction of a mathematical model for goods handling in a warehouse. This report, which concludes the first stage of the project, defines a mathematical model for warehouses in which goods are stored in unit locations. A practical example would be a store using pallets in pallet racking.

1.2. We have been careful to base the model only on such data as are generally available in warehouse operations, and believe that the formulae we put forward can be applied directly in practical problems, provided our assumption is justified, that handling effort, however expressed (e.g. cost of handling, time taken per unit), is proportional to quantity and distance over which goods have to be moved during the warehousing process.

1.3. Our main findings are that handling effort can be minimised, within the framework of a given time dependent storage policy such as FIFO or LIFO, by:

i) imposing a rule of always storing incoming goods in the nearest free location and issuing outgoing goods from the nearest full location.

ii) by forming storage blocks of unit locations with dimensions given by the dimensions of the unit location and the performance characteristics of the handling equipment, and

iii) by storing each commodity within the warehouse in accordance with a calculation, which uses as its data the average rate of throughput and the maximum expected storage capacity for that commodity.

1.4. By the application of the above rule one can decide on the capacity, shape and lay-out of a warehouse for a variety of available handling equipment and thence choose that combination of equipment and lay-out, that minimises total cost.

1.5. Sections 2 and 3 of this report are introductory. Section 4, develops the main formulae. Section 5 discusses their use.

Section 6 then considers the problem of order picking, and Section 7 discusses the effect of time dependent storage policies on the model.

1.6. These 7 sections form the main part of the report. We have added a short section (Section 8) on an attempt to devise a formula for building cost and its link to handling effort, and a section (Section 9) on the use of the model when considering warehouse automation. Section 10 deals with the mathematics.

1.7. We have tried throughout (except in Section 10) to keep the mathematics as simple as possible. Nevertheless by the nature of the project, we could not avoid using mathematical formulae.

In order to enhance the understanding of the formulae, we have attached a numerical example. Obviously, we have chosen small numbers which may be far from reality; we have however tried to construct the example in such a way, that the values we have choosen are in a relationship to each other similar to that found in practice.

2. THE PURPOSE AND APPLICABILITY OF A MATHEMATICAL MODEL

2.1. Contrary to general belief, mathematics is not concerned with calculation. Calculational procedures, such as, for example, the rules of arithmetic, have rather the characters of by-products of the formal theoretical work of mathematicians, which encompasses the field of logical relationships; such relationships can, but need not, be of a quantitative nature.

2.2. Any mathematical formula is a formal description of a logical relationship between two or more "things"; what the "things" are, need not be defined. It is the formula that matters, because mathematics is concerned with finding consistent rules that explore logical relationships and therefore permit statement about the collection of "things" that are in such a relationship.

2.3. Conversely, given a set of such consistent rules, if we can define formal relationships between known "things" we can apply the rules, and the consequent statements are then applicable to the "things" of interest.

2.4. For example, the term "centre of gravity", is a theoretical fiction, derived from observations in physical/mechanical studies. It can be expressed formally in mathematical terms.

In certain circumstances, the same formula gives the best location of a warehouse that has to supply a known number of retail outlets in a given area. Thus, in one case the "things" that are related by the formula are points and theoretical forces in an abstract geometrical configuration, in the other, they are real geographical locations and actual quantities of goods.

2.5. The art of constructing mathematical models is the derivation of formal relationships from real situations. Once the abstract formulations are written down, one can operate on the formulae by the rules of mathematics in order to solve problems that arise from the real situation. Clearly, there will always be a difference between reality and theoretical description, and the quality of a mathematical model depends on finding a formula, such that that difference does not affect the practical results.

2.6. The results that usually are of interest, and the whole purpose of constructing mathematical models is twofold. Firstly to gain insight into and understanding of complex situations or systems, and secondly, to derive rules that in some way optimise activities in such situations or the perforamce of such systems. One has, of course, to define what is meant by "optimising". In commercial studies, optimisation usually denotes either maximising profits or minimising costs.

2.7. In this first attempt to construct a mathematical model of a warehouse we had first of all to simplify reality in order to be able to set up our formulae, always with the proviso, that we must not depart from the real situation so far as to make our results inapplicable.

2.8. Thus we consider here only the type of warehouse in which direct access to all units stored exists. An example of such a warehouse, would be one in which all items come as standard pallets and are stored in pallet racking, one pallet per cubicle. The results are, therefore, not directly applicable to warehouses in which goods are stored in stacks. We belive, however, that from this first model of a simple type of warehouse, we shall be able to derive models for the more complicated storage system. These will be the subject of a second report.

2.9. Again, since in order to apply the model to any real situation, requires observations of the real situation, we took care that the data that will be required are of a type that are generally available to warehouse managers, as explained in the following chapters.

3. TERMINOLOGY, NOTATION AND BASIC ASSUMPTIONS

3.1. The project is concerned with materials handling in a warehouse. For the purpose of the research, the term "warehouse" covers all types of spatial storage, i.e. any location, building, construction etc. that is assigned to hold stocks of one or more difference types of items in unitised form. Whether these stocks are raw materials, in-process stocks or finished goods is irrelevant. Equally the research is not concerned with the reason why stocks are held, or with stock control.

We assume that stocks, in the amounts given by the stock control system, need to be stored and the project is concerned with the form of storing these stocks in an optimal way. We shall, however, use certain results as to stock distributions that have been obtained by research into stock control systems.

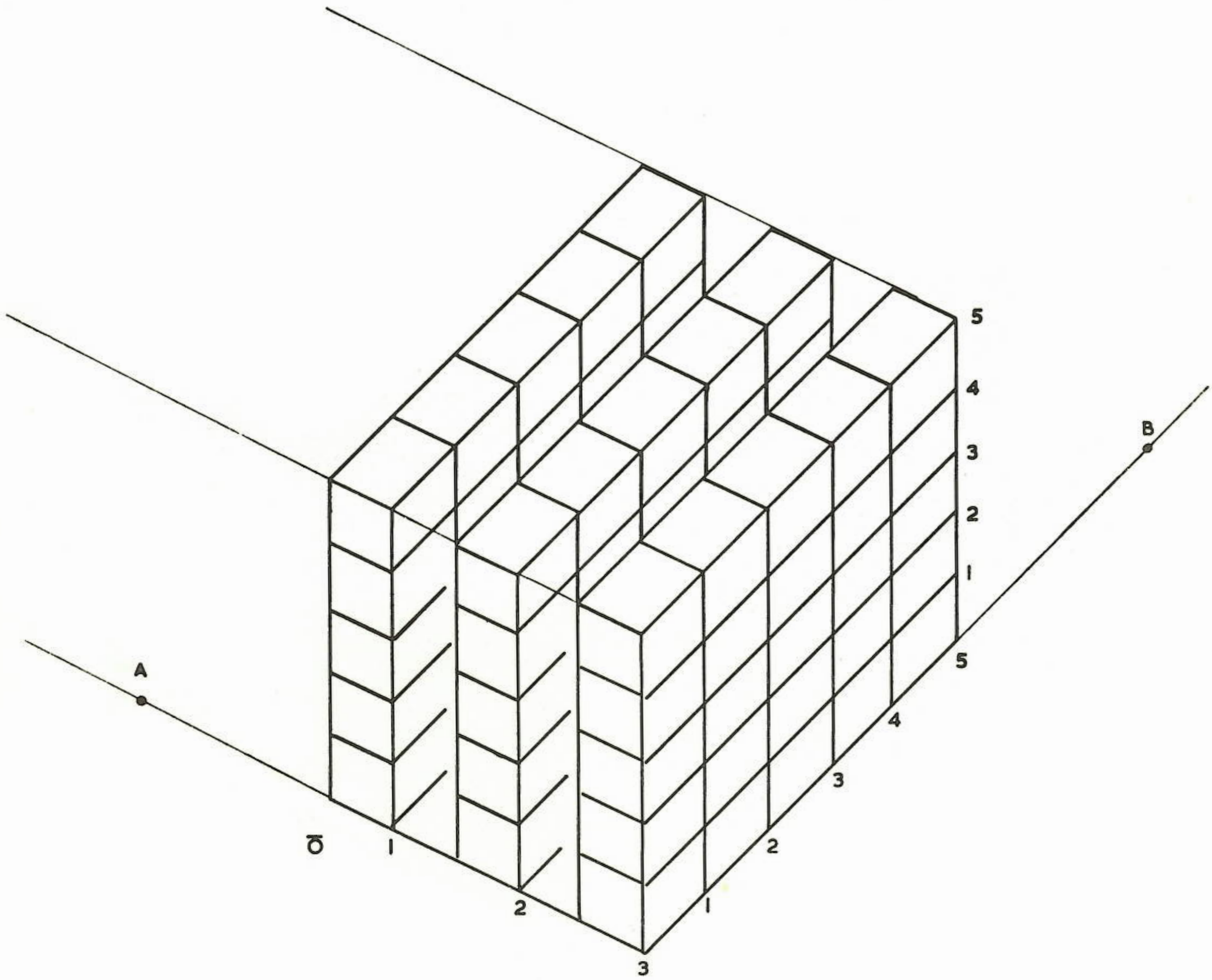
3.2. The term "distributions" as used in this report refers to probability distributions. Thus the stock of any item held in a warehouse will have a "distribution", i.e. it is possible to assign a probability that at any time the stock of the item may have a given value, and also state that stock will not exceed a certain value more than, say, x% of time. Further, we can then also speak of an average stock and a stock variance. Variance in connection with probability is a measure of variability, usually designated by the greek letter σ^2. For example, a stock that will never fall below say 10 units and never exceed say 20 units will have a smaller variance than one that fluctuates between, say, 0 and 50 units. The square root of the variance called "standard deviation" is also often used in statistical calculation and since the notation for variance is σ^2 the notation for the standard deviation is σ.

3.3. Otherwise, in this report we shall use greek letters other than σ to denote proportions, i.e. greek letters stand for ratios, which, unless otherwise indicated, will lie between 0 and 1.

3.4. Unitised goods are usually stocked in rectangular blocks within a storage area, e.g. in racking or stacks of boxes or pallets, etc. For the first step we chose a very simple model, for which the output and input is always in the same unit, where any unit within the stack or block can be extracted and the handling equipment can move only one unit at a time. An example would be a warehouse which receives and issues goods in integral multiples of pallet loads, and where pallets are held in racking, which is subdivided into cells holding a single pallet.

FIG. 3.1.

WAREHOUSE BLOCK OF STORE CELLS



In Section 6 below we consider the case of output units different from input units, i.e. order picking.

Fig. 3.1 shows such an arrangement of 125 cells, 3 cells wide 5 cells long and high. Each location could be identified by a triplet of integral numbers, giving position in the two directions and height. In such a block arrangement, movement is possible only in directions at right angles to each other.

Thus the distance from a given reference point to any storage cell is given by the sum of three numbers, the distance from reference point along, across and upwards. These three distances can serve as the identification triplet of numbers for each storage cell.

3.5. For the purpose of the model, we can take as reference point one bottom corner of the block (marked $\bar{0}$ in Fig. 1), as the true reference points (say A and B in Fig. 3.1) i.e. receiving and despatch bays, will only add a known distance to all locations. This may be a surprising statement, and it is therefore, worth while to go into greater detail.

Consider the bottom layer of cells in a 3 x 3 arrangement. The distance of each cell from the theoretical reference point is shown in each cell in Fig. 3.2a.

Fig.3.2a

2	3	4
1	2	3
0	1	2

$\bar{O}$

B Fig.3.2b

0	1	2
1	2	3
2	3	4

$\bar{O}$

B Fig.3.2c

2	4	6
2	4	6
2	4	6

$\bar{O}$

Fig.3.2d B

4	4	4
4	4	4
4	4	4

$\bar{O}$

Assume now that the entrance is at $\bar{O}$ and the exit at the top left corner at B, (Fig.3.2b). The distances of the cells, from the exit are also shown in Fig.3.2b. Since total distance over which any item of goods has to travel, is the sum of coming into a cell, and out, the total distance of interest for each cell is the sum of the distances from entry and exit point. This is shown in Fig. 3.2c. If we move the exit to the diagonally opposite corner, the cell distances become equal in each layer (Fig. 3.2d). Observe, that the average distance remains the same, no matter where exit and entrance are.

If entrance and exit are shifted to the middle of one side as in Fig.3.2e

Fig. 3.2e

	1	2	3
$\bar{O}$	0	1	2
	1	2	3

a saving in total and hence average distance accrues, but again the individual cell distances change by fixed amounts, which are known once the exact location of entrance/exit is known.

3.6. If one further considers, that a warehouse is completely filled only on relatively rare occasions, there is a definite advantage in having a single point as entrance and exit. If, for example in the nine cell warehouse of Fig. 3.2, we impose the rule that goods are always stored in the nearest empty cell and taken from the nearest full cell, and on average only 5 of the nine cells are occupied then on average movement will take place in the area of the "nearest" 5 cells, and the average distance over which any single item moves will be 2.4, 2.8 and 4, for the arrangement in 3.2a, 3.2c and 3.2d respectively. The greater the proportion that is usually occupied the smaller the difference between the possible arrangements of entrance and exit. This is true, for every size of warehouse, and, therefore can be taken, as the first result, obtained from our model.

Whether, this result is in practice applicable, depends, of course, on whether goods can be moved into and out of a storage area via the same doorway. There are many situations in which this is possible, but even where this is not possible, the results obtained by assuming a single reference point are applicable, because the effect of separating entrance from exits only adds known fixed amounts to the distances.

3.7. We further assume that input and output is variable, and that there exists a stock control system which permits us to estimate the distribution of stocks, average stock and variance and any correlation between stocks of any two items. Vassian (JORSA 1955 3 (3) 272-282) has shown that, given the replenishment system: Order = Forecast of demand - Stock available + Safety Stock and fixed lead time, the stock has the same distribution as the Forecast Error, independent of the forecasting formula used. A number of other authors, including the writer of this report, have shown that the use of an exponential smoothing formula for forecasting, results in an unbiased distribution of forecast errors with average 0, and furthermore, that such a formula adapts itself quickly to any changes in demand, so that over time the distribution of errors can be taken as stable and symmetrical.

Usually lead time variability does not materially alter this distribution. Thus, for certain results we shall assume that the stock control system is such that stocks are distributed symmetrically around an average equal to a stipulated safety stock.

3.8. Finally, we have to decide the question of optimisation. Warehousing is usually a commercial operation. A priori, therefore, profit maximisation would seem appropriate. On the other hand, more often than not, warehousing forms only a part of a business, and furthermore our remit is restricted to the consideration of handling. For these reasons we selected "handling effort" as the criterion, and define it as dependent on distance over which a unit item has to be moved. In mathematical symbols, since movement within the simple warehouse model we stipulate, can take place only at right angles, this handling effort can be expressed as

$$H = f(x) + g(y) + d(z) \tag{3.1}$$

with: x = distance along, y = distance across, z = vertical distance. In words: handling effort, per unit, is related to the sum of movements along, across and vertical. Further consideration, in particular study of published performance characteristics of handling equipment and work study results on goods handling, indicates that the relationship between distance and effort is linear, and can therefore be expressed as

$$H = ax + by + cz \tag{3.2}$$

The handling effort itself can be expressed as a cost, - in which case a, b, and c, represent cost of moving one item one unit in each direction -, or as time, or as power consumption, whichever is appropriate.

The object is, of course, to minimise handling effort.

3.9. We do not say, that handling effort is always proportional to distance moved as stated in formula (3.2), only that in our preliminary investigations we have not found any contrary example, and that, therefore, in our first attempt to construct a mathematical model of movement in a warehouse the formula is appropriate.

LIST OF SYMBOLS

α_i = Ratio of maximum capacity for i^{th} commodity to total maximum capacity

β_i = Ratio of average stock to maximum capacity for i^{th} commodity

$$\left.\begin{array}{lll}\gamma & = & \text{Ratio of width to length} \\ \partial & = & \text{Ratio of height to length}\end{array}\right\} \text{of storage cell}$$

μ = Ratio of average minimum stock to average stock

a = Unit effort in x direction (length)

b = Unit effort in y direction (width)

c = Unit effort in z direction (height)

$$\left.\begin{array}{lll}g & = & \text{Unit effort in y direction} \\ d & = & \text{Unit effort in z direction}\end{array}\right\} \text{when measuring in terms of cell length}$$

e, f, h, as a, g, d, for sequential order picking

M_i = Average rate of throughput

$\bar{K}, \bar{K}_i$ = Average stock, total and i^{th} commodity

K, K_i = Maximum required capacity, total and i^{th} commodity

$$\left.\begin{array}{lll}A & = & \text{Unit cost for floor, roof area} \\ B & = & \text{Unit cost for wall area}\end{array}\right\} \text{of building}$$

ℓ = Average number of lines per Order.

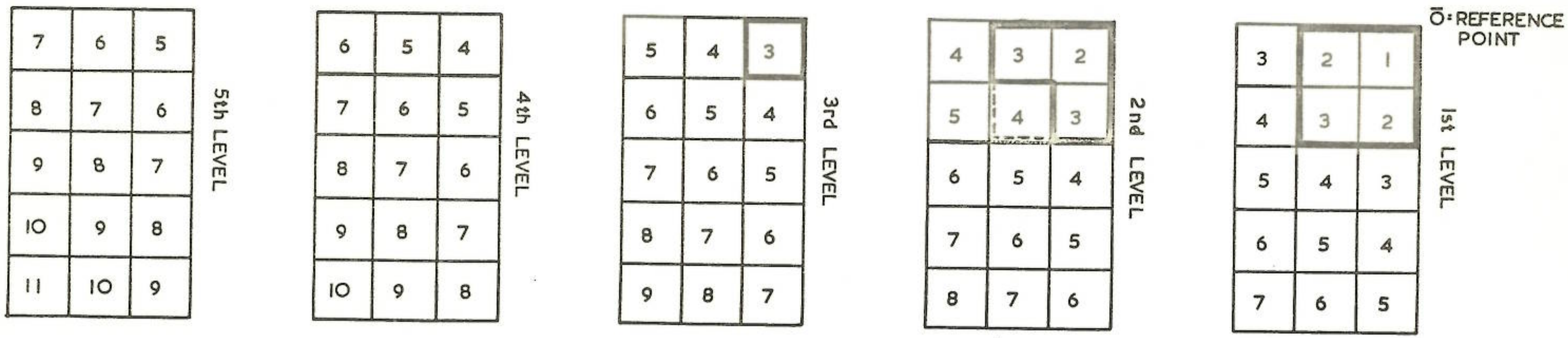
4. MOVEMENT WITHIN THE WAREHOUSE

4.1. The first and perhaps most important conclusion that we have arrived at, by considering the simple model described in the previous section, is that movement in a warehouse depends on stock distributions rather than possible forms of demand and supply, provided that such stock distributions are not time dependent. This proviso seems at first to be important, but research into stock control systems has shown that statistical methods of adaptive demand forecasting coupled with efficient re-order rules leads to stable stock distributions that are independent of time, except for known seasonal variations for which provision can be made.

4.2. To illustrate this point in greater detail, movement and hence handling in a warehouse, consisting of storage cells as in Fig. 1., depends on the amount of cells that are full and their spatial distribution within the block of cells at any given time. The number of full cells will fluctuate in accordance with the stock distribution. If we impose a rule, that incoming goods are always assigned to the nearest free cells and outgoing goods always taken from the nearest full cells, we shall achieve a clustering of full cells that will tend to the form of a cube, one corner of which will be the reference point, if the cells themselves are cubes. This is illustrated in Fig. 4.1, where the values marked in the cells are the distances from the reference point. Thus for 8 cells

FIG 4.1.

FLOOR PLAN OF WAREHOUSE WITH ALL DISTANCES



full, the arrangement marked by the thick lines gives the minimum total distance. No other arrangement of 8 cells can give a lower value.

4.3. As, in our simple model, we define a cell to be capable of holding one unit only, the term "nearest" is unambiguous since each cells distance from a reference point can be measured, and if, as is in practice often true, there are two separate reference points, incoming and outgoing, the cell distance is the sum of the two distances, as any incoming unit will in due course, become an outgoing unit.

Thus in order to estimate the average distance over which units have to be moved, we need only consider the extent of the cluster of cells given by the average stock in the warehouse.

4.4. If we have a warehouse of K cells, arranged as a block m cells long, n cells wide and p cells high, and the cells themselves are cubes of side 1, then the average distance to the reference point is given by

$$\bar{d} = \frac{m + n + p}{2} \qquad (4.1)$$

The minimum total distance of all cells is achieved if, as near as possible, $m = n = p$.

4.5. If, as is usual, the cells are rectangular with their sides in ratio of $1:\gamma:\delta$, for length to width to height, the average distance, taking the longest side as unit of measurement is given by

$$\bar{d} = \frac{m + \gamma n + \delta p}{2} \tag{4.1a}$$

Again minimum total distance is achieved if the block is a cube, i.e. if the cells are arranged as near as possible so that we have a cube: w cells long, $\frac{1}{\gamma}w$ cells across and $\frac{1}{\delta}w$ cells high, with $w = \sqrt[3]{K\gamma\delta}$ In practice w is the nearest integer so that $w^3 \geq K\gamma\delta$ and similarly $\frac{1}{\gamma}w$, $\frac{1}{\delta}w$ must be rounded up to its nearest whole number.

4.6. Our interest, however, is in a handling function that is a linear function of distance, of the form $ax + by + cz$. Thus to find the average value of the handling function per unit in a block of: $m \times n \times p = K$ cells, in which every cell has the chance of being full or empty, we multiply the average distance in the block by a, b, c, respectively. Setting $g = b\gamma$, $d = c\delta$ we have an average unit handling value as per Expression (4.1b), an approximation, that suffices for the purposes of minimisation. The exact formulae is derived in Sub-section 10.

$$\bar{H} = \frac{am + gn + dp}{2} \tag{4.1b}$$

Again, a minimum handling value, be it cost or any other expression of handling effort, is achieved if the K cells are arranged so that we have, as near as possible, a block, $\frac{w}{a}$ cells long, $\frac{w}{g}$ cells wide and $\frac{w}{d}$ cells high, with $w = \sqrt[3]{Kagd}$

4.7. The rule of always using nearest cells forces movement to take place within a cluster of cells of the order of stock in hand. On average, therefore, movement takes place in a space equivalent to that required by the average stock in hand, say, $\bar{K}$.

If then there are no spatial restrictions, that is, if the linear dimensions of the warehouse are greater than or equal to $\frac{w}{a}, \frac{w}{g}, \frac{w}{d}$ cells, respectively, and one commodity only is stored, the minimum average handling effort per unit is given by

$$H = \frac{3}{2}\bar{w} \tag{4.2}$$

where $\bar{w} = \sqrt[3]{\bar{K}agd} = \sqrt[3]{\bar{K}G}$

The above implies that we now re-define "nearest", as least handling effort.

4.8. Consider now a warehouse that has a capacity of K units. It has to store N different commodities, of which the stock distributions are known. Accordingly the required maximum capacity for the i^{th} commodity is K_i, the sum of the capacities adding to the total K, thus $\sum_{i=1}^{N} K_i = K$. The proportion of capacity taken up by the maximum required for the i^{th} item is $\alpha_i = K_i/K$. Further the average stock of the i^{th} item is given by $\bar{K}_i$, with the total average stock $\bar{K}$, thus $\sum_{i=1}^{N} \bar{K}_i = \bar{K}$. Finally the average rate of throughput for the i^{th} commodity is M_i units, and the total average rate of throughput for all commodities is M, so that $\sum_{i=1}^{N} M_i = M$. We shall assume that

$M_i \geq M_{i+j}$; that is we rank the N commodities in descending order of movement rate. Thus the first item has the highest and the N^{th} item the lowest rate of throughput.

4.9. There are essentially only two methods of assigning commodities to storage cells.

Method 1 separates the K cells into N groups. This partition could be achieved by dividing the axis of the block in proportion to the required maximum capacities. In its simplest form, one could divide just one axis, say the longest one, so that the i^{th} commodity would be assigned a space $\alpha_i m$ cells long, n cells wide and p cells high.

Alternatively partition could be carried out along two or three axes, so that the i^{th} part would be $\mu_i m$ cells long, $\nu_i n$ cells wide p cells high with $\mu_i \nu_i = \alpha_i$ or $\mu_i m$ cells long $\nu_i n$ cells wide and $\pi_i p$ cells high with $\mu_i \nu_i \pi_i = \alpha_i$ respectively. (See Fig.4.2 for two commodities).

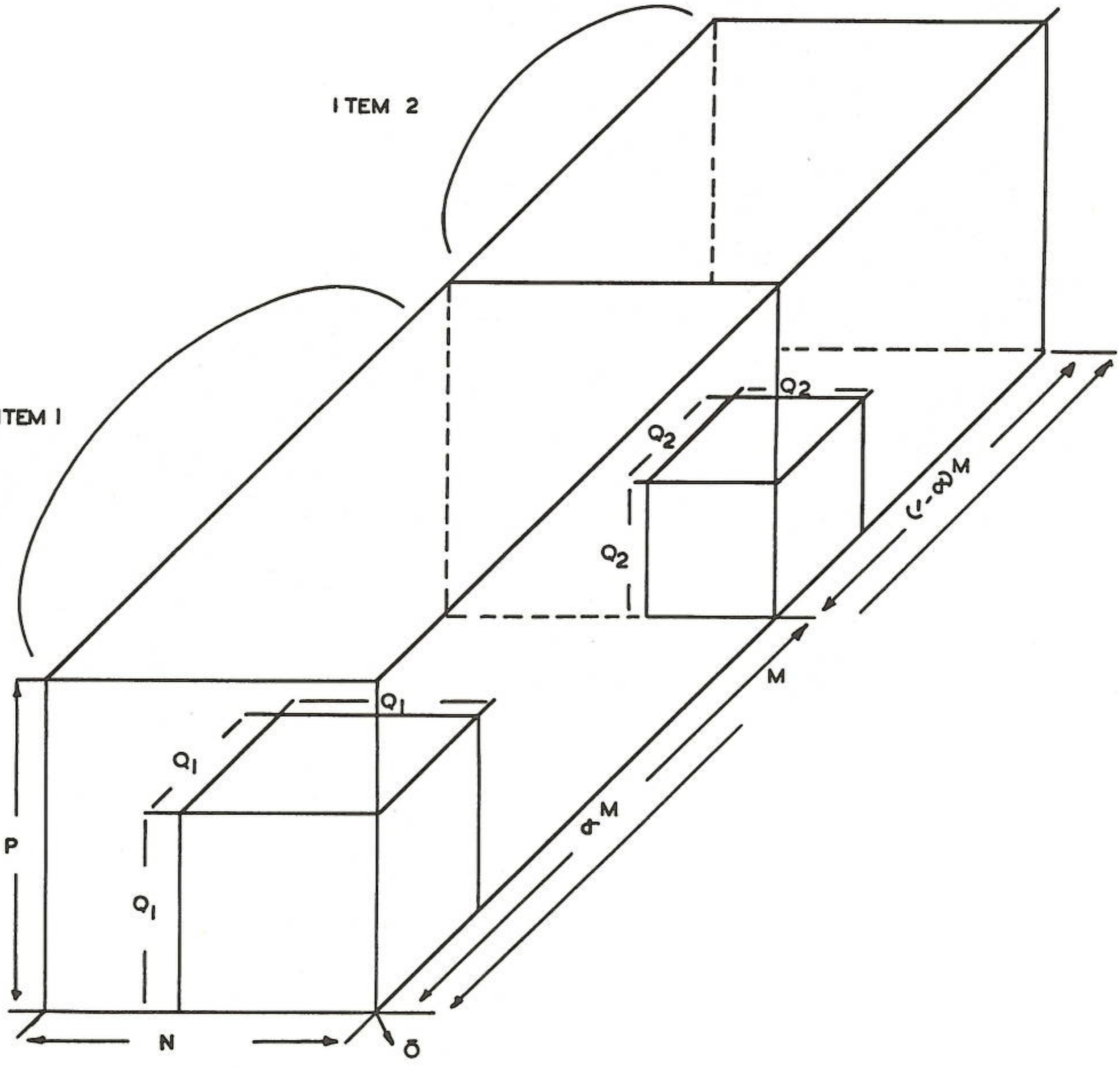
Method 2 consists of storing all items together.

4.10. Method 1, with the commodities ranked in descending order of throughput, is the wellknown rule of Thumb: "Put the fast moving items nearest the door".

Within the space allocated to the i^{th} commodity, movement

FIG.4.2.

WAREHOUSE POSITIONING BY ITEM



will on average be within a rectangular block $\bar{w}_i$ long, $\frac{\bar{w}_i}{g}$ wide and $\frac{\bar{w}_i}{d}$ high. One corner of this block will be nearest the reference point. The average minimal handling effort for the i^{th} commodity is thus given by

$$\bar{H}_i = M_i \left[\frac{3}{2} \bar{w}_i + d_i \right] \tag{4.3}$$

d_i = handling effort per unit over the distance from the corner of the space, assigned to the i^{th} commodity, nearest to reference point, to the reference point. The first part of (4.3) within the square brackets gives the average handling effort per unit, within the space assigned to the i^{th} commodity. Fig. 4.2. shows the arrangement for two commodities.

It is clear, that no matter how many commodities there are, d_1 = 0 always, and all other d_i will be greater or equal w_1, if, at least, the space assigned to the first commodity, which is the fastest moving, is dimensioned so that handling effort for it is a minimum.

4.11. For the total average handling effort for all commodities say H (1) under Method 1 of layout we have therefore

$$H(1) = \frac{3}{2} \sum_{i=1}^{N} M_i \bar{w}_i + \sum_{i=1}^{N} M_i d_i \tag{4.4}$$

and

$$H(1) > \frac{3}{2} \sum_{i=1}^{N} M_i \bar{w}_i + w_i \sum_{i=2}^{N} M_i \tag{4.4a}$$

Now let:

$$\alpha_i = K_i/K$$

$$\beta_i = \bar{K}_i/K_i$$

$$\beta = \bar{K}/K$$

$$agd = G$$

then

$$\beta = \sum_{i=1}^{N} \alpha_i \beta_i$$

$$\bar{w}_i = G^{1/3} \alpha_i^{1/3} \beta_i^{1/3} K^{1/3}$$

$$w_i = G^{1/3} \alpha_i^{1/3} K^{1/3}$$

and (4.4a) can be rewritten

$$H(1) > \frac{3}{2} G^{1/3} K^{1/3} \sum_{i=1}^{N} (\alpha_i \beta_i)^{1/3} M_i + G^{1/3} \alpha_i^{1/3} K^{1/3} \sum_{i=2}^{N} M_i \tag{4.4b}$$

Under Method 2 the total average handling effort, say, $H(2)$, is given by

$$H(2) = \frac{3}{2} \bar{w} M = \frac{3}{2} G^{1/3} K^{1/3} \beta^{1/3} M \tag{4.5}$$

4.12. Clearly, if the difference between (4.4) and (4.5) is positive, then Method 2 is better, i.e. requires less handling effort, otherwise Method 1 is preferable. We cannot write down the values of d_i, other than d_1, explicitly, since there are a large number of ways of assigning space to the commodities. If we use (4.4b) instead (4.4) in calculating the difference, we may err in favour of Method 1. We show below (Section 5) how this error can be avoided.

Let Δ be the difference between (4.4b) and (4.5). The sign of Δ is therefore the criterion which decides between Method 1 and Method 2.

$$\Delta = \frac{3}{2} G^{1/3} K^{1/3} \sum_{i=1}^{N} (\alpha_i \beta_i)^{1/3} M_i + G^{1/3} \alpha_1^{1/3} K^{1/3} \sum_{i=2}^{N} M_i - \tag{4.6}$$
$$- \frac{3}{2} G^{1/3} \beta^{1/3} K^{1/3} M$$

Since we are only interested in whether Δ is positive or negative we can simplify (4.6) to

$$= \frac{3}{2}\left[\sum_{i=1}^{N} (\alpha_i \beta_i)^{1/3} M_i - M \left(\sum_{i=1}^{N} \alpha_i \beta_i\right)^{1/3}\right] + \alpha_1^{1/3} (M - M_1) \tag{4.6a}$$

Expression (4.6a) shows that the choice between the methods depends on rate of throughput, relative capacity requirements, and stock distributions.

4.13. It may appear that all the above formulae neglect weight as affecting handling effort.

Now, weight of a storage unit does differ between commodities, thus a standard pallet load of one commodity will on the whole have the same dimensions as that of another commodity, but may weigh considerably more or less.

The effect of weight, however, is again a proportional one. Thus we only need to alter the expression for (3.2.) to

$$H = S\,(ax + by + cz)$$

where S is a weight factor.

We, therefore, need only define $N_i = S_i M_i$, and $N = \Sigma\, S_i M_i$, and substitute N_i, N, for M_i, M in the expressions of the preceding subsections to account for weight. In other words M_i, and M in these formulae can be rates of throughput either in units of quantity or units of weight, whichever may be appropriate.

4.14. The criterion of choice, given by (4.6a) is based on handling effort only. The total maximum capacity K of a warehouse, however, is not a fixed number, but depends on method of storage and stock distributions. In fixing a maximum capacity, the intention is clearly to assure with a high probability that one can store all the necessary stock. Given that one wishes to assure this with a probability of, say, 99%, then under Method 1, the part of the warehouse assigned to the i^{th} item must have a capacity of K_i, such that the probability that at any time stock of the i^{th} item exceeds K_i is less than 1%. Similarly for all other items, and K, the total capacity is the sum of the K_i. Under Method 2, however, one needs to assure only that 99% of maximum total stock i.e. stock of all items together, can be stored. Now if the items are independent, that is if they are not correlated, the probability that both item i and j exceed K_i and K_j respectively is 0.01% much less than the stipulated 1%. In other words, the probability that more than one item will at any given time have a very high stock is small compared to the probabilities of each item rising to such high stock separately. Hence the required total value K under Method 2 will for the same items be less than that for Method 1. This is true also if some of the items are correlated, since such a group can for this purpose be treated as a single item in the calculation of the required value of K.

4.15 If the stock control system is such that stock distributions are symmetric and approximately normal, this can be easily demonstrated as follows:

The maximum capacity required for the i^{th} commodity will be given by

$K_i = \bar{K}_i + k\sigma_i$; where k depends on the risk one is willing to take of being unable to find room for some incoming units. With k = 3 , for example, that risk is of the order of one tenth of one per cent. The total capacity required is therefore under Method 1

$$K = \sum_{i=1}^{N} \bar{K}_i + k \sum_{i=1}^{N} \sigma_i \qquad (4.7)$$

If the stock distribution for all commodities are symmetrical, then the distribution of the total stock, will approach the normal distribution with average equal to the sum of the averages and variances equal to the sum of the variances, i.e.

$$\bar{K} = \sum_{i=1}^{N} \bar{K}_i \qquad \sigma^2 = \sum_{i=1}^{N} \sigma_i^2 \qquad (4.8)$$

Under Method 2 the required capacity for the same risk factor k is therefore

$$K' = \sum_{i=1}^{N} \bar{K}_i + k\sqrt{\sum_{i=1}^{N} \sigma^2_i} \qquad (4.9)$$

In the difference

$$K - K' = k\left\{\sum_{i=1}^{N} \sigma_i - \sqrt{\sum_{i=1}^{N} \sigma^2_i}\right\} \tag{4.10}$$

the first term in the bracket of (4.10) is a sum of roots, the second term is the root of a sum of the same elements. Hence the second term is less than the first; therefore, the difference $K - K'$ is always positive.

Thus under Method 2, there will always be less space required than under Method 1.

5. APPLICABILITY OF THE SIMPLE MODEL

5.1. The formulae of Section 4, above, decide the optimal shape and method of layout of a warehouse for which input and output is in the same unit and in which every unit stored is directly accessible. Optimal here means least hadnling effort.

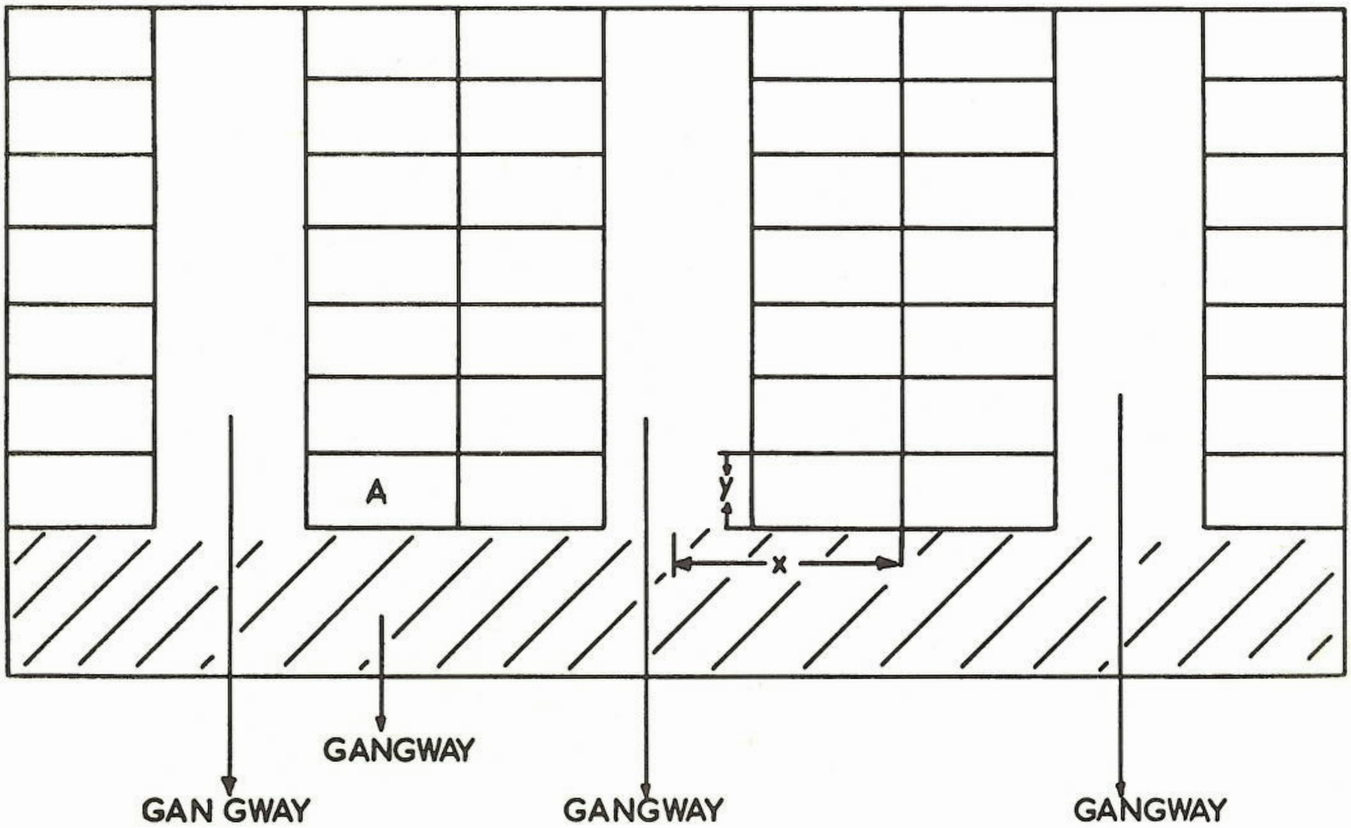
We now discuss how in practice these formulae can be applied.

5.2. To the reader, not used to handling mathematical models, the formulae, as given, must appear to have a grave defect. We have stated that handling effort depends on the distance over which any unit must be moved into and out of stock, but nowhere have we explicitly mentioned that part of the distance that in practice must be taken up by gangways; but gangways there must be, as is obvious from Fig.3.1.

The point of the formulae, however, is that they include that part of the distance attributable to gangways, on the assumption that all gangways have the same width and distance is measured in units of one side of a cell plus a proportion of gangway width, as shown in Fig. 5.1. Throughout we calculate distances from a reference point, $\bar{O}$, which is the first cell. Thus the distances calculated are less than the true distances. This does not effect the derivation of shape and layout that assures minimal handling effort, as the true distances and handling

FIG. 5.1.

LAY-OUT OF GANGWAYS



SHADED PART REPRESENTS LATERIAL GANGWAY

A. ACTUAL CELL

x, y. CELL DIMENSIONS AS USED IN FORMULAE effort differ from the calculated one only by a fixed constant amount, which is the handling effort from entrance/exit to the first cell.

For the calculation of the actual area required, one can therefore use the extended cell dimensions. Now the cell dimensions are x, γx and δx. For K cells the total area required is therefore

$$K \; \gamma x^2 \; + \; K\gamma \; x \; r, \tag{5.1}$$

but the distance from the reference point to, for example, the cell marked A in Fig. 5, is given by

$$D = 5x \; + \gamma x \tag{5.2}$$

Thus distance and handling effort is measured in units related to the cell inclusive of gangway required for access. As can be seen from Fig. 5.1, this is the only part of the total area of gangways that enters distance calculation within the warehouse area.

Since the dimensions of a unit of item to be stored, - for instance, one pallet load - are known, the cell dimensions and minimal gangway widths can be calculated. Hence units based on the above described cell dimension, can be translated, whenever required, into the more common measures of length, area and volume.

5.3. Expression (4.2) gives the minimal average handling effort per unit of a commodity for which the average stock and, by implication, the maximum stock is known.

This minimum can be achieved only, if the total number of cells required for that commodity, K_i, are arranged into a block of $\frac{w_i}{a}$ cells x $\frac{w_i}{g}$ cells x $\frac{w_i}{d}$ cells. In practice, $w_i = \sqrt[3]{K_i agd}$ is hardly ever a whole number, hence one must choose the nearest whole numbers.

5.4. Nevertheless, even the approximation to the ideal minimum that is necessary in practice, will not disturb greatly the shape of the block of cells that ensures minimal handling effort, and that shape can in practice be achieved, certainly in new buildings, that are designed on the basis of the above formulae, and possibly in old buildings. If, in any existing buildings, it should not be possible to arrange the cells into the required minimal block, the formulae will permit calculation of the excess handling effort over the theoretical minimum, that is due to the effect of the building, and thence can be used in evaluating the advantages of a move to a different building.

5.5 The expression (4.6a) is the criterion for layout. Consideration of (4.6a) leads one to believe that more often than not Δ will be negative. On the other hand, in a multi-commodity warehouse, it is questionable whether the only possible decision is: either separate all commodities or store all together. It is far more practicable to investigate, which of the commodities ought to be stored together, which separate, and where. Thus the Δ is calculated in a step wise fashion. Starting with commodities 1 and 2 i.e.

$$\Delta = \frac{3}{2}\left[(\alpha_1\beta_1)^{1/3} M_1 + (\alpha_2\beta_2)^{1/3} M_2 - (M_1 + M_2)(\alpha_1\beta_1 + \alpha_2\beta_2)^{1/3}\right] + \alpha_1^{1/3} M_2 \tag{5.3}$$

If $\Delta > 0$, set $M_1' = M_1 + M_2$ $M_2' = M_3$

and substitute M's for M in (5.3). Clearly the α_i and β_i will also change, since in the first step $\alpha_1 = K_1/(K_1 + K_2)$, and $\alpha_2 = 1 - \alpha_1$: in the second step, using the M', $\alpha_1' = (K_1 + K_2)/(K_1 + K_2 + K_3)$; β_1 in the second step is $\beta_1' = \alpha_1\beta_1 + \alpha_2\beta_2$. The third step, if $\Delta > 0$ in the second step, is based on the sum of the first three commodities, $M_1'' = M_1 + M_2 + M_3$ with consequent changes in the α_i's and β_i's. This procedure is repeated till Δ becomes negative. If this happens at the i^{th} step, the first i-1 commodities are to be stored together.

It is clear that in calculating Δ by (5.3) the d_i of (4.4) equals the w_1 of (4.4a), hence the inquality of (4.4a) does not apply.

As soon as a negative Δ is reached, we restart the process for the (n - i + 1) remaining commodities, so that in the end, there will be groups of commodities, that are to be stored together. This calculation does not, of course, exclude the result that each commodity ought to be stored separately.

An illustration of the above process is given in the numerical example on pages 76 ff.

5.6. The savings in handling effort and space due to storing different commodities together must, of course, be set off against increased cost of data processing, that such a storage pattern requires. It is obvious that, if a computer is used for stock control or order processing, then the cost of the additional data processing will be relatively low, as all that will be required is a modification to the existing programmes. The cost of keeping track of all units in a completely manual data processing system, may be higher than the savings achieved by mixed storage. No generally valid formula can be given, but again, as in the case of building imposed restrictions, the calculation of achievable savings in handling and space can help to judge possible changes in any existing data processing system.

5.7. The values of a, g and d, depend on the dimension of the cell and gangway width, and the handling equipment used. Thus, to give a practical example, in a warehouse storing palletised goods, moved by fork lift trucks, different performance characteristics of the trucks will lead to different shape of blocks and different layouts. Again the cost of the trucks will vary with their performance characteristics.

If a, g and d is expressed in monetary terms, the total cost of handling will be given by the handling effort as calculated from the above given formulae plus the cost of the equipment. Hence one can calculate the total costs for all available types of fork lift trucks and thence decide on the most suitable (Example on page 76ff.)

6. ORDER PICKING

6.1. Order picking becomes necessary when the unit of input is different, and greater, than the unit of output. There are essentially only two methods: Commodity directed, which we shall call "parallel picking" and Order directed, or "sequential picking".

6.2. Parallel picking means that a number of orders are dealt with in parallel, by selecting the total quantity of one commodity at a time required for that number of orders and distributing it to the orders.

6.3. Sequential order picking deals with the orders in sequence that is selecting all the commodities required for a single order, at a time.

6.4. There is also a hybrid case, where commodities are transferred in quantities required for a number of orders to a marshalling area, and sequential order picking takes place there. This method is often found in warehouses of the pallet rack type, where the lowest cells are used as "live" store, from which sequential picking takes place, whilst the rest of the warehouse is in effect one in which unit of input equals unit of output.

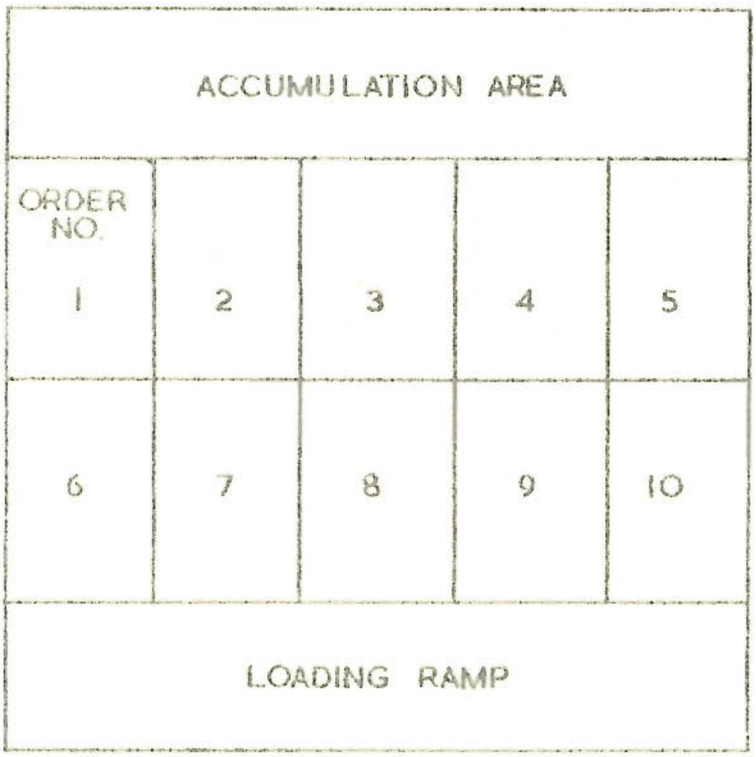
6.5. Total average handling effort under parallel picking is, of course, equal to handling effort within a warehouse, where input and output units are equal, plus the effort required to distribute each commodity to a set of orders. It is not possible to find a general mathematical expression for measuring the handling effort due to assigning of a given quantity of a single commodity to a given number of orders, as this can be done in many different ways, dependent on circumstances particular to trade, transport methods, etc. One simple way might be to transport a quantity of input units, greater than, but as near as possible equal to, the quantity of a commodity required for a set of orders to an accumulation area and thence distribute the order quantities to adjacent order areas which themselves are adjacent to loading bays, as shown in Fig. 6.1. The accumulation area must be at least of sufficient capacity to hold the largest total quantity of the commodity in highest demand for a set of orders.

Obviously, once the system of assigning the total quantity picked to a set of orders is given, one can estimate the handling effort involved.

6.6. It should be noted that under parallel picking, the handling effort within the warehouse or storage area is independent of the effort required in assigning items to orders. Thus inability to find a general optimisation procedure for the latter does in no way obstruct optimisation of the former by the methods given in Sections 4 and 5.

FIG. 6.1

MARSHALLING AREA FOR ORDERS



6.7. In sequential picking we shall first consider the case where any cubicle of the warehouse may hold one major unit (input unit) of any of the commodities stored, i.e. a layout as per Method 2 of Section 4.

We assume that any single orders, to which the picking system applies, can be put together in one picking round through the warehouse. We shall further assume that output units can be picked from any cell; if this were not so, - for example, if it were impossible to pick from the high cells - we would not have a true sequential picking procedure, but the hybrid case of subsection 6.4 above.

6.8. On these assumptions, the handling effort is again related to distance, along the three axis, provided that, once a cell is reached, any number of output units, up to the total contained in the cell, can be taken without further movement. Of course, quantity picked does influence total handling effort, in the sense, that a picker taking, say, 5 units from a cell, will spend more time at that location, than if he takes only one unit. Since, however, we are interested in the total average handling effort in the warehouse per unit of time, and the actual transfer of a unit from cell to, say, collecting pallet, can be taken to require the same effort at any cell, the total average effort ascribable to quantity picked is a constant dependent only on the average rate of output, which must be of the order of half the average rate of throughput, i.e. $M/2$, in terms of input units. Since it is a constant, it does not enter in any mathematical minimisation procedure.

6.9. If the average number of orders is, say P, and the average number of lines is ℓ, on average ℓ different cells will have to be visited per collecting round. On the assumptions of sub-section 6.7. the major units of the commodities will be randomly distributed over the warehouse. It is shown in the section on Mathematics, that in these conditions the ℓ cells will on average lie along the diagonal from cell 0 to the farthest cell, and will be $\frac{1}{\ell+1}x$, $\frac{1}{\ell+1}y$, $\frac{1}{\ell+1}z$ apart Thus the average distance from point of reference to the furthest cell to be visited is given by:

$$\frac{\ell}{\ell+1}\left(\bar{X} + \bar{Y} + \bar{Z}\right), \text{ where}$$

$\bar{X}$, $\bar{Y}$, $\bar{Z}$ is the length, width and height of the average occupied block of cells.

If any cell is directly accessible from any other cell, the total average handling effort of picking could be written

$$H(p) = \frac{M}{2}t + \frac{2P\ell}{\ell+1}\left[e\bar{X} + f\bar{Y} + h\bar{Z}\right] \tag{6.1}$$

and the total handling effort in the warehouse, i.e. input and output together as:

$$H(T) = \frac{M}{2}t + \frac{M}{4}(a\bar{X} + g\bar{Y} + d\bar{Z}) + \frac{2P\ell}{\ell+1}\left[e\bar{X} + f\bar{Y} + h\bar{Z}\right] \quad (6.2)$$

$$= \frac{M}{2}t + A\bar{X} + B\bar{Y} + C\bar{Z}$$

$$A = \frac{Ma}{4} + \frac{2P\ell}{\ell+1}e, \quad B = \frac{Mg}{4} + \frac{2P\ell}{\ell+1}f, \quad C = \frac{Md}{4} + \frac{2P\ell}{\ell+1}h$$

Note: h is the unit handling effort in the vertical direction, if picking can proceed from one level to a higher level, without the necessity of returning to ground level. Otherwise

$h = \ell h'/2$, where h' = unit handling effort in vertical direction

A minimum handling effort would again require a block of cells $\frac{W'}{A}$ long, $\frac{W'}{B}$ wide, and $\frac{W'}{C}$ high with $W' = \sqrt[3]{KABC}$

In short the procedure is the same as described in Section 4.

6.10. Unfortunately it is difficult to imagine a real warehouse in which each cell is directly accessible from any other cell. Even if space or building costs were nil, such a layout would require free standing racking, each rack being exactly one cell long and wide, and Z cells high, the cost of such racking would be significantly higher than the more usual arrangements, such as that in Fig. 5.1

The layout of Fig. 5.1 is the most economical in terms of space and racking, but the worst from the point of sequential order picking, as it requires the maximum amount of back-tracking during a collecting round. Any other arrangement, that is, putting in more lateral gangways, will improve the situation at the expense of space.

No matter what the layout of the racking, the average location of the cells to be visited, will be as described in sub-section 6.9

The case of interest is, of course, when ℓ, the number of picking points per collecting round, is relatively large; we can take as representative example the case where $X/(\ell+1) = 1$, that is on average a cell in each rack has to be visited during one round. In this case the total average picking effort can be written as

$$H\ (p) = \frac{M}{2}t + P\left\{\frac{2\ell}{\ell+1}\left[e\bar{\bar{X}} + h\bar{Z}\right] + \left[2\ell(\ell+1) + \frac{\ell}{\ell+1}\right]\bar{Y}\right\} \qquad (6.1a)$$

if ℓ is an even number

$$\text{or } H\ (p) = \frac{M}{2}t + P\left\{\frac{2\ell}{\ell+1}\left[e\bar{\bar{X}} + f\bar{Y} + h\bar{Z}\right] + 2\ell(\ell-1)\bar{Y}\right\}$$

if ℓ is odd.

6.11. We could again add handling effort for input to (6.1a) and thence derive a cell block that would minimise total handling effort. We feel, however, that this would not be of great practical value. Firstly, as already said, the expression (6.1a) applies only to the racking layout of Fig. 5.1, which, though common, is not necessarily the best. Picking effort could be improved by adding lateral gangways, but the amount of improvement depends on the number ℓ. Secondly, if the factor $X/(\ell+1)$ is smaller or greater than 1, that is, if on average more than one cell per rack, or less than one cell per rack respectively, are visited per collecting round, the expressions, equivalent to (6.1a), become rather cumbersome and a large variety both of rack layouts, and interval factors $X/(\ell+1)$ would have to be considered before any general conclusions could be stated. It is doubtful, if such general statements would be close enough to reality to be directly applicable to any given case. On the whole, now that we have laid down the procedure, it seems more economical to carry out the actual calculations from case to case.

6.12. There is, however, one further reason. If there are, say, V varieties stocked, the maximum number of cells required to form a picking face is of the order of V, i.e. it is exactly V, if any cell can be replenished as soon as it becomes empty or slightly more than V, if it takes more than the time spent on a picking round to replenish the cells of those relatively few varieties, which are in high demand.

Now, V must be considerably less than $\bar{K}$. Even, if on each collecting round the entire picking face of V cells has to be traversed and the cells are arranged only at one level the total picking effort per collecting round is given by

$$H'(p) = \frac{M}{2p}t + fV \qquad (6.3)$$

In picking through the entire warehouse, the picking effort per round, even if there is direct access from all cells to all cells, is given by

$$H''(p) = \frac{M}{2p}t + \frac{2\ell}{\ell+1}(e\bar{X} + f\bar{Y} + h\bar{Z}) \tag{6.3a}$$

Since V is much smaller than $\bar{K} = \bar{X}\,\bar{Y}\,\bar{Z}$ it is entirely likely that $H'(p)$ is much smaller than $H''(p)$. It is therefore, also likely that the total handling effort in a "hybrid" warehouse, consisting of the type dealt with in Sections 4 and 5, plus a V cell picking face, will be less, than that for sequential picking within a warehouse as given in the previous subsections.

6.13. The unidirectional, one level arrangement of the V cells for a picking face is not necessarily the best. A unidirectional arrangement of cells z high, and y long, where

$$z = a\sqrt{V} \qquad y = \frac{1}{a}\sqrt{V}$$

with $a = \sqrt{\frac{2f}{h\ell}}$

will be optimal, if varieties are assigned to the cells on the picking face randomly.

This can still be further improved by assigning varieties in order of frequency of demand, that is in order of frequency of a variety appearing on orders, with the high frequency varieties occupying the lowest cells.

6.14. The values of f and h will again vary according to the handling equipment used. We can therefore, choose between various available equipment, by recalculating the picking effort in a similar way as illustrated on pages 76 ff.

6.15. Thus, there is some evidence, that a warehouse with true sequential picking will require more handling effort, than the hybrid arrangement, though, of course, we cannot at this stage show, that this is true in all cases. We have, however, shown that in each particular case, the necessary calculations, on which to base a choice between, parallel, sequential and hybrid picking can be made.

6.16. During the above discussion we have assumed that all varieties are stored together, as per Method 2 of Section 4. We have also shown in the preceding sections that this method requires less total space.

Let us now, in order to simplify the argument, assume that height does not matter in sequential picking. For any given height the area under Method 2, will therefore be less than the area under Method 1. It is clear that, since a collecting round consists effectively of a round trip over the area with, or without back-tracking, the smaller the area, the shorter the collecting round. On the whole, therefore for sequential picking, Method 2 must be preferable to Method 1; this is true for whatever height can be usefully and economically employed in storing.

As from the discussion in Sections 4, 5 and the example, it appears that, when many varieties are stored, Method 2 is often preferable at least for a significant part of the total storage requirement, we feel that discussion of sequential picking under Method 1 is unnecessary.

7. TIME DEPENDENT STORAGE POLICIES

7.1. The previous sections dealt with a simple warehouse model in a static sense. This was sufficient, the only consideration being optimisation of handling, i.e. the problem posed was spatial.

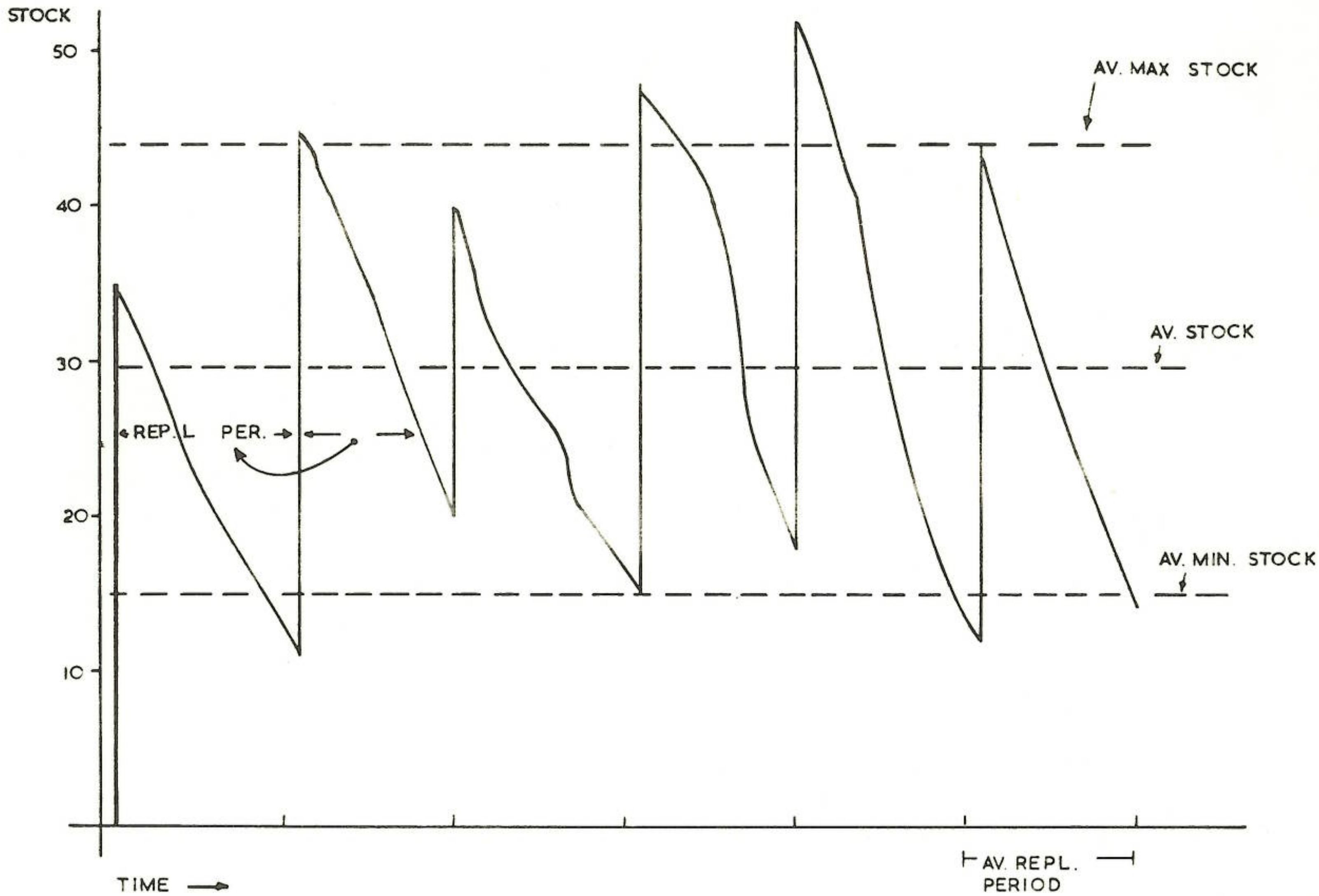
The prob em becomes a dynamic one as soon as stock policies are taken into consideration, that is, as soon as the time dimension enters the problem.

We discuss below how the imposition of a time dependent policy effects the model.

7.2. The "Saw Tooth Diagram" (Fig. 7.1 is familiar to all stock controllers. It depicts the stock level over time. The interval between two adjacent peaks is the "replenishment period". The average stock in any replenishment period is given by the mid-point of the line connecting the peak with the lowest point, i.e. the end point, Thus, if the beginning stock level is K_1 and the stock at the end of the period, that is the stock just before a further replenishment arrives, K_2, the average stock during the period is given by $\frac{1}{2}(K_1 + K_2)$

7.3. The average stock level over time can be estimated by averaging the mid-points of all periods. Similarly one can estimate an average maximum stock, say $\bar{K}^*$ and an average minimum stock $\bar{\bar{K}}$, by averaging all the peaks and lowest points

FIG. 7.I.
SAW-TOOTH DIAGRAM



on the diagram respectively.

7.4. Conversely, give $\bar{K}^*$ and $\bar{\bar{K}}$, the average stock level over time, $\bar{K}$, is given by $\frac{1}{2}(\bar{K}^* + \bar{\bar{K}})$, and this must, of course equal to the average stock as derived from other sources.

If we write $\mu = \bar{\bar{K}}/\bar{K}$, then $\bar{K}^* = (2 - \mu)\bar{K}$.

7.5. The total average movement, in and out, in the average replenishment period is, therefore, given by $M = (1 - \mu)\bar{K}$, and the average volume occupied by this movement is $(2-2\mu)\bar{K}$. The value of μ can easily be obtained from stock records, by evaluating the ratio $\bar{\bar{K}}/\bar{K}$ or from the expression $M = (1 - \mu)\bar{K}$.

7.6. Consider now what happens in the average period, given all movements are average and given the rule of always filling the nearest empty cell and taking from the nearest full cell, modified by FIFO. This means that the shortest distance rule applies only within groups of equally old stock units. Assume first that $\mu = \frac{2}{3}$ and we start with a full store, that is the stock equals $1\frac{1}{3}\bar{K}$, all equally old and located in the nearest $1\frac{1}{3}\bar{K}$ cells. During the first replenishment period $(2-2\mu)\bar{K} = \frac{2}{3}\bar{K}$ units are issued and according to rule are taken from the nearest location, as the entire stock is of the same age.

At the start of the second period $\frac{2}{3}\bar{K}$ units are received and go into the free locations. The $\frac{2}{3}\bar{K}$ issues, however, are taken from the farther half of the store as they are older. At the start of the third period, the $\frac{2}{3}\bar{K}$ incoming stock units go into the locations vacated by issues during the second period, and issues during the third period are the receipts at the start of the second period.

7.7. Obviously, this process repeats itself continuously and equally obviously, the average distance over which a unit, in or out, must be handled is given by formula (4.1) which explicitly becomes

$$\bar{d}_{FIFO} = \frac{3}{2}\sqrt[3]{(2 - \mu)\bar{K}} \tag{7.1}$$

if the cells are cubical.

The statements about non-cubical cells and handling functions in Sections 4 and 5 above are valid in this case provided one substitutes $(2 - \mu)\bar{K}$ for $\bar{K}$ in all expressions. The movements in and out are graphically depicted for $1\frac{1}{3}\bar{K} = 26$ in figure 7.2a

7.8. Imagine now the same operation with a $\mu = \frac{1}{2}$, again starting from a stock equal to $(2 - \mu)\bar{K}$, that is $1\frac{1}{2}\bar{K}$, all equally old. During the first period issues will empty the nearest $\frac{2}{3}$ of the cells, which will be filled at the start of the second period with new receipts. During the second period issues are taken first from the furthest $\frac{1}{3}$ of cells and then from the nearest

$\frac{1}{3}$. Receipts at the start of the third period go into the cells vacated during the second period, and issues are taken from the nearest $\frac{2}{3}$ of cells, and so on (See Fig. 7.2b)

7.9. Clearly in these conditions the average distance an unit moves is now less than that given in formula (7.1), but not much less. In fact (7.1) gives an upper limit for the average distance, and can be used as a conservative estimate of that average.

7.10. Similar considerations of average movement, in and out, when the ratio μ is greater than $\frac{2}{3}$, illustrated in Figs. 7.2c and 7.2d, lead us to the same conclusion. Section 10 specifies the exact formulae, but for all practical purposes formula (7.1) is sufficiently accurate.

7.11. We have coined the term MINDIS to describe the policy of minimising movement in the warehouse regardless of age of stock. The same analysis that was used for FIFO in the preceding subsections leads directly to an appropriate expression for the average distance over which units are moved in the case of MINDIS. Although it may appear that this case has been treated in Sections 4 and 5, this is not so as there the static case was considered. In actual fact stock movements have a time dimension, i.e. the saw tooth diagram is a proper representation of stock movement under all policies.

7.12. If no account of age of stock is taken and only the minimum distance rule applies, issues will always be taken from the nearest full cells and receipts stored in the nearest empty cells. Thus, on average movement will take place in a cube of volume $(2 - 2\mu)\bar{K}$ with origin at the reference point if the cells are cubical and we have

$$d_{MINDIS} = \frac{3}{2}\sqrt[3]{(2-2\mu)\bar{K}} \qquad (7.2)$$

7.13. The expressions for handling effort in Section 4 are also valid in this case, if one substitutes $(2-2\mu)\bar{K}$ for $\bar{K}$ in all expressions.

7.14. It is not possible to consider movements under LIFO purely on the basis of average movement and average stocks. If indeed there were no fluctuations from the average, LIFO would be equivalent to MINDIS. One can, however, fairly easily imagine the average picture of a LIFO store.

Starting from a store whose nearest cells are full, a new receipt will occupy the farther locations. During the ensuing period, if issues exceed receipts, all the cells filled at the start of the period will be emptied, and some of the nearest cells as well. The next batch of receipts will occupy these empty cells if it does not exceed the previous issue; if it does some further far cells will be filled. Issues then will be taken first from the cells filled during that period, then from cells filled in the preceding period and so on. Thus, there will be a tendency for old stock to accumulate in the cells which are about average distance from origin.

7.15. The average maximum stock is given again as $(2-\mu)\bar{K}$. The old stock which moves relatively rarely, will on average be $\mu\bar{K}$, and movement will take place in the space in front and behind this barrier of old stock, which is centred on the mean line of the cube of volume $(2-\mu)\bar{K}$. See fig. 7.3.

The average distance for cubical cells can thus be approximated by averaging the average distance in a cube of volume $(1-\mu)\bar{K}$ and the average distance of the $(1-\mu)\bar{K}$ farthest cells in a cube of volume $(2-\mu)\bar{K}$.

$$d_{LIFO} = \tfrac{3}{4}\ \bar{K}^{1/3} \left\{ (1-\mu)^{1/3} + (2-\mu)^{1/3} + 1/\ \ (2-\mu)^{2/3} + (2-\mu)^{1/3} + 1 \right\} \quad (7.3)$$

Details of derivation of (7.2) are given in section 10. Again the expressions in Section 4 for handling effort apply, given the proper substitution for $\bar{K}$ derived from (7.2)

Thus the effect of time, alters the expression for unit average handling effort only in the sense that the value $\bar{w}$ becomes $^{3}\sqrt{SG}$ instead of $\sqrt[3]{\bar{K}G}$

$S = (2-\mu)\bar{K}$ for FIFO $\quad (2-2\mu)\bar{K}$ for MINDIS

Similarly any $\bar{w}_i = {}^{3}\sqrt{S_i G} = \sqrt[3]{(2-\mu)\bar{K}G}$ and $\sqrt[3]{(2-2\mu)\bar{K}G}$

with β and β_i becoming S/K and S_i/K_i respectively.

For LIFO the substitution is somewhat more complex as movement takes place on average in two, spatially separated, parts of the storage volume, so that one cannot directly substitute S for $\bar{K}$ in (4.2), but has to rewrite 4.2 as

$$H = \frac{3}{4}\left[w' + w''\,(1+\tau)\right]$$

$$w' = \sqrt[3]{(1-\mu)\bar{K}agd} \qquad (7.4)$$

$$w'' = \sqrt[3]{(2-\mu)\bar{K}agd}$$

$$\tau = 1/\left[(2-\mu) + (2-\mu)^{2/3} + (2-\mu)^{1/3}\right]$$

The development to the decision criterion Δ from (4.2) to (4.6) remains the same using the form of (7.4) for handling effort.

MOVEMENT IN AND OUT OF WAREHOUSE
under FIFO

Notes to Figs. 7.2a, b, c, d

Each line of the grid shows the complete number of cells available in each period. Cells are identified by number and distance from reference point.

↙ denotes movement out

↗ denotes movement in

A number in the cell denotes the period of receipt of stock that does not move during the period.

Fig. 7.2a $\eta = 2/3$ $(2-\eta)\bar{K} = 26$

Cell No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Dist	0	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	3	4	4	4	4	4	4	5	5	5	6
Period 1		↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	0	0	0	0	0	0	0	0	0	0	0	0	0	
2	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	
3	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	
4	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	
5	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	
6	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	
7	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	
8	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	
9	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	
10	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	
11	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	
12	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	

Fig. 7.2b $\mu = \frac{1}{2} \quad (2-\mu)\,\bar{K} = 27$

Cell No		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Dist		0	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	3	4	4	4	4	4	4	5	5	5	6
Period	1	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	0	0	0	0	0	0	0	0	0
	2	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙
	3	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↗	↗	↗	↗	↗	↗	↗	↗	↙
	4	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙
	5	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↗	↗	↗	↗	↗	↗	↗	↗	↗
	6	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙
	7	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↗	↗	↗	↗	↗	↗	↗	↗	↗
	8	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙
	9	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↗	↗	↗	↗	↗	↗	↗	↗	↗
	10	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙
	11	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↗	↗	↗	↗	↗	↗	↗	↗	↗
	12	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗↙	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙

Fig. 7.2c $\mu = 4/5$ $(2-\mu)\bar{K} = 27$

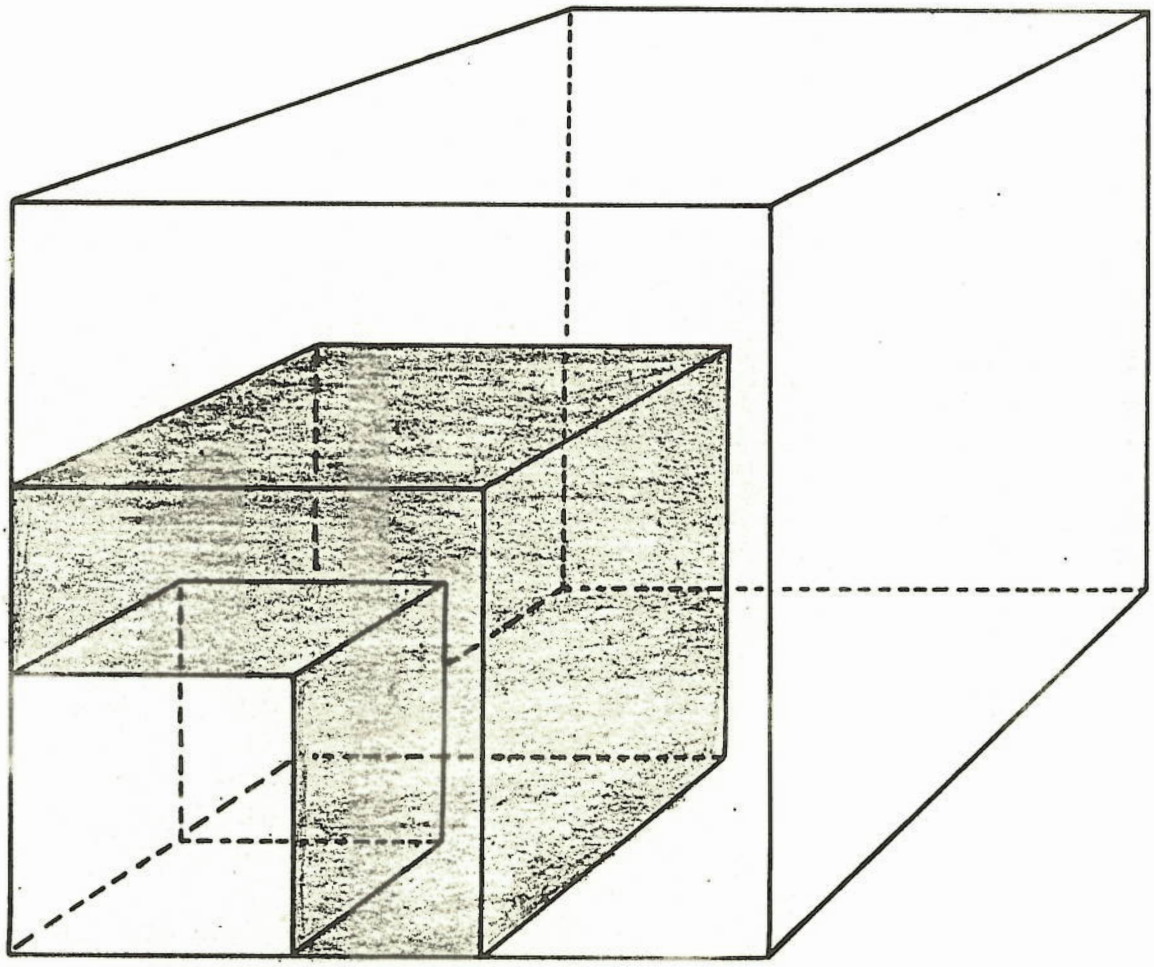
Cell No	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Dist.	0	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	3	4	4	4	4	4	4	5	5	5	6
Period 1	↙	↙	↙	↙	↙	↙	↙	↙	↙	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	0	0	0	0	0	0	0	0	0
3	2	2	2	2	2	2	2	2	2	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙
4	↙	↙	↙	↙	↙	↙	↙	↙	↙	3	3	3	3	3	3	3	3	3	↗	↗	↗	↗	↗	↗	↗	↗	↗
5	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	4	4	4	4	4	4	4	4	4
6	5	5	5	5	5	5	5	5	5	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙
7	↙	↙	↙	↙	↙	↙	↙	↙	↙	6	6	6	6	6	6	6	6	6	↗	↗	↗	↗	↗	↗	↗	↗	↗
8	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	7	7	7	7	7	7	7	7	7
9	8	8	8	8	8	8	8	8	8	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙
10	↙	↙	↙	↙	↙	↙	↙	↙	↙	9	9	9	9	9	9	9	9	9	↗	↗	↗	↗	↗	↗	↗	↗	↗
11	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	10	10	10	10	10	10	10	10	10
12	11	11	11	11	11	11	11	11	11	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙

Fig.7.2d $\mu = 5/7$ $(2-\mu)\bar{K} = 27$

Cell No.		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Dist		0	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	3	4	4	4	4	4	4	5	5	5	6
Period	1	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	2	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	0	0	0
	3	↙	↙	↙	↙	↙	↙	↙	↙	↙	2	2	2	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙
	4	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	3	3	3	↗	↗	↗
	5	↙	↙	↙	↙	↙	↙	↙	↙	↙	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	4	4	4
	6	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	5	5	5	↗	↗	↗	↙	↙	↙
	7	↙	↙	↙	↙	↙	↙	↙	↙	↙	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	6	6	6	↗	↗	↗
	8	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	↗	↗	↗	↙	↙	↙	7	7	7
	9	↙	↙	↙	↙	↙	↙	↙	↙	↙	↗	↗	↗	↗	↗	↗	↗	↗	↗	8	8	8	↗	↗	↗	↙	↙	↙
	10	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	↙	9	9	9	↗	↗	↗
	11	↙	↙	↙	↙	↙	↙	↙	↙	↙	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	10	10	10
	12	↗	↗	↗	↗	↗	↗	↗	↗	↗	↙	↙	↙	↙	↙	↙	↙	↙	↙	11	11	11	↗	↗	↗	↙	↙	↙

FIG.7.3.

SHADED VOLUME REPRESENTS SPACE, IN WHICH ON AVERAGE NO MOVEMENT OCCURS UNDER LIFO



8. THE COST OF STORAGE SPACE

8.1. The analysis of handling effort in a warehouse, discussed in the foregoing sections is based on the assumption, that handling effort is a function of the distance over which stock items have to be moved. It showed that to minimise handling effort the storage space must have certain dimensions, related to maximum volume required and the characteristics of handling equipment. Nothing was said about the effort of obtaining a space of such dimensions. In simpler language, if we agree to measure handling effort in monetary terms, we minimised the cost of handling but neglected in the process the cost of space, by stating that for a given storage volume, minimal handling cost can be achieved if the storage space is of a certain length, width and height.

8.2. The cost of space is a capital charge that has to be serviced at regular intervals. It is, therefore, correct to apportion that charge to the total average handling cost, per time unit, and to attempt to minimise the total cost of running a warehouse, particularly when considering opening a new warehouse, rather than storage in an existing building.

8.3. A warehouse structure is usually a rectangular box. Though architects, rather surprisingly, are reluctant to accept mathematical cost formulae, for the purpose of this investigation an approach to construct such a formula must be made on the best evidence available.

8.4. It appears that the building cost for a boxlike warehouse can be subdivided into two main items, one dependent on the floor area, the second on the wall area required. Cost of floors per square unit depends again on the load bearing requirements, floor coverings etc. Similarly roofing is a function of floor area, and so, it appears, are part of the services such as lighting, drainage etc. There is thus a part of the total cost, that is roughly proportional to the floor area, or in symbols

$$C_1 = A\,x\,y \tag{8.1}$$

where x = length, y = width

8.5. The cost of walls, stanchions and other supports and part of the services are similarly related to wall area, in symbols

$$C_2 = 2B\,(x+y)z, \text{ where } z = \text{height} \tag{8.2}$$

The proportionality factors A and B, which effectively are prices per square unit, depend on the requirements of the particular building, and can be expressed in terms of £. s. d.

In the main, the total capital cost of such a boxlike structure can thus be expressed as

$$C = A\,x\,y + 2B\,(x + y)z \qquad (8.3)$$

and given that we wish the box to contain a given volume of stock items, say, K, then

$K' = x'\,y'\,z'$ where K' is the volume taken up by K storage units, and x', y', z' are linear measures

From the expressions (8.3) and (8.4) we can derive the dimensions of the building of volume K', that would have minimum cost. This turns out to be

$$x' = y' = \sqrt[3]{2\theta K'} \qquad (8.4)$$

$$z' = \sqrt[3]{K'/4\theta^2} \qquad \theta = \frac{B}{A}$$

8.6. Clearly, the dimensions that minimise capital costs for a given volume, are not the same as those that minimise handling costs. In order to obtain the dimensions that minimise total handling costs, including space costs, we must combine the expressions for space cost and handling effort. In the simplest case, that is the store layout by Method 2, of Section 4 and gangway layout as per Fig. 5.1 we have

$$TC = Axy + 2B\,(x + y)z + C\,(ax + by + cz - br) \qquad (8.5)$$

where: r is the depth of the transverse gangway in Fig. 5.1, a known constant, and $C = \frac{M}{2} \beta^{1/3}$,

We want to find the values of x', y', and z', that minimise the expression (8.6), subject to $x'y'z' = K'$

Unfortunately it is not possible to find these values by direct analytical methods but a numerical solution is always possible, and a method is explained in para. 10.10.

9. AUTOMATION IN WAREHOUSES

9.1. In the preceding sections we have described procedures to minimise handling effort within warehouses, on the assumption that handling effort is a function of the average distance over which one unit has to be moved during the storage process. We have based our deductions on data that are normally available to warehouse managers, and have established that the physical shape of the storage space is dependent on the characteristics of the handling equipment, and the layout within that space, that is the assignment of storage locations to commodities, is dependent on the average throughput rates and maximal storage capacities. It was shown that variety, as such, does not influence handling effort.

It is natural to ask whether and how can our analysis help in deciding on automation of warehouse processes.

9.2. Before we can discuss this question, we must define the meaning of the term "automation". Warehouse processes are of two kinds: Input and Output. Input starts with the arrival of goods to be stored at the warehouse entrance. The arrival and the identification of the goods as to quantity and commodity is notified to the warehouse controller. On the basis of this information and in accordance with given rules, the controller decides where in the warehouse each unit of the goods is to be stored and orders the movement of the goods into the warehouse. This order initiates the physical transport of the goods.

Output starts with a demand to the controller for a quantity of goods to be issued. The controller, again in accordance with given rules, decides which particular units from which store location are to be used and orders their removal. His order initiates physical transport of the designated items to the exit.

There are thus two different types of activity involved in the warehouse processes: information processing and physical handling of goods, the link between the two activities being the controller, and it should be noted that all the controller's decisions are made in accordance with a given set of rules. We shall define as automation the exclusion of the human element, inclusive of the controller, from both activities.

9.3. We now consider whether the results described in the preceding sections, apply to automated warehouse processes. The short answer to this is yes, but with a different emphasis.

For example: it is entirely possible with our present technical resources to automate a warehouse, storing pallet units in racks, by substituting a computer for clerks and controller, and black boxes, linked to the computer, for the drivers of the handling equipment. In fact this is exactly the form of automation that has been realised in the warehouse of "The Kitchen of Sarah Lee". In such a set-up, handling effort and thence handling cost are still linked to distance over which an item has to be moved, and our model would apply. But pure handling-cost, i.e. costs dependent on the distance, are only part of the total handling cost. There are other costs, which are time dependent, such as amortisation rates of building and capital equipment, and in the case of automation the initial capital outlay will be high.

If we write total handling cost per time unit

$$H(T) = A + B,$$

where A represents amortisation of capital and B is given by

$$B = \frac{M}{2}(aX + gY + dZ)$$

minimisation of B in accordance with the procedures described in the preceding sections, will have increasingly smaller effect on H(T) as A increases. Now that part of A, that is due to automated handling equipment will increase according to the maximum distance over which any unit may have to be moved; in other words automated handling equipment must be able to operate over the entire extent of the warehouse, and its cost increases with increasing warehouse capacity. Once such equipment is installed the actual operating cost, which is a function of distance, will constitute a relatively small part of total cost per unit time.

It is thus more important to minimise the maximum required warehouse capacity than to minimise handling effort. The procedures of Sections 4 and 5, therefore, do not entirely apply, as clearly the minimum of required capacity is achieved when all items are stored together as shown in subsection 4.14 above.

Using conveyors and automatically operated gates on the storage cells, instead of driverless handling equipment, enforces the above argument, and such equipment is more likely to be used in automation.

9.4. It should be noted, that in minimising required capacity, variety enters in an indirect way. In itself variety does not matter in store automation, as the information processing equipment - in effect, a computer - can be presumed to recognise varieties by relevant item codes, for a very wide range of items, and all that is required is the knowledge that a given storage cell is either empty or contains one unit of a known variety, and the handling equipment is indifferent to the contents of the box or pallet it moves.

What matters is that each variety has a different stock variance. Stock variance is often in inverse proportion to movement rate, usually due to longer replenishment cycles for the items in low demand. This will result in relatively larger capacity requirements for slow movers.

Variety, therefore, forces a choice between the following alternatives:

i) exclude the slow moving commodities from automation

ii) attempt to decrease stock variances of these items by altering their replenishment system, or

iii) a mixture of i) and ii) above.

Once such a decision has been made the range of varieties does not enter the minimisation procedure.

9.5. The problems and costs of information processing for automation are in our experience often overestimated. Most organisations, controlling warehouses with sufficient throughput to warrant a feasibility study on automation, already have a computer for stock accounting and control; the additional routines required to control movement into and out of the warehouse represent only a small addition to the total of programmes, and furthermore all the variable data required for this purpose are identical with those required for the other stock routines. Indeed, one cannot think about warehouse automation except within the framework of a stock control system. What would, however, be required is a change in timing of supplying data to the various computer routines. For example, goods receipts into the warehouse are now usually given to the computer after the relevant goods have been stored, whilst in an automated warehouse they would have to be notified before the goods enter the warehouse. But the same information is necessary both for stock control and control of movement.

Because of this timing requirement, there may exist a problem of data capture. This, however, falls outside the remit of this research project.

A further problem area in the information processing activity, is the transmission of orders from the controlling machine, i.e. the computer, to the handling equipment. Again this is outside our project, and all that needs to be said here, is that a variety of electronic equipment for this purpose is already in existence. The same cannot be said about handling machinery and it appears that the real difficulties of warehouse automation are the design and the cost of automated handling equipment.

9.6. In the foregoing subsections we have again concentrated on a warehouse in which input and output is in the same units. From what has been said in Section 6 on true sequential order picking, and the difficulties in designing automated handling machinery, it appears that sequential order picking cannot easily be automated. The choice thus lies between parallel picking and a live store, that is, the system we termed hybrid.

One particular form of live store seems eminently suitable for automation, namely the type often called "dispenser". This consists of a number of parallel storage conveyors, either powered or, more usually, gravity controlled, feeding on to a transverse conveyor. Each of the parallel conveyors holds a number of units of a given commodity, and an order is made up by sequentially releasing the requisite quantity of each commodity on to the transverse conveyor; thus, at the end, the transverse conveyor will hold a complete order.

The problems with this type of order picking are: Firstly, the number of varieties directly effects the cost of the system, as each variety requires a separate storage conveyor. Therefore the smaller the average number of lines per order, for a given variety range, the greater the cost per line.

Secondly, the average quantity per order of a line differs between the commodities, hence for a regular replenishment cycle of the dispenser, variable lengths of storage conveyor are required, which may cause layout problems.

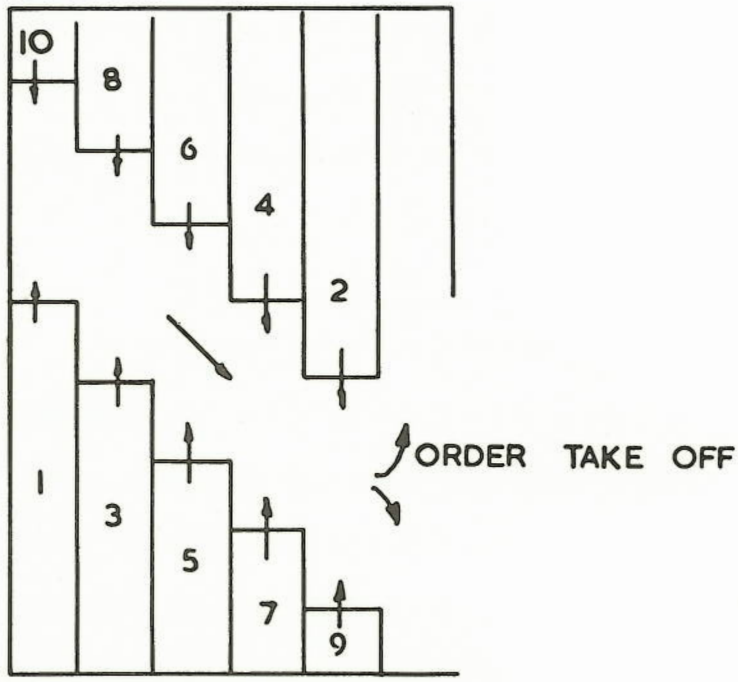
Nevertheless, the dispenser can easily be automated, by gating each storage conveyor. The automatic opening of a gate for a time interval proportional to the quantity required does not represent technical difficulties, neither does the feed back, to assure that the right quantity has been released. Furthermore, the gates of all lines requested by a single order can be opened simultaneously, so that the total time of picking an order depends mainly on the largest line quantity per order. The handling effort in such a system then depends first of all on the distribution of the quantity per order of the fastest moving line. Distance is of relatively little importance, particularly if, the layout of the dispenser is designed to minimise space taken up by it.

For example in Fig. 9.1 the numbers in the parallel storage cells identify commodities in order of their throughput; Thus 1 designates the fastest moving commodity. It is clear, that that part of the time taken to complete one order, which is ascribable to traversing the length of the transverse conveyor will tend to a constant value per order.

Capital cost will again be a function of capacity requirement, with variety having a direct influence on this cost.

FIG. 9.1

DISPENSER
LAY-OUT



COMMODITIES ARE NUMBERED
IN DESCENDING ORDER OF OUTPUT RATE

10 THE MATHEMATICS OF THE MODEL

10.1 The points $(x_i\ y_i\ z_i)$,

with $0 \leq x_i \leq a$

$0 \leq y_i \leq b$

$0 \leq z_i \leq c$

form a rectangular block of volume a.b.c. Distance is defined as in section 3, para. 4, that is distance between points p_i and p_j is given by

$$d(ij) = |x_i - x_j| + (y_i - y_j) + |z_i - z_j|$$

The average distance of the points from origin is then clearly:

$$\bar{d} = \frac{a + b + c}{2} \qquad (10.1)$$

The minimum $\bar{d}$, subject to a.b.c. = K is given by the condition $a = b = c = \sqrt[3]{K}$ as can be easily shown by setting the partial derivatives

$$\frac{\partial u}{\partial a}, \frac{\partial u}{\partial b}, \frac{\partial u}{\partial c}, \frac{\partial u}{\partial \lambda} = 0, \text{ where}$$

$$u = \tfrac{1}{2}(a + b + c) - \lambda(abc - K)$$

and solving for a, b, c, λ.

10.2 If we evaluate distance in a block of cubicle cells of side 1, in terms of number of cells along, across and upwards, and there are $K = a \times b \times c$ cells, the average cell distance from origin is given by:

$$\bar{d} = \left[bc \sum_{o}^{a-1} x_i + ac \sum_{o}^{b-1} y_i + ab \sum_{o}^{c-1} z_i \right] \Big/ abc$$

$$= \left[\frac{bc(a - 1)a}{2} + \frac{ac(b - 1)b}{2} + \frac{ab(c - 1)c}{2} \right] \Big/ abc \qquad (10.1a)$$

$$= \frac{a + b + c - 3}{2} \qquad (a,b,c,x_i,y_i,z_i,\text{integers}).$$

Again the minimum distances is given by the condition

$$a \simeq b \simeq c \simeq \sqrt[3]{K}$$

10.3 From (10.1) it follows that expression (4.1b) should correctly read

$$\bar{H} = \tfrac{1}{2} \left[am + gn + dp - (a + g + d \right] \qquad (10.2)$$

Since $\frac{a + g + d}{2}$ is a constant, it does not affect the search for minimal handling effort, hence (4.1b) is good enough for our purpose.

10.4 Derivation of (4.2) is given by:

Minimise $\frac{1}{2} \left[ax + gy + dz \right]$ subject to

$$xyz = K$$

Hence $F = ax + gy + dz - \lambda\ (xyz - K)$ (1 0.3)

$$\partial F/\partial x = a - \lambda yz = 0$$

$$\partial F/\partial y = g - \lambda xz = 0$$

$$\partial F/\partial z = d - \lambda xy = 0$$

$$\partial F/\partial \lambda = -xyz + K = 0 \qquad (10.4)$$

The equations (10.4) give

$$x = K^{1/3}\, g^{1/3}\, d^{1/3} \;/\; a^{2/3} = K^{1/3}(agd)^{1/3}/a$$

$$y = K^{1/3}\, a^{1/3}\, d^{1/3} \;/\; g^{2/3} = K^{1/3}(agd)^{1/3}/g$$

$$z = K^{1/3}\, a^{1/3}\, g^{1/3} \;/\; d^{2/3} = K^{1/3}(agd)^{1/3}/d \qquad (10.5)$$

Applying the usual tests we find that the values (10.5) are minimal.

10.5 For the derivation of the expressions in Section 7, we have:

Given a cube of K cubical cells, and side k, we wish to derive the average distance of the (K-Y) cells farthest away from origin, where the Y cells form a cube, of side y, based on the origin. The volume of the cube K, is thus divided into four parts: The cube Y, and three rectangular blocks of sides:

k x k x (k-y), y x y x (k-y), and k x y x (k-y), respectively.

10.6 We can write the average distance of the cells not in the cube Y, $\bar{d}_y$ as

$$\begin{aligned}\bar{d}_y = \tfrac{1}{2}\Big\{ & \frac{k^2\,(k-y)}{k^3 - y^3}\left[k + k + (k-y) - 3\right] + \\ + & \frac{y^2(k-y)}{k^3 - y^3}\left[y + y + (k-y) - 3\right] + \\ + & \frac{ky(k-y)}{k^3 - y^3}\left[k + y + (k-y) - 3\right]\Big\} + y \end{aligned} \qquad (10.6)$$

(10.6) is the properly proportioned average distance of cells in each of the rectangular blocks to the corner in each block nearest to the origin, plus the distance from these corners to the origin. It simplifies to:

$$\bar{d}_y = \tfrac{1}{2}\left\{\frac{k - y}{k^3 - y^3}\, k^2(k-y) + y^2(k-y) + ky\,(k-y) + \right.$$

$$+ \frac{1}{k^3 - y^3}\, 2k^3(k-y) + 2y^3(k-y) + ky\,(k+y)(k-y) -$$

$$\left. - 3 + 2y\right\} =$$

$$= \left\{\tfrac{1}{2}\; k+y - 3 + \frac{2k^3 + 2y^3 + k^2y + ky^2}{k^2 + ky + y^2}\right\} = \qquad (10.6a)$$

$$= \tfrac{1}{2}\left\{2k + y - 3 + \frac{k^3 + 2y^3}{k^2 + ky + y^2}\right\}$$

$\bar{d}_y \geq \bar{d}(y)$ for all $0 \leq y < k$ since $2k + y - 3 + \frac{k^3 + 2y^3}{k^2 + ky + y^2} > 3k-3$,

as $y^3 > 0$. The condition $y = k$ is meaningless and the condition $y = 0$ gives the formula for $\bar{d}$ from (10.1a).

10.7 Set $y/k = \tau$ The average distance $\bar{d}_y$ now becomes $\bar{d}_\tau$

$$\bar{d}_\tau = \tfrac{1}{2}\left\{2k + \tau k - 3 + \frac{k^3 + 2\tau^3 k^3}{k^2 + \tau k^2 + \tau^2 k^2}\right\} =$$

$$= \tfrac{1}{2}\left\{(2 + \tau)\, k - 3 + \frac{1 + 2\tau^3}{1 + \tau + \tau^2}\, k\right\} = \qquad (10.7)$$

$$= \frac{3k - 3}{2} \quad \frac{3}{2} \quad \frac{k\tau^3}{1 + \tau + \tau^2}$$

10.8 In what follows we shall designate by $\bar{d}\,(X)$ the average distance of all cells in a cube of volume X, and by $\bar{d}_\tau(X)$

the average distance of the $(X - \tau^3 X)$ cells farthest away from origin. Considering first the statements in section 7.8 and 7.9 above, a glance at the Figs. 7.2a, 7.2b makes it obvious that the average distance under FIFO is, for $0 < \mu \leq \frac{2}{3}$.

$$\bar{d}_{FIFO} = \tfrac{1}{2}\left\{\bar{d}\ (N) + \eta\bar{d}\ (N') + (1 - \eta)\ \bar{d}_{\tau}(\bar{K}^*)\right\} \tag{10.8}$$

$$\text{where } N = (2-2\mu)\ \bar{K}$$

$$N' = (2-3\mu)\ \bar{K}$$

$$\bar{K}^* = (2-\mu)\ \bar{K}$$

$$\tau = \frac{\sqrt[3]{2-2\mu}}{2-\mu}$$

$$\eta = \frac{2-3\mu}{2-2\mu}$$

Substituting the explicit expressions in (10.8) we have

$$d_{FIFO} = \tfrac{3}{4}\left\{\left[(2-2\mu)^{1/3}\bar{k}-1\right] + \frac{2-3\mu}{2-2\mu}\left[(2-3\mu)^{1/3}\ \bar{k} - 1\right] + \right. \tag{10.8a}$$

$$\left. + \frac{\mu}{2-2\mu}\left[(2-\mu)^{1/3}\ \bar{k} - 1\right]\right\} + \frac{\mu}{2-2\mu}\ \bar{k}\ \frac{(2-2\mu)/(2-\mu)}{1 + \left(\frac{2-2\mu}{2-\mu}\right)^{1/3} + \left(\frac{2-2\mu}{2-\mu}\right)^{2/3}}$$

10.9 It was stated in Section 7 that (10.8) can be approximated by, say, $\tilde{d}_{FIFO}$, as

$$\tilde{d}_{FIFO} = \frac{3}{2}\left[(2-\mu)^{1/3}\ \bar{k} - 1\right] \tag{10.9}$$

Let R be a ratio

$$R = \frac{d_{FIFO} + 3/2}{\tilde{d}_{FIFO} + 3/2}$$

From (10.8a) and (10.9) we have

$$R = \frac{\frac{3}{4}\left\{\left[(2-2\mu)^{1/3} + \frac{2-3\mu}{2-2\mu}(2-3\mu)^{1/3} + \frac{\mu}{2-2\mu}(2-\mu)^{1/3}(1+B)\right]\right\}}{\frac{3}{2}(2-\mu)^{1/3}} =$$

$$= \tfrac{1}{2}\left[\tau + \eta\left(\frac{2-3\mu}{2-\mu}\right)^{1/3} + (1-\eta)\quad(1+B)\right] \qquad (10.10)$$

where $B = \dfrac{\tau^3}{1+\tau+\tau^2}$

From (10.10) we have

$$2R = \tau + \eta a + (1-\eta)\quad(1+B) = 1 + \tau + B + \eta(a-1-\tau) =$$

$$= \tau + 1 + \tau^3/(1+\tau+\tau^2) + \eta\left[a-1-\tau^3/(1+\tau+\tau^2)\right] =$$

$$= \frac{1}{1+\tau+\tau^2}\left[1 + \eta a - \eta + (2+\eta a-\eta)\ (\tau+\tau^2) + (2-\eta)\tau^3\right]$$

Now $2 - \eta = = -\dfrac{2-3\mu}{2-2\mu} = \dfrac{2-\mu}{2-2\mu} = 1/\tau^3$

$$\therefore 2R = \frac{2 + \eta a - \eta(1+\tau+\tau^2)}{1+\tau+\tau^2} = 1/\tau^3 + a =$$

$$= \frac{(2-\mu)^{4/3} + (2-3\mu)^{4/3}}{(2-2\mu)\quad(2-\mu)^{1/3}} \qquad (10.10a)$$

Obviously, if $\mu = 0$, $R = 1$

If $\mu = \frac{2}{3}$, $R = 1$, by substitution in (10.10) or (10.10a).

The maximum deviation from $R = 1$ occurs at $\mu = \frac{1}{2}$, where it is of the order of 0.94. The maximum overestimate in using $\tilde{d}_{FIFO}$ instead of d_{FIFO} is therefore of the order of 7%.

10.10 When $\frac{2}{3} < \mu < 1$, we first consider the case for which

$$\frac{2-\mu}{2-2\mu} = n \quad \text{an integer} > 2$$

This implies that the average volume of movement divides the average maximum stock. Hence $2 - \mu = n\,(2-2\mu) = 2n-2n\mu$

$$(2n-1)\mu = 2n-2$$

$$\mu = \frac{2n-2}{2n-1}$$

In this case we have from Fig.7.2c

$$d_{FIFO} = \frac{1}{n} \sum_{i=1}^{n} d_{\tau_i} (iN) \tag{10.11}$$

$$\text{with } \tau_i = \frac{\sqrt[3]{i-1}}{i}, \quad \tau_1 = 0 \qquad \therefore d_{\tau_i}(N) = d(N)$$

$$N = (2-2\mu)\,\bar{K} = \frac{2\bar{K}}{2n-1}$$

Substituting explicit expressions into (10.11) we have

$$d_{FIFO} = \frac{1}{n} \sum_{i=1}^{n} \frac{3}{2} \bar{K} \left(\frac{2i}{2n-1}\right)^{1/3} (1 + B_i) - \frac{3}{2} \qquad =$$

$$= \frac{3\sqrt[3]{2}\,\bar{k}}{2n\sqrt[3]{2n-1}} \sum_{i=1}^{n} (1 + B_i)\sqrt[3]{i} - \frac{3}{2} \tag{10.11a}$$

$$\text{with } B_i = \frac{\tau_i^{\,3}}{1 + \tau_i + \tau_i^{\,2}}$$

Now $2 - \mu = 2n/(2n-1)$, hence

$$d_{FIFO} = \frac{3}{2}\left[\bar{k}\left(\frac{2n}{2n-1}\right)^{1/3} - 1\right] \tag{10.12}$$

The ratio R now becomes

$$R = \frac{3\ \sqrt[3]{2}\ \bar{k}\ \left[(1 + B_i)\ \sqrt[3]{i}\ \right]/2n\sqrt[3]{2n-1}}{3\ \sqrt[3]{2}\ \bar{k}\ \sqrt[3]{n}\ \ /2\ \sqrt[3]{2n-1}} =$$

$$= \frac{1}{n\sqrt[3]{n}} \sum_{i=1}^{n} (1 + B_i)\sqrt[3]{i} \qquad (10.13)$$

10.11 We now show that

$$\sum_{i=1}^{n} (1 + B_i)\ \sqrt[3]{i} = n\sqrt[3]{n} \qquad (10.14)$$

$$B_i = \frac{(i-1)/i}{1 + \left(\frac{i-1}{i}\right)^{1/3} + \left(\frac{i-1}{i}\right)^{2/3}}$$

Let $a_i = \sqrt[3]{i}$, $b_i = \sqrt[3]{i-1}$

The expression under the Summation sign in (10.14) now becomes

$$\left(\frac{1 + b_i^3/a_i^3}{1 + b_i/a_i + b_i^2/a_i^2}\right) a_i \qquad (10.15)$$

$$= \frac{a_i^3 + b_i a_i^2 + b_i^2 a_i + b_i^3}{a_i^2 + b_i a_i + b_i^2}$$

Multiplying both Numerator and Denominator of (10.15) by

(a-b) we have

$$\frac{a_i^4 - b_i^4}{a_i^3 - b_i^3} = \frac{i\ \sqrt[3]{i} - (i-1)\ \sqrt[3]{i-1}}{i - (i-1)} =$$

$$= i\ \sqrt[3]{i} - (i-1)\ \sqrt[3]{i-1} \qquad (10.15a)$$

(10.14) can thus be written

$$\sum_{i=1}^{n} \left[i\,^3\sqrt{i} - (i-1)\ ^3\sqrt{i} - 1 \right] = n\ ^3\sqrt{n} \qquad (10.14a)$$

Substituting the first n natural numbers in (10.14a) it is clear that (10.14a) holds for all positive integers n. Hence R = 1 for all cases for which $(2-\mu)/(2-2\mu) = n$

10.12 In the case where the ratio $(2-\mu)/(2-2\mu) = n + x/m$, x/m a proper fraction, the integer $n' = mn + x$ can be substituted for n in (10.11a) to estimate the average distance. Hence again the ratio R = 1, and $\tilde{d}_{FIFO} = d_{FIFO}$. It is obvious that in cases where the ratio $(2-\mu)/(2-2\mu)$ is neither n nor n' the ratio R will differ from 1. It would require a complicated analysis to estimate this difference, but it is clear that, considering the underlying process, the deviation of R from 1 can be only very small.

10.13. From para. 7.15 the average distance under LIFO is obtained as follows

$$\begin{aligned} d_{LIFO} &= \tfrac{1}{2}\left\{ d\left(\left[1-\mu\right]\bar{K} + d\ \frac{1}{2-\mu}\left(\left[2-\mu\right]\bar{K}\right)\right\}\right. \\ &= \tfrac{1}{2}\left\{\frac{3}{2}\left[\bar{k}\ (1-\mu)^{1/3} - 1\right] + \frac{3}{2}\left[\bar{k}\ (2-\mu)^{1/3} - 1\right] + \right. \\ &\quad \left. + \frac{3}{2}\,\bar{k}\ (2-\mu)^{1/3} B \right\} = \\ &= \tfrac{3}{4}\left\{\bar{K}\ \left[(1-\mu)^{1/3} + (2-\mu)^{1/3}\ (1 + B)\right] - 2\right\} \end{aligned} \qquad (10.16)$$

$$\text{with } B = \frac{1/(2-\mu)}{1+\left(\frac{1}{2-\mu}\right)^{1/3}+\left(\frac{1}{2-\mu}\right)^{2/3}} = 1/\ (2-\mu) + (2-\mu)^{2/3} + (2-\mu)^{1/3}$$

Substitution and re-arrangement in(10.16) leads to (7.2)

10.14 The cost of building (Section 8) is given by

$$Axy + 2B\ (x + y)z, \qquad xyz = K \tag{10.17}$$

It is obvious that for any value of z, (10.17) is a minimum if $x = y$. Minima for x,z can be obtained in the usual way and are

$$x = (2\theta K)^{1/3} \qquad \theta = B/A$$

$$z = (K/4\theta^2)^{1/3}$$

An overall cost under Method 2 of Section 4 is given by

$$Axy + 2B\ (x + y)z + C(ax + by + cz) \tag{10.18}$$

(10.18) is a minimum for arbitrary z if x and y are in the ratio

$$\frac{x}{y} = \frac{2Bz + Cb}{2Bz + Ca} \tag{10.19}$$

Obviously if $a = b$, $\frac{x}{y} = 1$ and (10.19) becomes

$$F = AKz^{-1} + 4BK^{\frac{1}{2}}z^{\frac{1}{2}} + 2CaK^{\frac{1}{2}}z^{-\frac{1}{2}} + Ccz \tag{10. 18a}$$

$\frac{dF}{dz}$ gives a polynomial in z of degree 6 and we cannot therefore solve it analytically. There is, however, no difficulty in evaluating (10.18a) for a range of values of z and thence establish a minimum. A similar procedure can be established when $a \neq b$, giving

$$F = AKz^{-1} + 2BK^{\frac{1}{2}}z^{\frac{1}{2}}(\omega + \frac{1}{\omega}) + CaK^{\frac{1}{2}}z^{\frac{1}{2}}\omega + CbK^{\frac{1}{2}}z^{-\frac{1}{2}}\frac{1}{\omega} + Ccz \tag{10.18b}$$

with $\omega = \sqrt{\frac{2Bz + Cb}{2Bz + Ca}}$

$$x = \omega\sqrt{\frac{K}{z}}$$

$$y = \frac{1}{\omega}\sqrt{\frac{K}{z}}$$

10.15 Assuming that commodities are randomly stored together as under Method 2 of Section 4, and in sequential order picking ℓ cells have to be visited, the average coordinates of the m^{th} cell ($m = 1$ to ℓ) to be visited can be derived from the distribution of the m^{th} smallest value in a sample of ℓ taken from a rectangular distribution of range 0 to A.

The general distribution of the m^{th} smallest value in a sample of ℓ is given by

$$f(x) = \frac{\ell!}{(m-1)!(\ell-m)!}\left\{\left[P(x)\right]^{m-1}\left[1-P(x)\right]^{\ell-m} p(x)\ dx\right\} \tag{10.20}$$

where $p(x)\ dx$ is the probability density function

and $P(x)$ the probability distribution of the sampled population

The average of $f(x)$, given $p(x)dx = \frac{1}{A}$, $0 \leq x \leq A$,

and $P(x) = x/A$, is:

$$M(x) = \int_{o}^{A} \frac{\ell!}{(m-1)!(\ell-m)!}\left\{\left(\frac{x}{A}\right)^{m-1}\left(1-\frac{x}{A}\right)^{\ell-m}\frac{x}{A}\right\} dx = \tag{10.21}$$

$$= \frac{\ell!}{(m-1)!(\ell-m)!}\int_{o}^{A}\left(\frac{x}{A}\right)^{m}\left(1-\frac{x}{A}\right)^{\ell-m} dx =$$

$$= \frac{\ell!A}{(m-1)!(\ell-m)!}\ B\left[(m+1),(\ell-m+1)\right]$$

B designating the Beta-Function.

Hence the average is $A\ \frac{m}{\ell+1}$, $m \leq \ell$

11. CONCLUDING REMARKS

11.1 We opened this report with a discussion of the character and use of mathematical models. It seems right to end it by considering what has been achieved by constructing the particular model of the preceding sections.

11.2 Movement in warehouses is complex. In order to abstract and discover the underlying logical relationships we had to simplify this complex picture. We did this by making two assumptions, namely, unitary storage locations, and proportionality of handling effort to distance. The first assumption restricts the model to a certain, albeit fairly large, class of warehouses; and by working out the model for this class, we also have been given valuable guide lines for constructing models for other types.

The second assumption is justified, though not proven, by considering the published characteristics of handling equipment and work study results.

11.3 We then created a further simplification by imposing a movement rule, which forces all movement into a predictable pattern. This rule of always using nearest available storage location seems eminently justified, on the grounds that the aim is to minimise distance over which goods have to be moved. It is true that in practice this rule is hardly anywhere obeyed, but we cannot see any reason, why it should not be implemented in warehouse operations. It can, therefore, form the basic conditions under which the model operates.

11.4 Once this condition and assumptions are given, the relationships between the variable quantities and hence minimisation could be dervied by fairly simple and well established mathematical procedures.

As a result we found the warehouse dimensions and lay-out, as between commodities, which will minimise handling effort.

The calculations required to apply the model for practical purposes are easy and based on data that are normally available to warehouse managers.

11.5 Perhaps it should be pointed out, that at first the model was constructed for a situation that cannot arise in practice, by excluding the time dimension. This was done in order to simplify the mathematics. We then introduced time into the model, and showed that on average, the logical relationships, as derived from the static situation, still hold.

11.6 The use of averages is justified, because we are concerned with long term warehouse operations and if the average is a minimum, then, despite short term fluctuation, the total over time will be a minimum as well.

11.7 In all the formulae, we constructed, maximum capacity requirements are an important element, and increasingly important as one moves to automated handling. We must, therefore, point out, that in any application of the model, it is of great importance to get a good estimate of maximal expected capacities.

E. NUMERICAL EXAMPLE

E.1. As mentioned in the Introduction, we attach a numerical example to demonstrate the use of the formulae. Obviously we cannot take our data from an actual case, as this would require too great an amount of calculation for the reader to follow easily. On the other hand the figures we use must be as close to reality as possible. We have, therefore, decided to restrict our illustration to a warehouse stocking five commodities only, with the following characteristics.

TABLE E.1.

Commodity No. i	Rate of Throughput/ week pallets M_i	Average Stock in hand pallets K_i	Standard deviation σ_i	Average weeks stock in hand
1	500	1000	100	2
2	300	600	133	2
3	100	300	100	3
4	60	240	120	4
5	40	200	100	5

The relationship exhibited by the figures in the above table E.1 is one that is often found in real warehouse situations. The stock profile with the 40% of variety accounting for 80% of throughput is perhaps optimistic, - more often, one finds that $\frac{1}{5}$ of variety account for $\frac{4}{5}$ of turnover - but has been forced on us by the restriction that more than 5 commodities could lead to too much calculating effort on the part of the reader.

Again, average stocks being relatively less for high throughput items than for slow moving ones, is a common phenomenon. Similarly, it is easier to forecast demand on fast moving items hence their relative variability round average stock, measured by the "coefficient of variation", that is the ratio of standard deviation to average, is less than that of slow movers.

We further stipulate for this example that stocks for all commodities are symmetrically distributed around their average, that the commodities do not exhibit seasonal fluctuations, and that the risk of running out of space is of the order of 0.1 of 1%.

Further, the unit is a standard pallet load 48" x 40", 48" high, and the weight per unit is nominally the same for all commodities.

The data of table E.1. are usually directly available in an organisation concerned with keeping stocks of various commodities. If not, they can be deduced from stock accounts.

E.2. From table E.1 one can derive the values for maximum space requirements.

TABLE E.2

Commodity No. i	Maximum Space Requirement in Pallet Loads $K = \bar{K}_i + 3\sigma$	Ratio average to maximum β_i
1	1300	.7692
2	1000	.6000
3	600	.5000
4	600	.4000
5	500	.4000
Total	4000	

We further assume that we have a choice between 3 possible types of handling equipment, say, fork lift trucks, with the following characteristics. All three move at an average speed of 3 m.p.h. require an isle width of 100", but the first, F_1, works at an average vertical speed of 26.4 ft/min; the second F_2, at 35.2 ft/min; and the third F_3, at 52.8 ft/min. Average vertical speed is taken as the average speeds of lifting full load, going up unloaded, lowering full load lowering empty.

The cell dimensions we take as 50 x 50 x 60 so that

$\gamma = .5, \partial = .6$

With $a = b = 1$, $c = 10$, 7.5, and 5 for F_1, F_2, F_3, respectively

Hence $a = 1$ $g = 0.5$ $d = 6$, 4.5 and 3, for F_1, F_2, F_3 respectively

and $\sqrt[3]{agd} = \sqrt[3]{G} = 1.4422$, 1.3104 and 1.1447.

E.3 We now proceed to test which items ought to be stored together, by formulae (5.3)

Starting with items 1 and 2 we have

$$\alpha_1 = K_1 (K_1 + K_2) = 0.5652$$

$$\alpha_2 = 1 - {}_1 = 0.4348$$

$$(\alpha_1 \beta_1)^{1/3} = 0.7576$$

$$(\alpha_2 \beta_2)^{1/3} = 0.6390$$

$$(\alpha_1 \beta_1 + \alpha_2 \beta_2)^{1/3} = \beta^{1/3} = 0.8860$$

$$(\alpha_1)^{1/3} = 0.8268$$

$$\Delta_{1,2} = \frac{3}{2} \left[0 \cdot 7576.500 + 0 \cdot 6390.300 - 0 \cdot 8860.800\right] + 0 \cdot 8268.300 =$$

$$= - 207 \cdot 45 + 248 \cdot 04 = 40 \cdot 59$$

$\Delta_{1,2}$ positive, hence Item 1 and Item 2 to be stored together.

Maximum storage requirement for items 1 and 2 together is $1600 + 3 \sqrt{100^2 + 133^2} \approx 2100$

E.4. Next, test whether items 1, 2 and 3 can be stored together

$\alpha'_1 = 2100/2700 = 0.7778$

$\alpha_3 = 600/2700 \quad 0.2222$

$\beta'_i = 1600/2100 \quad 0.7619$

$\beta_3 = \quad 0.5000$

$(\alpha'_1\beta'_1)^{1/3} = (0.5926)^{1/3} = 0.8400$

$(\alpha_3\beta_3)^{1/3} = (0.1111)^{1/3} = 0.4808$

$(\alpha'_1\beta'_1 + \alpha_3\beta_3)^{1/3} = (0.7037)^{1/3} = 0.8895$

$(\alpha'_1)^{1/3} = 0.9196$

$\Delta_{(1,2),3} = \frac{3}{2}\left[0.8400.800 + 0.4808.100 - 0.8895.900\right] + 0.9196.100 =$

$= - 126.71 + 91.96 = - 34.75,$

$\Delta_{(1,2),3}$ negative

E.5. Next, test whether items 3 and 4 should be stored together

$\alpha_3 = 0.5000$

$\alpha_4 = 0.5000$

$\beta_3 = 0.5000$

$\beta_4 = 0.4000$

$(\alpha_3\beta_3)^{1/3} = (0.2500)^{1/3} = 0.6210$

$(\alpha_4\beta_4)^{1/3} = (0.2000)^{1/3} = 0.5848$

$(\alpha_3\beta_3 + \alpha_4\beta_4)^{1/3} = (0.4500)^{1/3} = 0.7663$

$\alpha_3^{1/3} = 0.7937$

$\Delta_{3,4} = \frac{3}{2}\left[0{\cdot}6210.100 + 0{\cdot}5848.60 - 0{\cdot}7663.160\right] + 0{\cdot}7937.60 =$

$= - 38{\cdot}13 + 47{\cdot}62 = 9{\cdot}49$

$\Delta_{3,4}$ positive, hence items 3 and 4 to be stored together.

Maximum capacity requirement for items 3 and 4 together is

$$540 + 3\sqrt{100^2 + 120^2} \simeq 1005$$

Next, test whether items 3,4 and 5 should be stored together

$\alpha'_3 = 1005/1505 = 0.6678$

$\alpha_5 = 500/1505 = 0.3322$

$\beta'_3 = 540/1005 = 0.5373$

$\beta_5 = \qquad = 0.4000$

$(\alpha'_3\beta'_3)^{1/3} = (0.3588)^{1/3} = 0.7106$

$(\alpha_5\beta_5)^{1/3} = (0.1329)^{1/3} = 0.5104$

$(\alpha'_3\beta'_3 + \alpha_5\beta_5)^{1/3} = (0.4917)^{1/3} = 0.7893$

$(\alpha'_3)^{1/3} = 0.8741$

$$\Delta_{(3,4),5} = \frac{3}{2}\left[0{\cdot}7106.160 + 0{\cdot}5104.40 - 0{\cdot}7893.200\right] + 0{\cdot}8741.40 =$$

$$= -35{\cdot}62 + 34{\cdot}96 = -0{\cdot}658$$

Although $\Delta_{(3,4),5}$ is negative, it is only just so. It does indicate that it is strictly better to store item 5 separately.

Since, however, this is an example to demonstrate the use of the formulae, we shall say that $\Delta_{(3,4),5} = 0$ and store items 1 and 2 together and Items 3,4 and 5 together.

E.6. Turning now to the dimensions of the warehouse required using F_1 we have:

w_1 = the factor for storing Items 1 and 2 = 18.47

This gives a block of 18 x 39 x 3 cells

a total of 2106 cells, with an average unit handling cost of 27.75 as the nearest approach to the optimum.

If we then store items 3,4 and 5 together we have w_2 = 15.72, which gives a block of 15 x 29 x 3 cells, a total of 1305 cells at an unit average handling cost of 23.75 to the point of the block nearest to reference point. Note that 15.72 ÷ 6, that is the value that determines height equals 2.62, so before deciding on a height of 3 one must try out a height of 2. This gives a block of 18 x 36 x 2 = 1296 at an unit average handling cost of 24.00. As our criterion is handling cost, the 3 high block should be chosen, with a total average handling cost of

$$800.27{\cdot}25 + 200\ (23{\cdot}75 + 18) = 30550$$

On the other hand the block of dimension 18 x 36 x 2 when fitted on top of the block 18 x 39 x 3, would result in a total handling effort of 30600. The latter arrangement appears to be the best if the two storage blocks have to be fitted into a warehouse that is a rectangular box. The total number of cells required is 2106 + 1296 = 3402, and the shape 18 x 39 x 5 can hold 3510. Setting a general value of moving a unit distance equal to 1^d, $a = 1^d$, $g = \frac{1}{2}^d$, $d = 6^d$ then the total handling cost per time unit at an absolute minimum, i.e. two blocks 18 x 39 x 3 and 15 x 29 x 3 fitted together is £127.29, which would, if fitted into a rectangular box, give room for at least 261 superfluous cells. The total cost per time unit of the second arrangement, requiring a box of only 3510 cells is £127.50.

At this stage we must refer the reader back to subsections 3.5 ff. of the report, that the above calculation gives a value based on a reference point at one corner of the rectangular storage block. To estimate the true minimal value, the reference point should be suitable placed, as for example in Fig. 2e and the necessary adjustments made. We must, however, again emphasise that the conclusions as to layout that we derived by the above calculations are valid for all possible shifts of the reference point.

E.7. To the value of handling effort we must add a capital and other fixed charges per time unit for the equipment incl. personnel. Thus for F_1 we have a total handling cost of

$$TC(1) = C_1 + 127.50$$

Going through the same calculation for F_2 we have

$w_1 = 16.78$, $w_2 = 14.28$ which gives us to nearest integral numbers blocks of

$$\left.\begin{array}{l} 16 \times 33 \times 4 \text{ for Items 1 and 2} \\ 16 \times 27 \times 3 \text{ for items 3,4 and 5} \end{array}\right\} \begin{array}{l} \text{at a unit handling effort of: } 25.25 \\ \hspace{10em} 21.50 \end{array}$$

or

$$\left.\begin{array}{l} 15 \times 35 \times 4 \text{ for Items 1 and 2} \\ 15 \times 29 \times 3 \text{ for Items 3,4 and 5} \end{array}\right\} \begin{array}{l} \text{at a unit handling effort of: } 25.15 \\ \hspace{10em} 21.50 \end{array}$$

From the point of view of handling effort both arrangements are equivalent and the choice would then depend on the costs of shapes of warehouse that is to contain them.

The total handling effort is in either arrangement equal 27867, hence for F_2 we have at the same evaluation a handling cost of

$$TC(2) = C_2 + 116.11$$

Finally for F_3 we have

$w_1 = 14.66$ $w_2 = 12.48$ which gives cell blocks of

14 x 30 x 5 at a unit handling effort of 22.00 for items 1 and 2

13 x 25 x 4 " " " 18.75 for items 3,4 and 5

with a total handling effort of 24350, hence a total handling cost for F_3

$$TC(3) = C_3 + 101.46$$

The choice of store layout and equipment is then given by the smallest of the values TC(1), TC(2) and TC(3).